New Geometical Approaches to Amplitudes:

Stephen Parke
Fermilab

http://www.simonsfoundation.org/quanta/20130917-a-jewel-at-the-heart-of-quantum-physics/

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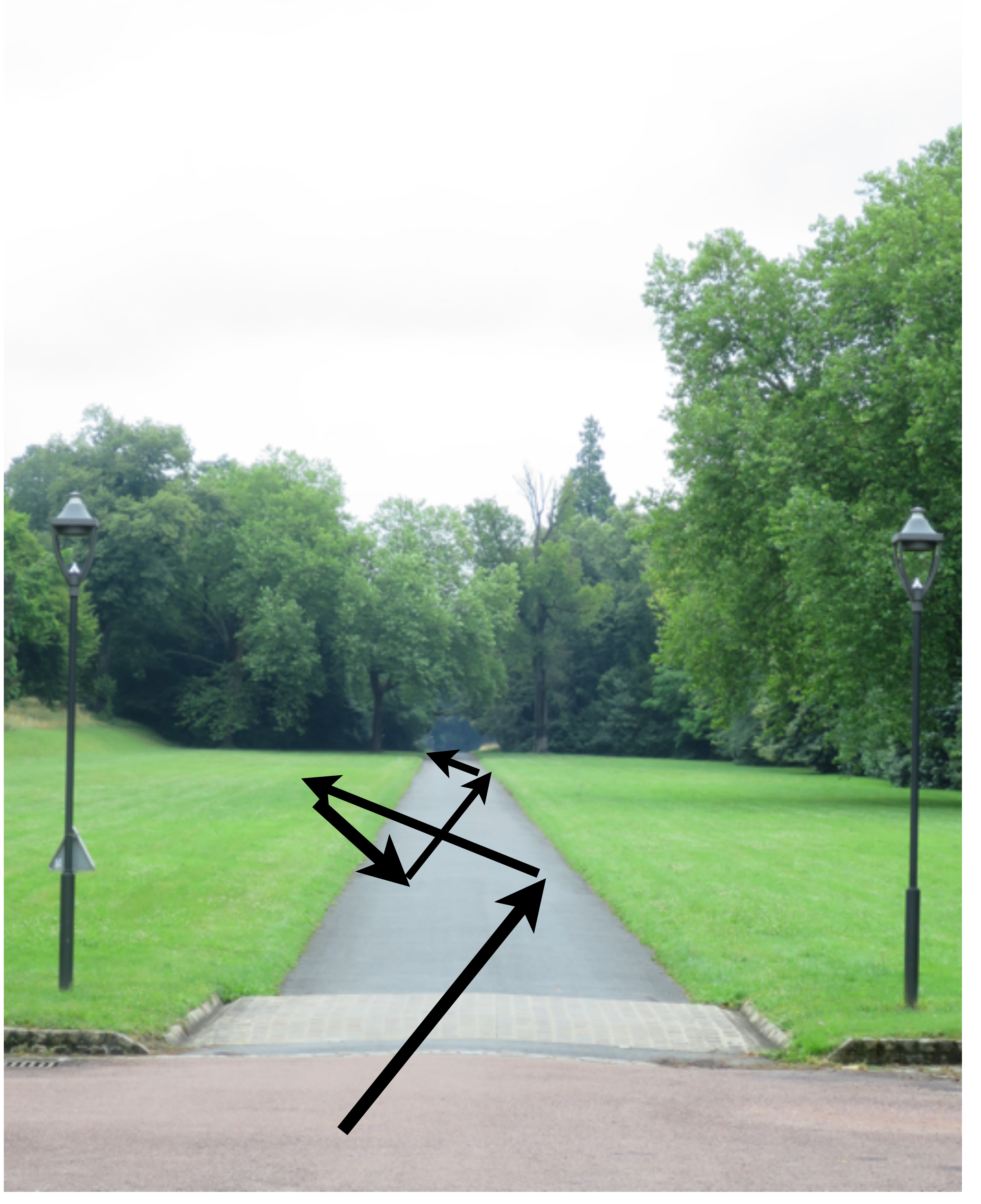

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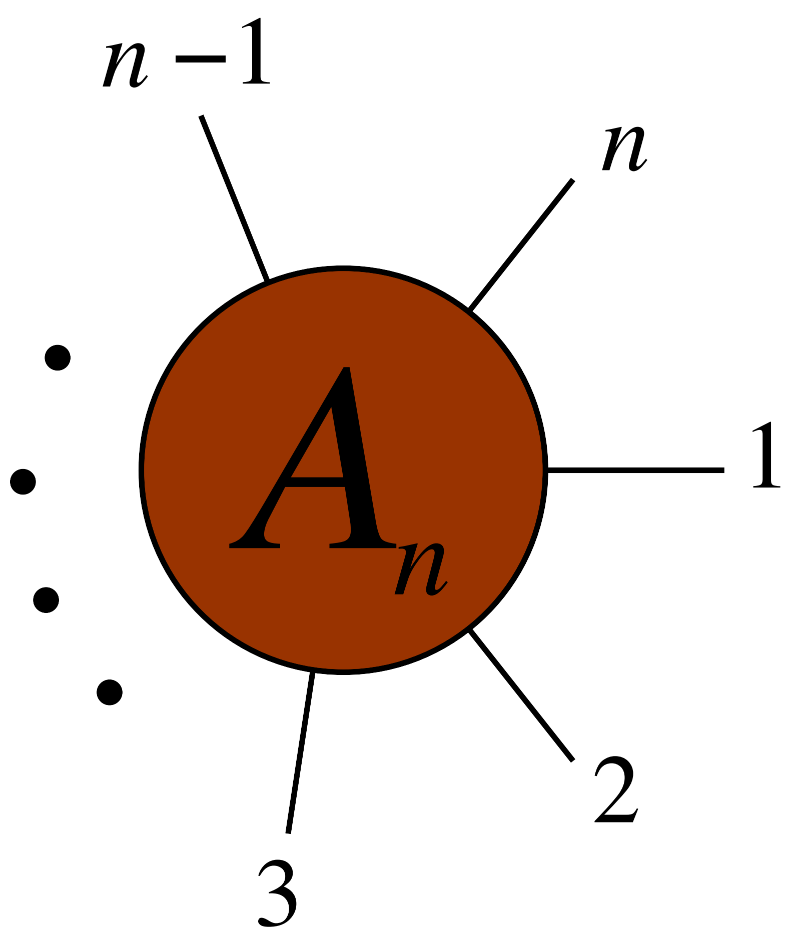
Amplitudes = on mass shell scattering amplitudes



massless external particles

all particles outgoing

Outline:

- Motivations
- Some Amplitudes
- Twistor String Theory
- Amplituhedron
- Summary

I. Movitations

Why are Amplitudes Interesting?

- Phenomenology: LHC processes
- Structure of the Theory
- Principals

Principals:

- Classical Mechanics:

Principals:

- Classical Mechanics:
- Quantum Mechanics:

Principals:

- Classical Mechanics:
- Quantum Mechanics:
- Quantum Field Theory:

Principals:

Principal of Least Action:

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- What is the correct way to incorporate Gravity ?

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????

???

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Space-Time, Quantum Mechanics and Scattering Amplitudes

Nima Arkani-Hamed

Scattering amplitudes in gauge theories and gravity have extraordinary properties that are completely invisible in the textbook formulation of quantum field theory using Feynman diagrams. In the standard approach--going back to the birth of quantum field theory--space-time locality and quantum-mechanical unitarity are made manifest at the cost of introducing huge gauge redundancies in our description of physics. As a consequence, apart from the very simplest processes, Feynman diagram calculations are enormously complicated, while the final results turn out to be amazingly simple, exhibiting hidden infinite-dimensional symmetries. This strongly suggests the existence of a new formulation of quantum field theory where locality and unitarity are derived concepts, while other physical principles are made more manifest. Rapid advances have been made towards uncovering this new picture, especially for the maximally supersymmetric gauge theory in four dimensions. These developments have interwoven and exposed connections between a remarkable collection of ideas from string theory, twistor theory and integrable systems, as well as a number of new mathematical structures in algebraic geometry. In this talk I will review the current state of this subject and and describe a number of ongoing directions of research.

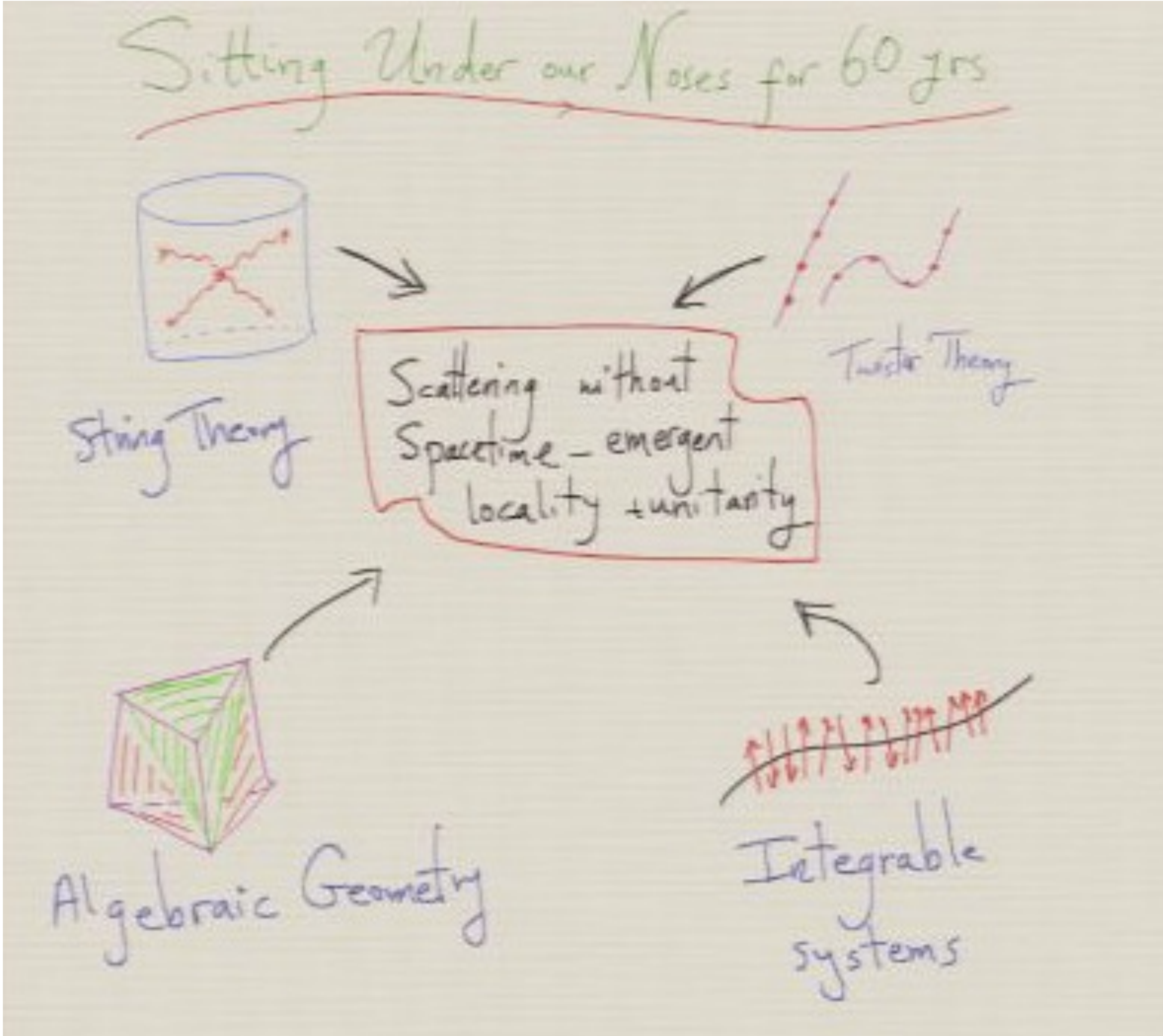
http://streamer.perimeterinstitute.ca/Flash/418223e8-4fae-4ebc-8de4-458f84364e3c/viewer.html

Quantum Mechanics
and
Gravity

“Space-Time is Doomed”

Nina Arkani-Hamed

Space-Time has to emerge from “the” fundamental description !


Sitting Under our Noses for 60 yrs
String Theory
Twistor Theory
Scattering without Spacetime – emergent locality + unitarity
Algebraic Geometry
Integrable systems

Summary: We are after a theory for

$$M_{n,k}[\lambda_a, \tilde{\lambda}_a, \tilde{\eta}_a; \ell_i]$$

Without Unitary evolution through Spacetime

$$\left\{ \text{Emergent Space-time}, \text{Emergent QM} \right\}$$

II. Amplitudes in Feynman Perturbation Theory

Supercollider physics

Rev. Mod. Phys. **56**, 579 – Published 1 October 1984

E. Eichten, I. Hinchliffe, K. Lane, and C. Quigg

Multijet Phenomena:

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W+W- pairs in their nonleptonic decays. **The cross sections for the elementary two to four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future.** It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

2 gluons to 4 gluons:

for each gluon: momentum p_i, polarization vector ϵ_i and color charge a_i

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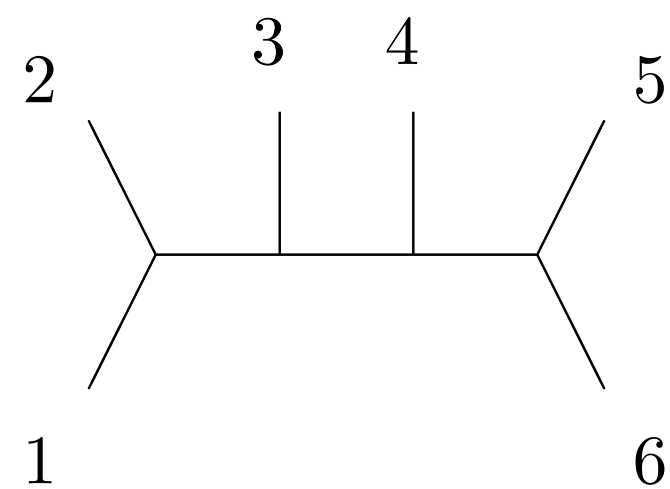

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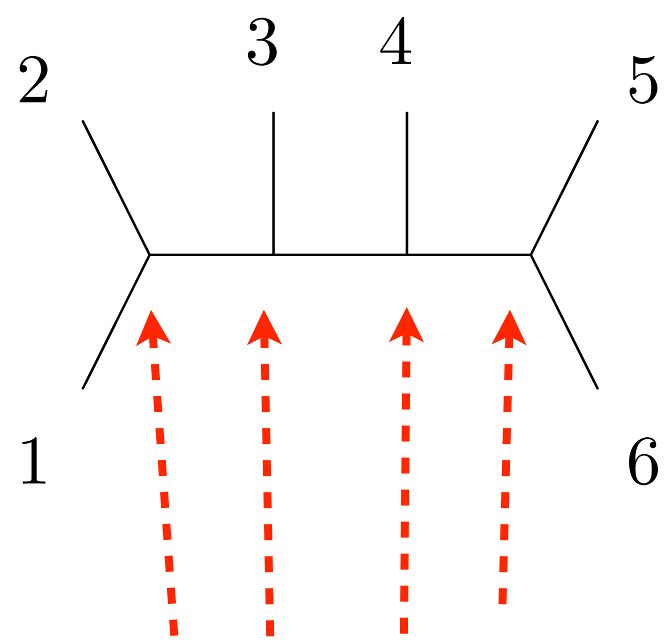

$$ig\ f_{a_1 a_2 a_3}\ [g_{\mu_1\mu_2}(p_1 - p_2)_{\mu_3} + g_{\mu_2\mu_3}(p_2 - p_3)_{\mu_1} + g_{\mu_3\mu_1}(p_3 - p_1)_{\mu_2}]$$

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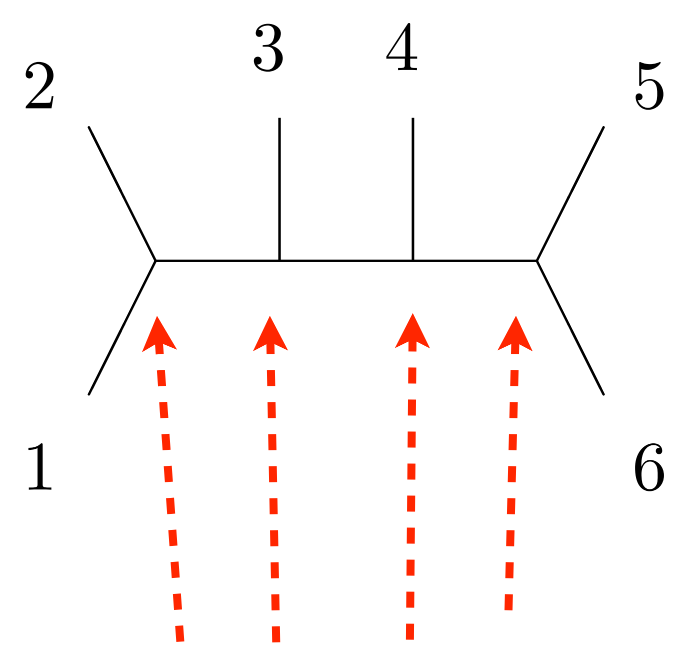

$$ig\ f_{a_1a_2a_3}\ [g_{\mu_1\mu_2}(p_1-p_2)_{\mu_3}+g_{\mu_2\mu_3}(p_2-p_3)_{\mu_1}+g_{\mu_3\mu_1}(p_3-p_1)_{\mu_2}]$$

$$f_{a_1a_2X}f_{Xa_3Y}f_{Ya_4Z}f_{Za_5a_6}\ \epsilon_1^{\mu_1}\epsilon_2^{\mu_2}\epsilon_3^{\mu_3}\epsilon_4^{\mu_4}\epsilon_5^{\mu_5}\epsilon_6^{\mu_6}$$

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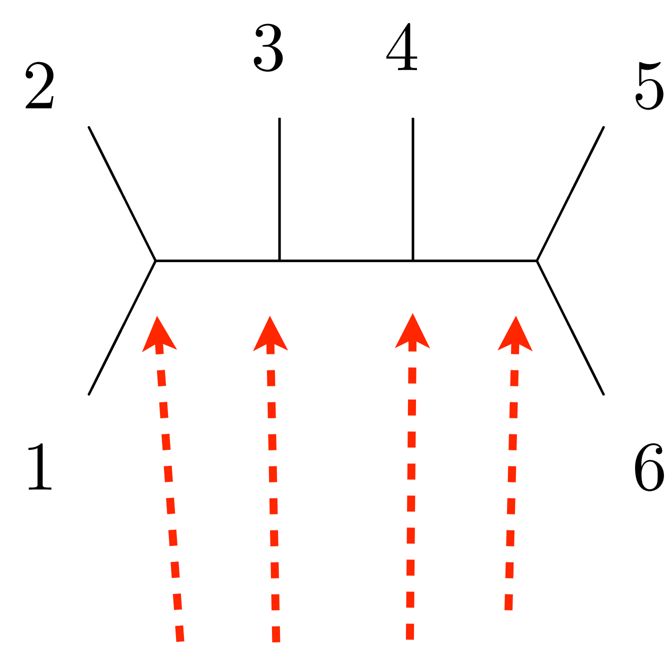

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$$f_{a_1 a_2 X} f_{X a_3 Y} f_{Y a_4 Z} f_{Z a_5 a_6}\ \epsilon_1^{\mu_1}\epsilon_2^{\mu_2}\epsilon_3^{\mu_3}\epsilon_4^{\mu_4}\epsilon_5^{\mu_5}\epsilon_6^{\mu_6} \quad \left\{\frac{\mathrm{4th\ order\ polyn\ of\ p's}}{\mathrm{s_{12}\ s_{123}\ s_{56}}}\right\}$$

where $s_{ij\ldots n} \equiv (p_i + p_j + \cdots p_n)^2$

the Amplitude:

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$\sum_{colors} \sum_{polarizations} |\mathcal{M}|^2$ 10^{11} terms ! ! !

before you start using identities like:

$f_{a_1XY} f_{a_2XY} = N\delta a_1 a_2$ & $\sum_{helicities} \epsilon_i^\mu \epsilon_i^{\nu *} = -g^{\mu\nu} + \frac{p_i^\mu q^\nu + p_i^\nu q^\mu}{(p_i \cdot q)}$

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- but the answer you say must permutation symmetric — true ! ! !
- you still have 10^8 terms !
- but you have to identify which of the 6! of each term leads to simplifying the result.

a use for SUSY:

SP and T. Taylor

- Embedd Yang-Mills or QCD in a SuperSymmetric Theory, N=1,2 or 4

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- relationships between pure gluon amplitudes and amplitudes were some of gluons are replaced with gluinos

$$|\mathcal{M}(g_{1+}, g_{2+}; \lambda_{3-}, g_{4+}, \lambda_{5+})| = \sqrt{\frac{(3\cdot 4)}{(4\cdot 5)}}\, |\mathcal{M}(g_{1+}, g_{2+}; g_{3-}, g_{4+}, g_{5+})|$$

$$|\mathcal{M}(g_{1-}, g_{2-}, g_{3-}, g_{4-}, g_{5+}, g_{6+})| = \frac{s_{56}}{s_{23}}\, |\mathcal{M}(g_{1-}, \phi_{2-}, \phi_{3-}, g_{4-}, \phi_{5+}, \phi_{6+})|$$

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- compact code for calculating 2 gluons to 4 gluons: wasn't a neat algebraic expression !

PT Amplitudes:

- To leading order in N_c

$$\sum_{colors} |\mathcal{M}(+++\cdots+)|^2 \quad \sim 0$$

$$\sum_{colors} |\mathcal{M}(-++\cdots+)|^2 \quad \sim 0$$

$$\sum_{colors} |\mathcal{M}(--+\cdots+)|^2 \quad \sim \quad (1\cdot 2)^4 \sum_{perms} \frac{1}{(1\cdot 2)(2\cdot 3)\cdots(n\cdot 1)}$$

using $(i\cdot j) \equiv 2p_i\cdot p_j = s_{ij}$

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$\Rightarrow$ tremendous number of cancellations ! ! !

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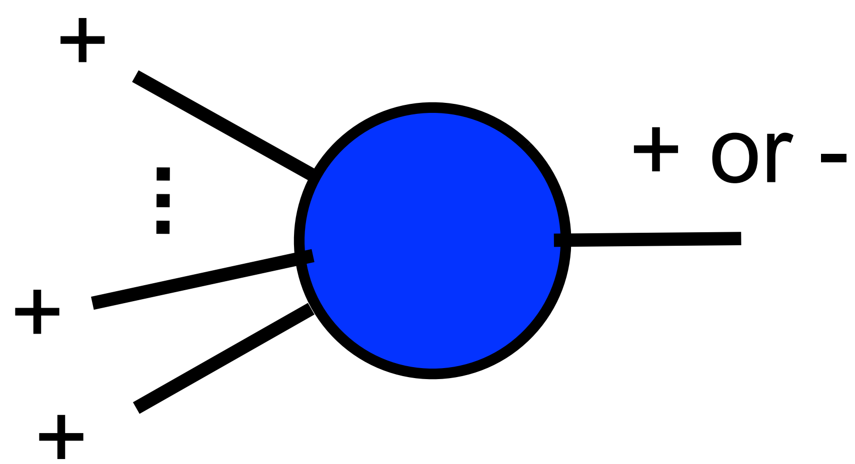

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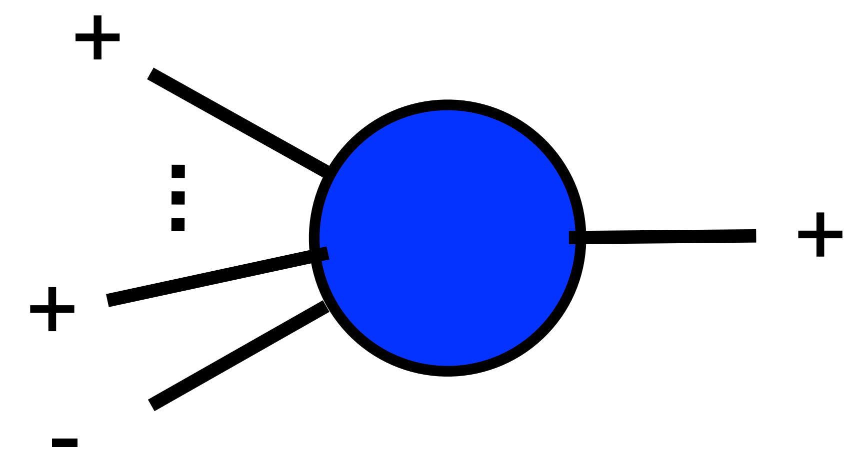

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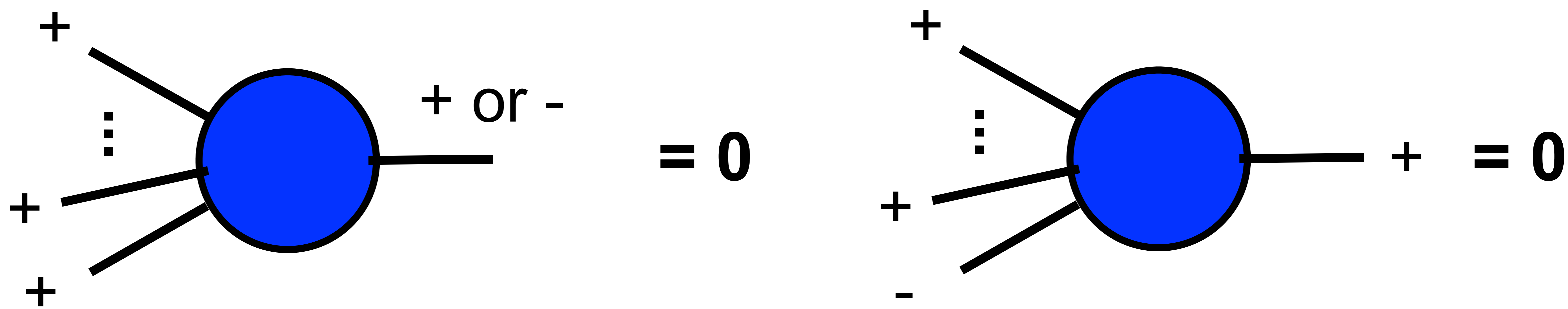
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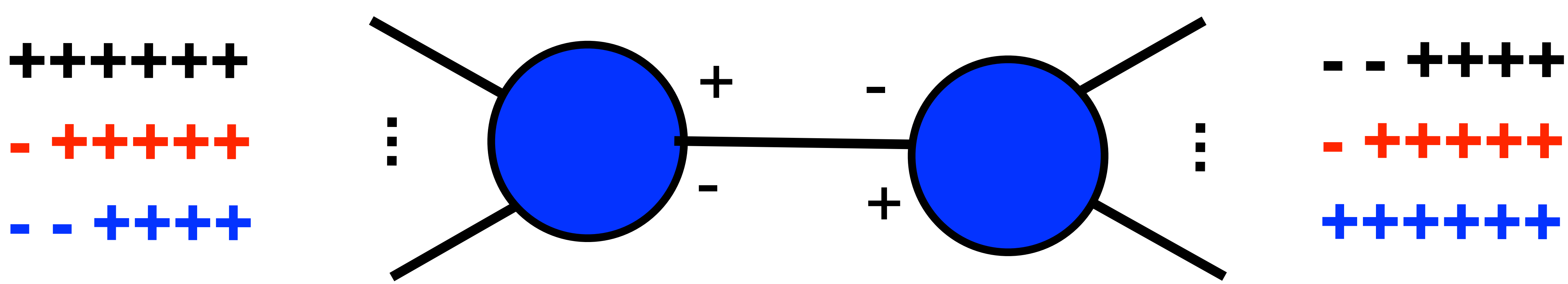
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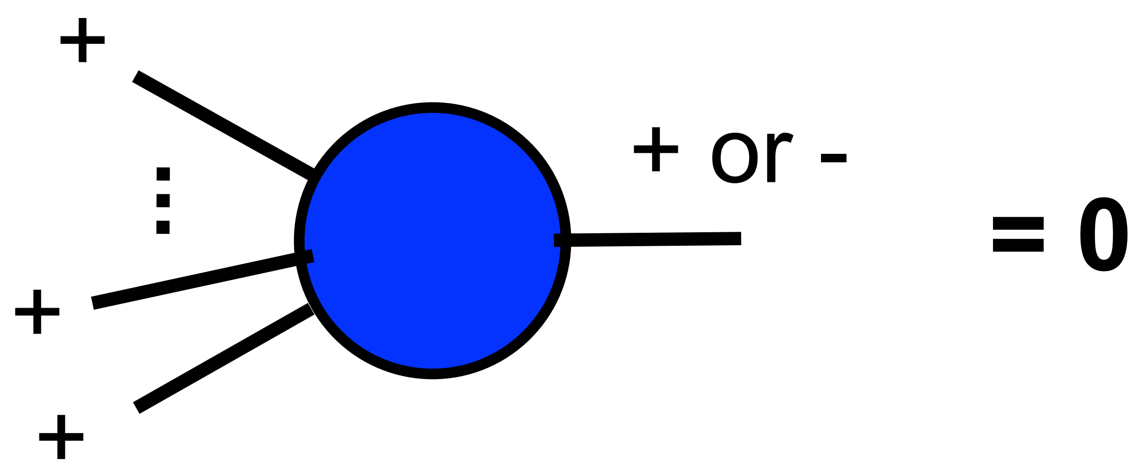
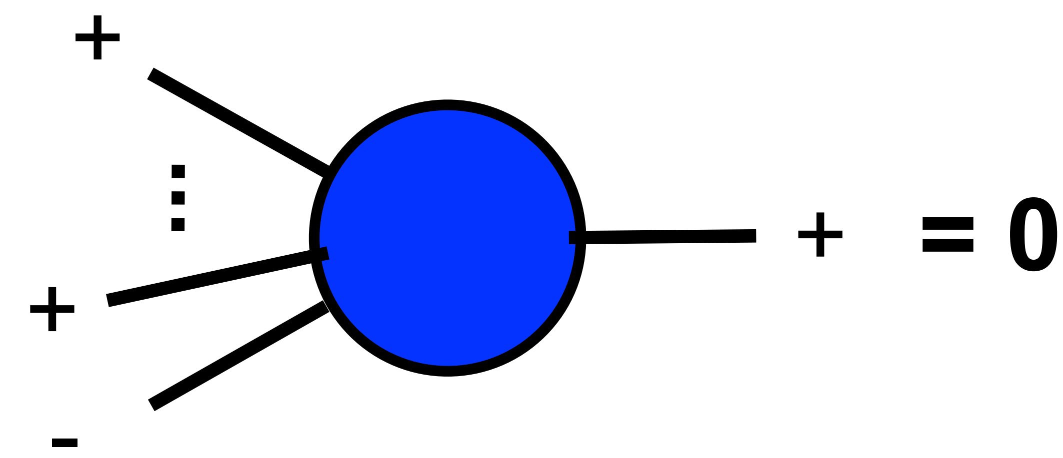
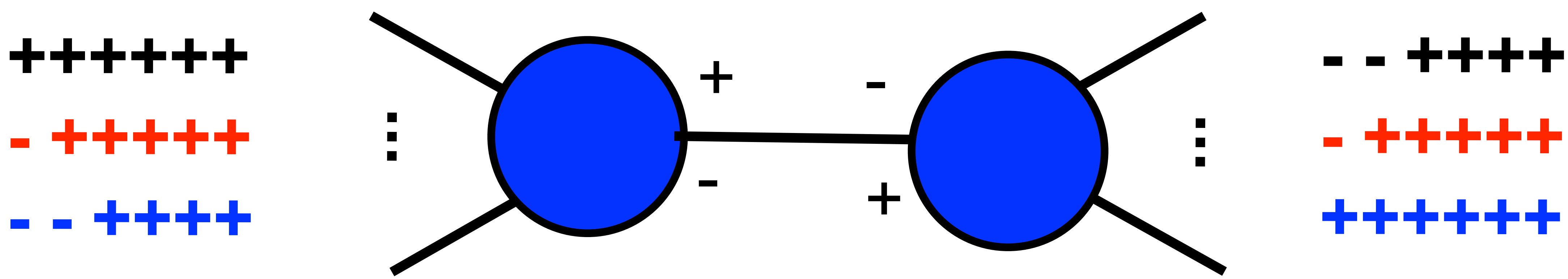






No poles with more than 2 particles ! ! ! i.e. s_{123}

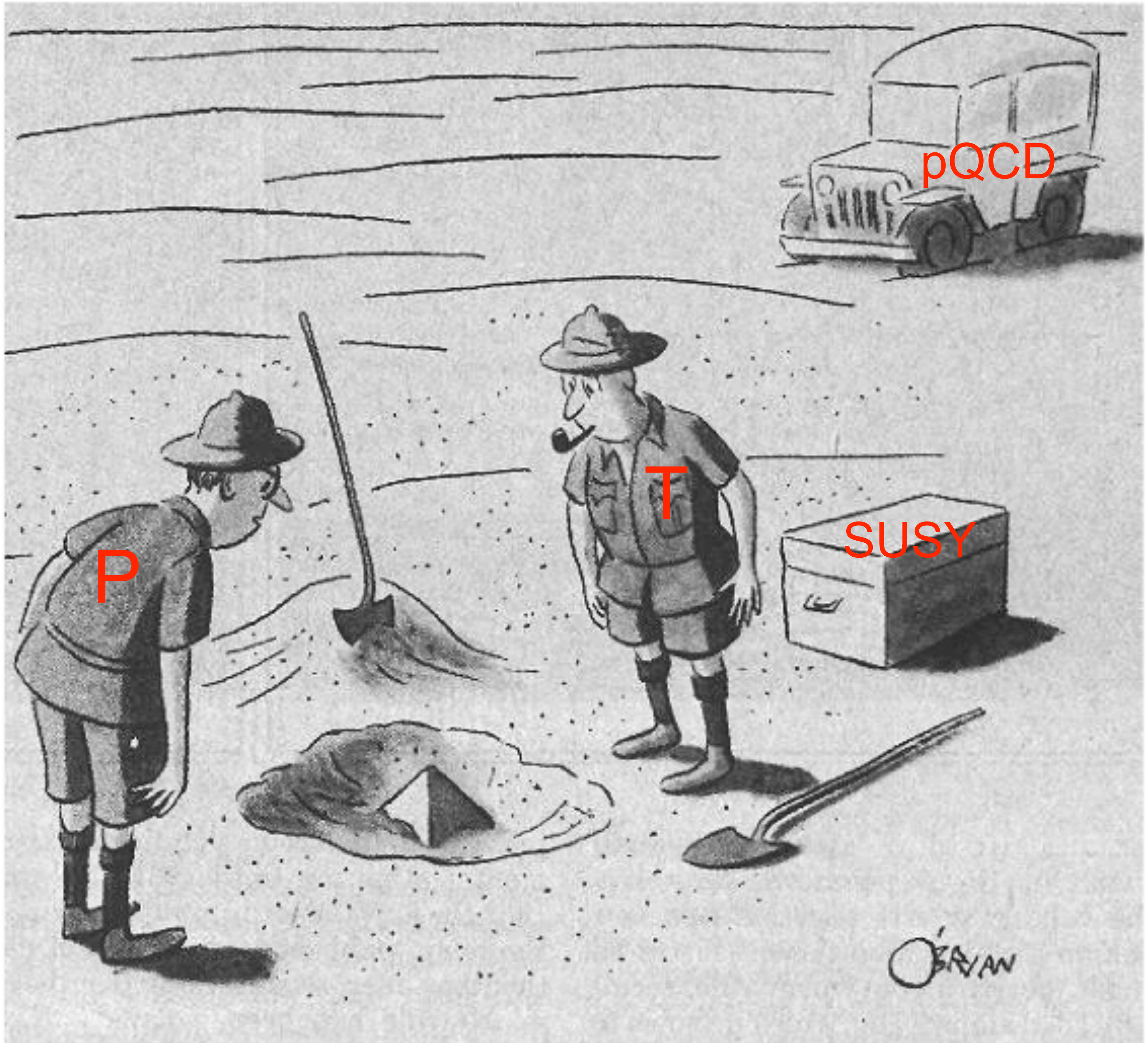
Altarelli & Parisi

$$|\mathcal{M}_n(--+++\cdots)|^2 \xrightarrow[1\|2]{} 0,$$

$$|\mathcal{M}_n(--+++\cdots)|^2 \xrightarrow[2\|3]{} 2g^2N\frac{z^4}{z(1-z)}\frac{1}{s}|\mathcal{M}_{n-1}(--++\cdots)|^2,$$

$$|\mathcal{M}_n(--+++\cdots)|^2 \xrightarrow[3\|4]{} 2g^2N\frac{1}{z(1-z)}\frac{1}{s}|\mathcal{M}_{n-1}(--++\cdots)|^2,$$

also soft gluon conditions also satisfied.


pQCD
T
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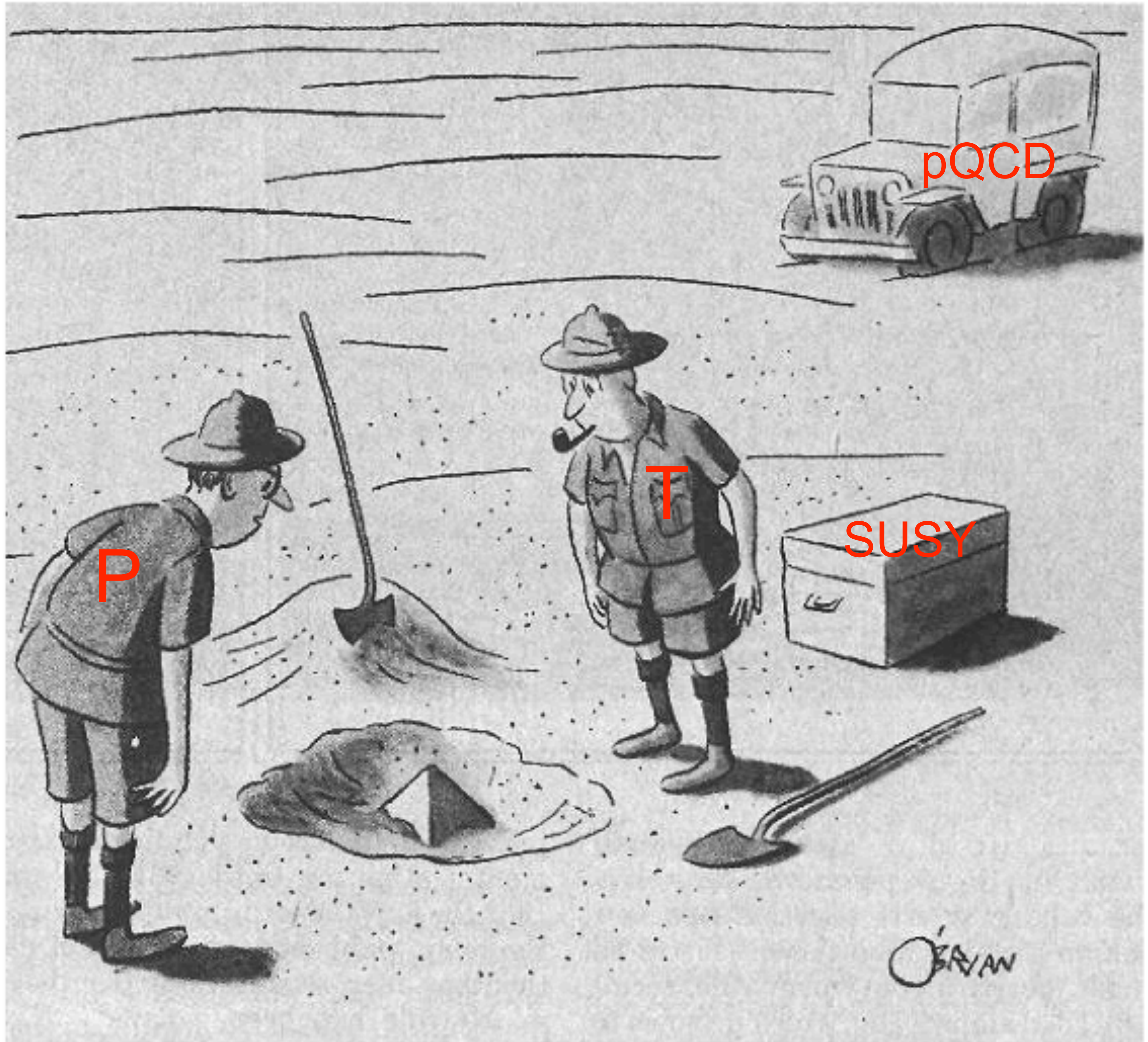
SUSY
O'BRYAN



“This could be a great discovery. Depending, of course, on how far down it goes.”

Amplitude for n-Gluon Scattering:

VOLUME 56, NUMBER 23 PHYSICAL REVIEW LETTERS 9 JUNE 1986

Amplitude for *n*-Gluon Scattering

Stephen J. Parke and T. R. Taylor
Fermi National Accelerator Laboratory, Batavia, Illinois 60510
(Received 17 March 1986)

A nontrivial squared helicity amplitude is given for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors.

PACS numbers: 12.38.Bx

Computations of the scattering amplitudes for the vector gauge bosons of non-Abelian gauge theories, besides being interesting from a purely quantum-field-theoretical point of view (determination of the S matrix), have a wide range of important applications. In particular, within the framework of quantum chromodynamics (QCD), the scattering of vector gauge bosons (gluons) gives rise to experimentally observable multijet production at high-energy hadron colliders. The knowledge of cross sections for the gluon scattering is crucial for any reliable phenomenology of jet physics, which holds great promise for testing QCD as well as for the discovery of new physics at present (CERN S$p\bar{p}$S and Fermilab Tevatron) and future (Superconducting Super Collider) hadron colliders.[1]

In this short Letter, we give a nontrivial squared helicity amplitude for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors. To our knowledge this is the first time in a non-Abelian gauge theory that a nontrivial, on-mass-shell, squared Green's function has been written down for an arbitrary number of external points. Our result can be used to improve the existing numerical programs for the QCD jet production, and in particular for the studies of the four-jet production for which no analytic results have been available so far. Before presenting the helicity amplitude, let us make it clear that this result is an educated guess which we have compared to the existing computations and verified by a series of highly nontrivial and nonlinear consistency checks.

For the n-gluon scattering amplitude, there are $(n+2)/2$ independent helicity amplitudes. At the tree level, the two helicity amplitudes which most violate the conservation of helicity are zero. This is easily seen by the embedding of the Yang-Mills theory in a supersymmetric theory.[2,3] Here we give an expression for the next helicity amplitude, also at tree level, to leading order in the number of colors in SU(N) Yang-Mills theory.

If the helicity amplitude for gluons $1, \ldots, n$, of momenta $p_1, \ldots, p_n$ and helicities $\lambda_1, \ldots, \lambda_n$, is $\mathcal{M}_n(\lambda_1, \ldots, \lambda_n)$, where the momenta and helicities are labeled as though all particles are outgoing, then the three helicity amplitudes of interest, squared and summed over color, are

$$|\mathcal{M}_n(+++++\cdots)|^2 = c_n(g,N)[0+O(g^4)], \tag{1}$$

$$|\mathcal{M}_n(-++++\cdots)|^2 = c_n(g,N)[0+O(g^4)], \tag{2}$$

$$|\mathcal{M}_n(--+++\cdots)|^2 = c_n(q,N)[(p_1\cdot p_2)^4 \times \textstyle\sum_P [(p_1\cdot p_2)(p_2\cdot p_3)(p_3\cdot p_4)\cdots(p_n\cdot p_1)]^{-1} + O(N^{-2}) + O(g^2)], \tag{3}$$

where $c_n(g,N) = g^{2n-4}N^{n-2}(N^2-1)/2^{n-4}n$. The sum is over all permutations P of $1, \ldots, n$.

Equation (3) has the correct dimensions and symmetry properties for this n-particle scattering amplitude squared. Also it agrees with the known results[4,5] for $n=4$, 5, and 6. The agreement for $n=6$ is numerical.[5,6] More importantly, this set of amplitudes is consistent with the Altarelli and Parisi[7] relationship for all n, when two of the gluons are made parallel. This is trivial for the first two helicity amplitudes but is a highly nontrivial statement for the last amplitude, as shown here:

$$|\mathcal{M}_n(--+++\cdots)|^2 \underset{1\parallel 2}{\longrightarrow} 0, \tag{4}$$

$$|\mathcal{M}_n(--+++\cdots)|^2 \underset{2\parallel 3}{\longrightarrow} 2g^2N\frac{z^4}{z(1-z)}\frac{1}{s}|\mathcal{M}_{n-1}(--++\cdots)|^2, \tag{5}$$

$$|\mathcal{M}_n(--+++\cdots)|^2 \underset{3\parallel 4}{\longrightarrow} 2g^2N\frac{1}{z(1-z)}\frac{1}{s}|\mathcal{M}_{n-1}(--++\cdots)|^2, \tag{6}$$

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where s is the corresponding pole and z is the momentum fraction. The result for particles 2 and 3 nearly parallel, Eq. (5), is only simple because $\mathcal{M}_{n-1}(-+++\cdots)$ is zero to this order in g so that there is no interference term and therefore azimuthal averaging is not required.

The surprise about this result is that all denominators are simple dot products of two external momenta. The Feynman diagrams for n-gluon $(n>5)$ scattering contain propagators $(p_i+p_j+p_k)^2$, $(p_i+p_j+p_k+p_m)^2, \ldots$. These propagators must cancel for Eq. (3) to be correct; this occurs for $n=6$. Of course, Altarelli and Parisi have taught us that many cancellations are expected.

We do not expect such a simple expression for the other helicity amplitudes. Also, we challenge the string theorists to prove more rigorously that Eq. (3) is correct.

Fermilab is operated by the Universities Research Association Inc. under contract with the United States Department of Energy.

[1] E. Eichten, I. Hinchliffe, K. Lane, and C. Quigg, Rev. Mod. Phys. **56**, 579 (1984).

[2] M. T. Grisaru, H. N. Pendleton, and P. van Nieuwenhuizen, Phys. Rev. D **15**, 997 (1977); M. T. Grisaru and H. N. Pendleton, Nucl. Phys. **B124**, 81 (1977).

[3] S. J. Parke and T. R. Taylor, Phys. Lett. **157B**, 81 (1985).

[4] T. Gottschalk and D. Sivers, Phys. Rev. D **21**, 102 (1980); F. A. Berends, R. Kleiss, P. de Causmacker, R. Gastmans, and T. T. Wu, Phys. Lett. **103B**, 124 (1981).

[5] S. J. Parke and T. R. Taylor, Fermilab Report No. Pub-85/118-T, 1985 (to be published); Z. Kunszt, CERN Report No. TH-4319, 1985 (to be published).

[6] Another numerical fact worth mentioning is that to leading order in g but to all orders in N, the amplitude $|\mathcal{M}_{n=6}(--++++)|^2$ is permutation symmetric apart from the factor $(p_1\cdot p_2)^4$. This allows all permutations of this amplitude to be trivially calculated from one such permutation.

[7] G. Altarelli and G. Parisi, Nucl. Phys. **B126**, 298 (1977).

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For the n-gluon scattering amplitude, there are $(n+2)/2$ independent helicity amplitudes. At the tree level, the two helicity amplitudes which most violate the conservation of helicity are zero. This is easily seen by the embedding of the Yang-Mills theory in a supersymmetric theory.[2,3] Here we give an expression for the next helicity amplitude, also at tree level, to leading order in the number of colors in SU(N) Yang-Mills theory.

If the helicity amplitude for gluons $1,\ldots,n$, of momenta $p_1,\ldots,p_n$ and helicities $\lambda_1,\ldots,\lambda_n$, is $\mathcal{M}_n(\lambda_1,\ldots,\lambda_n)$, where the momenta and helicities are labeled as though all particles are outgoing, then the three helicity amplitudes of interest, squared and summed over color, are

$$|\mathcal{M}_n(+++++\cdots)|^2 = c_n(g,N)[0+O(g^4)], \tag{1}$$

$$|\mathcal{M}_n(-++++\cdots)|^2 = c_n(g,N)[0+O(g^4)], \tag{2}$$

$$|\mathcal{M}_n(--+++\cdots)|^2 = c_n(g,N)[(p_1\cdot p_2)^4 \times \textstyle\sum_P[(p_1\cdot p_2)(p_2\cdot p_3)(p_3\cdot p_4)\cdots(p_n\cdot p_1)]^{-1}+O(N^{-2})+O(g^2)], \tag{3}$$

where $c_n(g,N)=g^{2n-4}N^{n-2}(N^2-1)/2^{n-4}n$. The sum is over all permutations P of $1,\ldots,n$.

Equation (3) has the correct dimensions and symmetry properties for this n-particle scattering amplitude squared. Also it agrees with the known results[4,5] for $n=4$, 5, and 6. The agreement for $n=6$ is numerical.[5,6] More importantly, this set of amplitudes is consistent with the Altarelli and Parisi[7] relationship for all n, when two of the gluons are made parallel. This is trivial for the first two helicity amplitudes but is a highly nontrivial statement for the last amplitude, as shown here:

$$|\mathcal{M}_n(--+++\cdots)|^2 \underset{1\parallel 2}{\longrightarrow} 0, \tag{4}$$

$$|\mathcal{M}_n(--+++\cdots)|^2 \underset{2\parallel 3}{\longrightarrow} 2g^2N\frac{z^4}{z(1-z)}\frac{1}{s}|\mathcal{M}_{n-1}(--++\cdots)|^2, \tag{5}$$

$$|\mathcal{M}_n(--+++\cdots)|^2 \underset{3\parallel 4}{\longrightarrow} 2g^2N\frac{1}{z(1-z)}\frac{1}{s}|\mathcal{M}_{n-1}(--++\cdots)|^2, \tag{6}$$

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where s is the corresponding pole and z is the momentum fraction. The result for particles 2 and 3 nearly parallel, Eq. (5), is only simple because $\mathcal{M}_{n-1}(-+++\cdots)$ is zero to this order in g so that there is no interference term and therefore azimuthal averaging is not required.

The surprise about this result is that all denominators are simple dot products of two external momenta. The Feynman diagrams for n-gluon $(n>5)$ scattering contain propagators $(p_i+p_j+p_k)^2$, $(p_i+p_j+p_k+p_m)^2$, ... These propagators must cancel for Eq. (3) to be correct; this occurs for $n=6$. Of course, Altarelli and Parisi have taught us that many cancellations are expected.

We do not expect such a simple expression for the other helicity amplitudes. Also, we challenge the string theorists to prove more rigorously that Eq. (3) is correct.

Fermilab is operated by the Universities Research Association Inc. under contract with the United States Department of Energy.

[1] E. Eichten, I. Hinchliffe, K. Lane, and C. Quigg, Rev. Mod. Phys. **56**, 579 (1984).
[2] M. T. Grisaru, H. N. Pendleton, and P. van Nieuwenhuizen, Phys. Rev. D **15**, 997 (1977); M. T. Grisaru and H. N. Pendleton, Nucl. Phys. **B124**, 81 (1977).
[3] S. J. Parke and T. R. Taylor, Phys. Lett. **157B**, 81 (1985).
[4] T. Gottschalk and D. Sivers, Phys. Rev. D **21**, 102 (1980); F. A. Berends, R. Kleiss, P. de Causmacker, R. Gastmans, and T. T. Wu, Phys. Lett. **103B**, 124 (1981).
[5] S. J. Parke and T. R. Taylor, Fermilab Report No. Pub-85/118-T, 1985 (to be published); Z. Kunszt, CERN Report No. TH-4319, 1985 (to be published).
[6] Another numerical fact worth mentioning is that to leading order in g but to all orders in N, the amplitude $|\mathcal{M}_{n=6}(--++++)|^2$ is permutation symmetric apart from the factor $(p_1\cdot p_2)^4$. This allows all permutations of this amplitude to be trivially calculated from one such permutation.
[7] G. Altarelli and G. Parisi, Nucl. Phys. **B126**, 298 (1977).

The original paper was shorter but the editor insisted we add somethings ! ! !

How to Proceed?

- How to organize the color factors?

  Can one organize the terms into gauge invariant sub-amplitudes?

- How to deal with the polarization vectors?

  How to write ϵ's that are explicitly Lorentz Invariant?

Mangano, Parke and Xu

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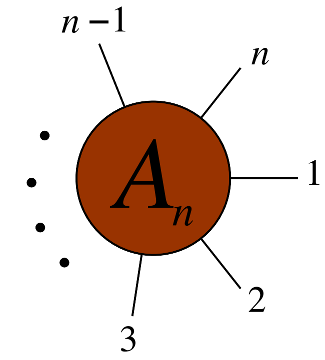
CHINESE MAGIC:

Mangano, Parke and Xu

COLOR: Color Ordered Sub-amplitudes:

$$\mathcal{M}_n = \sum_{perm'} tr\left(\lambda^{a_1}\lambda^{a_2}\ldots\lambda^{a_n}\right) m(p_1,\epsilon_1;p_2,\epsilon_2;\cdots;p_n,\epsilon_n),$$

Chan-Paton factors



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$$\mathcal{M}_n \quad = \quad \sum_{perm'} tr\left(\lambda^{a_1}\lambda^{a_2}\ldots\lambda^{a_n}\right) m(p_1,\epsilon_1;p_2,\epsilon_2;\cdots;p_n,\epsilon_n),$$

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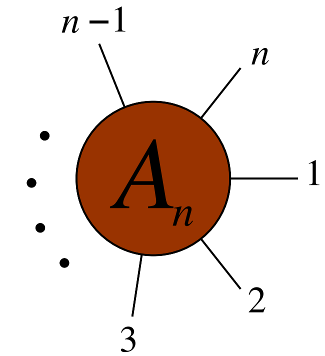

$$[\lambda^a,\lambda^b] = i f_{abc}\lambda^c$$

$$f_{abc} \;=\; -2i\; tr(\lambda^a\lambda^b\lambda^c \;-\; \lambda^c\lambda^b\lambda^a)$$

$$tr(\lambda^a\lambda^b) = \tfrac{1}{2}\delta^{ab}$$

COLOR: Color Ordered Sub-amplitudes:

$$\mathcal{M}_n \quad = \quad \sum_{perm'} tr\left(\lambda^{a_1}\lambda^{a_2}\ldots\lambda^{a_n}\right)\, m(p_1,\epsilon_1;p_2,\epsilon_2;\cdots;p_n,\epsilon_n),$$

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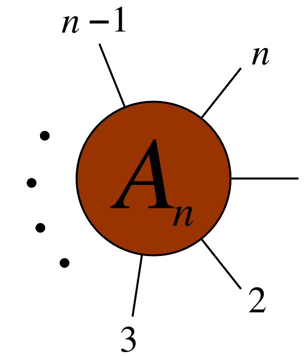

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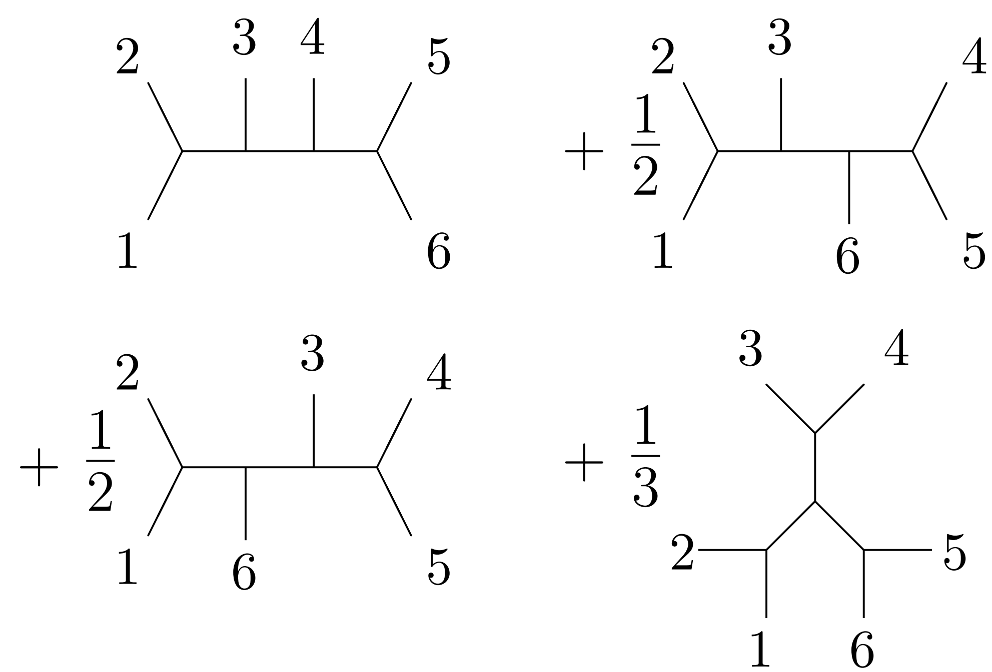
$$tr(\lambda^a\lambda^b) = \tfrac{1}{2}\delta^{ab}$$

6 gluon example:



Color Ordered Sub-amplitudes:

$$\mathcal{M}_n \quad = \quad \sum_{perm'} tr\left(\lambda^{a_1}\lambda^{a_2}\ldots\lambda^{a_n}\right) m(p_1,\epsilon_1;p_2,\epsilon_2;\cdots;p_n,\epsilon_n),$$

(1) $m(1,2,\ldots,n)$ is gauge invariant.
(2) $m(1,2,\ldots,n)$ is invariant under cyclic permutations of $1,2,\ldots,n$
(3) $m(n,n-1,\ldots,1) = (-1)^n\, m(1,2,\ldots,n)$
(4) The Ward Identity:

$$m(1,2,3,\ldots,n) + m(2,1,3,\ldots,n) + m(2,3,1,\ldots,n) + \cdots + m(2,3,\ldots,1,n) = 0$$

(5) Factorization of $m(1,2,\cdots,n)$ on multi-gluon poles.
(6) Incoherence to leading order in number of colors:

$$\sum_{colors} |\mathcal{M}_n|^2 = \frac{N^{n-2}(N^2-1)}{2^n} \sum_{perm'} \left\{|m(1,2,\cdots,n)|^2 + \mathcal{O}(N^{-2})\right\}.$$

Polarization Vectors: Spinor Dot Products:

notation: if $\psi(p)$ is a Dirac spinor, for massless particle, $p^2 = 0$, then define

$$|p\pm\rangle = \psi_\pm(p) = \frac{1}{2}(1 \pm \gamma_5)\psi(p) \qquad \langle p \pm| = \overline{\psi_\pm(p)}$$

$$\langle pq\rangle = \langle p - |q+\rangle \qquad [pq] = \langle p + |q-\rangle$$

$$\langle ij\rangle \equiv \sqrt{|S_{ij}|}\ \exp(i\phi_{ij}), \qquad \cos\phi_{ij} = \frac{(p_i^1 p_j^+ - p_j^1 p_i^+)}{\sqrt{p_i^+ p_j^+}}$$

$$[ij] \equiv \sqrt{|S_{ij}|}\ \exp(i\bar{\phi}_{ij}) \qquad \sin\phi_{ij} = \frac{(p_i^2 p_j^+ - p_j^2 p_i^+)}{\sqrt{p_i^+ p_j^+}}.$$

just complex numbers

$$S_{ij} \equiv \langle ij\rangle\ [ji].$$

Spinor Algebra:

$$\langle pq\rangle = \langle p-|q+\rangle \qquad [pq] = \langle p+|q-\rangle$$

$$\langle p+|q+\rangle = \langle p-|q-\rangle = \langle pp\rangle = [pp] = 0$$

$$\langle pq\rangle = -\langle qp\rangle, \qquad [pq] = -[qp]$$

$$2\,|p\pm\rangle\langle q\pm| \;=\; \frac{1}{2}(1\pm\gamma_5)\gamma^\mu\langle q\pm|\,\gamma_\mu\,|p\pm\rangle,$$

$$\langle pq\rangle^* = -sign(p\cdot q)[pq] = sign(p\cdot q)[qp]$$

$$|\langle pq\rangle|^2 = 2(p\cdot q),$$

$$\langle p\pm|\,\gamma_{\mu_1}\ldots\gamma_{\mu_{2n+1}}\,|q\pm\rangle \;=\; \langle q\mp|\,\gamma_{\mu_{2n+1}}\ldots\gamma_{\mu_1}\,|p\mp\rangle,$$

$$\begin{aligned}
Tr(\hat{P}_1\hat{P}_2\hat{P}_3\cdots\hat{P}_{2n}) &= [12]\langle 23\rangle\cdots\langle 2n1\rangle \;+\; \langle 12\rangle[23]\cdots[2n1] \\
&= 2\,\sqrt{S_{12}S_{23}\ldots S_{2n1}}\;\cos(\phi_{12}-\phi_{32}+\phi_{34}-\ldots-\phi_{1\;2n}).
\end{aligned}$$

$$\begin{aligned}
Tr(\hat{P}_1\hat{P}_2\hat{P}_3\cdots\hat{P}_{2n}\gamma_5) &= [12]\langle 23\rangle\cdots\langle 2n1\rangle \;-\; \langle 12\rangle[23]\cdots[2n1] \\
&= -2i\,\sqrt{S_{12}S_{23}\ldots S_{2n1}}\;\sin(\phi_{12}-\phi_{32}+\phi_{34}-\ldots-\phi_{1\;2n}).
\end{aligned}$$

Polarization Vectors: Helicity

$$\epsilon_+^\mu(k,q) = \frac{\langle q - |\gamma^\mu| k - \rangle}{\sqrt{2}\langle q - | k + \rangle}, \qquad \epsilon_-^\mu(k,q) = \frac{-\ \langle q + |\gamma^\mu| k + \rangle}{\sqrt{2}\langle q + | k - \rangle}.$$

where k is the momentum of the gluon
and q is a reference momentum $(q^2 = 0)$.

[6] Z. Xu, Da-Hua Zhang and L. Chang, Tsinghua University Preprints, Beijing, The People's Republic of China, TUTP–84/4, TUTP–84/5, TUTP–84/6 and Nucl. Phys. B291 (1987) 392.

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Color Ordered sub-amplitudes are independent of the q's

[6] Z. Xu, Da-Hua Zhang and L. Chang, Tsinghua University Preprints, Beijing, The People's Republic of China, TUTP–84/4, TUTP–84/5, TUTP–84/6 and Nucl. Phys. B291 (1987) 392.

Polarization Vector Properties:

The polarizations for vectors with momentum p, as defined in the text:

$$\epsilon_\mu^\pm(p,k) \;=\; \pm\frac{\langle p\pm|\,\gamma_\mu\,|k\pm\rangle}{\sqrt{2}\langle k\mp|p\pm\rangle},$$

$$\epsilon^\pm(p,k)\cdot\gamma \;=\; \pm\frac{\sqrt{2}}{\langle k\mp|p\pm\rangle}\left(\,|p\mp\rangle\langle k\mp|\;+\;|k\pm\rangle\langle p\pm|\,\right),$$

enjoy the following properties:

$$\epsilon_\mu^\pm(p,k) \;=\; (\epsilon_\mu^\mp(p,k))^*,$$

$$\epsilon^\pm(p,k)\cdot p \;=\; \epsilon^\pm(p,k)\cdot k \;=\; 0,$$

$$\epsilon^\pm(p,k)\cdot\epsilon^\pm(p,k') \;=\; 0,$$

$$\epsilon^\pm(p,k)\cdot\epsilon^\mp(p,k') \;=\; -1,$$

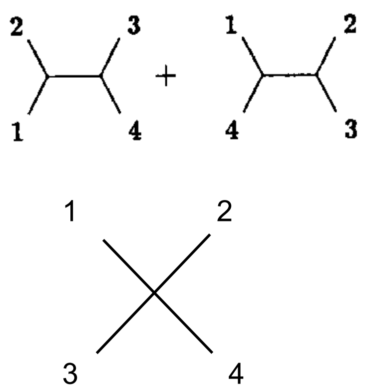
$$\epsilon^\pm(p,k)\cdot\epsilon^\pm(p',k) \;=\; 0,$$

$$\epsilon^\pm(p,k)\cdot\epsilon^\mp(k,k') \;=\; 0,$$

$$\epsilon_\mu^+(p,k)\,\epsilon_\nu^-(p,k) \;+\; \epsilon_\mu^-(p,k)\,\epsilon_\nu^+(p,k) \;=\; -g_{\mu\nu} \;+\; \frac{p_\mu k_\nu + p_\nu k_\mu}{p\cdot k}.$$

Exercise:

- Calculate matrix element for 2 gluons to 2 gluons



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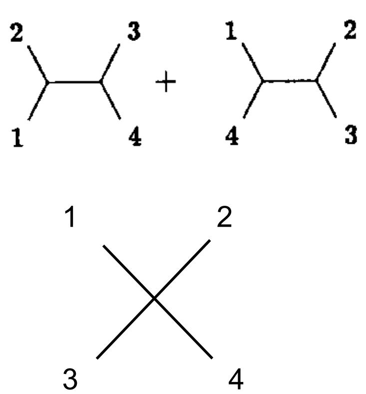


6 gluon result:

$$\mathcal{M}(g_1^-, g_2^-, g_3^+, g_4^+, \ldots, g_6^+) = ig^4 \langle 12\rangle^4 \sum_{\{1,2,3,4,5,6\}'} tr(\lambda^{a_1}\lambda^{a_2}\ldots\lambda^{a_6}) \frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle 61\rangle},$$

$$m_{3+3-}(g_1, g_2, \ldots, g_6) = ig^4 \left[\frac{\alpha^2}{t_{123}s_{12}s_{23}s_{45}s_{56}} + \frac{\beta^2}{t_{234}s_{23}s_{34}s_{56}s_{61}} + \frac{\gamma^2}{t_{345}s_{34}s_{45}s_{61}s_{12}} + \frac{t_{123}\beta\gamma + t_{234}\gamma\alpha + t_{345}\alpha\beta}{s_{12}s_{23}s_{34}s_{45}s_{56}s_{61}} \right],$$

	$1^+2^+3^+4^-5^-6^-$ $X = p_1 + p_2 + p_3$	$1^+2^+3^-4^+5^-6^-$ $Y = p_1 + p_2 + p_4$	$1^+2^-3^+4^-5^+6^-$ $Z = p_1 + p_3 + p_5$
α	0	$-[12]\langle 56\rangle\langle 4\|Y\|3\rangle$	$[13]\langle 46\rangle\langle 5\|Z\|2\rangle$
β	$[23]\langle 56\rangle\langle 1\|X\|4\rangle$	$[24]\langle 56\rangle\langle 1\|Y\|3\rangle$	$[51]\langle 24\rangle\langle 3\|Z\|6\rangle$
γ	$[12]\langle 45\rangle\langle 3\|X\|6\rangle$	$[12]\langle 35\rangle\langle 4\|Y\|6\rangle$	$[35]\langle 62\rangle\langle 1\|Z\|4\rangle$

+++---

$$A_6(1^+,2^+,3^+,4^-,5^-,6^-) = i\,\frac{([1\,2]\,\langle 4\,5\rangle\,\langle 6^-|\,(1+2)\,|3^-\rangle)^2}{s_{61}s_{12}s_{34}s_{45}s_{612}} + i\,\frac{([2\,3]\,\langle 5\,6\rangle\,\langle 4^-|\,(2+3)\,|1^-\rangle)^2}{s_{23}s_{34}s_{56}s_{61}s_{561}} + i\,\frac{s_{123}\,[1\,2]\,[2\,3]\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6^-|\,(1+2)\,|3^-\rangle\,\langle 4^-|\,(2+3)\,|1^-\rangle}{s_{12}s_{23}s_{34}s_{45}s_{56}s_{61}}\,.$$

++++++-+++++++-

$$A_{jn}^{\mathrm{MHV}} \equiv A_n^{\mathrm{tree}}(1^+,2^+,\ldots,j^-,\ldots,(n-1)^+,n^-) = i\,\frac{\langle j\,n\rangle^4}{\langle 1\,2\rangle\cdots\langle n\,1\rangle}\,.$$

Factorization Properties:

Soft:

$$m(1^+,2\ldots,n) \xrightarrow{1^+ \text{ soft}} \left\{\frac{g\ \langle n\ 2\rangle}{\langle n\ 1\rangle\langle 1\ 2\rangle}\right\}\ m(2,3\ldots,n)$$

$$m(1^-,2\ldots,n) \xrightarrow{1^- \text{ soft}} \left\{\frac{g\ [n\ 2]}{[n\ 1][1\ 2]}\right\}\ m(2,3\ldots,n).$$

collinear:

$$m(1^+,2^+,3,\ldots) \xrightarrow{1^+ \parallel 2^+} \left\{\frac{ig\ [12]}{\sqrt{z(1-z)}}\right\}\ \frac{-i}{S_{12}}\ m(P^+,3,\ldots)$$

$$m(1^+,2^-,3,\ldots) \xrightarrow{1^+ \parallel 2^-} \left\{\frac{ig\ z^2\langle 12\rangle}{\sqrt{z(1-z)}}\right\}\ \frac{-i}{S_{12}}\ m(P^+,3,\ldots)$$

$$+\left\{\frac{ig\ (1-z)^2\ [12]}{\sqrt{z(1-z)}}\right\}\ \frac{-i}{S_{12}}\ m(P^-,3,\ldots)$$

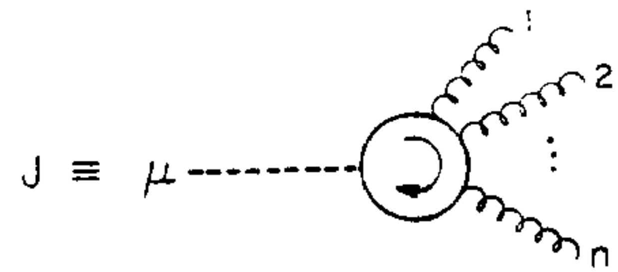
$$m(1^-,2^-,3,\ldots) \xrightarrow{1^- \parallel 2^-} \left\{\frac{ig\ \langle 12\rangle}{\sqrt{z(1-z)}}\right\}\ \frac{-i}{S_{12}}\ m(P^-,3,\ldots).$$

Multi-particle:

$$m(1,2,3,4,5,6) \rightarrow m(1,2,3,-P)\ \frac{-i}{P^2}\ m(P,4,5,6)$$

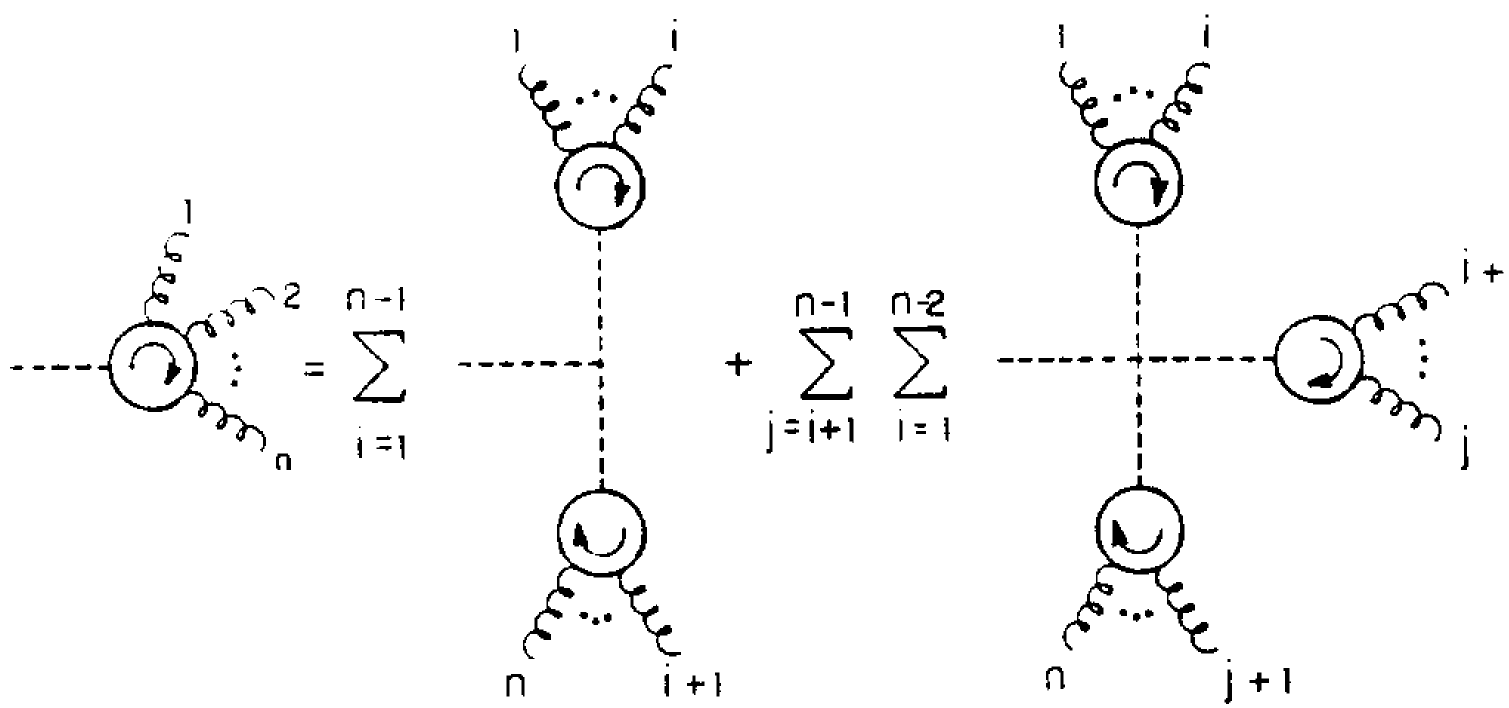
Recursion Relations:

off mass shell



on mass shell

color ordered current



Berends and Giele

What about Quarks, Squarks & Gluinos ?

$$A_q(\bar{q}_1^+, q_2^-, g_3^-, g_4^+, \ldots, g_n^+) = ig^{n-2}\langle 23\rangle^3\langle 13\rangle \sum_{\{3,\ldots,n\}} (\lambda_3\lambda_4\cdots\lambda_n)_{\hat{2}\hat{1}} \frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}.$$

$$A(\bar{\phi}_1^+, \phi_2^-, g_3^-, g_4^+, \ldots, g_n^+) = ig^{n-2}\langle 23\rangle^2\langle 13\rangle^2 \sum_{\{3,\ldots,n\}} (\lambda_3\lambda_4\cdots\lambda_n)_{\hat{2}\hat{1}} \frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}.$$

$$A_{\tilde{g}}(\Lambda_1^+, \Lambda_2^+, \Lambda_3^-, \Lambda_4^-, g_5^+, \ldots, g_n^+) = ig^{n-2}\langle 12\rangle\langle 34\rangle^3 \sum_{perm'} tr(\lambda_1\lambda_2\cdots\lambda_n) \frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}.$$

also for the other helicity amplitudes:

Mangano and Parke: Phys.Rept 200:301-367,1991

Witten's Twistor String Theory

``Perturbative gauge theory as a string theory in twistor space''

hep-th/0312171

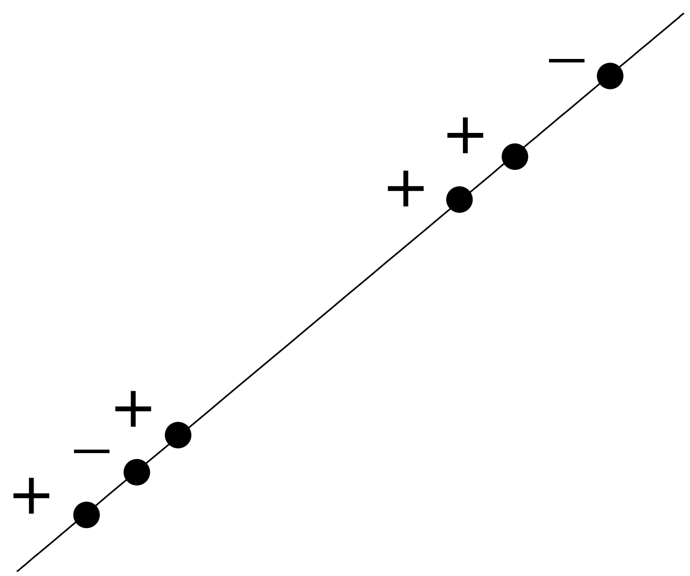
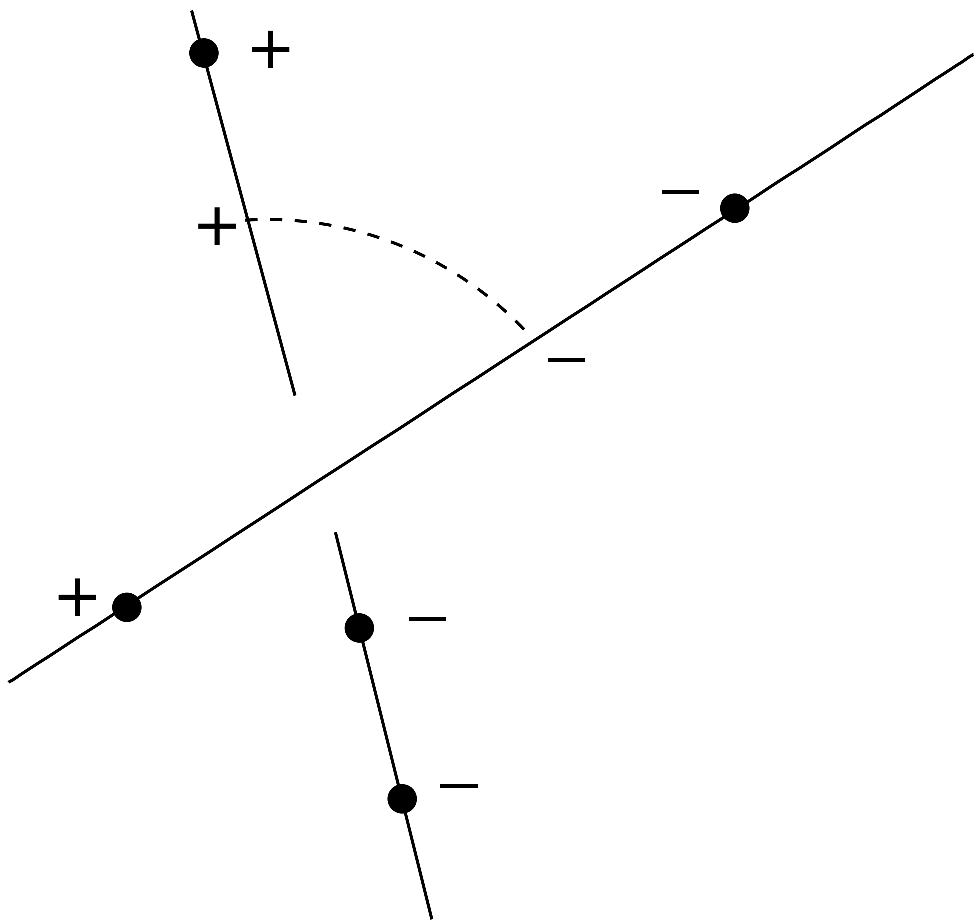


Fig. 2: The MHV amplitude for gluon scattering is associated with a collinear arrangement of points in $\mathbb{R}^3$.

(a)



6 gluons

and are quite complicated. There are three essentially different cases, namely helicities $+++---$, $++--+-$, or $+-+-+-$. These amplitudes can all be written

$$A = 8g^4 \left[\frac{\alpha^2}{t_{123} s_{12} s_{23} s_{45} s_{56}} + \frac{\beta^2}{t_{234} s_{23} s_{34} s_{56} s_{61}} + \frac{\gamma^2}{t_{345} s_{34} s_{45} s_{61} s_{12}} + \frac{t_{123}\beta\gamma + t_{234}\gamma\alpha + t_{345}\alpha\beta}{s_{12} s_{23} s_{34} s_{45} s_{56} s_{61}} \right]. \tag{3.31}$$

	$1^+2^+3^+4^-5^-6^-$ $X = 1+2+3$	$1^+2^+3^-4^+5^-6^-$ $Y = 1+2+4$	$1^+2^-3^+4^-5^+6^-$ $Z = 1+3+5$
α	0	$-[12]\langle 56\rangle\langle 4\|Y\|3\rangle$	$[13]\langle 46\rangle\langle 5\|Z\|2\rangle$
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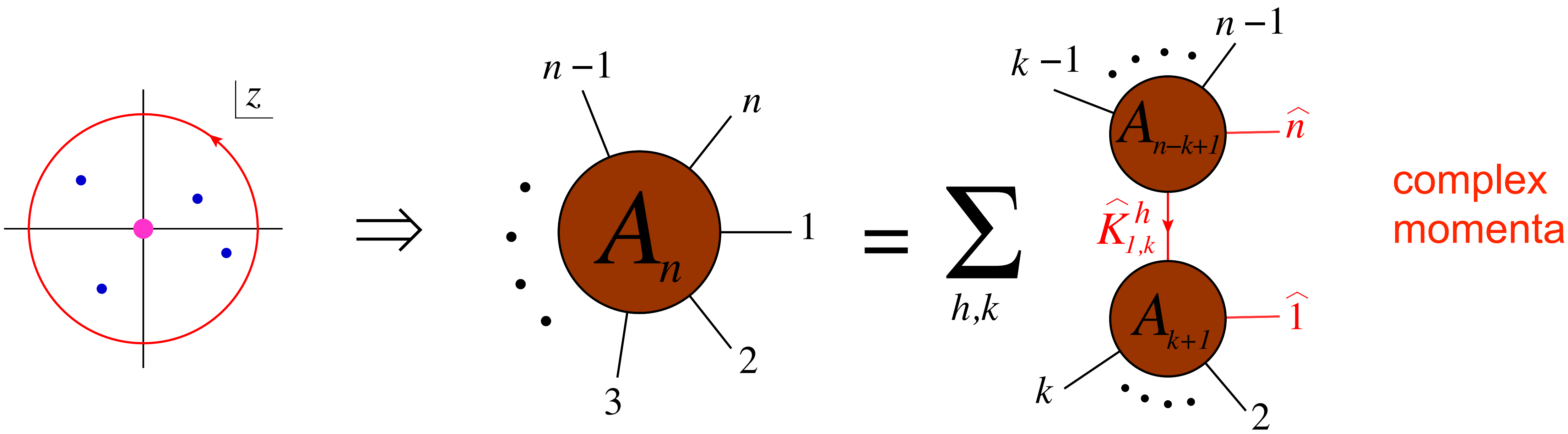
BCFW recursion:



On-mass shell recursion relations:

R. Britto, F. Cachazo, B. Feng and E. Witten, "Direct proof of tree-level recursion relation in Yang-Mills theory," Phys. Rev. Lett. **94**, 181602 (2005) [hep-th/0501052].

Scattering Amplitudes

Henriette Elvang, Yu-tin Huang

arXiv:1308.1697

A brief introduction to modern amplitude methods

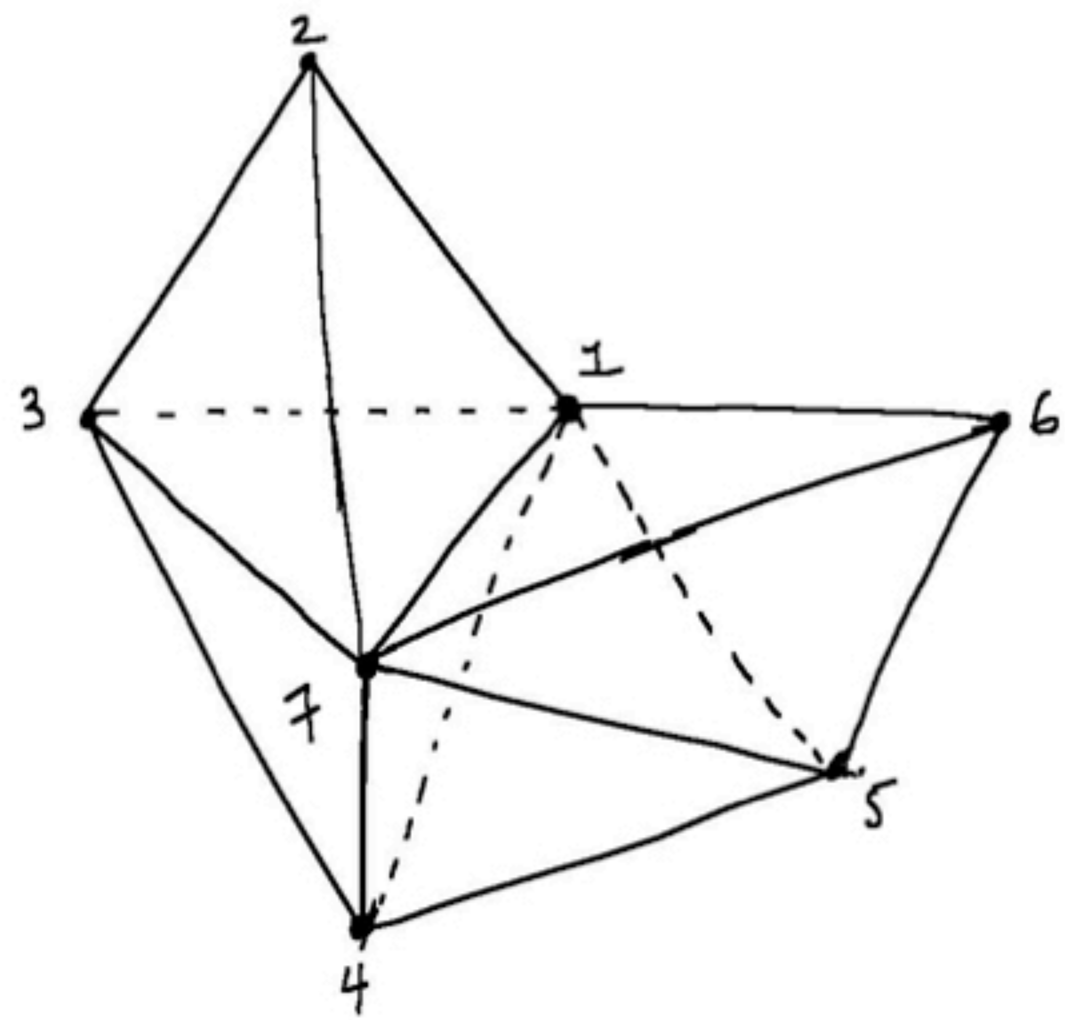
Lance Dixon

arXiv:1310.5353

Amplituhedron:

http://susy2013.ictp.it/video/05_Friday/2013_08_30_Arkani-Hamed_4-3.html

7 - point Amplituhedron in $\mathbb{P}^3$



Tree Amplitude for $[1^- 2^+ 3^+ 4^+ 5^+ 6^+ 7^- 8^-]$

Locality + Unitarity emerge

(N=4 planar SUSY Theory)

Principals:

Principal of Least Action:

- Classical Mechanics:
- Quantum Mechanics:
- Quantum Field Theory:

????
???
????

- What is the correct way to incorporate Gravity ?

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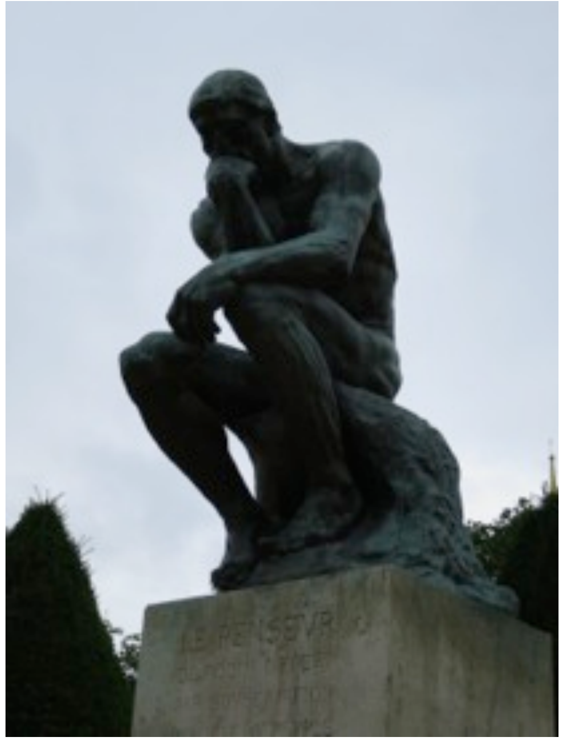

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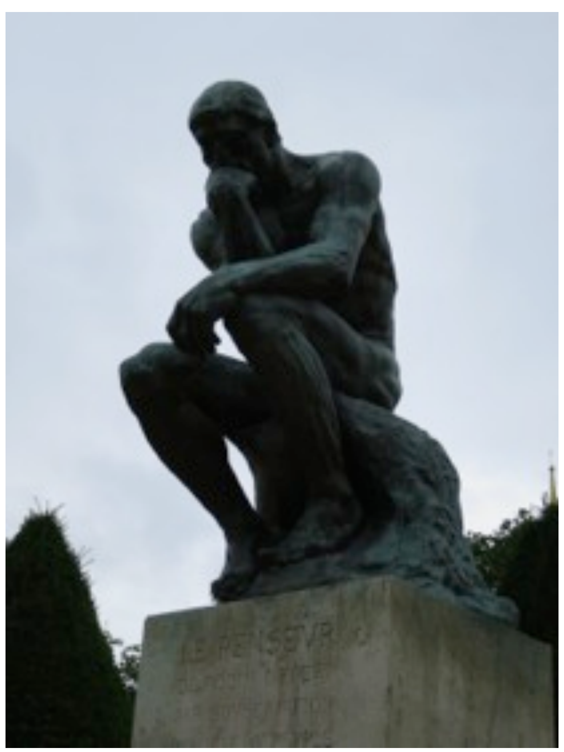

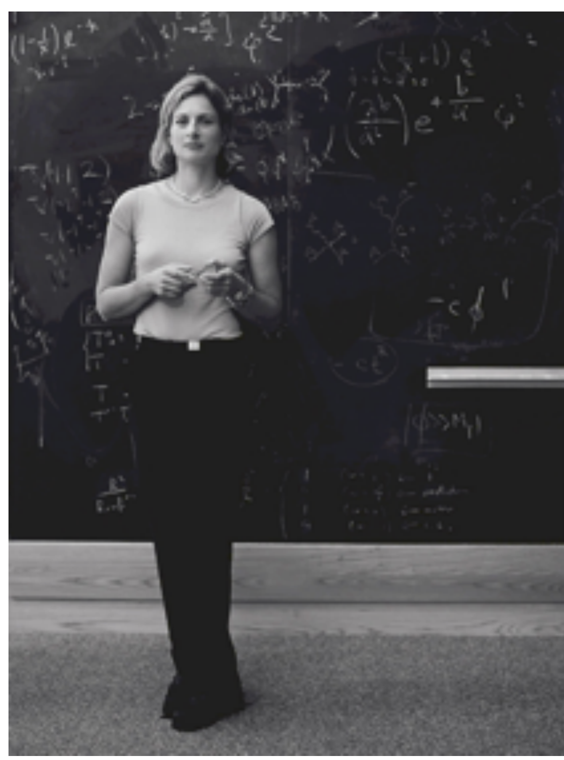

s the correct way

rporate Gravity ?