On ghost-free massive gravity

based on: C. Deffayet, JM, G Zahariade [arXiv:1207.6338],[arXiv:1208.4493]; JM, D. Steer ([arXiv:1310.6560], [arXiv:1405.1862].

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Plan

Outline

Free massive spin 2 field

The massless action

The Einstein-Hilbert action for $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ is

$$S_0 = \frac{1}{2} \int h_{\mu\nu} \mathcal{G}^{\mu\nu}, \quad (1)$$

with

$$\begin{aligned} \mathcal{G}^{\mu\nu} &= \mathcal{R}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\mathcal{R}^{\alpha}{}_{\alpha}, \\ \mathcal{R}^{\mu\nu} &= \Box h^{\mu\nu} - \partial^{(\mu}\partial_{\alpha} h^{\nu)\alpha} + \partial^{\mu}\partial^{\nu} h^{\alpha}{}_{\alpha}. \end{aligned} \quad (2)$$

The gauge summetry $\delta h_{\mu\nu} = \partial_{(\mu}\xi_{\nu)}$ is guaranteed by the Bianchi identity $\partial_{\mu}\mathcal{G}^{\mu\nu} = 0$ and removes $2 \times D$ dof from from the $D(D+1)/2$ components leaving $(D-2)(D-1)/2 - 1$ dof.

Free massive spin 2 field

The massive action

Addition of the most general mass term

$$S_1 = -\frac{m^2}{2} \int \left(h_{\mu\nu} h^{\mu\nu} - a (h^{\alpha}{}_{\alpha})^2 \right) \tag{3}$$

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gives the exchange amplitude

$$T.\Delta.T = \frac{T.T - \frac{1}{D-1}(T')^2}{\Box - m^2} - \frac{(T')^2}{(D-1)\left(\Box - \frac{m^2(1-aD)}{(D-2)(a-1)}\right)}, \tag{4}$$

with $T' = T^{\alpha}_{\alpha}$.

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so the action describes a massive spin 2 field and a massive scalar ghost.

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The equations of motion

$$\mathcal{G}_{\mu\nu} = m^2(h_{\mu\nu} - a\eta_{\mu\nu}h'). \tag{5}$$

- and the Bianchi identity give

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This removes a vectorial dof.

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- the trace of the equation

$$\frac{2-D}{2}\mathcal{R} = m^2(1-aD)h', \tag{7}$$

- Using the vectorial constraint in $\mathcal{R} = 2(\Box h' - \partial^\mu \partial^\nu h_{\mu\nu})$ gives

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So only for $a = 1$ we have $\mathcal{R} = 0$ and consequently $h' = 0$.

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Interacting massive spin 2 field

Fierz-Pauli theory can be non linearly completed by considering actions

$$S_{g,m} = M_g^2 \int d^4x \sqrt{-g} \left[R(g) - m^2 V(\mathcal{M}) \right], \tag{10}$$

- V is a scalar function of $\mathcal{M}^{\mu}{}_{\nu} = g^{\mu\sigma} f_{\sigma\nu}$,
- the theory contains, besides the dynamical metric $g_{\mu\nu}$, a non dynamical metric $f_{\mu\nu}$ usually considered to be flat.

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- the theory contains, besides the dynamical metric $g_{\mu\nu}$, a non dynamical metric $f_{\mu\nu}$ usually considered to be flat.
- V should be chosen such that
  - (i) when $f_{\mu\nu}$ is taken to be $\eta_{\mu\nu}$, $g_{\mu\nu} = \eta_{\mu\nu}$ is a solution of the field equations,
  - (ii) when expanded at quadratic order around this flat background, the action (10) has the Fierz-Pauli form .

There are infinitely many functions V that satisfy these requirements.

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until the work of de Rham, Gabadadze and Tolley (dRGT) (2010)

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ghostfree massive gravity

For a matrix X define F_k as

$$F_k(X) = \frac{1}{k!} X^{a_1}{}_{[a_1} \ldots X^{a_k}{}_{a_k]}, \tag{13}$$

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The dRGT action

$$S = M_P^2 \int d^4x \sqrt{-g} \left[R - m^2 \sum_{k=0}^{k=4} \beta_k F_k \left(\sqrt{g^{-1} f} \right) \right]. \tag{14}$$

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- The mass term depends on the matrix square root γ

$$\gamma^{\mu}{}_{\sigma} \gamma^{\sigma}{}_{\nu} = g^{\mu\sigma} f_{\sigma\nu}, \tag{15}$$

- proved to be ghostfree in the decoupling limit.

Outline

Ghostfree massive gravity- The moving frame formulation

In terms of dynamical one-forms θ^A and non-dynamical ℓ^A, the dRGT action reads

$$S[\theta^A] = \frac{1}{2}\int \Omega^{AB} \wedge \theta^*_{AB} + \sum_{n=0}^{D-1} \beta_n \int \ell^{A_1} \wedge \cdots \wedge \ell^{A_n} \wedge \theta^*_{A_1 \ldots A_n} , \qquad (16)$$

where

$$\theta^*_{A_1 \ldots A_n} \equiv \frac{1}{(D-n)!} \epsilon_{A_1 \ldots A_D} \theta^{A_{n+1}} \wedge \cdots \wedge \theta^{A_D} \qquad (17)$$

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The two actions coincide if

$$e^{A\mu} \ell^B{}_\mu - e^{B\mu} \ell^A{}_\mu = 0 \qquad (18)$$

and we set

$$\gamma^\mu{}_\nu = e_A{}^\mu \ell^A{}_\nu \qquad (19)$$

The equations of motion

$$G_A = t_A \ , \tag{20}$$

or equivalently $G_{AB} = t_{AB}$, with

$$t_A \equiv \sum_{n=0}^{D-1} \beta_n \ell^{A_1} \wedge \cdots \wedge \ell^{A_n} \wedge \theta^*_{AA_1 \ldots A_n} \equiv t_A{}^B \theta^*_B \tag{21}$$

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- Lorentz invariance imposes

$$G_{[AB]} = 0 = t_{[AB]} \ . \tag{23}$$

  - In many (but not all cases), this gives $e^{A\mu} \ell^B{}_\mu - e^{B\mu} \ell^A{}_\mu = 0$

Ghostfree massive gravity- The moving frame formulation

The scalar constraint ?

a "trace" of the equations of motion

$$m^A G_A = m^A t_A, \tag{24}$$

gives a scalar constraint if $m^A G_A$ has no second order derivatives.

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$$G_A = d\sigma_A + \tau_A, \quad \sigma_A = -\frac{1}{2}\omega^{BC}\theta^*_{ABC} \tag{25}$$

$$\tau_A = \frac{1}{2}\omega^B{}_{[A}\omega^{CD}\theta^*_{C]BD}. \tag{26}$$

the vectorial constraint $\mathcal{D}t_A = 0$ implies $\theta^A\sigma_A = 0$ and $\ell^A\sigma_A = 0$.
C. Deffayet, G. Zahariade, JM (2013)

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translation invariant solutions

In the linear FP theory, they are the Plane waves in the rest frame.

They capture the degrees of freedom.

A simplified hamiltonian framework, with exact (or numerical) solutions.

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They also generalise the Bianchi I solutions of GR.

$$e_{00} = -N\,, \qquad e_{0i} = -Nn_i\,, \qquad e_{ij} = \pi_{ij} - Nn_in_j\,, \tag{27}$$

and

$$\theta^{00} = -\left(\frac{1}{N} - \zeta^{ij}n_in_j\right)\,, \qquad \theta^{0i} = -\zeta^{ij}n_j\,, \qquad \theta^{ij} = \zeta^{ij}. \tag{28}$$

the $D(D+1)/2$ variables all depend on t, $\zeta = \pi^{-1}$.

β_1 mass term

$$ds^2 = -\frac{1}{N^2}dt^2 + (\zeta^2)_{ij}dx^i dx^j \equiv -dT^2 + (B^{-1})_{ij}dx^i dx^j$$

Bianchi

$$\mathrm{C}_1 \equiv \mathrm{tr}(\dot{\mathrm{B}}\mathrm{B}^{-1}) = 0 \tag{29}$$

Hamiltonian

$$n_i = 0, \quad \mathrm{C}_2 \equiv -\mathrm{tr}(\dot{\mathrm{B}}\mathrm{B}^{-1})^2 + 8\mathrm{m}^2\mathrm{tr}(1-\pi) = 0. \tag{30}$$

Equations of motion

$$-\partial_T(\dot{B}B^{-1}) = 2m^2\left[\pi + 2\mathrm{tr}(1-\pi) - \mathrm{N}\right] \tag{31}$$

β_1 mass term

The lapse

The trace of the equations

$$N = 1 - \frac{2D-3}{D-1}\mathrm{tr}(\pi - 1). \tag{32}$$

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Linearised limit $\pi = 1 + h$

$$\mathrm{tr}(\dot{\mathrm{h}}) = 0,\ \mathrm{tr}(\mathrm{h}) = 0,\ \mathrm{N} = 1. \qquad (33)$$

$$\ddot{h} + m^2 h = 0. \qquad (34)$$

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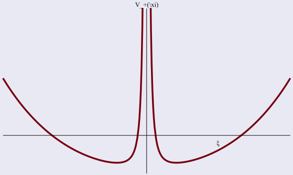
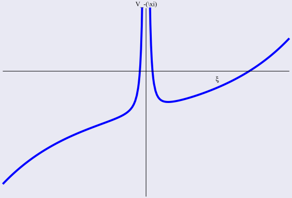
- well posed initial value problem and $N \geq 0$.
- sectors of solutions depending on signature of π, negative implies effective potential unbounded from below.

β_1 mass term

effective potential in 3D for positive and negative determinant





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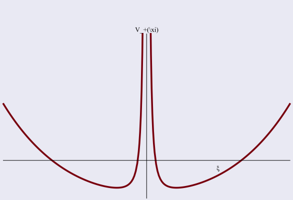
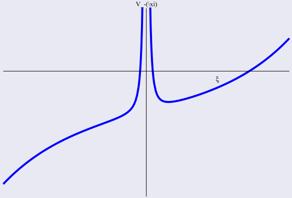
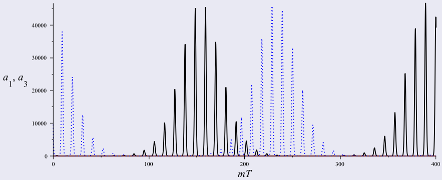




numerical solution in the 4D diagonal case



β_3 mass term

Bianchi

$$C_1 \equiv \mathrm{tr}\dot{\zeta} = 0. \tag{35}$$

Hamiltonian

$$n_i = 0,\ C_2 \equiv \left(\mathrm{tr}(\dot{B}B^{-1})\right)^2 - \mathrm{tr}\left((\dot{B}B^{-1})^2\right) + 8m^2(1 - \det\pi) = 0. \tag{36}$$

Equations of motion

$$\begin{aligned} -\partial_T(\dot{B}B^{-1}) + \partial_T \mathrm{tr}(\dot{B}B^{-1}) \quad &+ \quad \frac{1}{2}(\dot{B}B^{-1})\mathrm{tr}(\dot{B}B^{-1}) \\ -\frac{1}{4}\left[\left(\mathrm{tr}(\dot{B}B^{-1})\right)^2 + \mathrm{tr}\left((\dot{B}B^{-1})^2\right)\right] \quad &= \quad -2m^2\left[\zeta(N\det\pi) - 1\right], \end{aligned} \tag{37}$$

β_3 mass term

The lapse

- is not obtained from the trace of the eom.
- From the eom get $\ddot{\zeta}$ then use $\mathrm{tr}\dot{\zeta} = 0$ to obtain N.

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For $\pi = diag(e^{\delta}, \ldots, e^{\delta}, e^{\Delta})$

$$c = \mathrm{tr}\zeta = (\mathrm{D}-2)\mathrm{e}^{-\delta} + \mathrm{e}^{-\Delta}. \tag{38}$$

$$N = \frac{c\, e^{-(D-2)\delta}\left[(D-3)e^{(D-2)\delta} - 2c(D-3) + 2(D-2)^2 e^{-\delta}\right]}{(e^{-\delta} - e^{-\delta_*})^2 (D-2)^2 (D-1)^2}\,. \tag{39}$$

with

$$e^{-\delta_*} = \frac{c(D-3)}{(D-2)(D-1)}\,. \tag{40}$$

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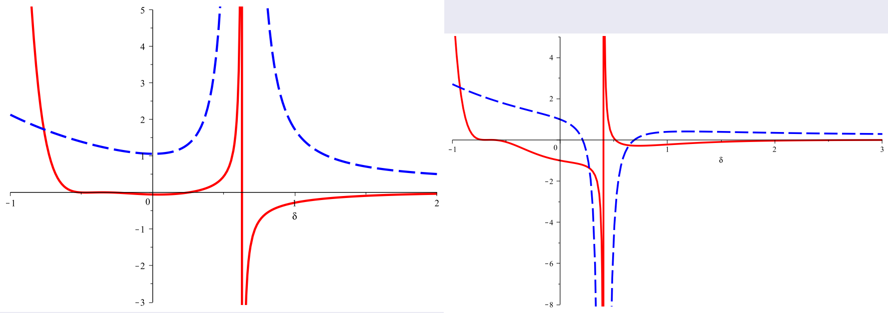
If $c > c_{\mathrm{crit}}$ with

$$c_{\mathrm{crit}} = \frac{D-1}{2^{1/(D-1)}} \left(\frac{D-2}{D-3}\right)^{\frac{D-2}{D-1}} \tag{41}$$

N changes sign and the evolution is singular at a finite time.

V and N



The potential $V(\delta)$ (red, solid lines) and $N(\delta)$ (blue dashed lines) in $D = 4$ dimensions for which $c_{\text{crit}} \simeq 3.78$. LH panel: $c = 3.3 < c_{\text{crit}}$; RH panel $c = 4 > c_{\text{crit}}$.

Outline

Conclusions

- Ghostfree massive gravity in the moving frame formulation :
  - no square roots
  - Constraints easy to obtain (β_1 and β_2).
- Classical instabilities in some sectors of the solutions β_1
- Singular time evolution for β_3
- More tests for β_1.