Introduction to On-Shell Methods in Quantum Field Theory

David A. Kosower
Institut de Physique Théorique, CEA–Saclay

Orsay Summer School, *Correlations Between Partons in Nucleons*
Orsay, France
July 2, 2014

Tools for Computing Amplitudes

- New tools for computing in gauge theories — the core of the Standard Model

- Motivations and connections
  - Particle physics: $SU(3) \times SU(2) \times U(1)$
  - $\mathcal{N} = 4$ supersymmetric gauge theories and strong coupling (AdS/CFT)
  - Witten's twistor string
  - Grassmanians
  - $\mathcal{N} = 8$ supergravity

Amplitudes

- Scattering matrix elements: basic quantities in field theory
- Basic building blocks for computing scattering cross sections

$$\mathcal{A}(gg \to gg \cdots g) \qquad \mathcal{A}(g^+ g^+ \to g^+ g^+ \cdots g^+)$$

- Using crossing

$$\mathcal{A}(0 \to gg\,g \cdots g) \qquad \mathcal{A}(0 \to g^- g^- g^+ \cdots g^+)$$

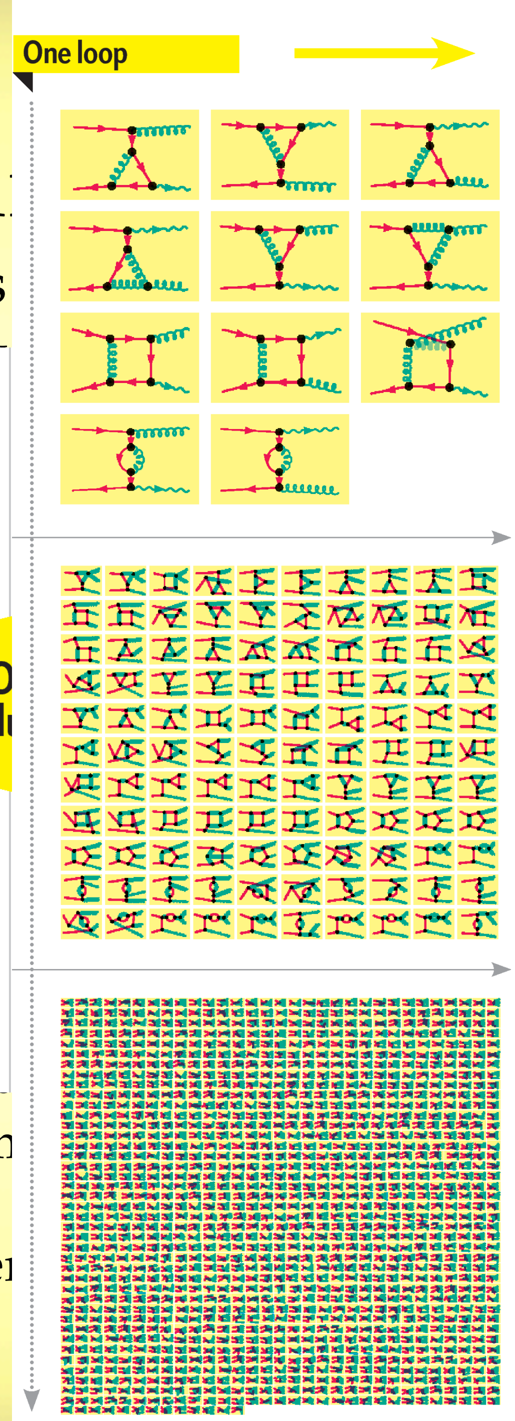
MHV

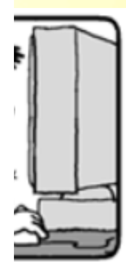
- Primary interest: in gauge theories; can derive all other physical quantities (*e.g.* anomalous dimensions) from them

- In gravity, they are the *only* physical observables

Trad… …oach

- Feynman Diagrams
  - Widely used fo…
  - Heuristic pictu…
  - Introduces idea… …ediate states
  - Precise rules fo…
  - Classic successe… discovery of asy…
- How it works
  - Pick a process
  - Grab a graduat…
  - Lock him or he…
  - Provide a copy … or at least of Peskin & Schroeder's Qu…
  - Supply caffeine, a n… …nt, and occasional instructions
  - Provide a compute… …ca, a copy of FORM & a C++ compiler



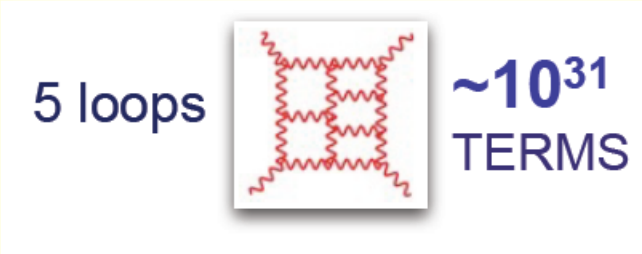


A Difficulty

- Huge number of diagrams in calculations of interest — factorial growth with number of legs or loops
- $2 \rightarrow 6$ jets: 34300 tree diagrams, ~ $2.5 \cdot 10^7$ terms

  ~$2.9 \cdot 10^6$ 1-loop diagrams, ~ $1.9 \cdot 10^{10}$ terms

- In gravity, it's even worse



Results Are Simple!

- Parke–Taylor formula for A^{MHV}

$$i\frac{\langle m_1\, m_2\rangle^4\,\delta^4(\sum_i k_i)}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle (n-1)\,n\rangle\,\langle n\,1\rangle}$$

Parke & Taylor; Mangano, Parke, & Xu

$$\sim\sqrt{2k_1\cdot k_2}$$

Even Simpler in $\mathcal{N}$=4 Supersymmetric Theory

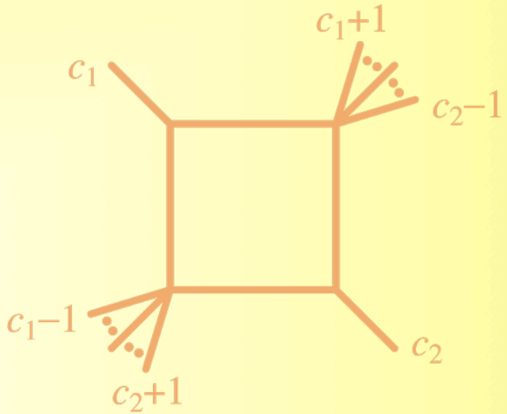
- Nair–Parke–Taylor formula for MHV-class amplitudes

$$i\frac{\delta^{4|8}\left(\sum_i \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} \middle| \sum_i \lambda_i^\alpha \eta_i^A\right)}{\langle 1\,2\rangle\,\langle 2\,3\rangle \cdots \langle (n-1)\,n\rangle\,\langle n\,1\rangle}$$

Answers Are Simple At Loop Level Too

One-loop in $\mathcal{N} = 4$:

$$-A^{\text{tree}}(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) \times \sum_{\text{easy 2 mass}} \text{Box} \cdot \tfrac{1}{2}(\text{its denominator})$$



- All-n QCD amplitudes for MHV configuration on a few Phys Rev D pages

On-Shell Methods

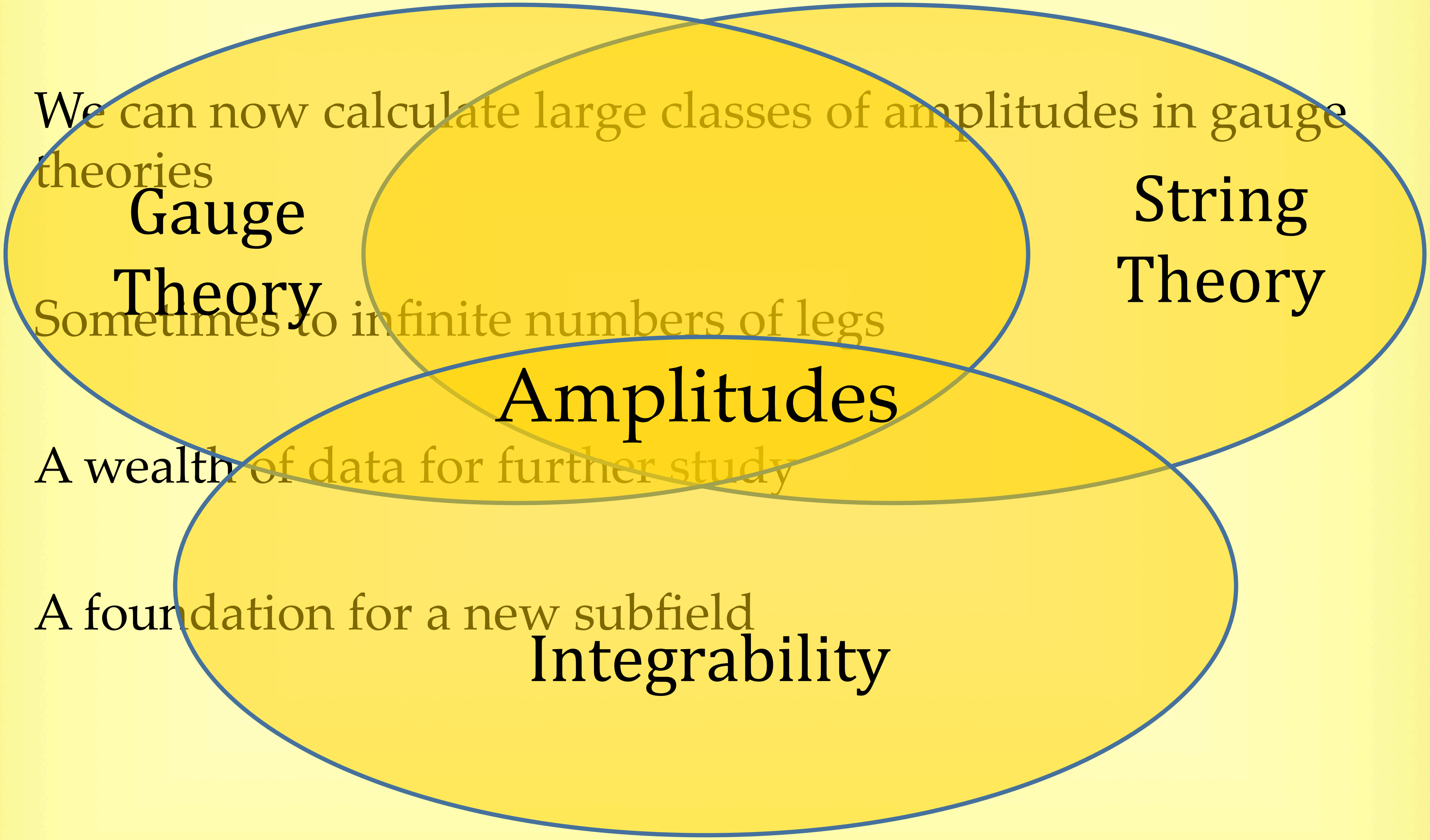
- All physical quantities computed
  - From basic interaction amplitude: A_3^{tree}
  - Using only information from physical on-shell states
  - Avoid size explosion of intermediate terms due to unphysical states
  - Without need for a Lagrangian
- Properties of amplitudes become tools for calculating
  - Kinematics
    - Spinor variables
  - Underlying field theory
    - Integral basis
  - Factorization
    - On-shell recursion relations (BCFW) for tree-level amplitudes
    - Control infrared divergences in real-emission contributions to higher-order calculations
  - Unitarity
    - Unitarity and generalized unitarity for loop calculations

We can now calculate large classes of amplitudes in gauge theories

Sometimes to infinite numbers of legs

A wealth of data for further study

A foundation for a new subfield



Spinor Variables

From Lorentz vectors to bi-spinors

$$p_\mu \qquad \longleftrightarrow \qquad p_{a\dot{a}} \equiv p \cdot \sigma = \begin{pmatrix} p^0 + p^3 & p^1 + ip^2 \\ p^1 - ip^2 & p^0 - p^3 \end{pmatrix}$$

$$p^2 \qquad \longleftrightarrow \qquad \det(p)$$

$$p' = \Lambda p \qquad \longleftrightarrow \qquad p' = upu^\dagger, \qquad u \in SL(2, C)$$

2×2 complex matrices
with det = 1

$$\boxed{p^2 = 0 \implies p = \lambda_a \tilde{\lambda}_{\dot{a}}}$$

Spinor Products

Spinor variables $\left|j^{+}\right\rangle = |j\rangle \equiv \lambda_j, \quad \left|j^{-}\right\rangle = |j] \equiv \tilde{\lambda}_j$

$$\left\langle j^{-}\right| \leftrightarrow \varepsilon^{\alpha\beta}\lambda_{j\beta}, \quad \left\langle j^{+}\right| \leftrightarrow \varepsilon^{\dot\alpha\dot\beta}\lambda_{j\dot\beta}$$

Introduce *spinor products*

$$\langle i\, j\rangle \;\equiv\; \left\langle i^{-}|j^{+}\right\rangle = \varepsilon^{\alpha\beta}\lambda_{i\alpha}\lambda_{j\beta}\,,$$

$$[i\, j] \;\equiv\; \left\langle i^{+}|j^{-}\right\rangle = \varepsilon^{\dot\alpha\dot\beta}\tilde\lambda_{j\dot\alpha}\tilde\lambda_{i\dot\beta}$$

Explicit representation

$$\lambda^{a} = \begin{pmatrix} \sqrt{k_-}e^{i\phi_k} \\ -\sqrt{k_+} \end{pmatrix}, \quad \tilde\lambda^{\dot a} = \begin{pmatrix} \sqrt{k_-}e^{-i\phi_k} \\ -\sqrt{k_+} \end{pmatrix}$$

where $e^{\pm i\phi_k} = \dfrac{k^1 \pm ik^2}{\sqrt{k_+k_-}}, \qquad k_\pm = k^0 \pm k^3$

Properties of the Spinor Product

- Antisymmetry $\langle j\, i\rangle = -\langle i\, j\rangle\,, \qquad [j\, i] = -[i\, j]$
- Gordon identity $\langle i^{\pm}|\, \sigma^{\mu}\, |i^{\pm}\rangle = 2k_i^{\mu}$
- Charge conjugation $\langle i^{-}|\, \sigma^{\mu}\, |j^{-}\rangle = \langle j^{+}|\, \sigma^{\mu}\, |i^{+}\rangle$
- Fierz identity $\langle i^{-}|\, \sigma^{\mu}\, |j^{-}\rangle \langle r^{+}|\, \sigma_{\mu}\, |q^{+}\rangle = 2\,\langle i\, q\rangle\, [r\, j]$
- Projector representation $|i^{\pm}\rangle\langle i^{\pm}| = \frac{1}{2}(1 \pm \gamma_5)\not{k}_i$
- Schouten identity $\langle i\, j\rangle\, \langle p\, q\rangle = \langle i\, q\rangle\, \langle p\, j\rangle + \langle i\, p\rangle\, \langle j\, q\rangle\ .$

Spinor Helicity

Gauge bosons also have only ± physical polarizations

Elegant — and covariant — generalization of circular polarization

$$\varepsilon_\mu^+(k,q) = \frac{\langle q^- | \sigma_\mu | k^- \rangle}{\sqrt{2}\,\langle q\,k\rangle}\,, \qquad \varepsilon_\mu^-(k,q) = \frac{\langle q^+ | \sigma_\mu | k^+ \rangle}{\sqrt{2}\,[k\,q]}\,,$$

'Chinese Magic'

Xu, Zhang, Chang (1984)

reference momentum ***q*** $\quad q \cdot k \neq 0$

Transverse $\quad k \cdot \varepsilon^{\pm}(k,q) = 0$

Normalized $\quad \varepsilon^+ \cdot \varepsilon^- = -1, \qquad \varepsilon^+ \cdot \varepsilon^+ = 0$

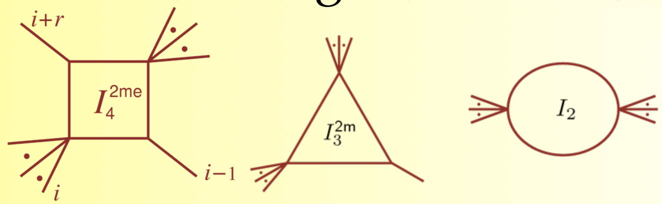
Color Decomposition

With spinors in hand, we can write a color decomposition formula

$$\mathcal{A}_n^{\text{tree}}(\{\lambda_i, \tilde{\lambda}_i, h_i(\pm), a_i\}) =$$
$$g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{Tr}\left(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \cdots T^{a_{\sigma(n)}}\right)$$
$$\times A_n^{\text{tree}}\left(\{\lambda_{\sigma(1)}, \tilde{\lambda}_{\sigma(1)}\}^{h_{\sigma(1)}}; \{\lambda_{\sigma(2)}, \tilde{\lambda}_{\sigma(2)}\}^{h_{\sigma(2)}}; \ldots\right.$$
$$\left.\{\lambda_{\sigma(n)}, \tilde{\lambda}_{\sigma(n)}\}^{h_{\sigma(n)}}\right)$$

Integral Basis

- At one loop
  - Tensor reductions — Brown–Feynman, Passarino–Veltman
  - Gram determinant identities
  - Boxes, triangles, bubbles, tadpoles



- At higher loops
  - Tensor reductions & Gram determinant identities
  - Irreducible numerators: Integration by parts — Chetyrkin–Tkachov
  - Laporta algorithm
  - AIR (Anastasiou,Lazopoulos), FIRE (Smirnov,Smirnov), Reduze (Manteuffel, Studerus), LiteRed (Lee)
  - \`Four-dimensional basis’: integrals with up to 4 L propagators

BCFW On-Shell Recursion Relations

Britto, Cachazo, Feng, Witten (2005)

- Define a shift $[j, l\rangle$ of spinors by a complex parameter z

$$\begin{aligned} |j] &\rightarrow |j] - z|l], \\ |l\rangle &\rightarrow |l\rangle + z|j\rangle \end{aligned}$$

- which induces a shift of the external momenta

$$\begin{aligned} k_j^\mu &\rightarrow k_j^\mu(z) = k_j^\mu - \frac{z}{2}\langle j|\gamma^\mu|l], \\ k_l^\mu &\rightarrow k_l^\mu(z) = k_l^\mu + \frac{z}{2}\langle j|\gamma^\mu|l] \end{aligned}$$

  and defines a z-dependent continuation of the amplitude $A(z)$
- Assume that $A(z) \rightarrow 0$ as $z \rightarrow \infty$

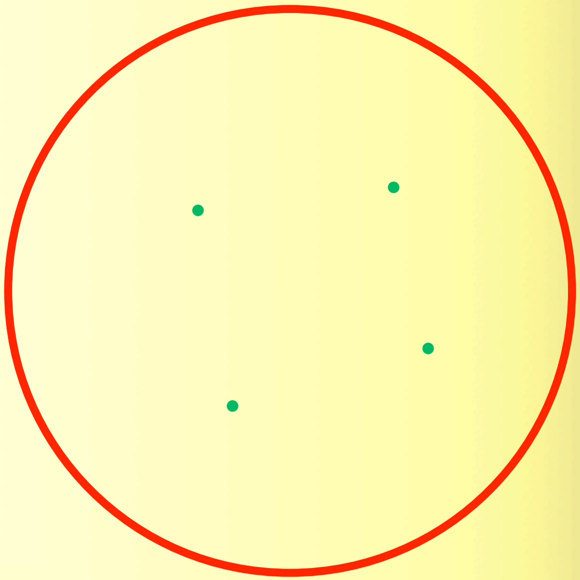
- Momenta are still on shell
- Momentum is still conserved

A Contour Integral

Consider the contour integral

$$\frac{1}{2\pi i}\oint_C \frac{dz}{z}\, A(z)$$

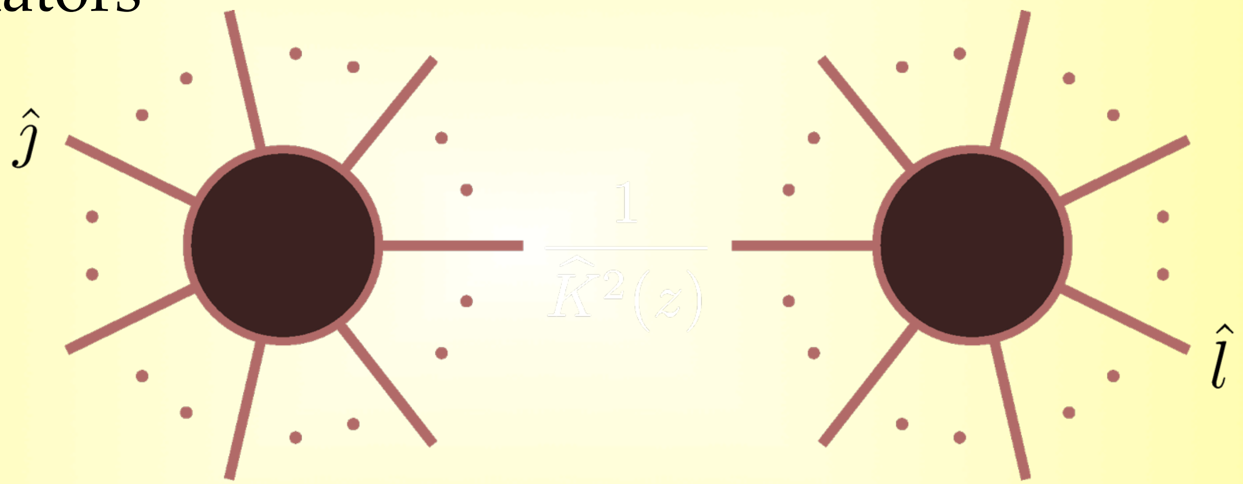
Determine A(0) in terms of other residues



$$A(0) = -\sum_{\text{poles }\alpha} \operatorname*{Res}_{z=z_\alpha} \frac{A(z)}{z}$$

Using Factorization

Other poles in z come from zeros of z-shifted propagator denominators



Splits diagram into two parts with z-dependent momentum flow

$$\longrightarrow \sum_{\text{partitions}} \begin{array}{c}\text{shifted legs on}\\ \text{opposite sides}\end{array}$$

Exactly factorization limit of z-dependent amplitude poles from zeros of

$$K^2_{a\cdots j\cdots b}(z) = K^2_{a\cdots b} - z \langle j|\, \not{K}_{a\cdots b}\, |l]$$

That is, a pole at

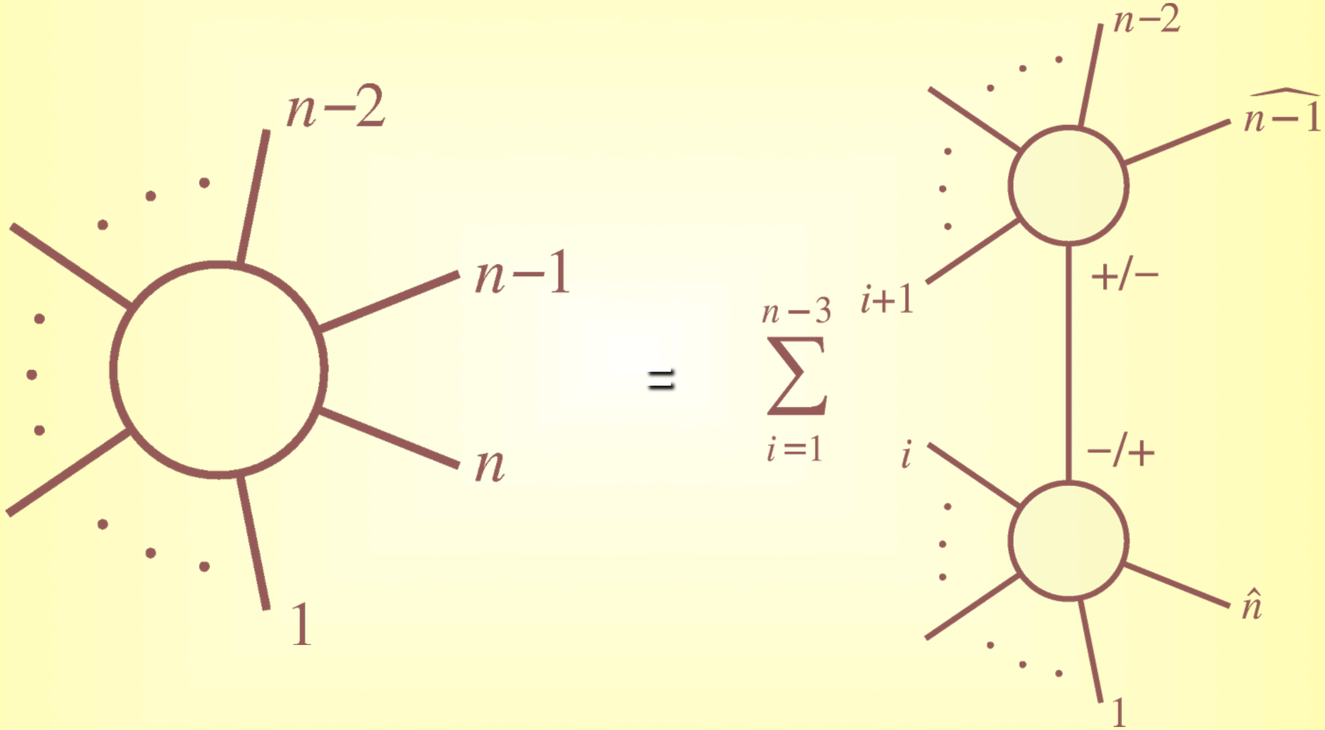
$$z_{ab} = \frac{K^2_{a\cdots b}}{\langle j|\, \not{K}_{a\cdots b}\, |l]}$$

Residue

$$\operatorname*{Res}_{z=z_{ab}} \frac{f(z)}{z\, K^2_{a\cdots b}(z)} = A_L(z_{ab}) \times \frac{i}{K^2_{a\cdots b}} \times A_R(z_{ab})$$

Notation $k = k(z_{ab})$

On-Shell Recursion Relation



- Partition *P*: two or more cyclicly-consecutive momenta containing *j*, such that complementary set $\overline{P}$ contains *l*,

$$
\begin{aligned}
P &\equiv \{P_1, P_2, \ldots, j, \ldots, P_{-1}\}, \\
\overline{P} &\equiv \{\overline{P}_1, \overline{P}_2, \ldots, l, \ldots, \overline{P}_{-1}\}, \\
P \cup \overline{P} &= \{1, 2, \ldots, n\}
\end{aligned}
$$

- The recursion relations are then

On shell

$$
\begin{aligned}
A_n(1, \ldots, n) &= \sum_{\substack{\text{partitions P} \\ \text{h}=\pm}} A_{\#P+1}(k_{P_1}, \ldots, \hat{j}, \ldots, k_{P_{-1}}, -\widehat{P}^{\,h}) \\
&\quad \times \frac{i}{P^2} \times A_{\#\overline{P}+1}(k_{\overline{P}_1}, \ldots, \hat{l}, \ldots, k_{\overline{P}_{-1}}, \widehat{P}^{\,-h})
\end{aligned}
$$

Unitarity

- Basic property of any quantum field theory: conservation of probability. In terms of the scattering matrix,

$$S^\dagger S = 1$$

In terms of the transfer matrix $iT = S - 1$ we get,

$$-i(T - T^\dagger) = T^\dagger T$$

or

$$2\,\text{“Im”}\,T_{fi} = (T^\dagger T)_{fi}$$

with the Feynman $i\varepsilon$

$$\mathrm{Disc}\, T = T^\dagger T$$

Diagrammatically, cut into two parts using Cutkosky rule

$$\frac{1}{\ell^2 - m^2 + i\delta} \quad \longrightarrow \quad -2\pi i \delta^{(+)}(\ell^2 - m^2)$$
$$= -2\pi i \delta(\ell^2 - m^2)\Theta(\ell^0)$$

Gedanken calculation

$$\underset{K^2}{\text{Disc}} \text{ One-Loop Amplitude} = \sum_{\substack{\text{One-Loop} \\ \text{Diagrams}} F} \underset{K^2}{\text{Disc}}\, F$$

Some diagrams are missing one or both propagators surrounding K^2:

$$\frac{\ell^2 - m^2 + i\delta}{\ell^2 - m^2 + i\delta}$$
$$\mapsto (\ell^2 - m^2 + i\delta)\delta(\ell^2 - m^2)$$
$$= 0$$

→ no contribution

Also fate of "off-shell" terms

$$\mathop{\text{Disc}}_{K^2} \text{One-Loop Amplitude} =$$

$$\sum_{\substack{\text{One-Loop}\\ \text{Diagrams}\\ \text{with both propagators}}F} \mathop{\text{Disc}}_{K^2} F$$

$$= \int d\text{Phase Space} \left(\sum_{\substack{\text{Left Tree}\\ \text{Diagrams}}L} L \right) \left(\sum_{\substack{\text{Right Tree}\\ \text{Diagrams}}R} R \right)$$

$$= \int d\text{Phase Space}\ (\text{Left Tree Amplitude})$$

$$\times (\text{Right Tree Amplitude})$$

Basic Unitarity

- Can reverse this approach to reconstruct amplitude from its discontinuities
- Look at all channels
- At one loop, each discontinuity comes from putting two propagators on shell, that is looking for all contributions with two specified propagators

Unitarity Method

Formalism

$$\text{Amplitude} = \sum_{j \in \text{Basis}} c_j \, \text{Int}_j + \text{Rational}$$

Rational function of spinors

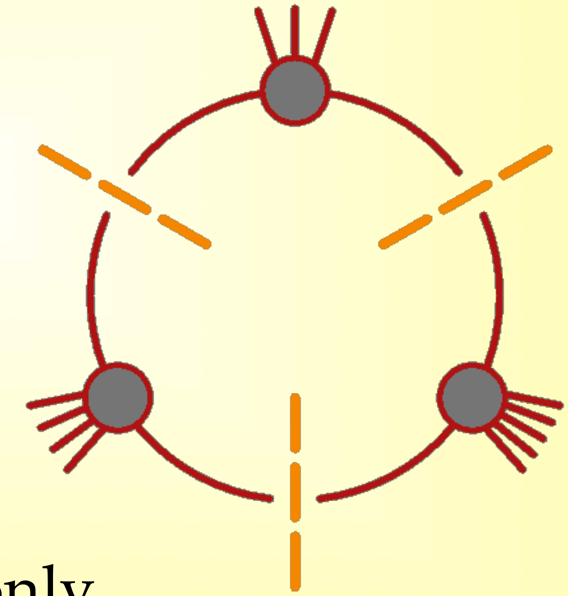
Known integral basis:

Unitarity in $D = 4$

On-shell Recursion;
D-dimensional unitarity
via ∫ mass

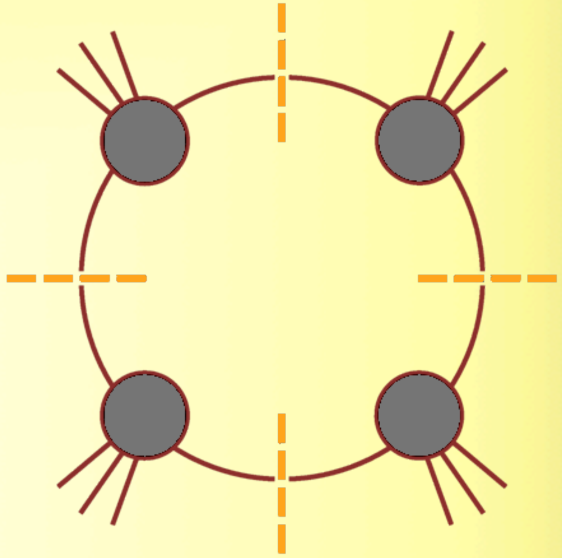
Generalized Unitarity

- Unitarity picks out contributions with two specified propagators
- Can we pick out contributions with *more* than two specified propagators?
- Yes — cut more lines



- Isolates smaller set of integrals: only integrals with propagators corresponding to cuts will show up
- Triple cut — no bubbles, one triangle, smaller set of boxes

- Can we isolate a single integral?
- D = 4 → loop momentum has four components
- Cut four specified propagators (quadruple cut) would isolate a single box



Quadruple Cuts

Work in D=4 for the algebra

$$\int \frac{d^4\ell}{(2\pi)^4}\, \delta^{(+)}(\ell^2)\, \delta^{(+)}((\ell-k_1)^2)\, \delta^{(+)}((\ell-K_{12})^2)\, \delta^{(+)}((\ell-K_{123})^2)$$

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2(\ell-k_1)^2(\ell-K_{12})^2(\ell-K_{123})^2}$$

Four degrees of freedom & four delta functions

… but are there any solutions?

A Subtlety

The delta functions instruct us to solve

$$\ell^2 = 0\,,\quad (\ell - k_1)^2 = 0\,,\quad (\ell - K_{12})^2 = 0\,,\quad (\ell + k_4)^2 = 0\,.$$

1 quadratic, 3 linear equations ⇒ 2 solutions

If k_1 and k_4 are massless, we can write down the solutions explicitly

$$\ell^\mu = \frac{\xi}{2} \left\langle 1^- \right| \mu \left| 4^- \right\rangle$$ solves eqs 1,2,4;

Impose 3rd to find $s_{12} = \frac{\xi}{2} \left\langle 1^- \right| 2 \left| 4^- \right\rangle \stackrel{m_2^2=0}{=} \frac{\xi}{2} \langle 1\, 2 \rangle\, [2\, 4]$

or $\ell^\mu = -\frac{[1\,2]}{2\,[2\,4]} \left\langle 1^- \right| \mu \left| 4^- \right\rangle$ $\qquad$ $\ell^\mu = \frac{\xi'}{2} \left\langle 4^- \right| \mu \left| 1^- \right\rangle$

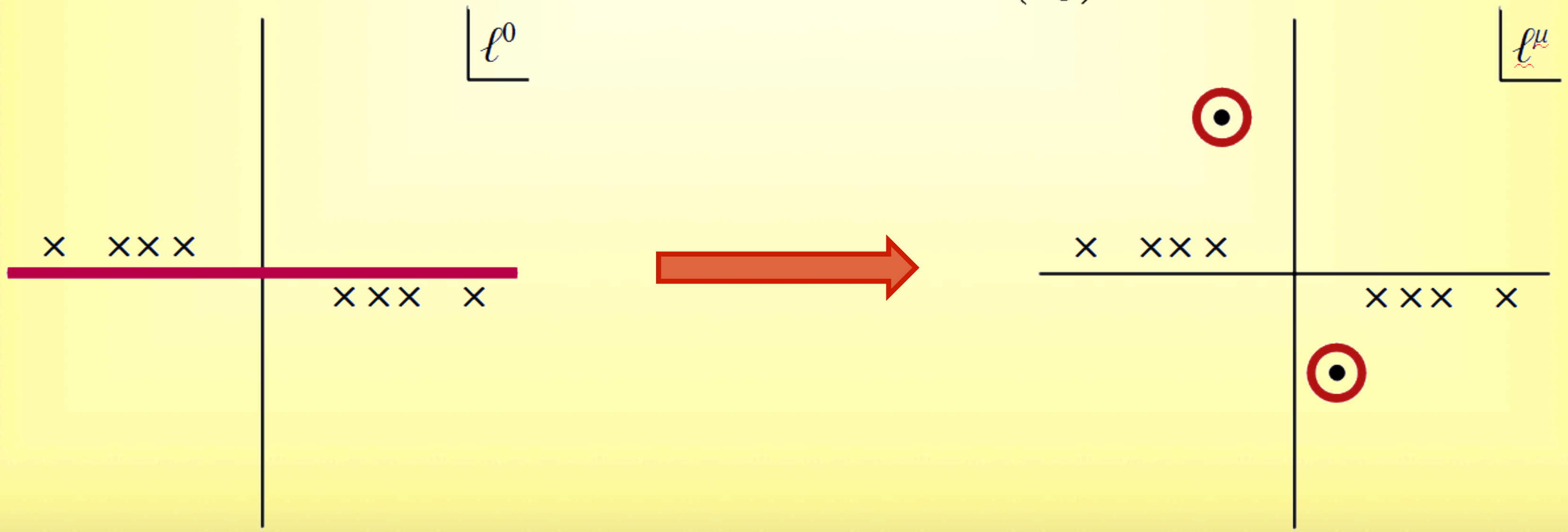
- Solutions are complex
- The delta functions would actually give zero!

Need to reinterpret delta functions as contour integrals around a global pole [other contexts: Vergu; Roiban, Spradlin, Volovich; Mason & Skinner]

Reinterpret cutting as contour modification

$$\oint_{C(z_0)} dz \, \frac{\mathrm{Poly}_1(z)}{\mathrm{Poly}_2(z) - a} = \frac{\mathrm{Poly}_1(z_0)}{\mathrm{Poly}_2'(z_0)} \qquad (\mathrm{Poly}_2(z_0) = a)$$

$$\int dz \, \mathrm{Poly}_1(z) \delta(\mathrm{Poly}_2(z) - a) \equiv \oint_{C(z_0)} dz \, \frac{\mathrm{Poly}_1(z)}{\mathrm{Poly}_2(z) - a}$$



- Global poles: simultaneous on-shell solutions of all propagators & perhaps additional equations

- Multivariate complex contour integration: in general, contours are tori

- For one-loop box, contours are T^4 encircling global poles

Two Problems

- Too many contours (2) for one integral: how should we choose the contour we use?

- Changing the contour can break equations:

$$0 = I_4[\varepsilon(\ell, k_1, k_2, k_4)]$$

  is no longer true if we modify the real contour to circle only one of the poles

Remarkably, these two problems cancel each other out

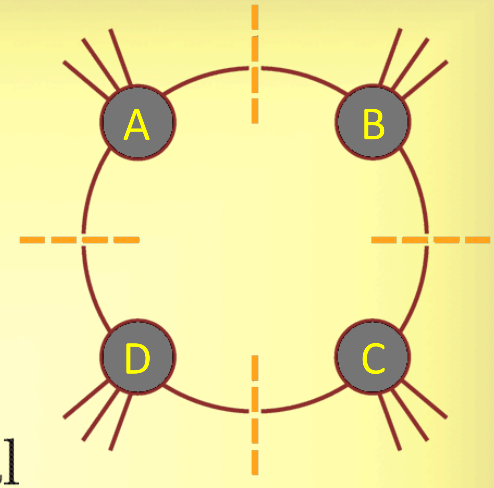
- Require vanishing Feynman integrals to continue vanishing on cuts

- General contour $\mathcal{C} = a_1\mathcal{C}_1 + a_2\mathcal{C}_2$

$$\int_{\mathcal{C}} d^4\ell \, \frac{\varepsilon(\ell, k_1, k_2, k_4)}{\ell^2(\ell - k_1)^2(\ell - K_{12})^2(\ell + k_4)^2} = (a_1 - a_2) f(k_1, k_2, k_4)$$

$\Rightarrow a_1 = a_2$

Box Coefficient



Go back to master equation

$$\text{Amplitude} = \sum_{j \in \text{Basis}} c_j \, \text{Int}_j + \text{Rational}$$

Apply quadruple cuts to both sides

$$\text{LHS} = \text{Jacobian} \times \sum_{\text{solutions}} \sum_{\substack{\text{species} \\ \text{helicities}}} A_A^{\text{tree}} A_B^{\text{tree}} A_C^{\text{tree}} A_D^{\text{tree}}$$

$$\text{RHS} = \text{coefficient} \times \text{Jacobian} \times \#\text{solutions}$$

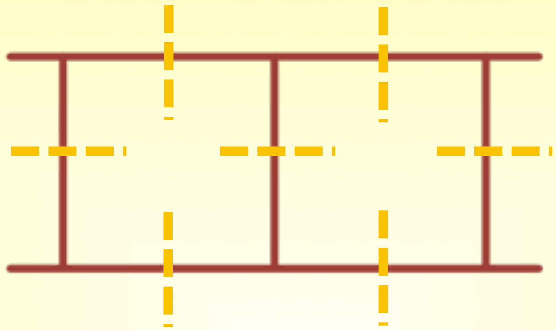
Solve: $\text{Box coefficient} = \frac{1}{\#\,\text{solutions}} \sum_{\text{solutions}} \sum_{\substack{\text{species} \\ \text{helicities}}} \prod_J A_J^{\text{tree}}$

Britto, Cachazo, Feng

No algebraic reductions needed: suitable for pure numerics

Planar Double Box

- Take a heptacut — freeze seven of eight degrees of freedom



- One remaining integration variable z
- Six solutions, for example S_2:

$$\ell_1^\mu = k_1^\mu + \frac{s_{12} z}{2 \langle 1\, 4\rangle\, [4\, 2]} \langle 1|\, \sigma^\mu\, |2] \,,$$

$$\ell_2^\mu = \frac{\langle 3\, 4\rangle}{2 \langle 3\, 1\rangle} \langle 1|\, \sigma^\mu\, |4]$$

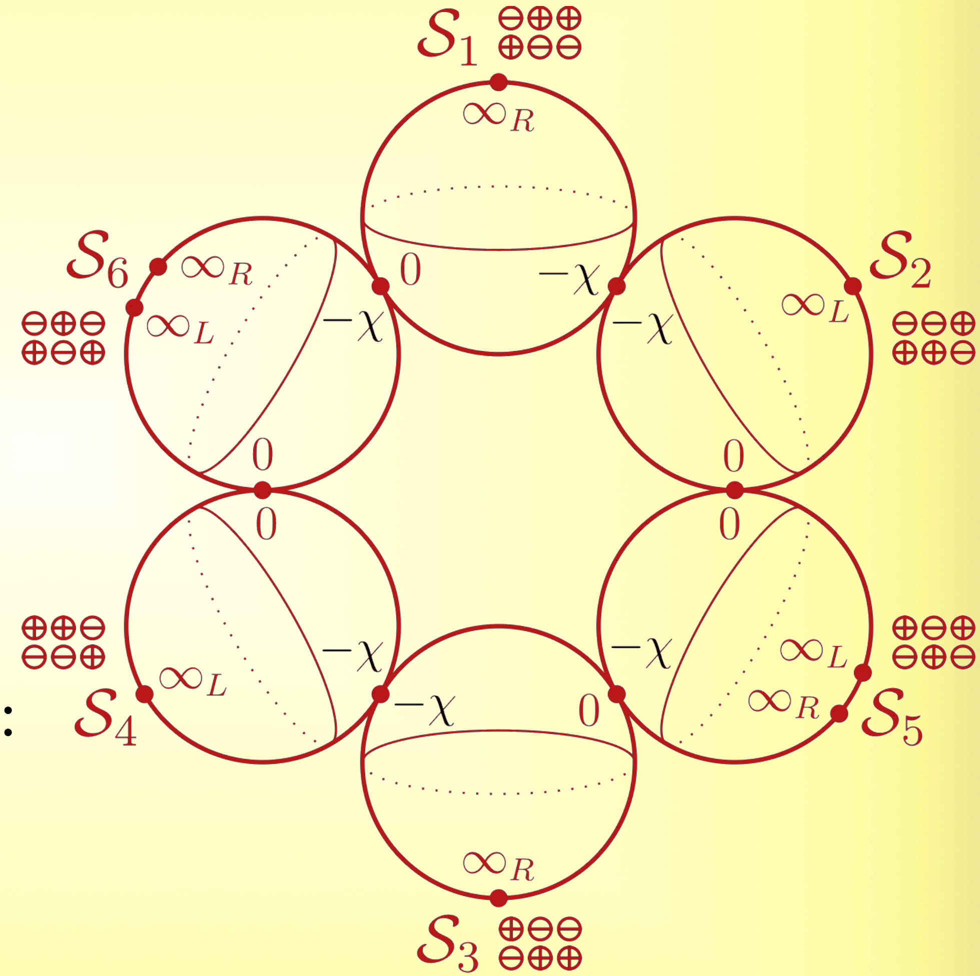
- Performing the contour integrals enforcing the heptacut ⇒ Jacobian

$$\mathcal{S}_2 : \quad \frac{1}{16 s_{12}^3\, z(z+\chi)} \qquad (\chi \equiv t/s)$$

- Localizes z ⇒ global pole ⇒ need contour for z within S_i

- But:
  - ➢ Solutions intersect at 6 poles
  - ➢ 6 other poles are redundant by Cauchy theorem ($\sum$ residues = 0)

- Overall, we are left with 8 global poles (massive legs: none; 1; 1 & 3; 1 & 4)
- Connections to algebraic geometry



- Two master integrals
- 4 ε constraint equations
- 2 IBP constraint equations

$\Rightarrow$ Two master contours – one for each integral

- Master formulæ for coefficients of basis integrals to $O(\varepsilon^0)$

$$c_1 = \frac{i\chi}{8w}\oint_{P_1}\frac{dz}{z(z+\chi)}\prod_{j=1}^{6}A_j^{\rm tree}(z)\,,\qquad c_2 = -\frac{i}{4s_{12}w}\oint_{P_2}\frac{dz}{z(z+\chi)}\prod_{j=1}^{6}A_j^{\rm tree}(z)$$

where $P_{1,2}$ are linear combinations of T^8s around global poles

$$P_1:\quad \begin{aligned} a_{1,1} &= w\,,\\ a_{2,1} &= -w\,,\\ a_{1,2} &= 0\,,\\ a_{1,3} &= w\,,\\ a_{2,3} &= -w\,,\\ a_{1,4} &= 0\,,\\ a_{3,5} &= -w\,,\\ a_{3,6} &= -w\,. \end{aligned}\qquad\qquad P_2:\quad \begin{aligned} a_{1,1} &= w\,,\\ a_{2,1} &= -w\,,\\ a_{1,2} &= -2w\,,\\ a_{1,3} &= w\,,\\ a_{2,3} &= -w\,,\\ a_{1,4} &= -2w\,,\\ a_{3,5} &= -3w\,,\\ a_{3,6} &= -3w\,. \end{aligned}$$

More explicitly,

$$c_1 = \frac{i}{8}\operatorname*{Res}_{z=0}\frac{1}{z}\prod_{j=1}^{6}A_j^{\rm tree}(z)\bigg|_{\mathcal{S}_1+\mathcal{S}_3} + \frac{i}{8}\operatorname*{Res}_{z=-\chi}\frac{1}{z+\chi}\prod_{j=1}^{6}A_j^{\rm tree}(z)\bigg|_{\mathcal{S}_1+\mathcal{S}_3} - \frac{i\chi}{8(1+\chi)}\operatorname*{Res}_{z=-\chi-1}\prod_{j=1}^{6}A_j^{\rm tree}(z)\bigg|_{\mathcal{S}_5+\mathcal{S}_6}\,,$$

$$c_2 = \frac{i}{4s_{12}\chi}\operatorname*{Res}_{z=0}\frac{1}{z}\prod_{j=1}^{6}A_j^{\rm tree}(z)\bigg|_{\mathcal{S}_1-2\mathcal{S}_2+\mathcal{S}_3-2\mathcal{S}_4} + \frac{i}{4s_{12}\chi}\operatorname*{Res}_{z=-\chi}\frac{1}{z+\chi}\prod_{j=1}^{6}A_j^{\rm tree}(z)\bigg|_{\mathcal{S}_1+\mathcal{S}_3} - \frac{3i}{4s_{12}(1+\chi)}\operatorname*{Res}_{z=-\chi-1}\prod_{j=1}^{6}A_j^{\rm tree}(z)\bigg|_{\mathcal{S}_5+\mathcal{S}_6}\,.$$

Summary

- Natural variables for kinematics: spinors
- Factorization can be exploited to obtain on-shell recursion relations
- Unitarity can be generalized to analytic structure, and exploited to compute loop amplitudes

Beyond the basics:

- Differential equation and symbol techniques for higher-loop integrals