Introduction to Perturbative QCD

partons, factorization, resummation, and all that

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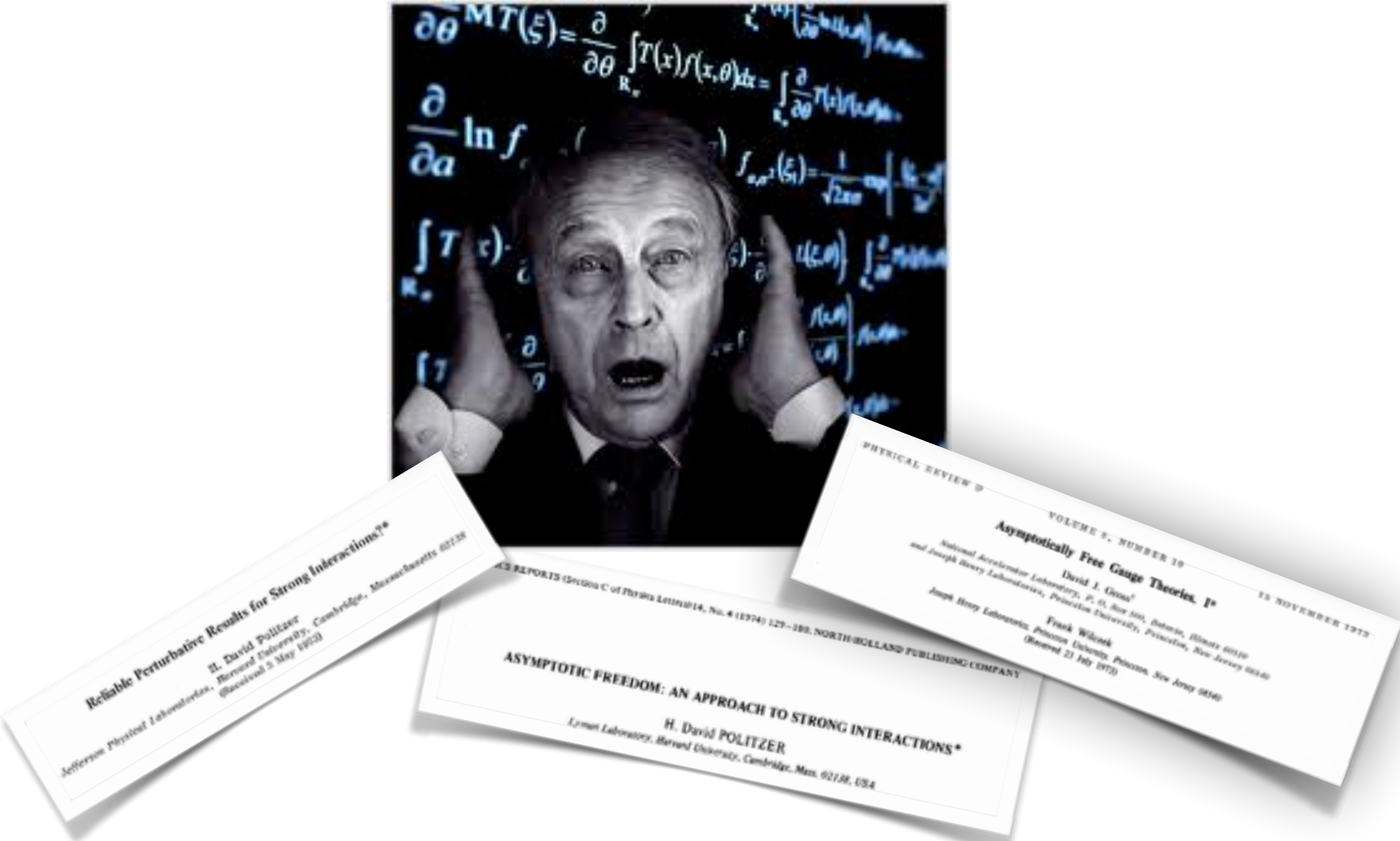
June 30th/ July 1st, 2014

disclaimer:

pQCD is about 40 years old – impossible to review in 6 hrs



Reliable Perturbative Results for Strong Interactions?*

H. David Politzer

Jefferson Physical Laboratories, Harvard University, Cambridge, Massachusetts 02138

(Received 5 May 1973)

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Asymptotically Free Gauge Theories. I*

David J. Gross†

National Accelerator Laboratory, P. O. Box 500, Batavia, Illinois 60510 and Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

Frank Wilczek

Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

(Received 23 July 1973)

CS REPORTS (Section C of Physics Letters) 14, No. 4 (1974) 129–180. NORTH-HOLLAND PUBLISHING COMPANY

ASYMPTOTIC FREEDOM: AN APPROACH TO STRONG INTERACTIONS*

H. David POLITZER

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topics & questions to be addressed

we will mainly concentrate on a few basics
and their consequences for phenomenology

- What are the foundations of QCD?
  keywords: color; SU(3) gauge group; local gauge invariance; Feynman rules
- What are the general features of QCD?
  keywords: asymptotic freedom; infrared safety; origin of "singularities"
- How to relate QCD to experiment?
  keywords: partons; factorization; renormalization group eqs. / evolution
- How reliable is a theoretical QCD calculation?
  keywords: scale dependence; NLO; small-x; all-order resummations
- What is the status of some non-perturbative inputs
  keywords: global QCD analysis

throughout this will be blended with discussions of some recent results and
advanced topics relevant for LHC, RHIC, HERA, COMPASS, EIC, …

bibliography – a personal selection

textbooks:

- the "pink book" on QCD and Collider Physics by R.K. Ellis, W.J. Stirling, and B.R. Webber always a good reference
- R.D. Field, Applications of pQCD detailed examples
- Y.V. Kovchegov, E. Levin, QCD at High Energy focus on small x physics
- J. Collins, Foundations of pQCD focus on formal aspects of evolution

lecture notes & write-ups:

- D. Soper, Basics of QCD Perturbation Theory, hep-ph/9702203
- Collins, Soper, Sterman, Factorization of Hard Processes in QCD, hep-ph/0409313
- G. Salam, Elements of QCD for Hadron Colliders, arXiv:1011.5131
- Particle Data Group, Review of Particle Physics, pdg.lbl.gov

talks & lectures on the web:

e.g. by D. Soper; G. Salam; G. Zanderighi; J. Campbell; G. Sterman; …

- annual CTEQ summer school, tons of material on www.cteq.org
- annual CERN/FNAL Hadron Collider Physics School hcpss.web.cern.ch/hcpss

tentative outline of the lectures

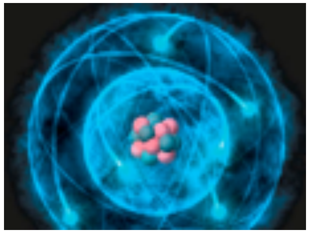
Part 1: the foundations

SU(3); color algebra; gauge invariance;
QCD Lagrangian; Feynman rules

Part 2: the QCD toolbox

asymptotic freedom; infrared safety;
the QCD final-state; jets; factorization

Part 3: inward bound: "femto spectroscopy"



QCD initial-state; DIS process; partons;
factorization; renormalization group; scales;
hadron-hadron collisions

Part 4: applications:

global analysis of PDFs; scales and theoretical
uncertainties; all-order resummations; ...

Part I

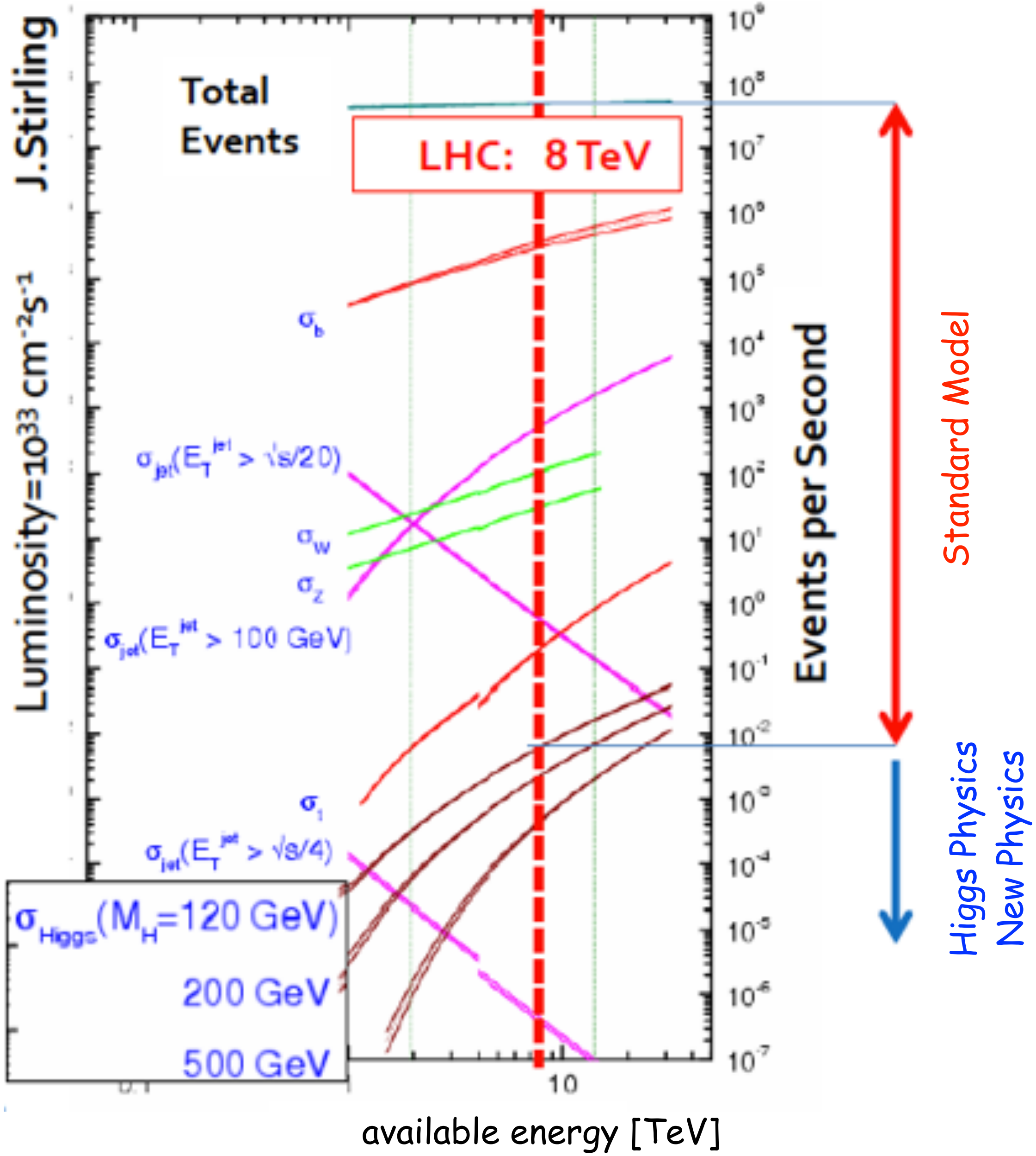
the QCD fundamentals

all about color

the concept of gauge invariance

shamelessly adapted
some material from
excellent QCD lectures by
G.Zanderighi & J.Campbell

QCD – why do we still care (or perhaps more than ever)



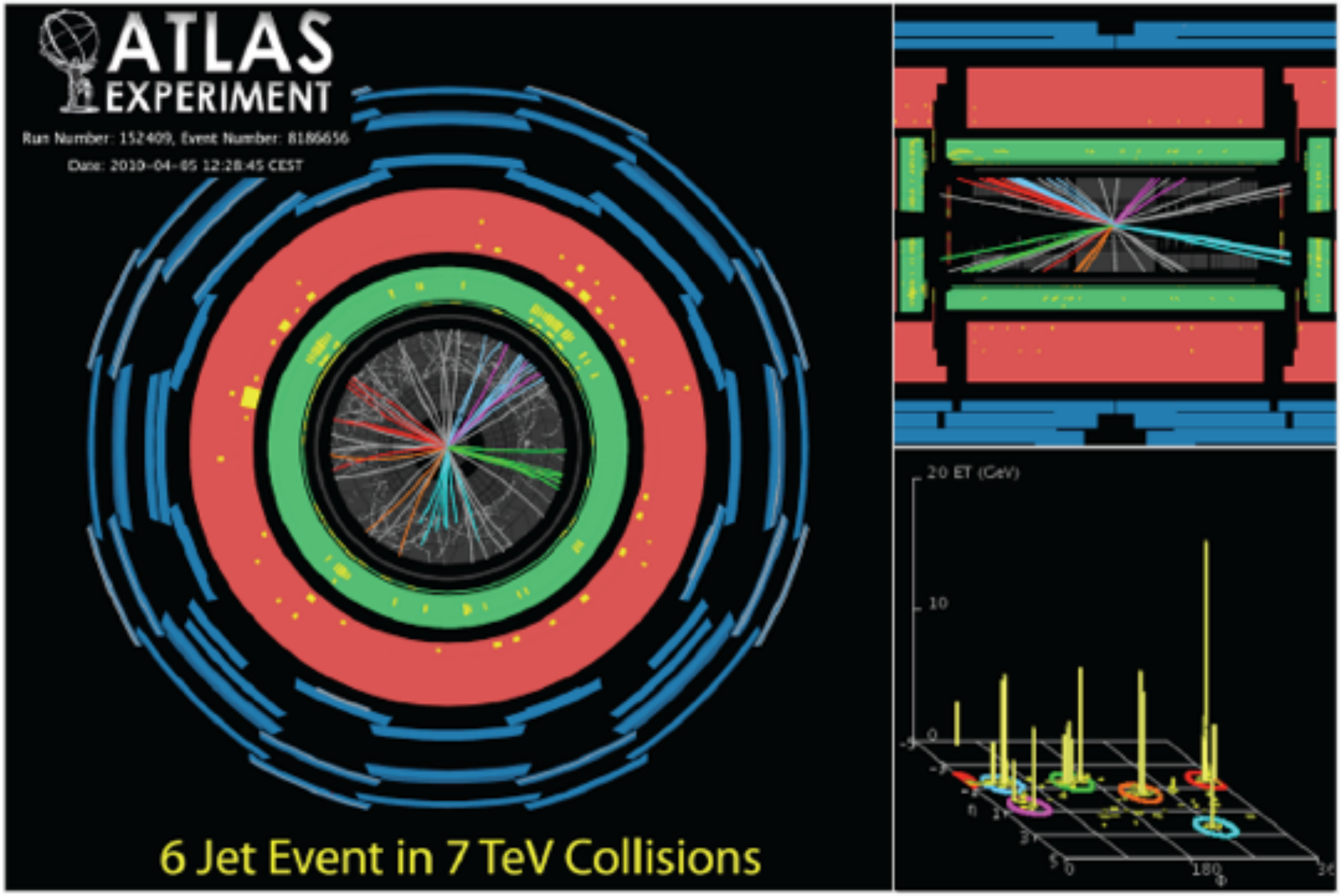
hadron colliders inevitably have to deal with QCD

studying the Higgs boson or discovering (perhaps) some New Physics requires a sophisticated quantitative understanding of QCD

P.W. Higgs, F. Englert (2013)

achieving that can be quite a challenge ...

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} F^{A}_{\mu\nu} F^{\mu\nu}_{A} + \sum_{\mathrm{flavors}} \bar{q}_i \left(i \not{D} - m \right)_{ij} q_j$$



QCD – the theory of strong interactions

a simple QED-like theory, leading to extremely rich & complex phenomena

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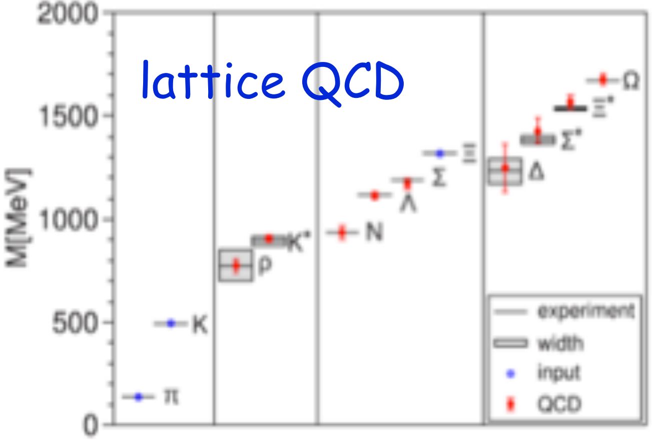
a simple QED-like theory, leading to extremely rich & complex phenomena

Durr et al., Science 322



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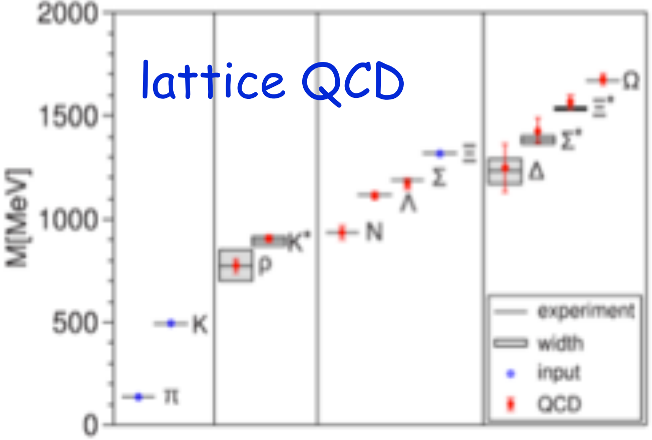
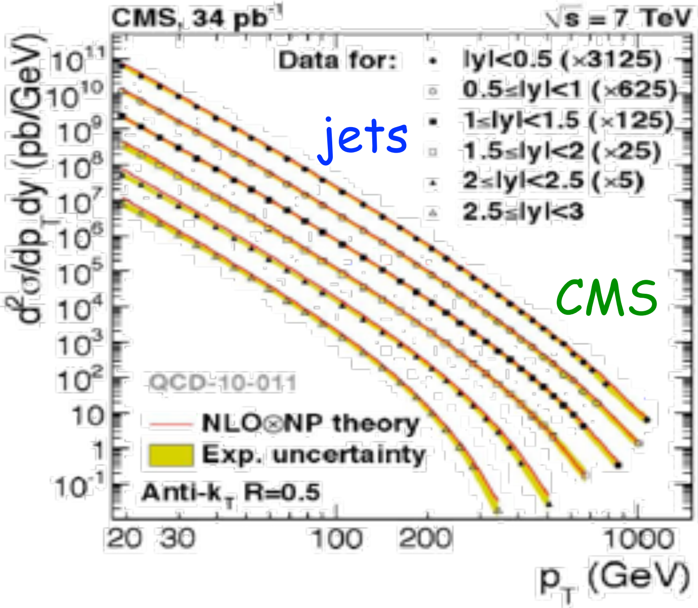


lattice QCD



jets

CMS

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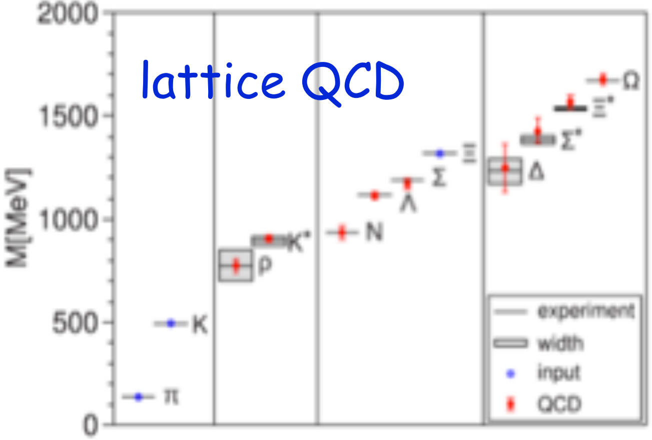

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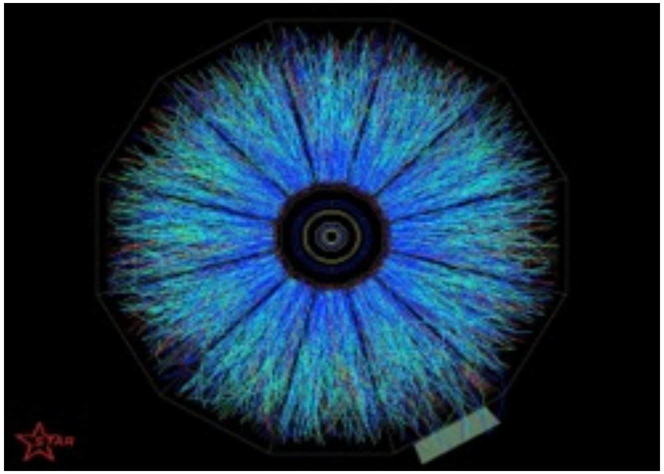
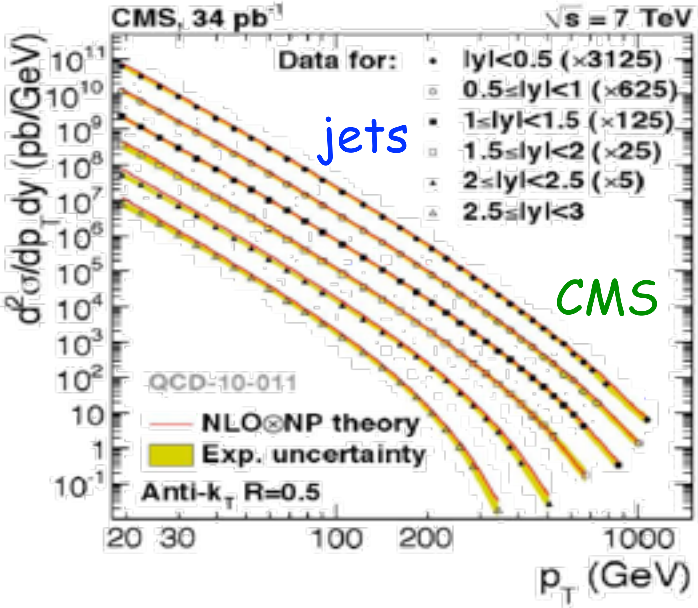


AuAu collision at STAR



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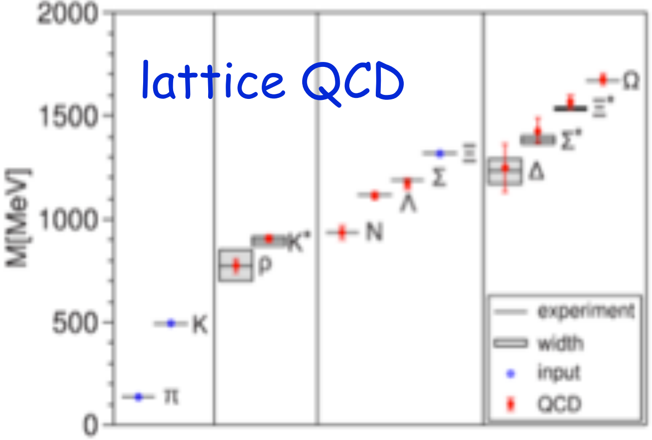
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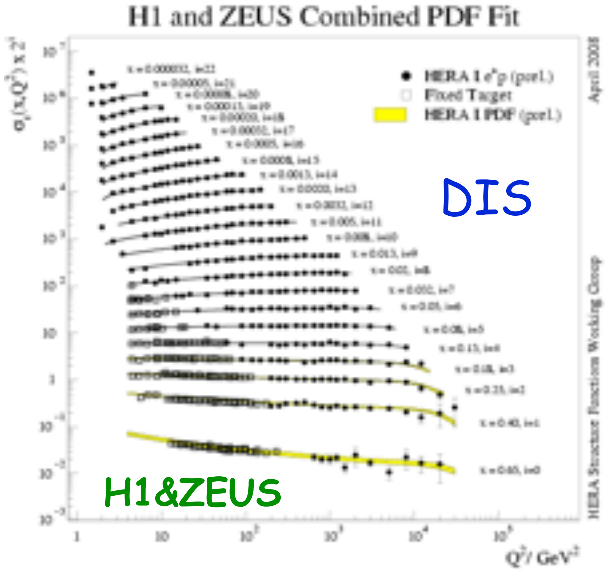
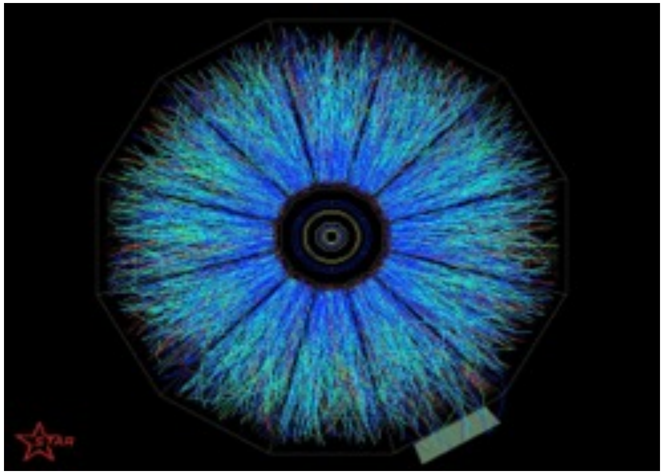

DIS
H1&ZEUS



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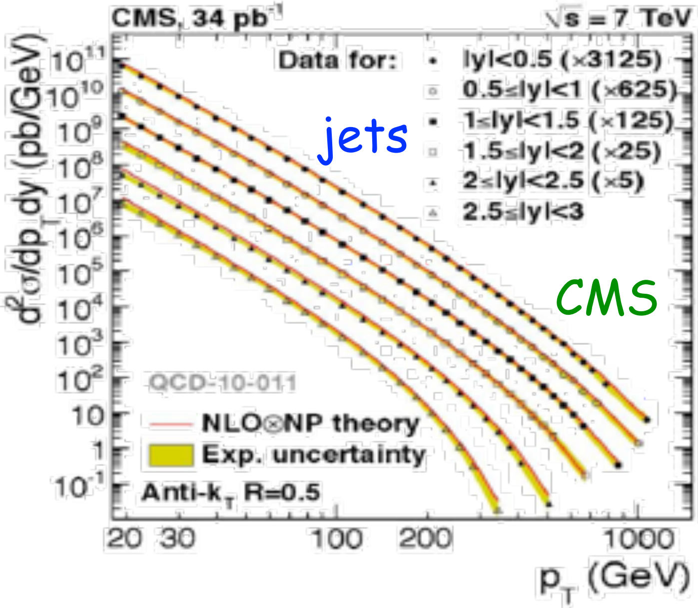
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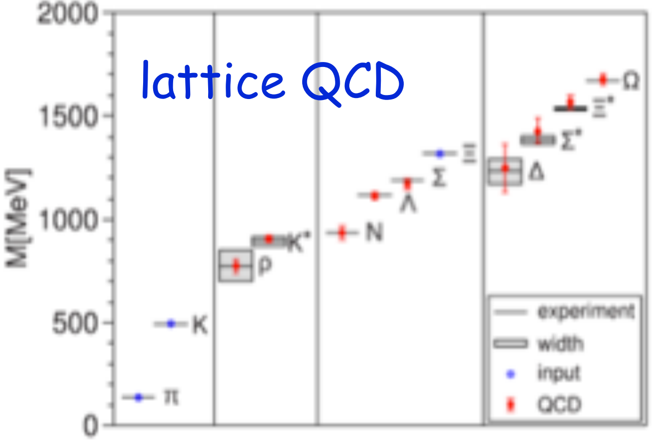
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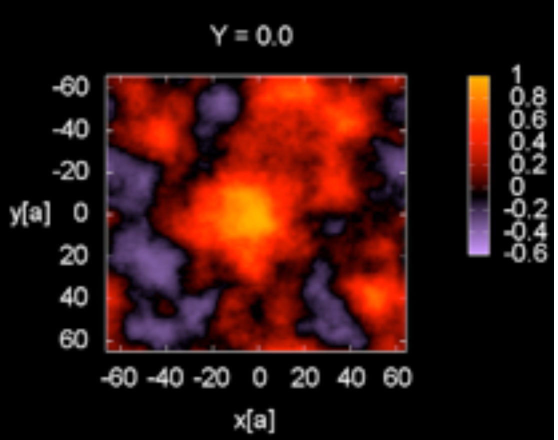
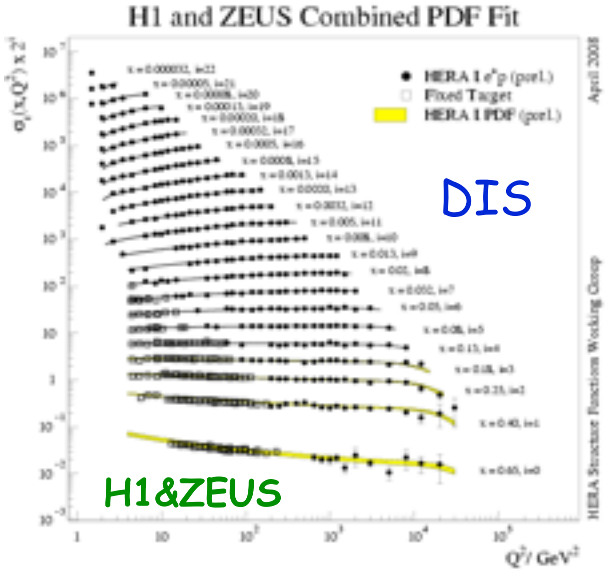


RGE evolution of gluon correlators
arXiv:1108.4764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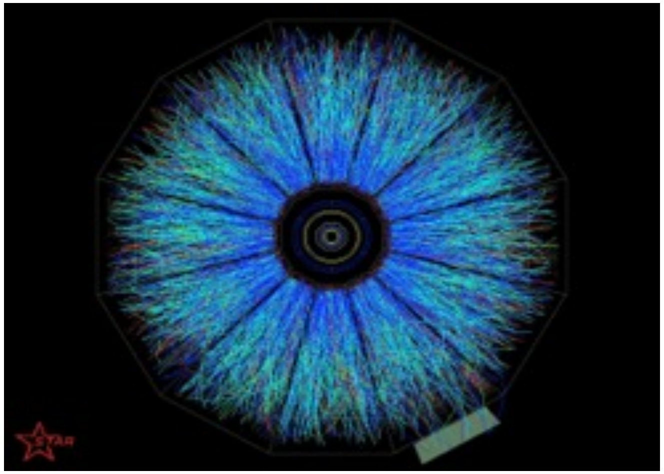

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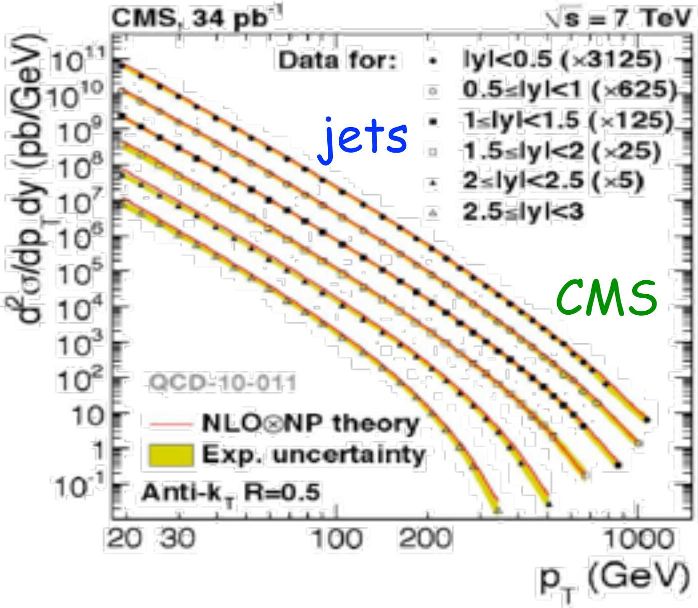
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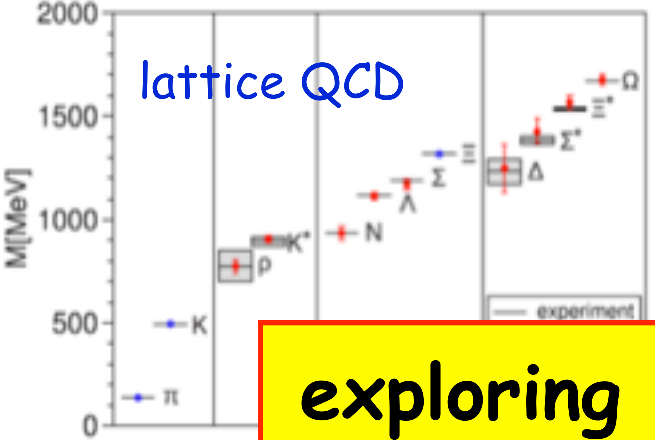

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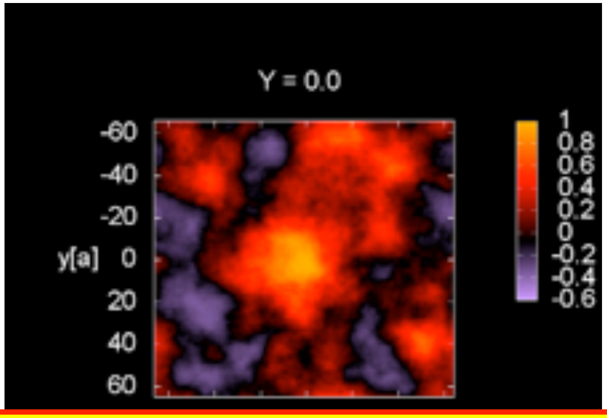
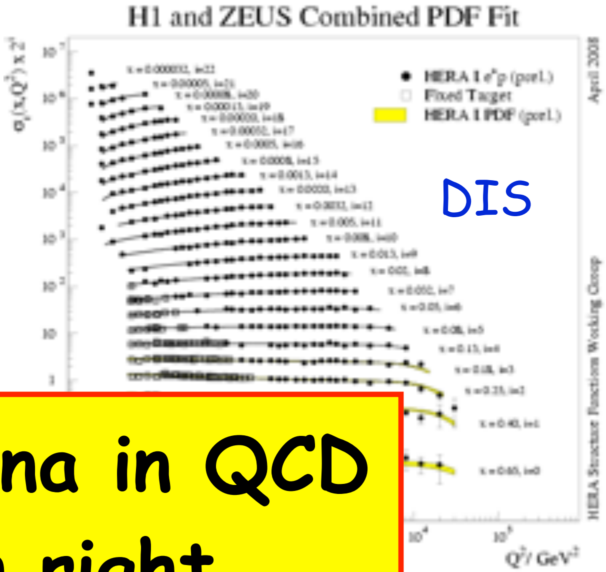
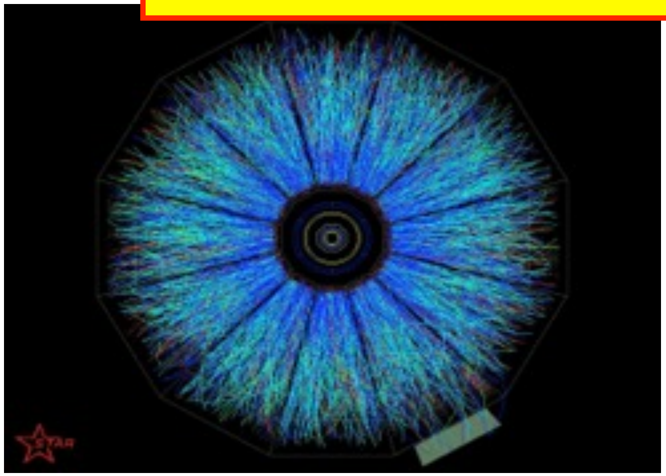




DIS

exploring all these phenomena in QCD is interesting in its own right



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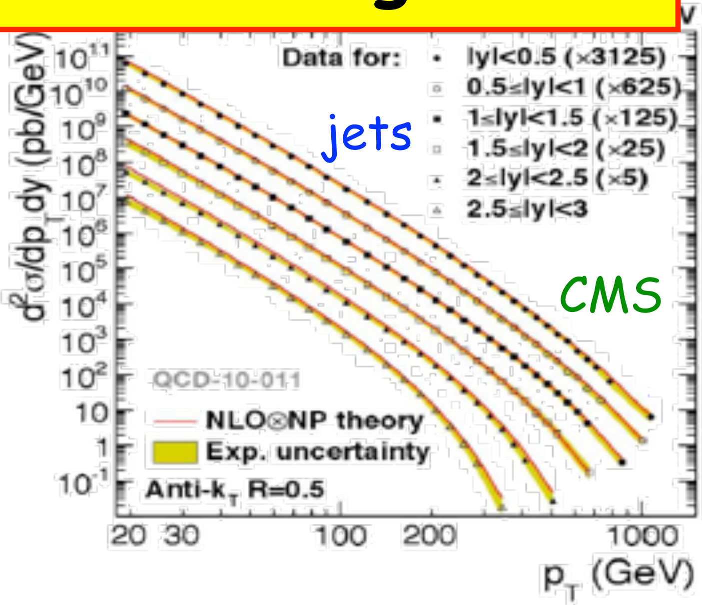
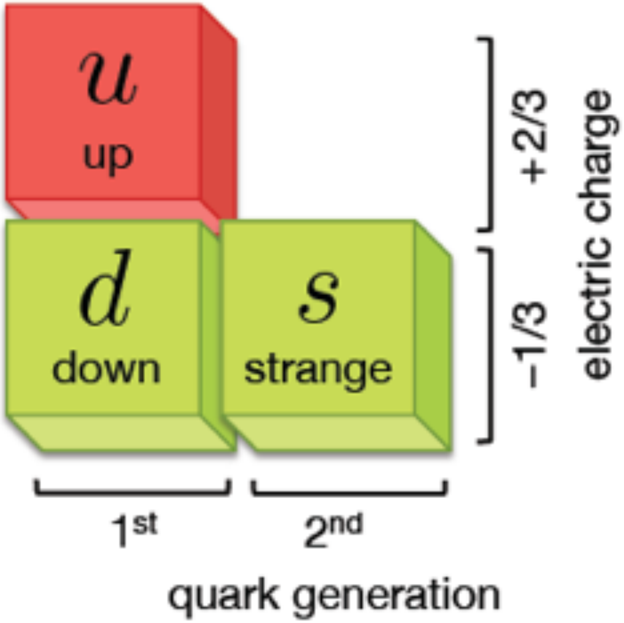
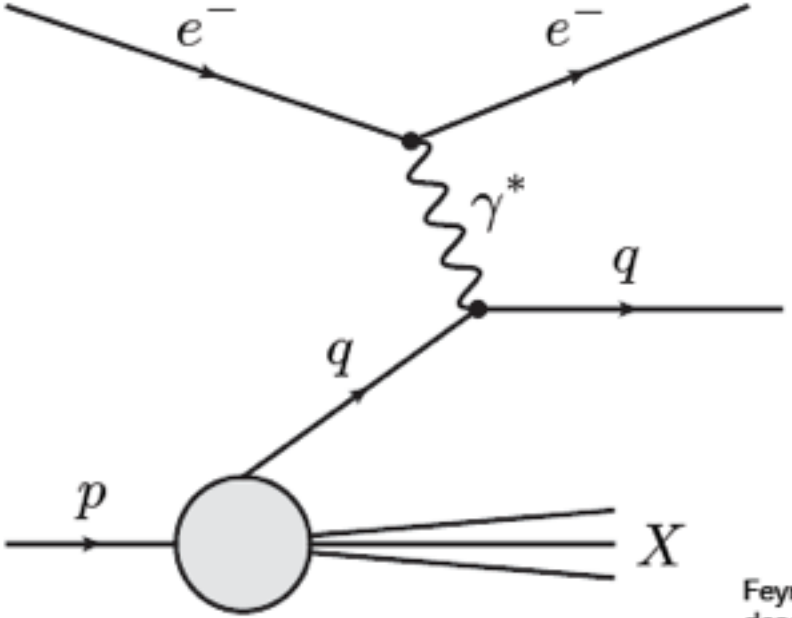


jets

CMS

QCD matter sector: Three Quarks for Muster Mark

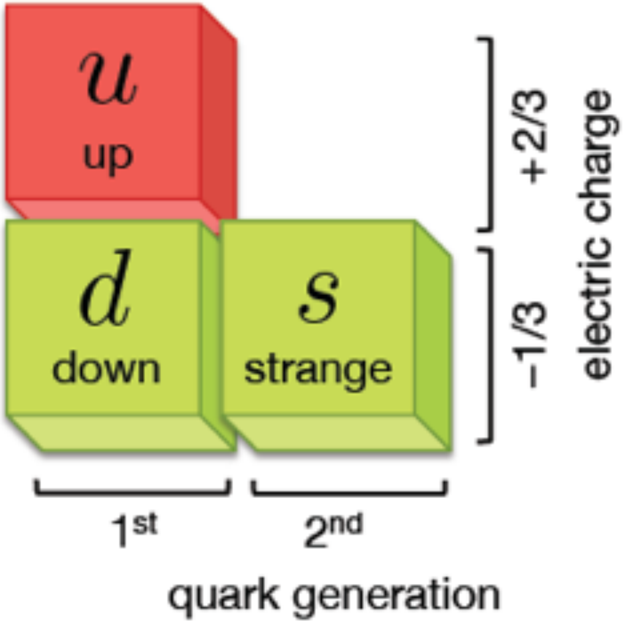


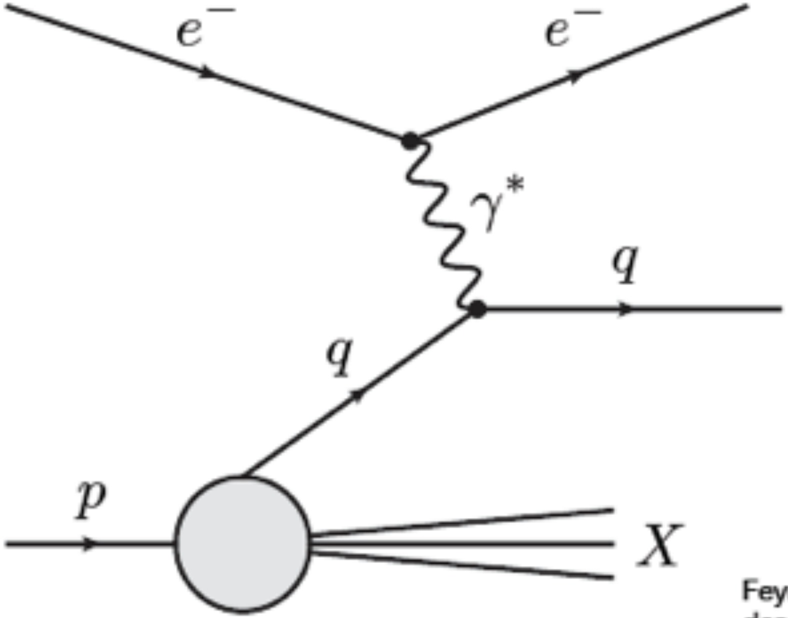


Feynman diagram describing DIS of an electron on a proton

existence of light quarks validated in deep-inelastic scattering (DIS) experiments carried out at SLAC in 1968

QCD matter sector: Three Quarks for Muster Mark





Feynman diagram describing DIS of an electron on a proton

existence of light quarks validated in deep-inelastic scattering (DIS) experiments carried out at SLAC in 1968

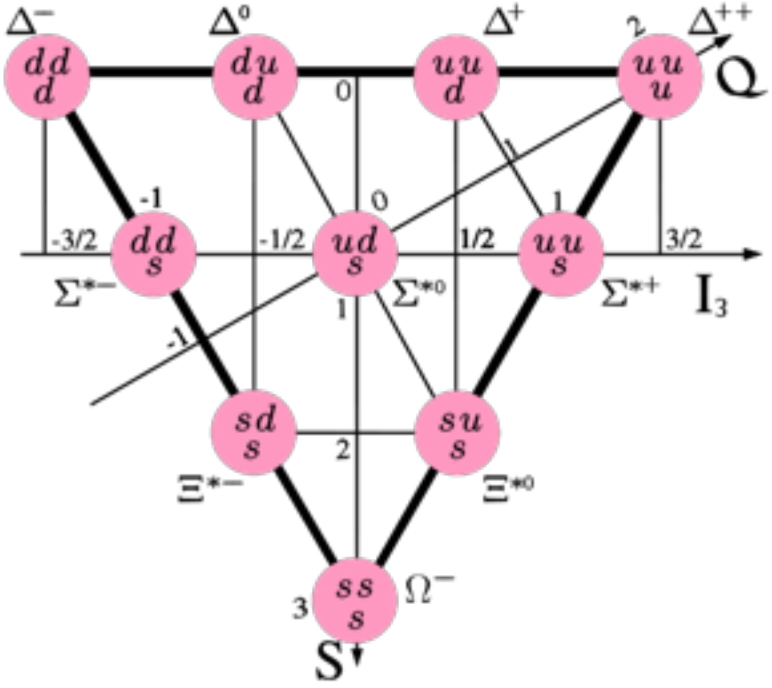
strange quarks necessary component in **quark model** to classify the observed slew of mesons/baryons Gell-Mann, Zweig (1964)

based on "**Eightfold Way**" (= $SU(3)_{flavor}$) Gell-Mann; Ne'eman (1961)

quark model: mesons and baryons

categorizes mesons (baryons) in terms of two (three) constituent quarks in **SU(3)$_{\text{flavor}}$ multiplets** = octets and decuplets

baryon decuplet



spectrum fully classified by assuming:

- quarks have spin $\frac{1}{2}$
- quarks have fractional charges
  (but combine into hadrons with integer charges)

quark model: mesons and baryons

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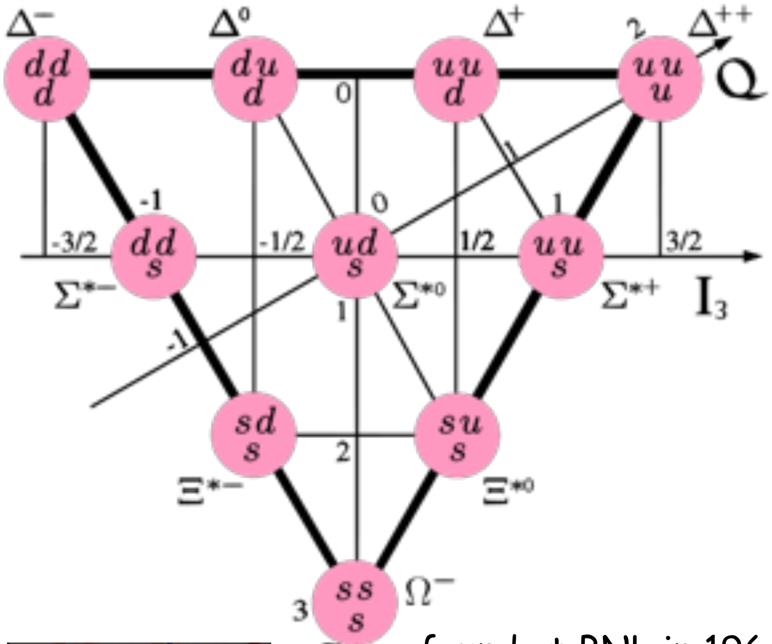


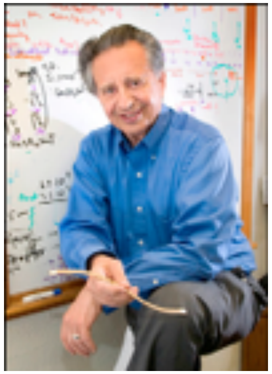
spectrum fully classified by assuming:

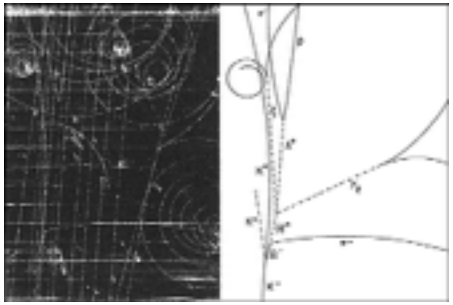
- quarks have spin ½
- quarks have fractional charges
  (but combine into hadrons with integer charges)

big success: prediction of Ω⁻ (sss)

found at BNL in 1964

N. Samios et al.





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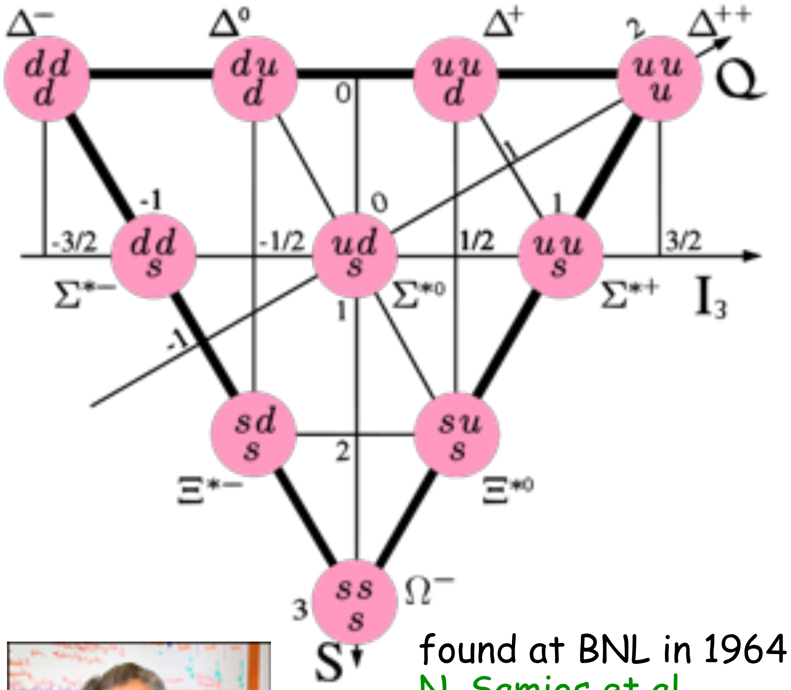

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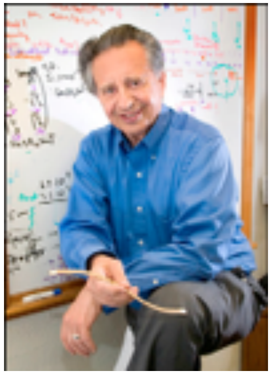
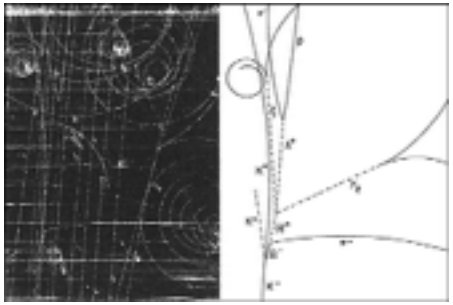




spectrum fully classified by assuming:

- quarks have spin ½
- quarks have fractional charges
  (but combine into hadrons with integer charges)

big success: prediction of Ω^- (sss)

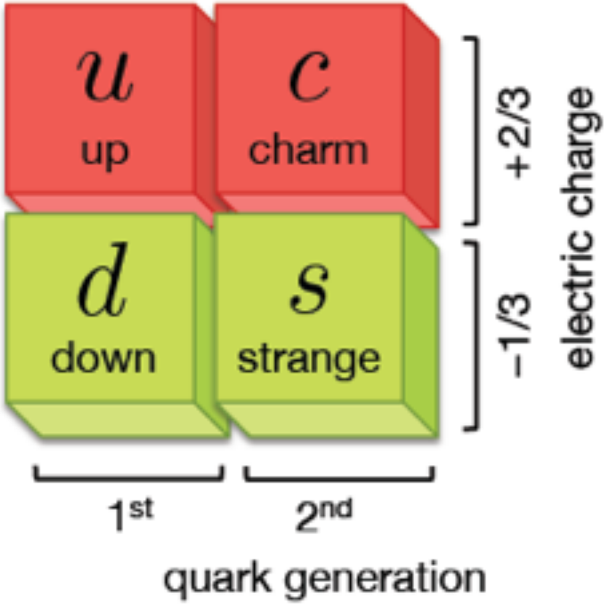
also, **first evidence of color**

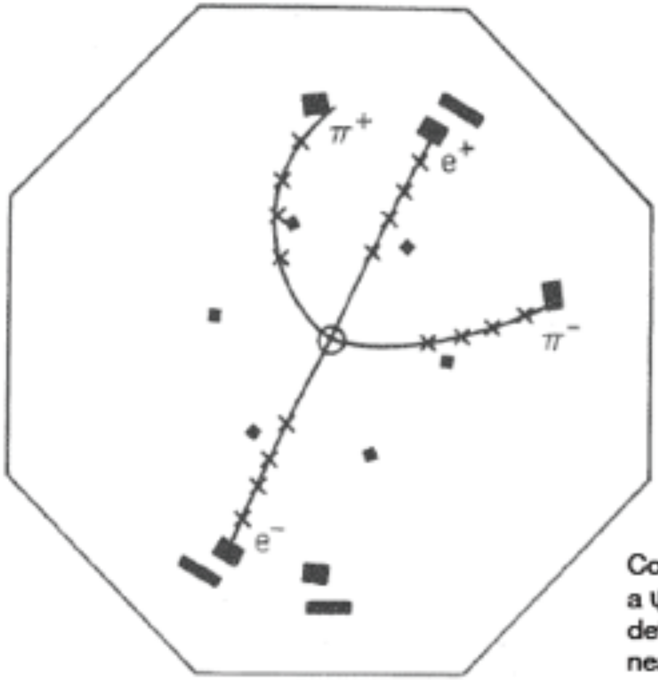
- Δ^{++} wave function |uuu> not anti-sym
  (violates Pauli principle)
- remedy: color quantum number
  but hadrons remain colorless/color singlets

$$\sim \sum_{ijk} \epsilon_{ijk} |q_i q_j q_k\rangle$$

$$\sim \sum_{i} |\bar{q}_i q_i\rangle$$

QCD matter sector: charm



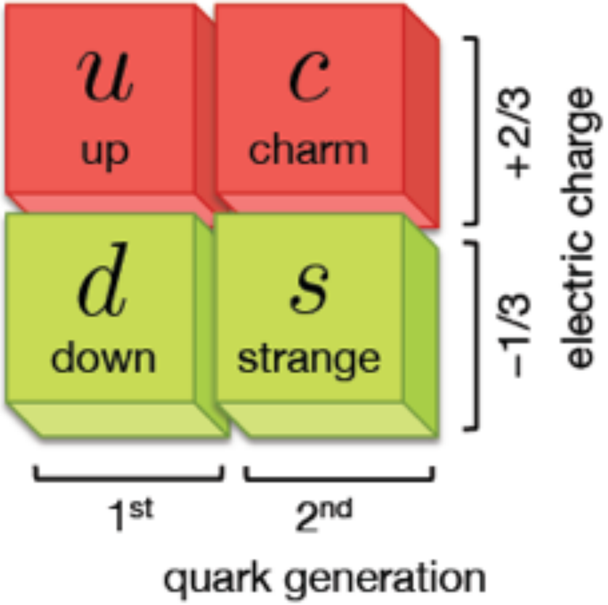


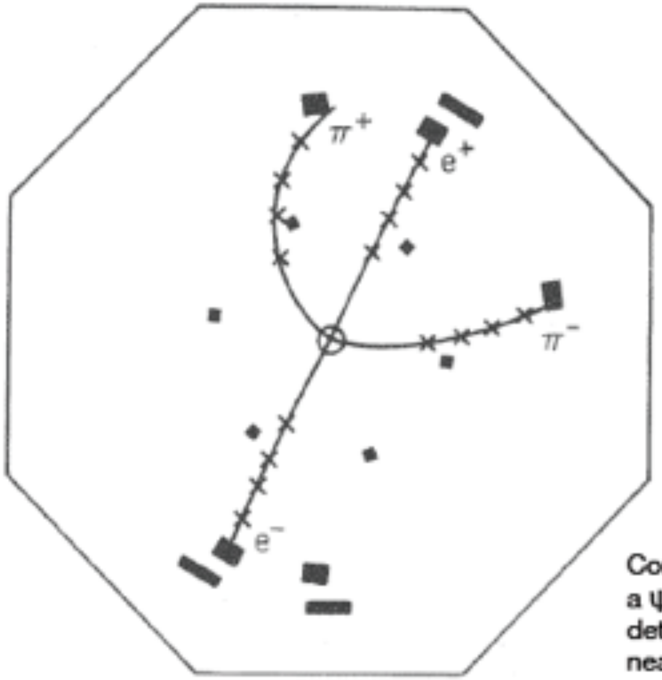
Computer reconstruction of a ψ′ decay in the Mark I detector at SLAC, making a near-perfect image of the Greek letter ψ

predicted on strong theoretical grounds (suppression of FCNC)

"GIM mechanism" in 1970 Glashow, Iliopolus, Maiani

QCD matter sector: charm





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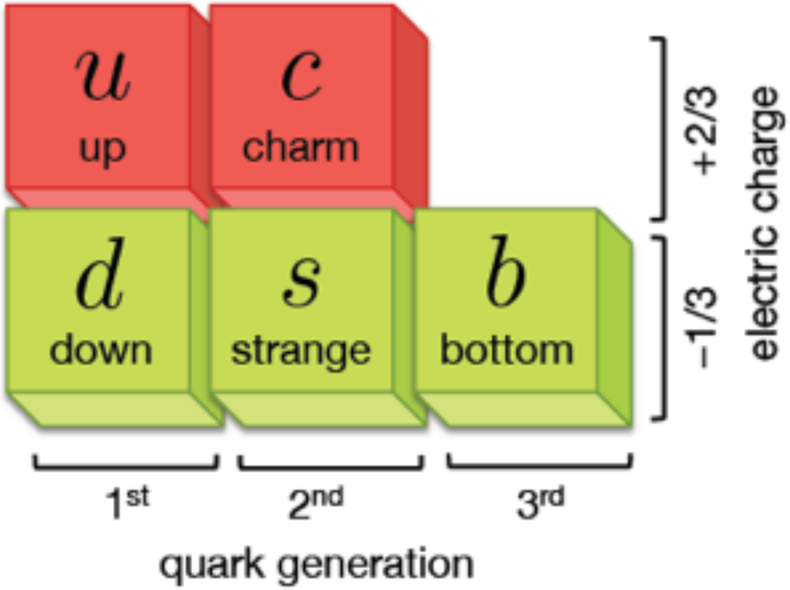
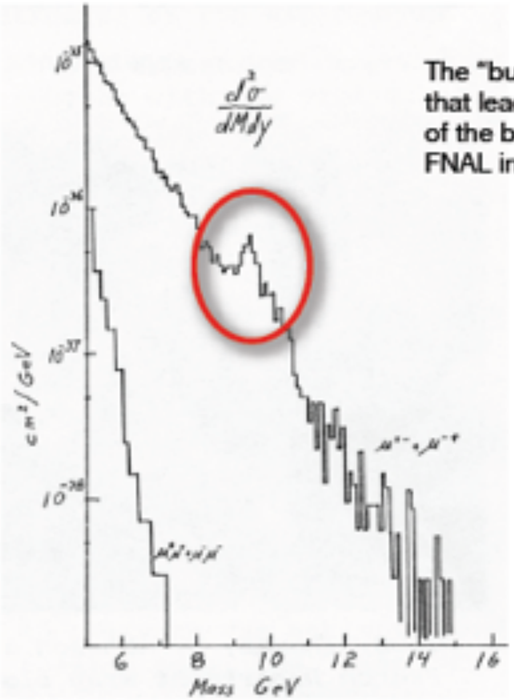
"GIM mechanism" in 1970 Glashow, Iliopolus, Maiani

observed during "November revolution" in 1974 both at SLAC (Richter et al.) and BNL (Ting et al.)
discovered meson became known as J/Ψ; Nobel Prize in 1976

QCD matter sector: bottom



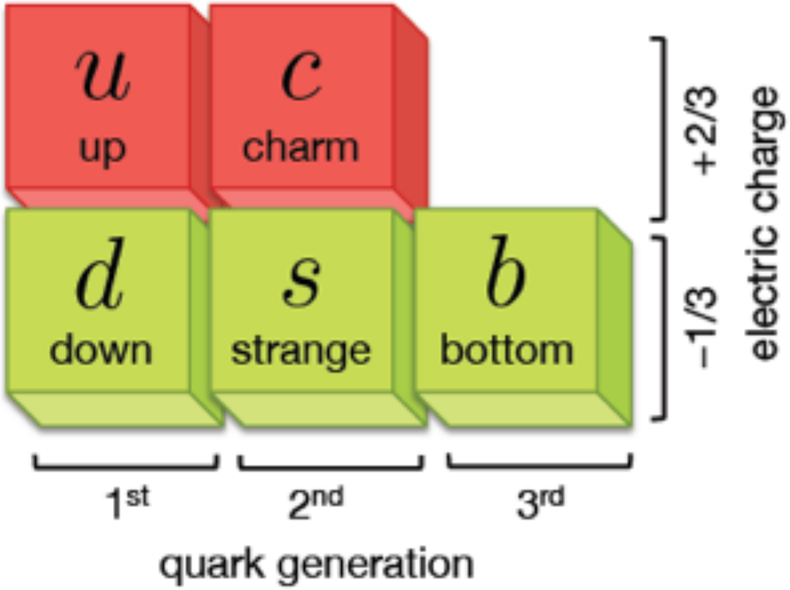


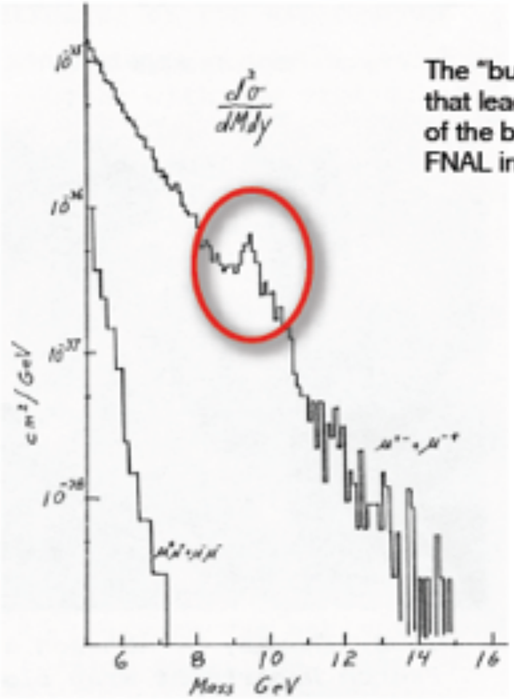
The "bump" at 9.5 GeV that lead to the discovery of the bottom quark at FNAL in 1977

theorized in 1973 in order to accommodate CP violation
(requires third generation)

Kobayashi, Maskawa Nobel Prize 2008

QCD matter sector: bottom





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Kobayashi, Maskawa Nobel Prize 2008

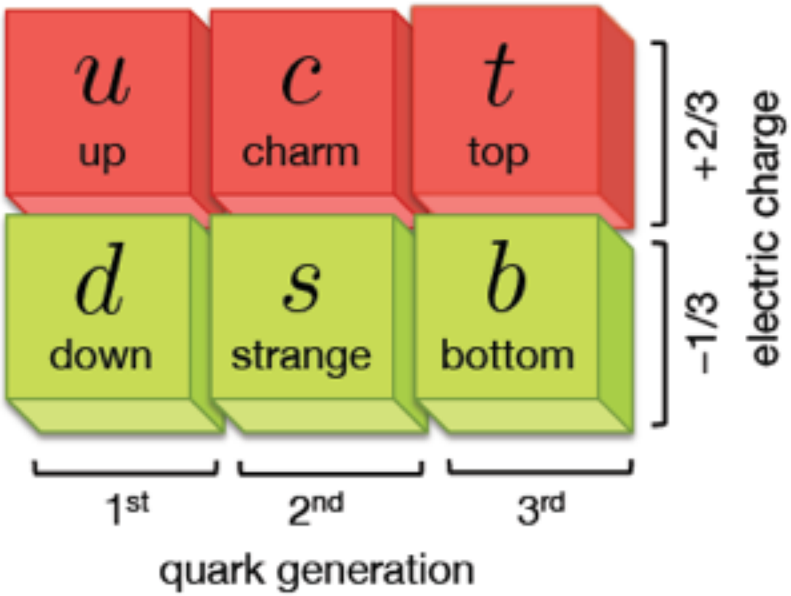
discovered in 1977 at FNAL (Υ meson or "bottomium")

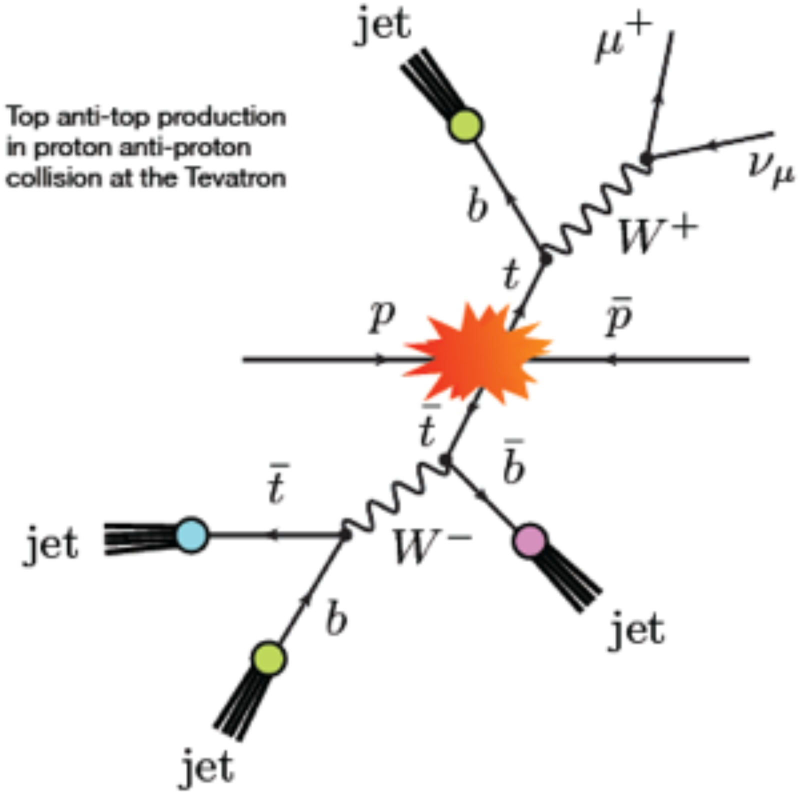
Ledermann et al.

L.L. coined also the term "God particle"

Nobel Prize in 1988 for muon neutrino

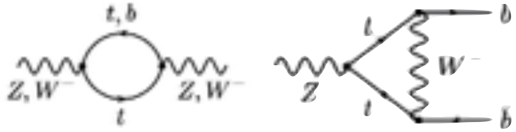
QCD matter sector: top




Top anti-top production in proton anti-proton collision at the Tevatron

by around 1994 electroweak precision fits point towards mass in range 145-185 GeV

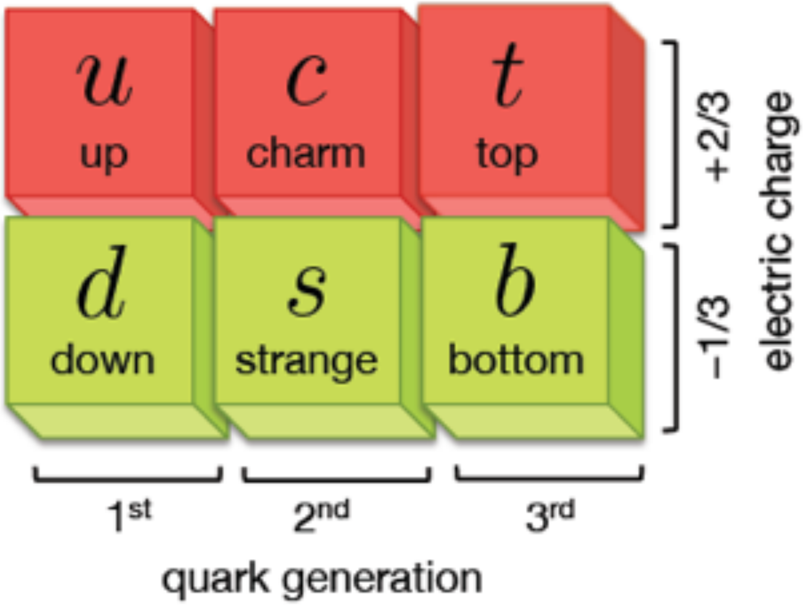
(vector boson mass and couplings are sensitive to top mass)

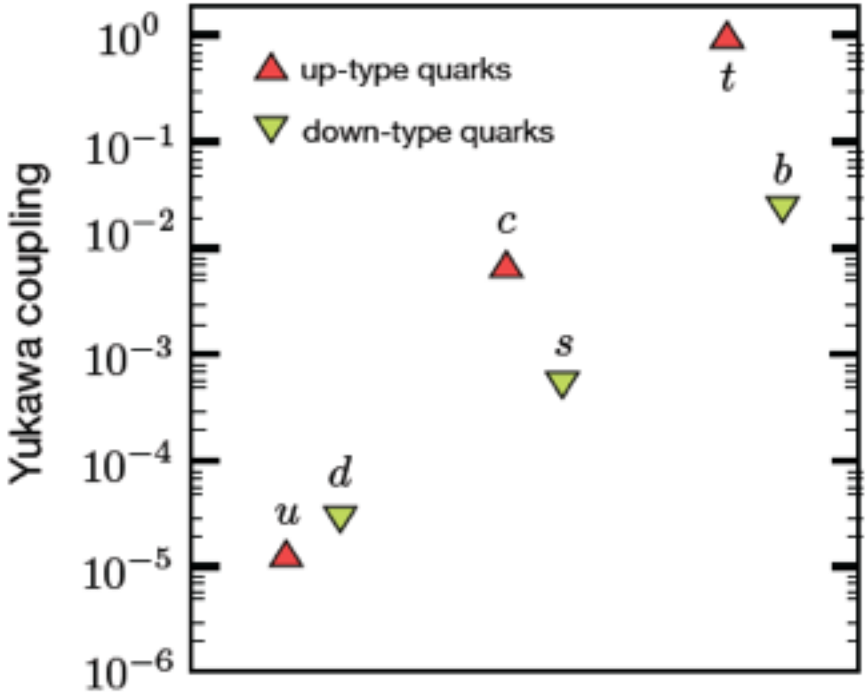


eventually discovered in 1995 by CDF and DØ at FNAL

(mass nowadays know to about 1 GeV)

QCD matter sector: 3 generations

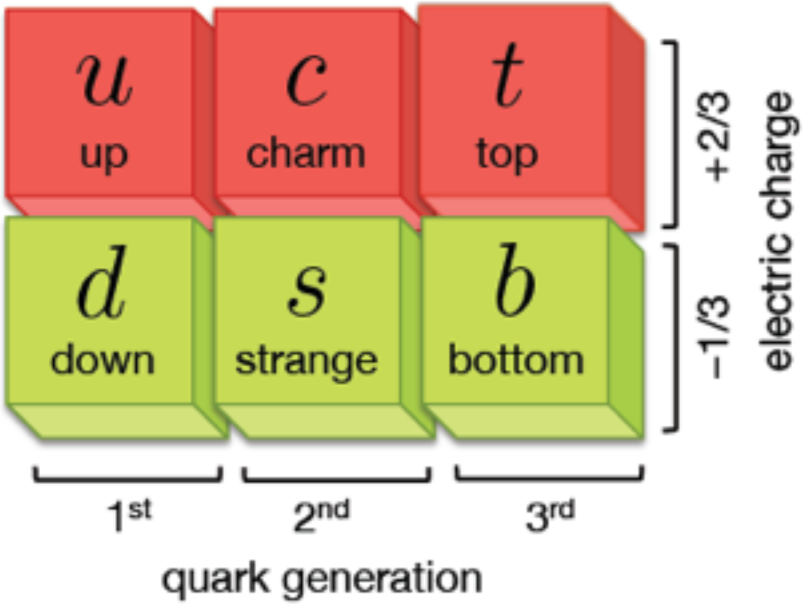


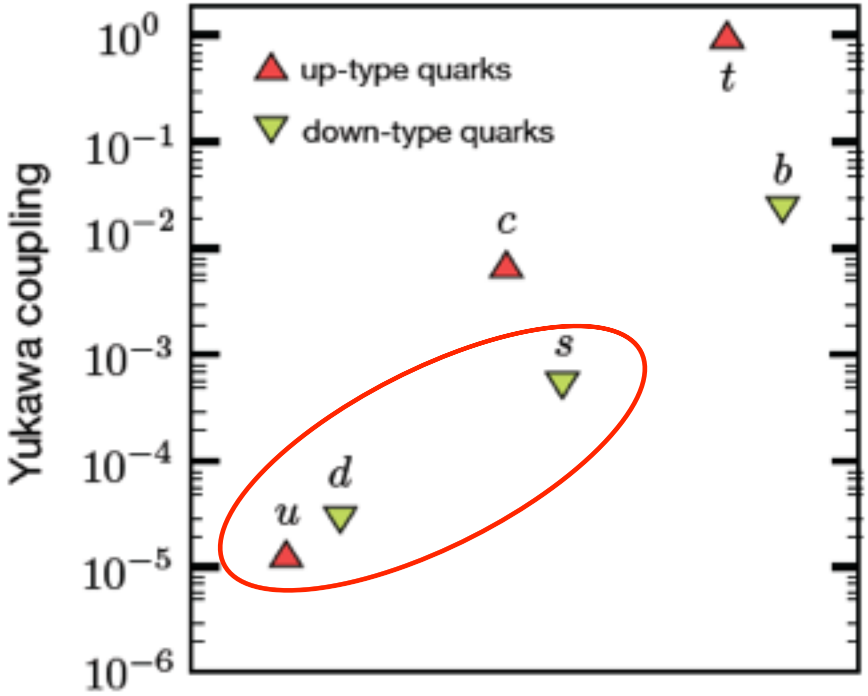


- masses of six quarks range from O(MeV) to about 175 GeV

  why the masses are split by almost six orders of magnitude remains a big mystery

QCD matter sector: 3 generations





- masses of six quarks range from O(MeV) to about 175 GeV
  why the masses are split by almost six orders of magnitude remains a big mystery

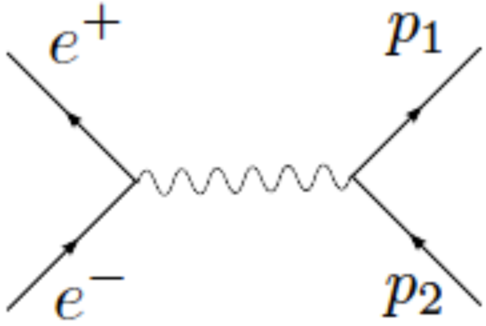
- masses of u, d, s quarks are lighter than 1 GeV (proton mass)
  in the limit of vanishing u,d,s masses there is an exact $SU(3)_{\text{flavor}}$ symmetry

further evidence for color quantum number

- **color can be probed directly in e^+e^- collisions**

  idea:

  production of fermion pairs (leptons or quarks) through a virtual photon sensitive to electric charge and number of degrees of freedom



further evidence for color quantum number

- **color can be probed directly in e^+e^- collisions**

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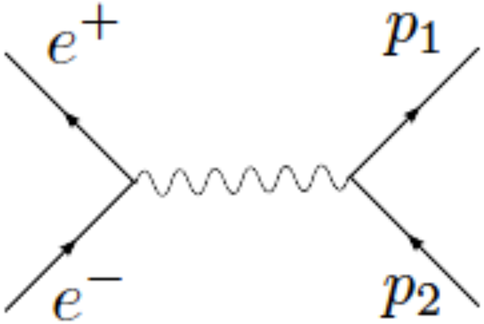

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$$R \equiv \frac{e^+e^- \to \text{hadrons}}{e^+e^- \to \mu^+\mu^-} \propto N_c \sum_f Q_f^2$$

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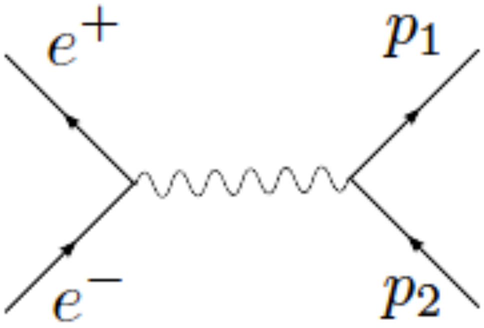

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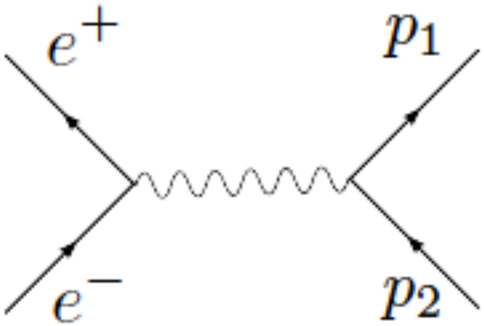

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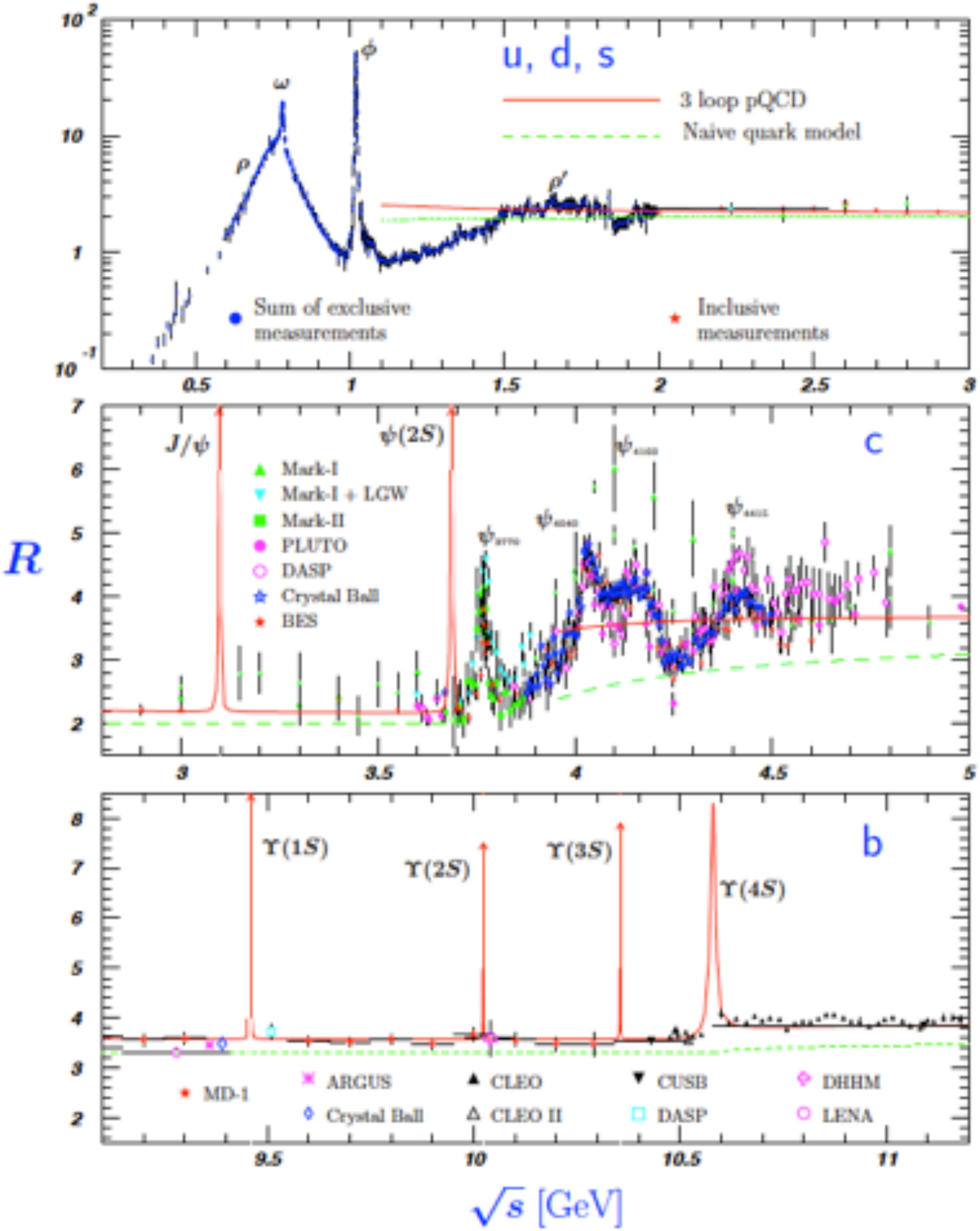
sum over active quarks

  - in LO described by process $\mathbf{e^+e^- \to q\bar{q}}$
  - each active quark is produced in one out of N_c colors above kinematic threshold

experimental results for R ratio



$$
\begin{aligned}
R_{u,d,s} &= 3\times\left[\left(\frac{2}{3}\right)^2+\left(-\frac{1}{3}\right)^2+\left(-\frac{1}{3}\right)^2\right]\\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
R_{u,d,s,c} &= R_{u,d,s}+3\times\left(\frac{2}{3}\right)^2\\
&= \frac{10}{3}
\end{aligned}
$$

$$
\begin{aligned}
R_{u,d,s,c,b} &= R_{u,d,s,c}+3\times\left(-\frac{1}{3}\right)^2\\
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caveats:

- higher order corrections
- mass effects near threshold

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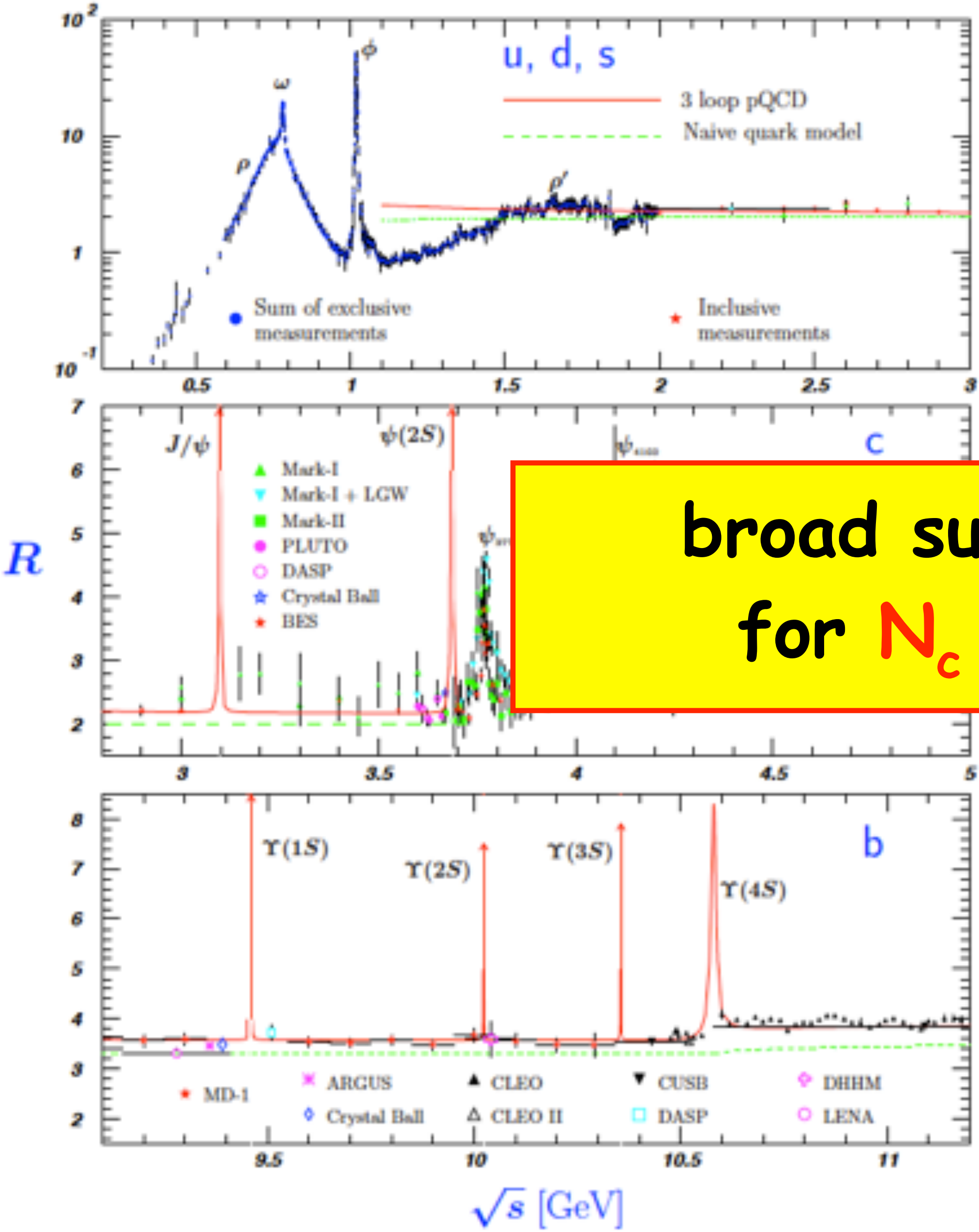

broad support for $N_c = 3$

$$R_{u,d,s} = 3\times\left[\left(\frac{2}{3}\right)^2+\left(-\frac{1}{3}\right)^2+\left(-\frac{1}{3}\right)^2\right] = 2$$

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QCD color interactions heuristically

- QCD color quantum number is mediated by the **gluon**

  analogous to the photon in QED
- gluons are changing quarks from one color to another

  as such they must also carry a color charge (unlike the charge neutral photon in QED)

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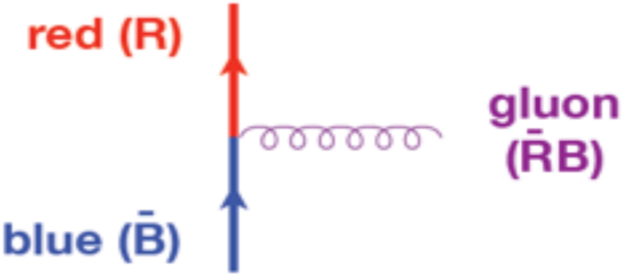

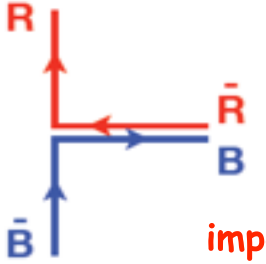

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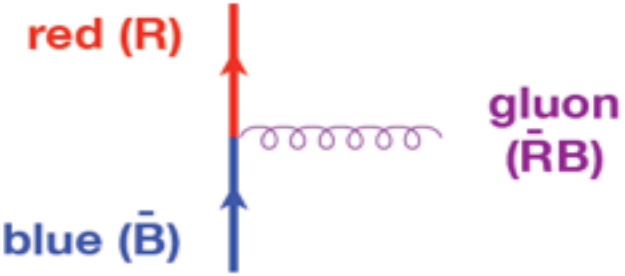

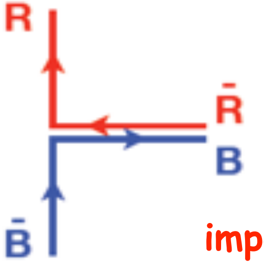

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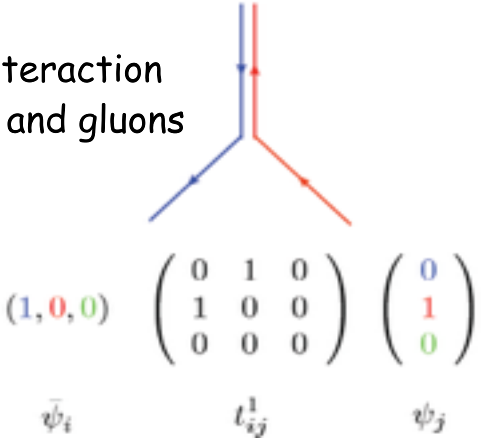
- color charge of each gluon represented by a 3x3 matrix in color space

  conventional choice: express t^a (a=1…8) in terms of **Gell-Mann matrices**

typical color interaction between quarks and gluons



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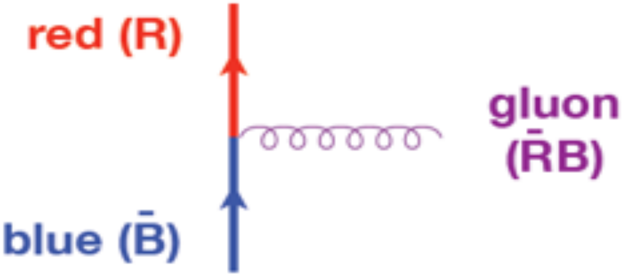

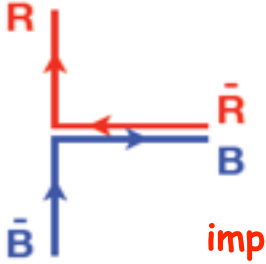

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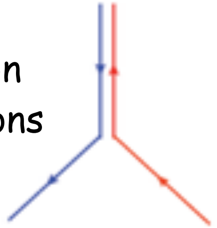
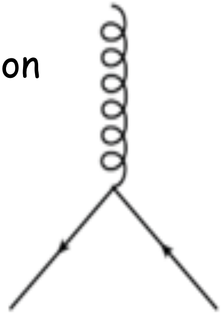


$$\underset{\bar{\psi}_i}{(1,0,0)} \underset{t^1_{ij}}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \underset{\psi_j}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

more formal expression as **Feynman rule**
[only color structure here]



$$\bar{\psi}_i \, t^A_{ij} \, \psi_j$$

QCD: an unbroken SU(3) Quantum Field Theory

guiding principle for all field theories: **local gauge invariance** of the underlying Lagrangian

i.e., redefining the quark and gluon fields independently at each space-time point has no impact on the physics

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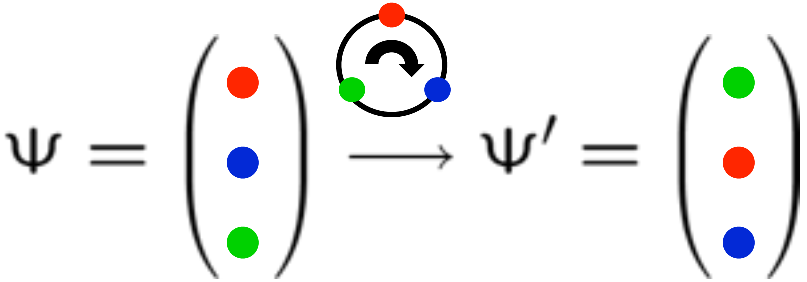

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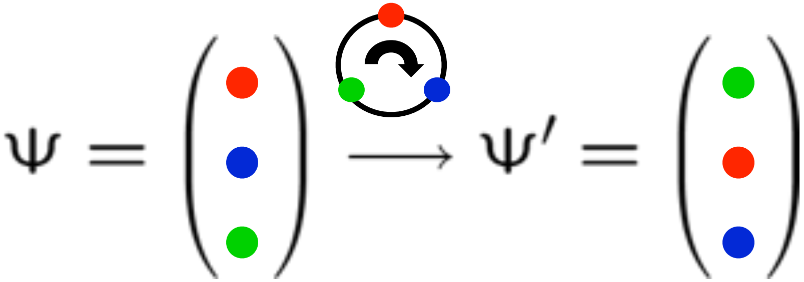

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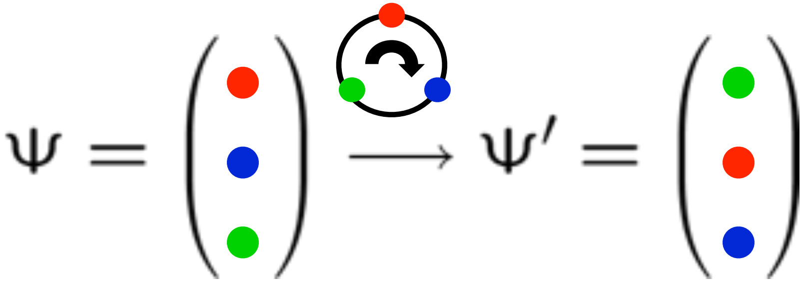

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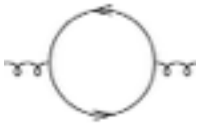
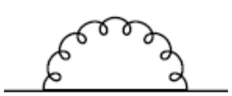
non-Abelian group structure:

- Lie algebra: $[t_a, t_b] = i\, f_{abc} t_c$

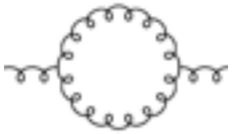
- invariants ("color factors") :



$T_F = 1/2$



$C_F = 4/3$



$C_A = 3$

the gauge group SU(N) with N=3

- choose special unitary group SU(3) as the gauge group for QCD
  - SU(N) is group of N x N matrices U
  - N x N generic complex matrix has N^2 complex (= 2 N^2 real) values

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properties can be studied from
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properties can be studied from infinitesimal transformations

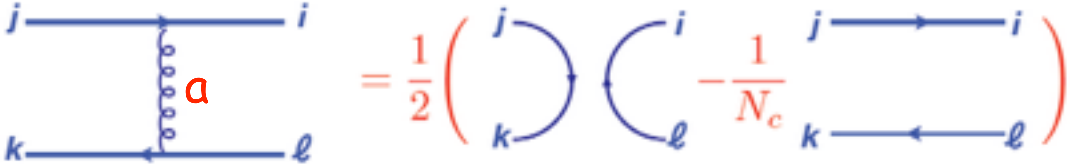
- why SU(3) ?

quarks and anti-quarks are different [rules out real SO(3)]

  - only compact simple Lie group with complex triplet representation

color algebra: Fierz identity, Casimir operators

- powerful Fierz identity $$\sum_{\mathbf{a}} (\mathbf{t^a})_{\mathbf{ij}} \, (\mathbf{t^a})_{\mathbf{kl}} = \frac{\mathbf{1}}{\mathbf{2}}\left(\delta_{\mathbf{il}}\delta_{\mathbf{jk}} - \frac{\mathbf{1}}{\mathbf{N}}\delta_{\mathbf{ij}}\delta_{\mathbf{kl}}\right)$$



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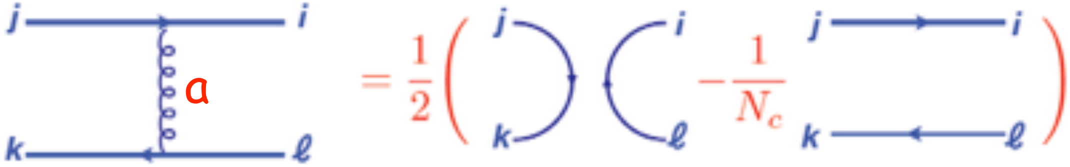
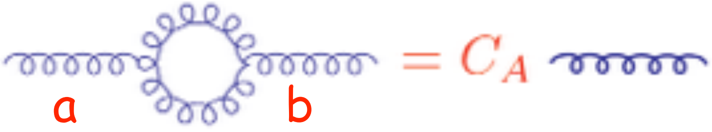


- N-1 Casimir operators (commute with all generators; proportional to identity)

  - fundamental representation

$$\sum_{a}\sum_{k}(t^a)_{ik}\,(t^a)_{kj} = C_F\delta_{ij} \qquad C_F = \frac{N^2-1}{2N}$$

  - adjoint representation (defined by $if_{abc} = 2\mathrm{Tr}([t^a, t^b]t^c)$ —> 8 (8x8) matrices)

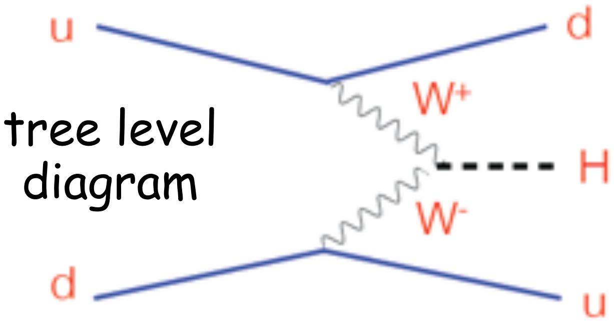


$$\sum_{cd} f_{acd}f_{bcd} = C_A\delta_{ab} \qquad C_A = N$$

color at work: a loop calculation

- vector boson fusion is an important Higgs search channel at the LHC

tree level diagram



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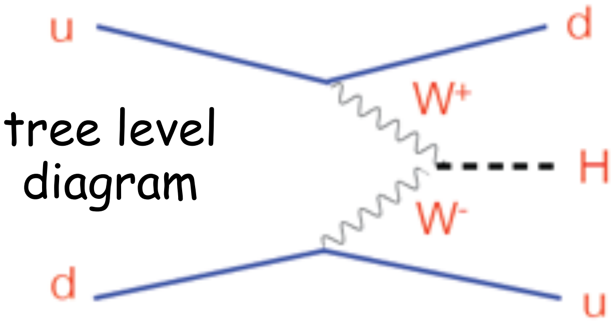
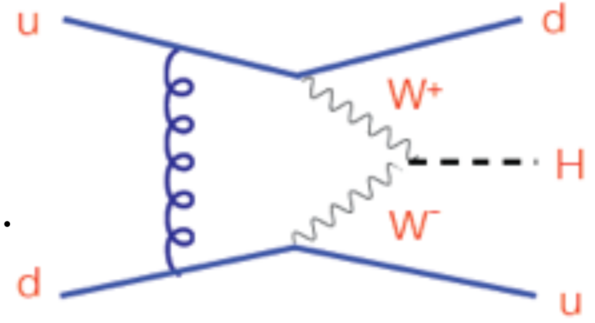


tree level
diagram

simple picture
receives NLO
corrections, e.g.



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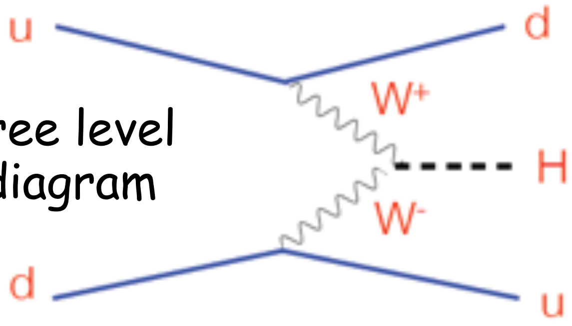

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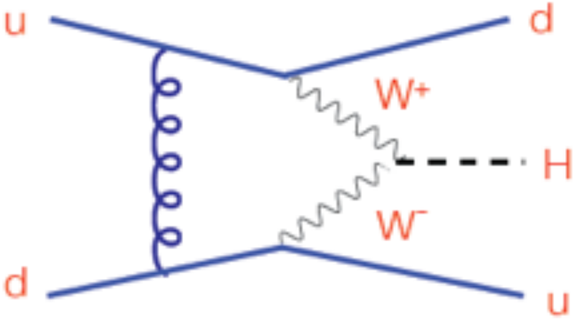

vanishes when interfered with LO diagram

WHY ?

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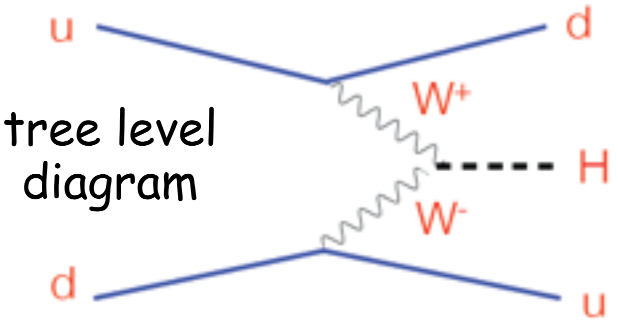

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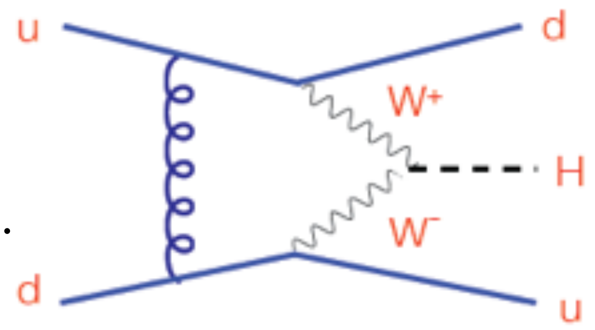

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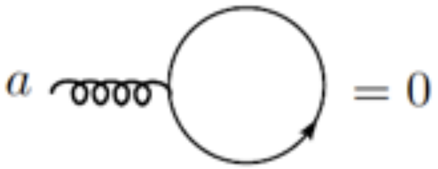

$$\mathrm{Tr}(t^a) = 0$$

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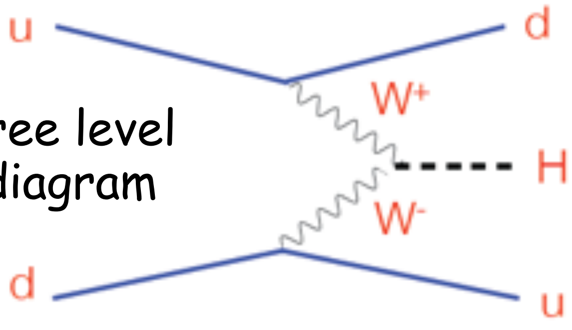

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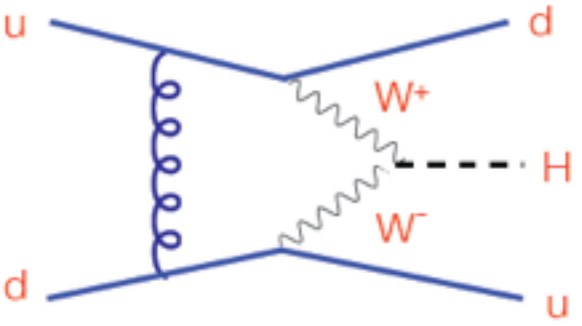

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- useful Jacobi identity

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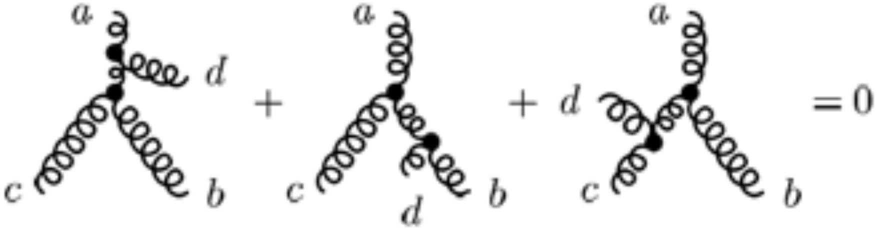

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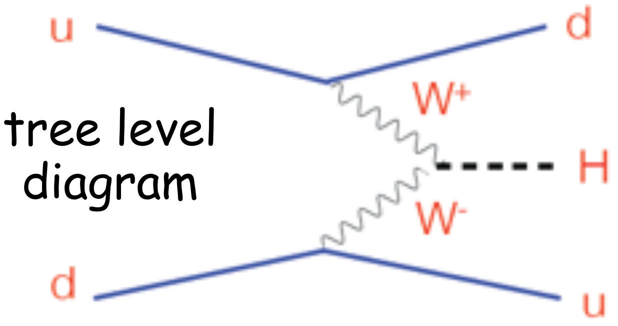

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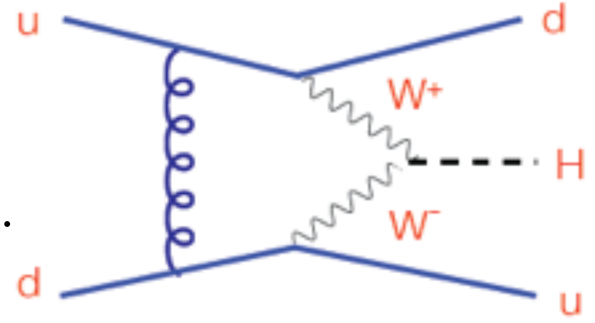

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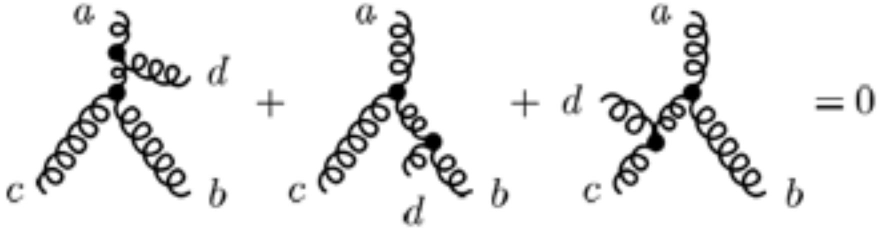
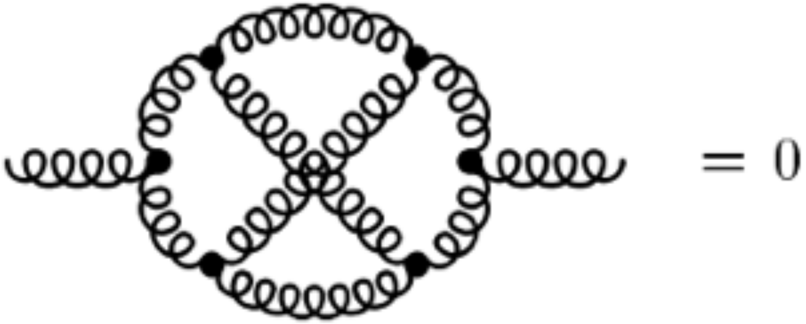


find



color at work: leading color approximation

example from J. Campbell's lectures

- to simplify large scale QCD calculations, one often works in the **leading color approximation**

what is it all about?

color at work: leading color approximation

example from J. Campbell's lectures

- to simplify large scale QCD calculations, one often works in the leading color approximation

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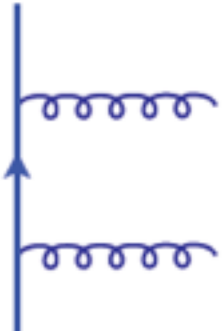
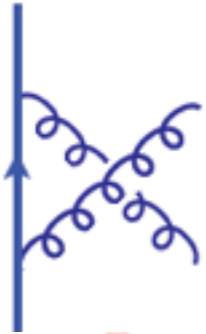
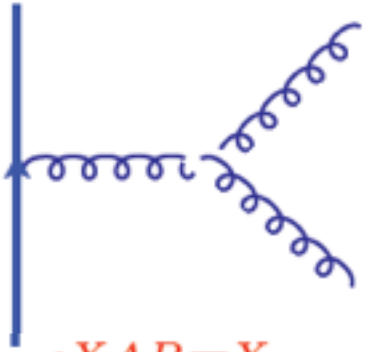
- simplest example: 2gluon + W production (W boson dropped - color neutral)



$C_1 : T^A T^B$



$C_2 : T^B T^A$



$C_3 : f^{XAB} T^X$

$= T^A T^B - T^B T^A$

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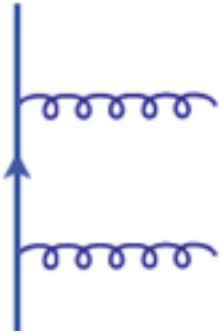

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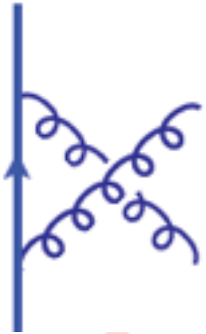

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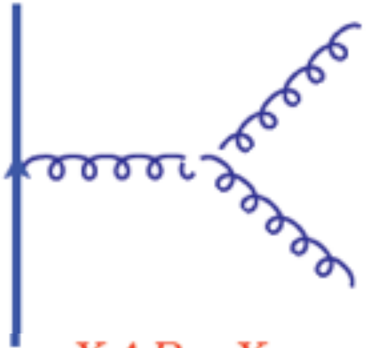

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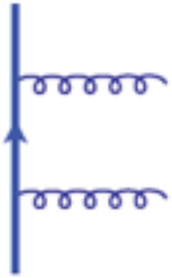
hence, only two color ordered structures

$C_1 + C_3 :$

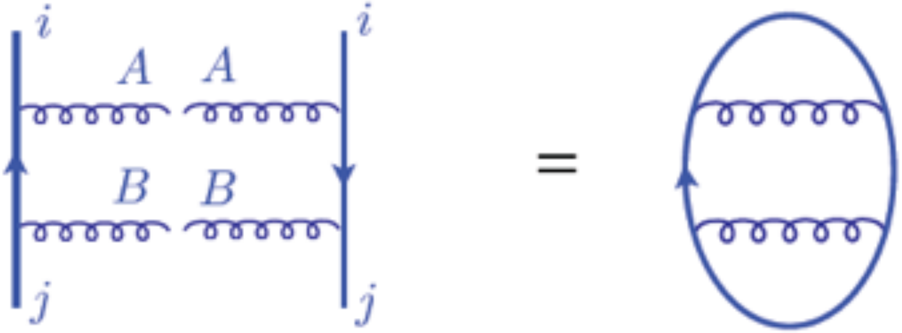


$C_2 - C_3 :$

leading color approximation (cont'd)

- need to square amplitudes to get cross section, e.g.:

$|\mathcal{C}_1 + \mathcal{C}_3|^2 :$



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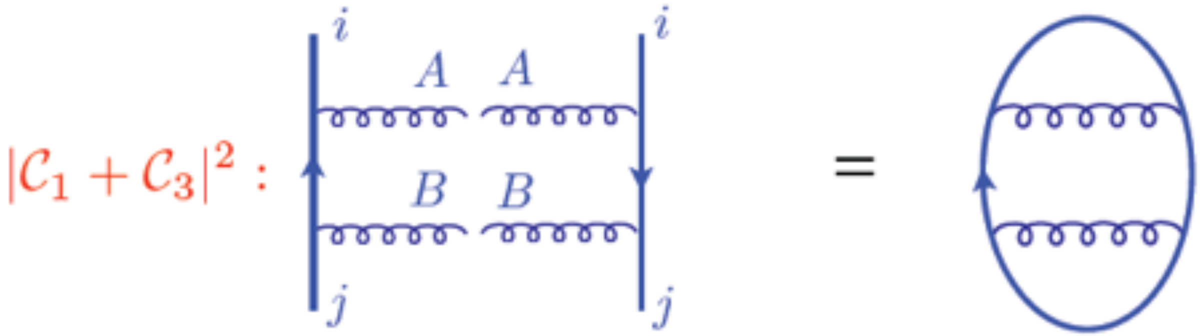
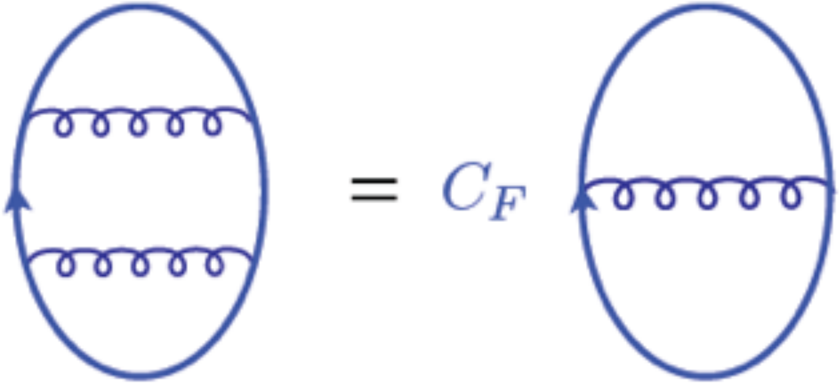


- apply powerful pictorial rules to compute



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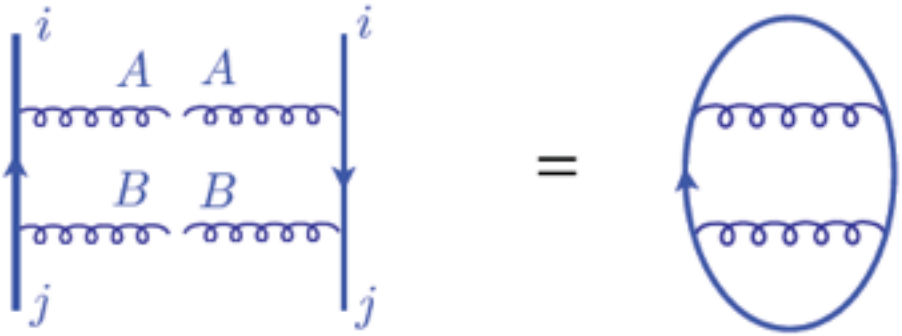

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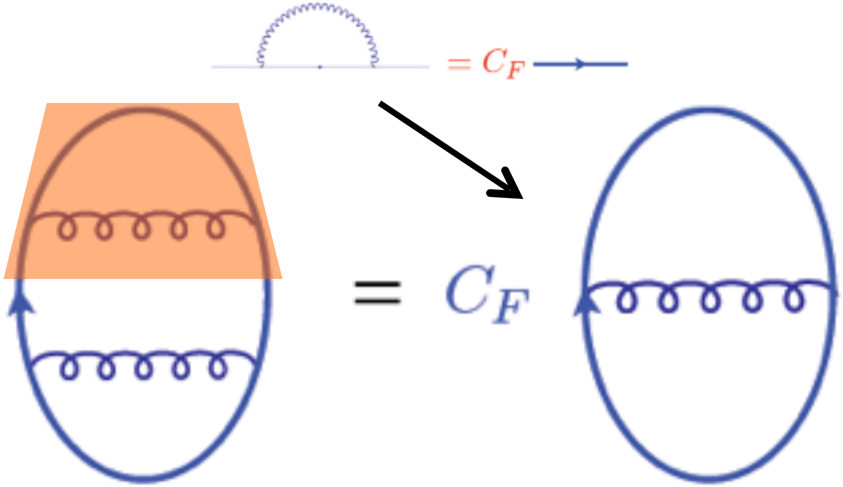

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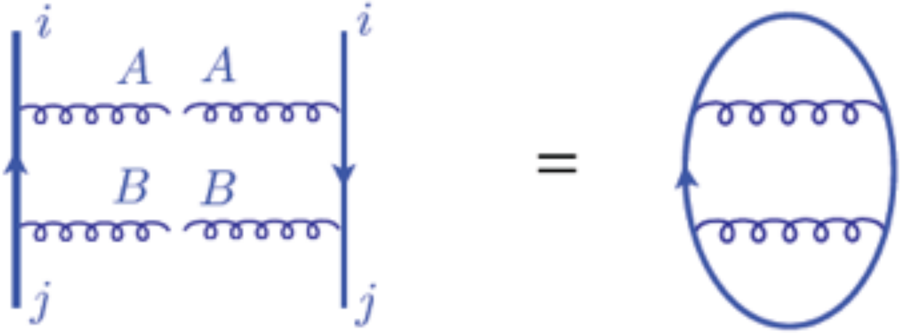

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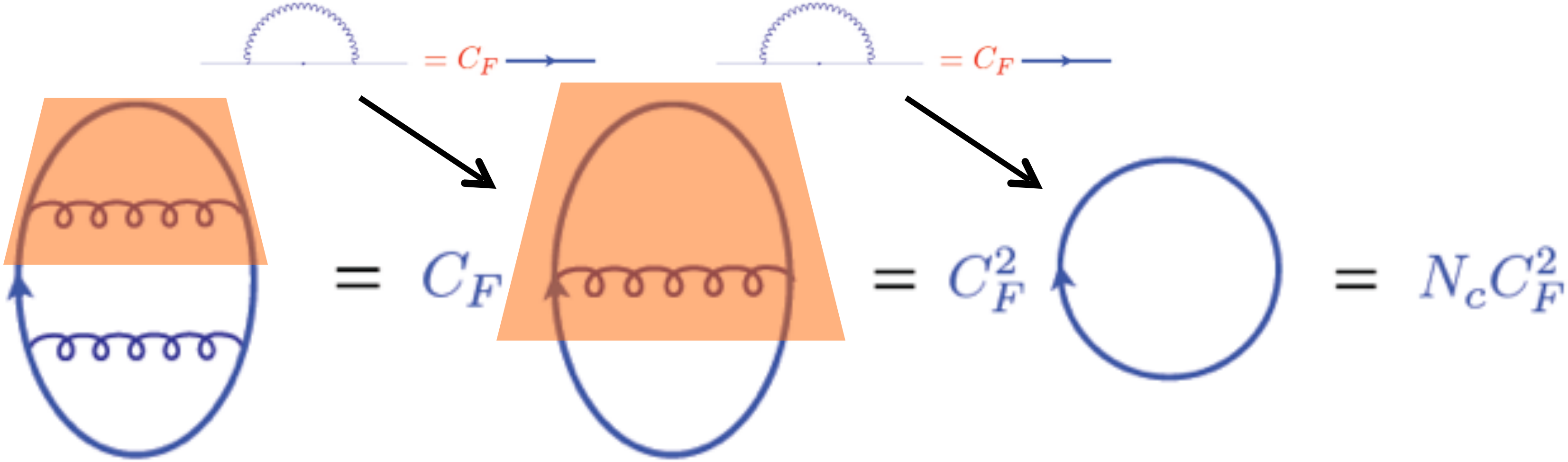

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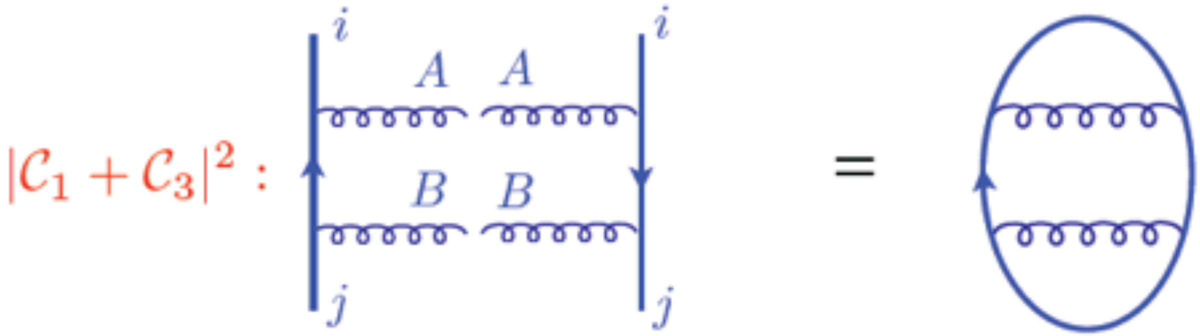

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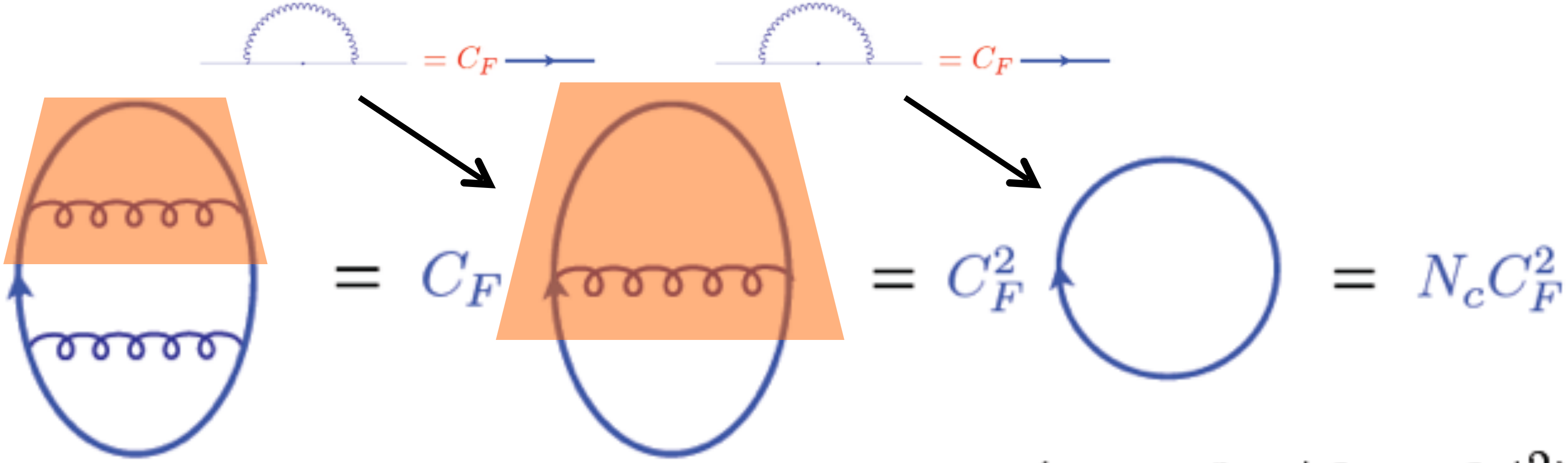


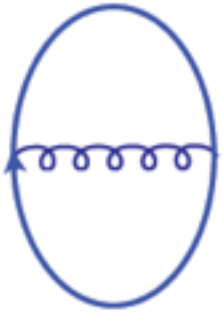
(same for $|\mathcal{C}_2-\mathcal{C}_3|^2$)

leading color approximation (cont'd)

- interference term needs to be massaged (use Fierz identity)

$(\mathcal{C}_1+\mathcal{C}_3)(\mathcal{C}_2-\mathcal{C}_3)^*$:

$= -\frac{1}{2N_c}$



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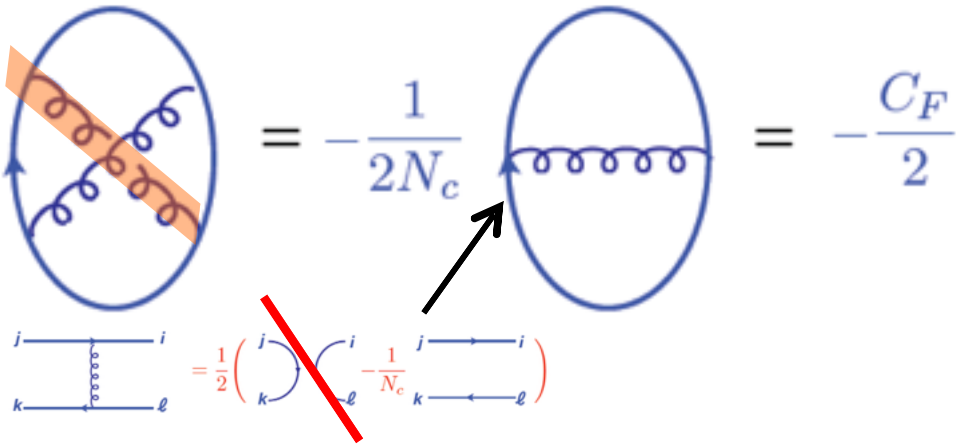

does not contribute: Tr $(t^a) = 0$

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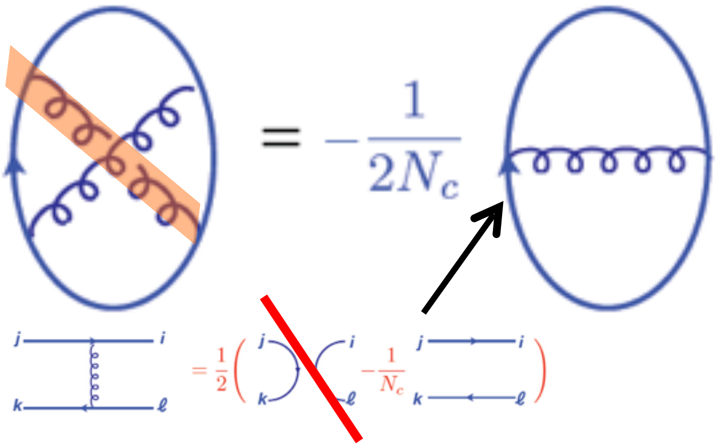


$$(C_1 + C_3)(C_2 - C_3)^* : \qquad = -\frac{1}{2N_c} \qquad = -\frac{C_F}{2}$$

does not contribute: Tr (t^a) = 0

- combine results for [after some reshuffling, use $N_c\, C_F^2 = (N_c^2\, C_F - C_F)/2$]

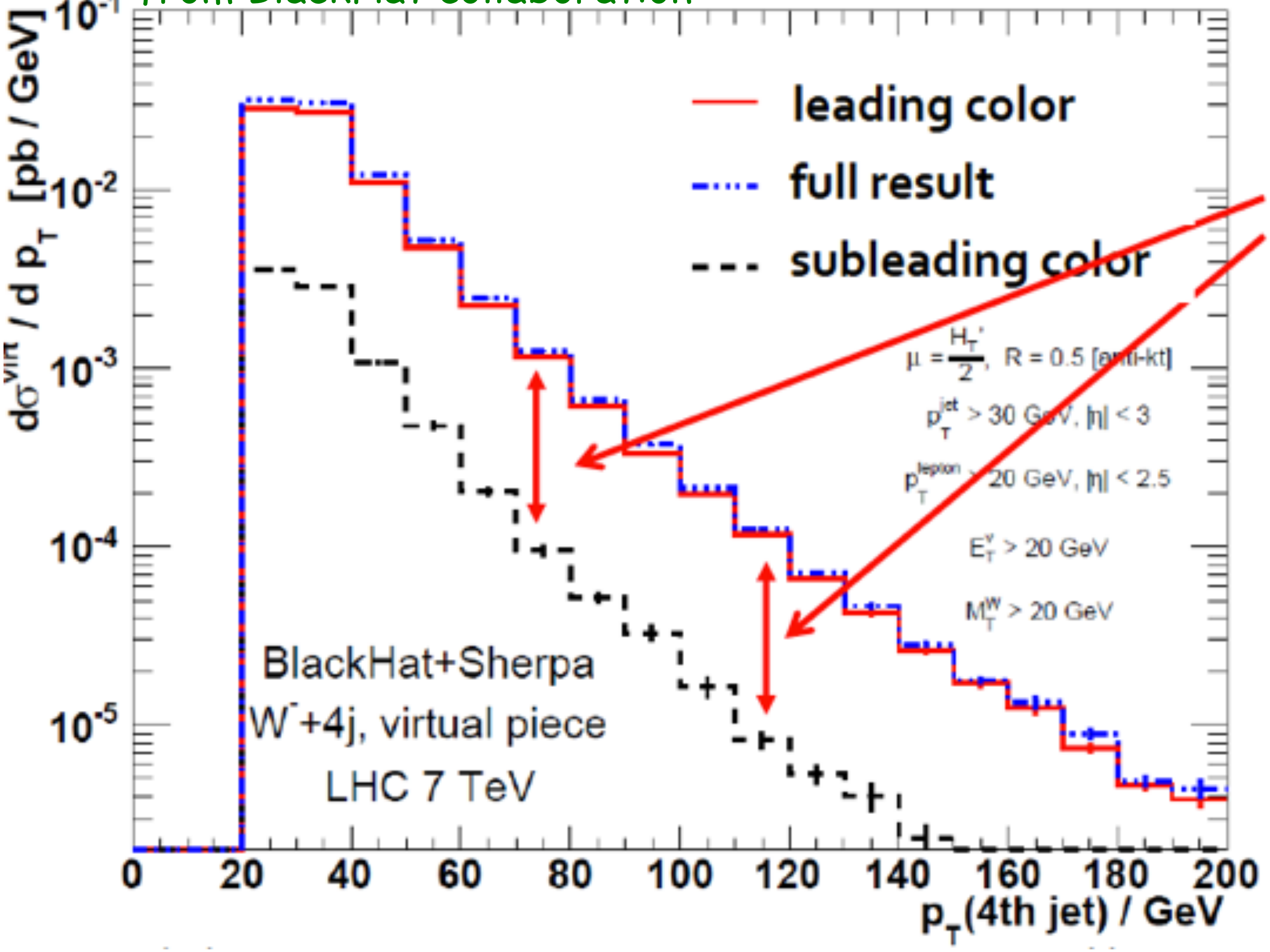
$$\frac{N_c^2 C_F}{2}\left(|C_1 + C_3|^2 + |C_2 - C_3|^2 - \frac{1}{N_c^2}|C_1 + C_2|^2 \right)$$

leading color contribution

$1/N_c^2$ suppressed

this structure only appears for QED-like diagrams w/o the three gluon vertex

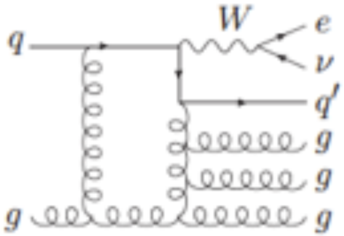
leading color approximation at work



from BlackHat collaboration

works very well

W+4jets @ NLO



experimental support for SU(3)

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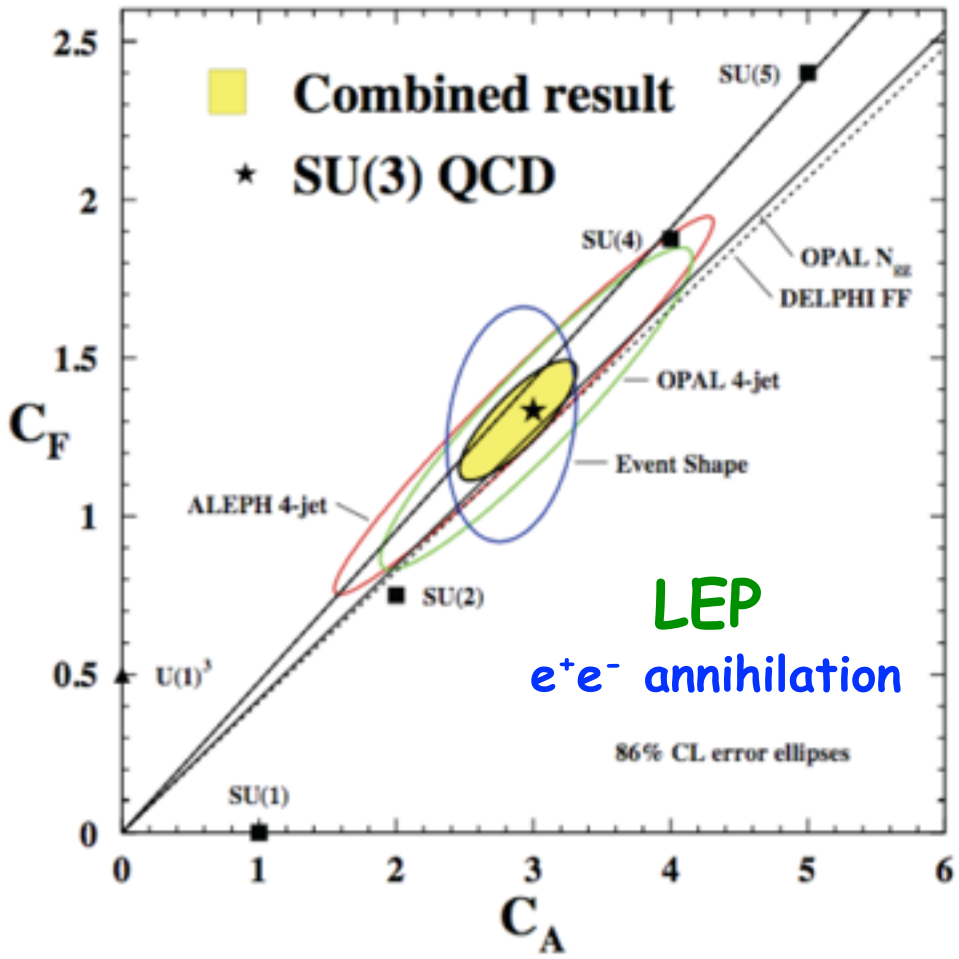
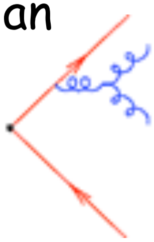


- angular correlations between four jets depend on C_A/C_F and T_F/C_F
- sensitivity to non-Abelian three-gluon-vertex

LO: Ellis, Ross, Terrano

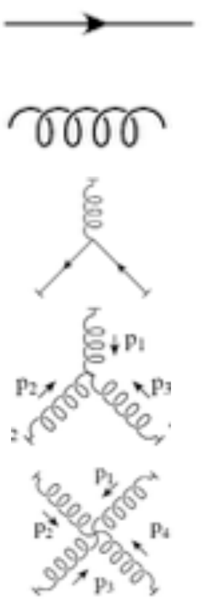


QCD Lagrangian & Feynman rules

L_{QCD} encodes all physics related to strong interactions

for perturbative calculations we simply read off the **Feynman rules**

$$
\begin{aligned}
\mathcal{L}_{\mathrm{QCD}} &= \bar{\Psi}(i\partial_\mu\gamma^\mu - m)\Psi \\
&- (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \\
&- g\bar{\Psi}A^a_\mu T_a \gamma^\mu \Psi \\
&- \frac{1}{2}g(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) f_{abc} A^{\mu b} A^{\nu c} \\
&- \frac{1}{4}g^2 f_{abc} A^b_\mu A^c_\nu f_{ade} A^{\mu d} A^{\nu e}
\end{aligned}
$$



like in QED

non abelian

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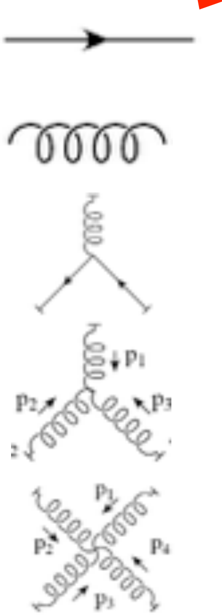

like in QED

- - - - - -

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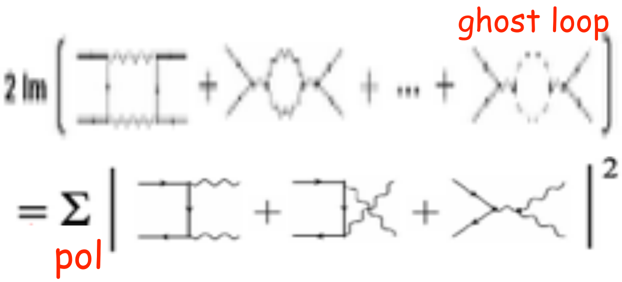
technical complications due to the gauge-fixing & ghost terms:

gauge-fixing: needed to define gluon propagator; breaks gauge-invariance but all physical results are independent of the gauge

ghosts: cancel unphysical degrees of freedom → unitarity



recall: gauge invariance in QED

$$\mathcal{L}_{\mathbf{QED}} = \mathcal{L}_{\mathbf{Dirac}} + \mathcal{L}_{\mathbf{Maxwell}} + \mathcal{L}_{\mathbf{int}}$$

$$= \bar{\mathbf{\Psi}}(\mathbf{i}\not{\partial} \;\; - \mathbf{m})\mathbf{\Psi} - \frac{1}{4}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} - \mathbf{q}\bar{\mathbf{\Psi}}\gamma_\mu\mathbf{\Psi}\mathbf{A}^\mu$$

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field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

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- photon mass term would violate gauge invariance

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electromagnetic vector potential A_μ

photon field carries no electric charge

field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

field strength itself gauge invariant

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"covariant" = $D_\mu\psi$ transforms as ψ

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electromagnetic vec

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more cumbersome to demonstrate for QCD

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a closer look at the QCD Lagrangian

- Yang and Mills proposed in 1954 that the local "phase rotation" in QED could be generalized to non Abelian groups such as SU(3)

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gluon field strength
a = 1,…,8

color index
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8 generators

- QCD interaction is flavor blind

- **coupling g_s is the only parameter** (masses have e-w origin)

QCD gauge transformations

- demand that QCD Lagrangian is invariant under local gauge transformations

  i.e., redefining the quark and gluon fields independently at each space-time point has no impact on the physics

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likewise $\bar{\Psi}' = \bar{\Psi}U^\dagger$

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  see QCD book by T. Muta for details

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- sufficient to demonstrate gauge invariance of quark term

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✔

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- covariant derivative must transform "with the quark field"

$$D_\mu\Psi \rightarrow (D'_\mu)_{ik}\Psi'_k = U_{ij}(x)(D_\mu)_{jk}(x)\Psi_k(x)$$

- sufficient to demonstrate gauge invariance of quark term

$$\mathcal{L} = -\frac{1}{4}F_a^{\mu\nu}F^a_{\mu\nu} + \sum_f \bar{\Psi}_i^{(f)}(i\not{D}_{ij} - m_f\,\delta_{ij})\Psi_j^{(f)}$$ ✓

- aside: gauge field transforms as $t^a A_a \rightarrow t^a A'_a = U\,t^a A_a\,U^{-1} + \frac{i}{g_s}(\partial U)\,U^{-1}$

  non Abelian part

QCD gauge transformations (cont’d)

- invariance of the first term more difficult to show

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- easiest to see by first re-writing field strength tensor as

$$\boxed{t_a F^a_{\mu\nu} = \frac{1}{ig_s}[D_\mu, D_\nu]}$$

to proof: use definition, commutation relation for generators, and consider action on a field

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- exploit that the commutator transforms as the covariant derivative itself

$$[\mathbf{D}_\mu, \mathbf{D}_\nu]_{\mathbf{ik}}\mathbf{\Psi}_{\mathbf{k}} \rightarrow \mathbf{U}_{\mathbf{ij}}\,[\mathbf{D}_\mu, \mathbf{D}_\nu]_{\mathbf{jk}}\mathbf{\Psi}_{\mathbf{k}}$$

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- find that **field strength is not gauge invariant** (unlike in QED)

$$(t_a\,F^a_{\mu\nu})_{ij} \rightarrow U_{ik}(t_a\,F^a_{\mu\nu})_{kl}U^{-1}_{lj}$$

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trick: use $\mathrm{Tr}(t_a t_b) = 1/2\ \delta_{ab}$

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use cyclicity of trace to remove U's ✔

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- like in QED, a gluon mass term is prohibited by gauge invariance

using the QCD Lagrangian: propagators

- the Lagrangian encodes all the rich physics phenomena of QCD
- in these lectures we are interested in **perturbative QCD**
  -> how to read off **Feynman rules** to compute cross sections?

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quark and gluon propagators

simple prescription:

- consider free, non-interacting theory ($g_s = 0$)

$$\mathcal{L}_{\mathbf{free}} = \bar{\mathbf{\Psi}}_{\mathbf{i}}(\mathbf{i}\partial_\mu\gamma^\mu - \mathbf{m})\delta_{\mathbf{ij}}\mathbf{\Psi}_{\mathbf{j}} - \frac{\mathbf{1}}{\mathbf{4}}(\partial_\mu \mathbf{A}^{\mathbf{a}}_\nu - \partial_\nu \mathbf{A}^{\mathbf{a}}_\mu)^{\mathbf{2}}$$

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inverse does not exist

encounter similar problem in QED
problem is freedom of gauge

gauge fixing and the gluon propagator

solution: add a **gauge fixing term** to the Lagrangian, e.g.,

$$\mathcal{L}_{\mathbf{gauge-fixing}} = -\frac{\mathbf{1}}{\mathbf{2}\lambda}(\partial^{\mu}\mathbf{A}^{\mathbf{a}}_{\mu})^{\mathbf{2}}$$

arbitrary
Lagrange multiplier

Lorenz condition $\partial^{\mu}\mathbf{A}^{\mathbf{a}}_{\mu} = \mathbf{0}$

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- leads to extra term such that an inverse now exists

$$\frac{i}{2\lambda} A^a_\mu p^\mu p^\nu A^a_\nu \qquad \frac{-i}{p^2}\left(g_{\mu\nu} - (1-\lambda)\frac{p^\mu p^\nu}{p^2}\right)\delta^{ab}$$

- particularly simple choice is **Feynman gauge** (λ=1)

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- particularly simple choice is **Feynman gauge** (λ=1)

- gauge fixing breaks explicitly gauge invariance though
  but since λ is arbitrary this leaves us with a powerful check of calculations

any dependence on λ must ultimately cancel in physical observables

another peculiarity: ghosts

- gauge fixing leads to consistent quantization of QED
- more trouble ahead for non Abelian theories:
    - covariant gauges introduce unphysical longitudinal d.o.f. for the gluon
      as for a photon only transverse d.o.f. are physical

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    - complex scalar field which obeys Fermi statistics
    - new Feynman rules: propagator and gluon-ghost-ghost coupling
    - eats unphysical degrees of freedom in polarization sum

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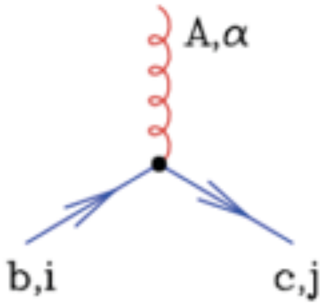
- alternatively one can choose a non-covariant (axial) gauge $\mathcal{L}_{\mathbf{axial}} = -\frac{1}{\xi}(\mathbf{n}^\mu \mathbf{A}^{\mathbf{a}}_\mu)^2$
  - at the expense of a more complicated gluon propagator

arbitrary direction

$$\frac{\mathbf{i}}{\mathbf{p}^2}\left(-\mathbf{g}_{\mu\nu} + \frac{\mathbf{n}_\mu \mathbf{p}_\nu + \mathbf{n}_\nu \mathbf{p}_\mu}{\mathbf{n}\cdot\mathbf{p}} + \frac{(\mathbf{n}^2 + \xi \mathbf{p}^2)\, \mathbf{p}_\mu \mathbf{p}_\nu}{(\mathbf{n}\cdot\mathbf{p})^2}\right)\delta_{\mathbf{ab}}$$

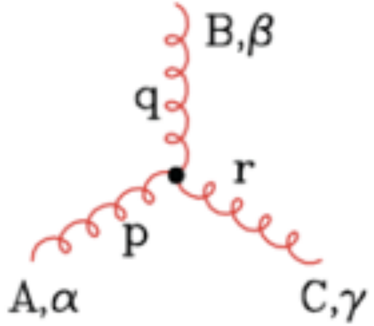
using the QCD Lagrangian: interactions

- interactions between quarks and gluons can be simply read off from the terms in the Lagrangian containing g_s



$$-ig\ \ (t^A)_{cb}\ \ (\gamma^\alpha)_{ji}$$

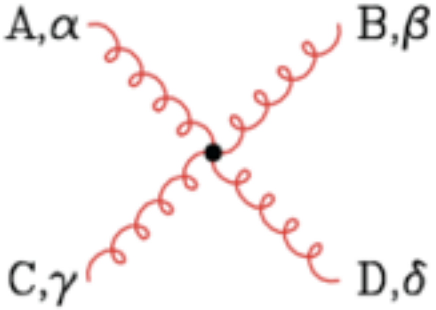
from the covariant derivative
as in QED except for color



$$-g\ \ f^{ABC}[(p-q)^\gamma g^{\alpha\beta}+(q-r)^\alpha g^{\beta\gamma}+(r-p)^\beta g^{\gamma\alpha}]$$

(all momenta incoming)

gluon self interactions



$$-ig^2\ \ f^{XAC}f^{XBD}\ \ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$
$$-ig^2\ \ f^{XAD}f^{XBC}\ \ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}]$$
$$-ig^2\ \ f^{XAB}f^{XCD}\ \ [g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$

from the g_s term in
the gluon field strength
no QED analogue

take home message for part I

THE FOUNDATIONS

QCD is based on a simple Lagrangian but has a rich phenomenology

QCD is based on the non Abelian gauge group SU(3)

- number of colors and group structure can be tested experimentally
- concept of local gauge invariance dictates interactions
- similarities to QED, yet profound differences (and more to come)
- color leads to self-interactions between "force carrying" gluons
- perturbation theory can be based on a short list of Feynman rules

color algebra decouples and can be performed separately

- color factors can be expressed in terms of two Casimirs: C_A and C_F
- powerful pictorial methods; possibility of "leading color approximation"

Part II

the QCD toolbox

asymptotic freedom, IR safety,
QCD final state, factorization

dichotomy of QCD

the gauge principle is elegant and powerful but any theory must ultimately stand (or fall) by its success (or failure)

QCD is the theory of **strong** interactions

– how can we make use of **perturbative** methods?

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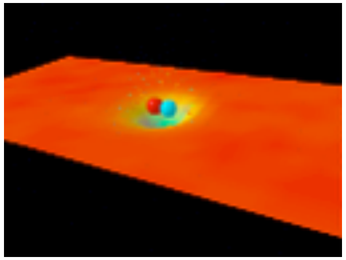
confinement



D. Leinweber

non-perturbative
structure of hadrons

e.g. through **lattice QCD**

asymptotic freedom

hard scattering
cross sections
and
renormalization group

with **perturbative methods**

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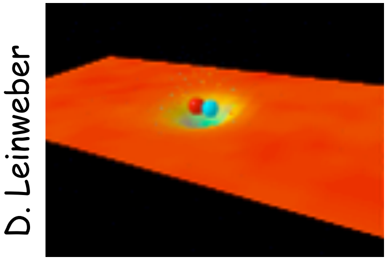

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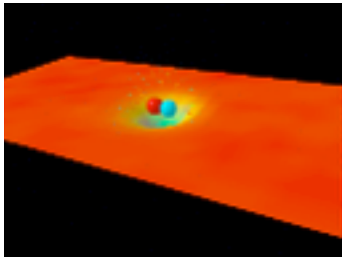

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interplay

probing hadronic structure with
weakly interacting quanta of asymptotic freedom

asymptotic freedom

Gross, Wilczek; Politzer ('73/'74)
Nobel prize 2004

value of strong coupling $\alpha_s = g^2/4\pi$ depends on distance r (i.e., on energy Q)

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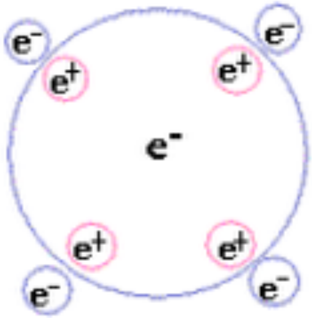
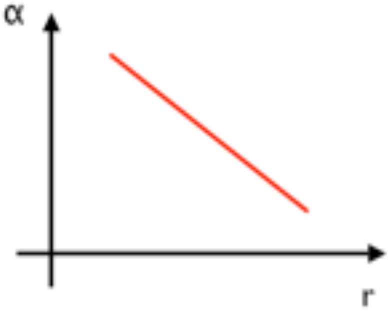
like QED

"screening" of the charge





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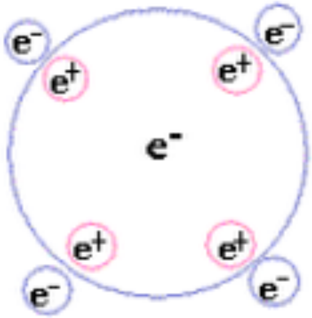
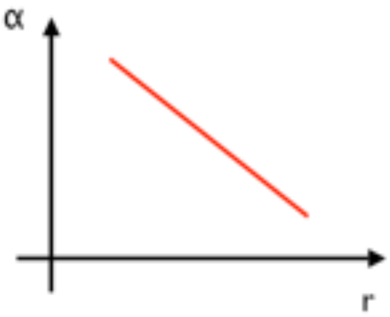
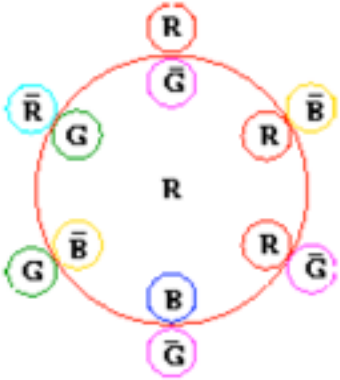
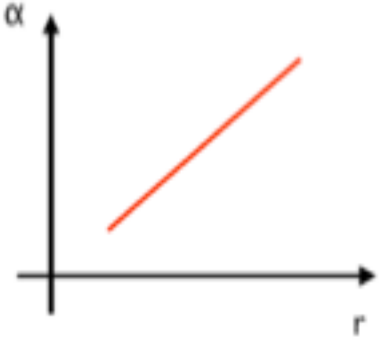
"anti-screening"

non
Abelian









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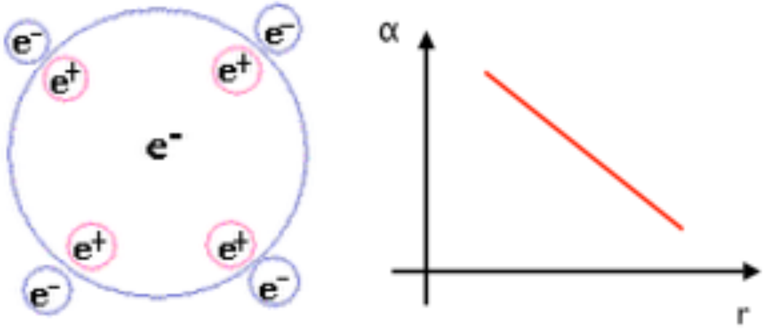

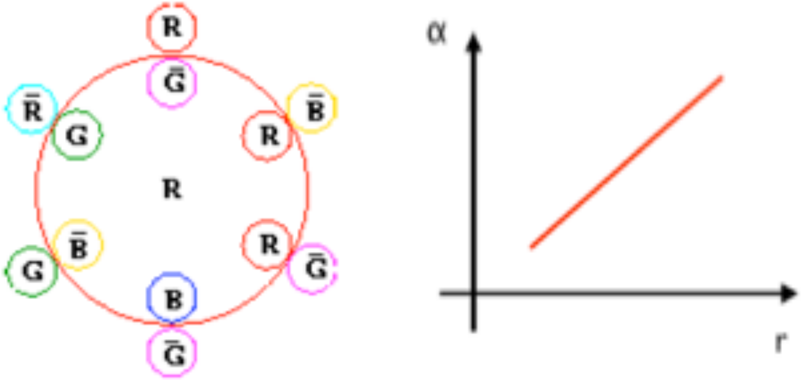

who wins ?

$$\boxed{\alpha_s(Q^2) \approx \frac{4\pi}{(\frac{11}{3}C_A - \frac{4}{3}T_F N_f)\ln(Q^2/\Lambda^2)}}$$

$Q \sim 1/r$

asymptotic freedom

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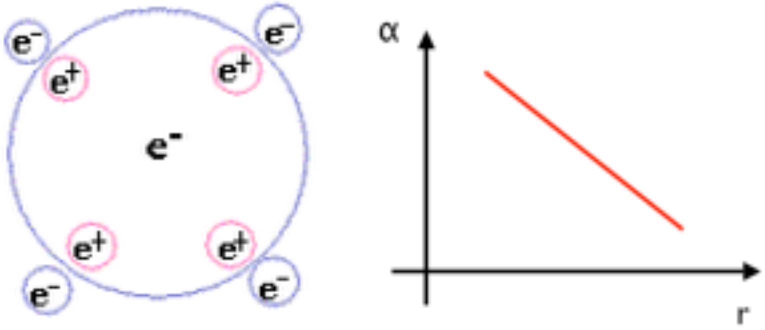

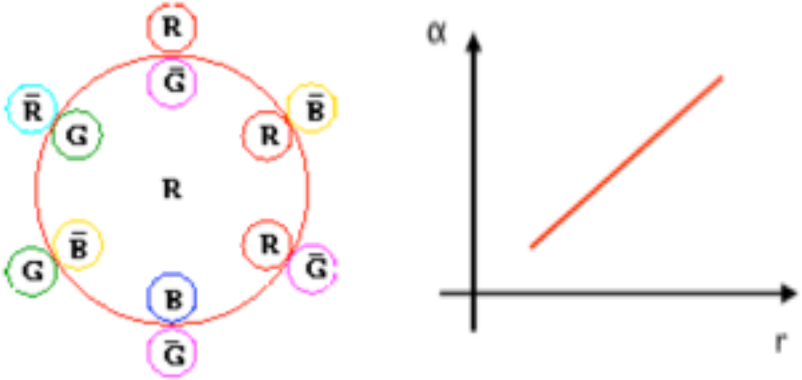

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$$\alpha_s(Q^2) \approx \frac{4\pi}{(\frac{11}{3}C_A - \frac{4}{3}T_F N_f)\ln(Q^2/\Lambda^2)}$$

$Q \sim 1/r$

asymptotic freedom

Gross, Wilczek; Politzer ('73/'74)
Nobel prize 2004

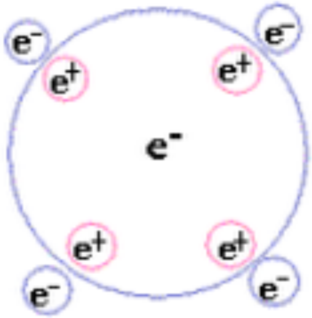
value of strong coupling $\alpha_s = g^2/4\pi$ depends on distance r (i.e., on energy Q)

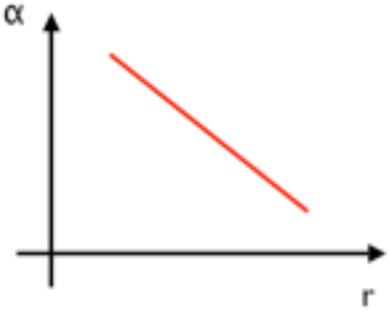
like QED

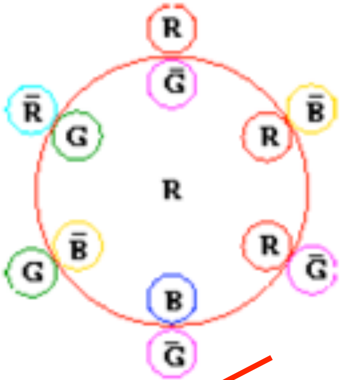
"screening" of the charge

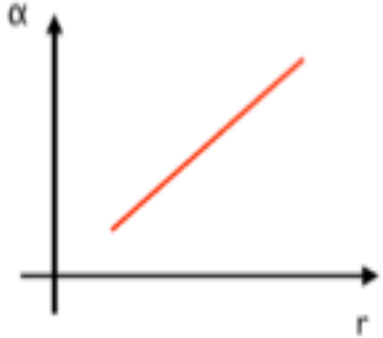
non Abelian

"anti-screening"









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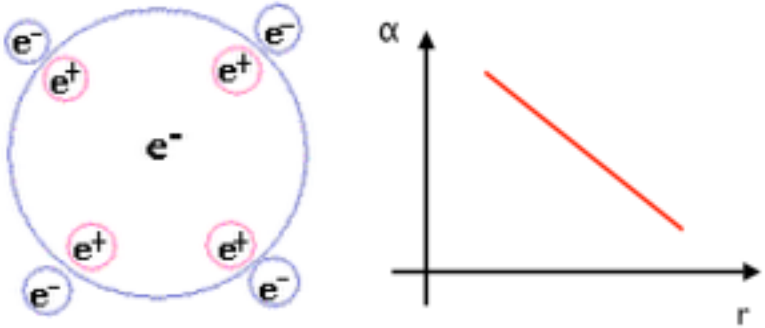

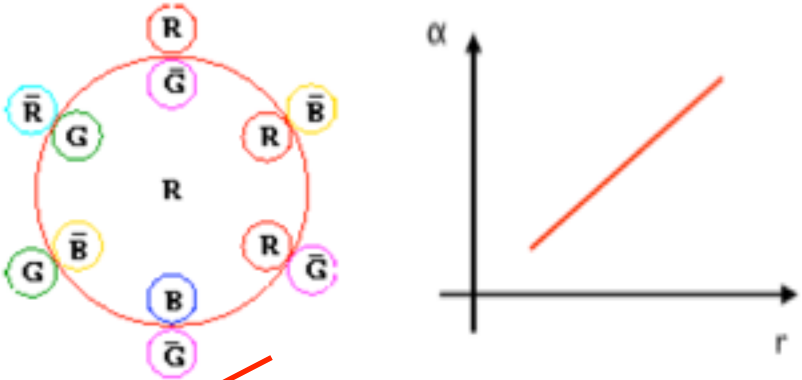

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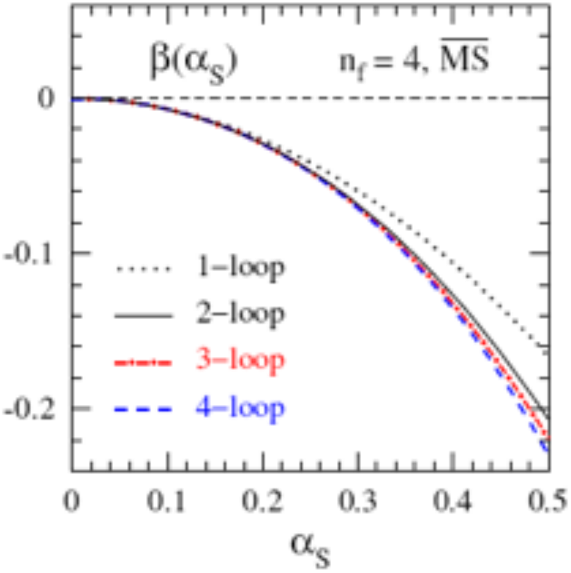
typical hadronic scale O(200 MeV)
Λ depends on N_f, pert. order and scheme

more formally: the QCD beta function

van Ritbergen, Vermaseren, Larin

$$Q^2 \frac{\partial a_s}{\partial Q^2} = \beta(a_s) = \underset{\text{LO}}{\overset{\text{('71), '73}}{-\beta_0 a_s^2}} \underset{\text{NLO}}{\overset{\text{'74}}{- \beta_1 a_s^3}} \underset{\text{NNLO}}{\overset{\text{'80}}{- \beta_2 a_s^4}} \underset{\text{N}^3\text{LO}}{\overset{\text{'97}}{- \beta_3 a_s^5}} + \ldots \qquad a_s \equiv \frac{\alpha_s}{4\pi}$$



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O(50000) diagrams !

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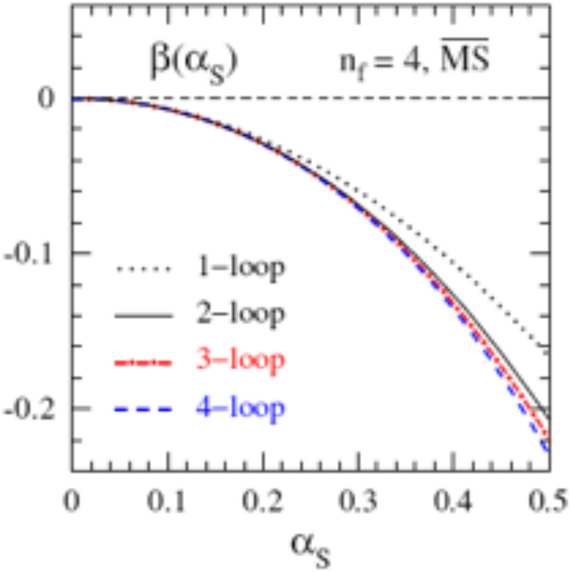
('71), '73 '74 '80 '97

$$Q^2\frac{\partial a_s}{\partial Q^2}=\beta(a_s)=-\beta_0 a_s^2-\beta_1 a_s^3-\beta_2 a_s^4-\beta_3 a_s^5+\dots \qquad a_s\equiv\frac{\alpha_s}{4\pi}$$

LO NLO NNLO N^{3}LO



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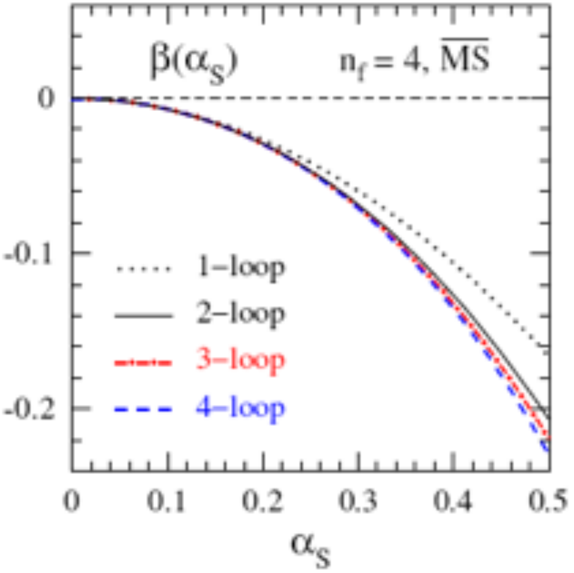

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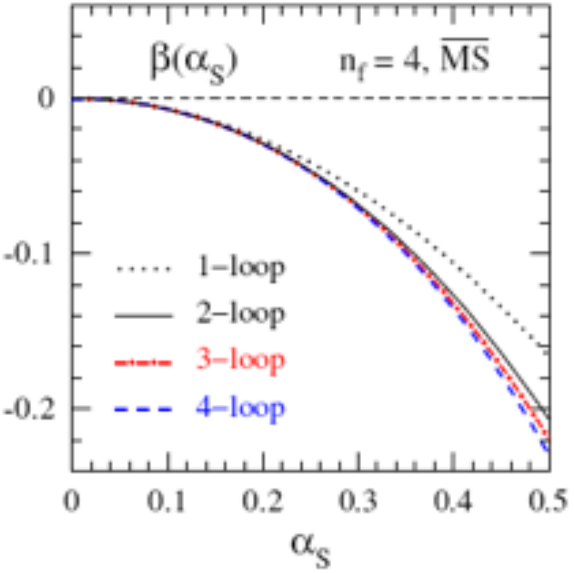

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tells us how α_s varies with scale but not its absolute value at μ_0

1st example of a renormalization group equation

some further observations

recap

$$\beta = -\alpha_s^2(\mu)\sum_i b_i \alpha_s^i(\mu)$$

$$b_0 = \frac{11N_c - 4n_f T_R}{12\pi}$$

- negative contribution to b_0 due to

- positive contribution to b_0 due to

- positive contribution larger —> $b_0 > 0$ (—> overall: negative beta function)

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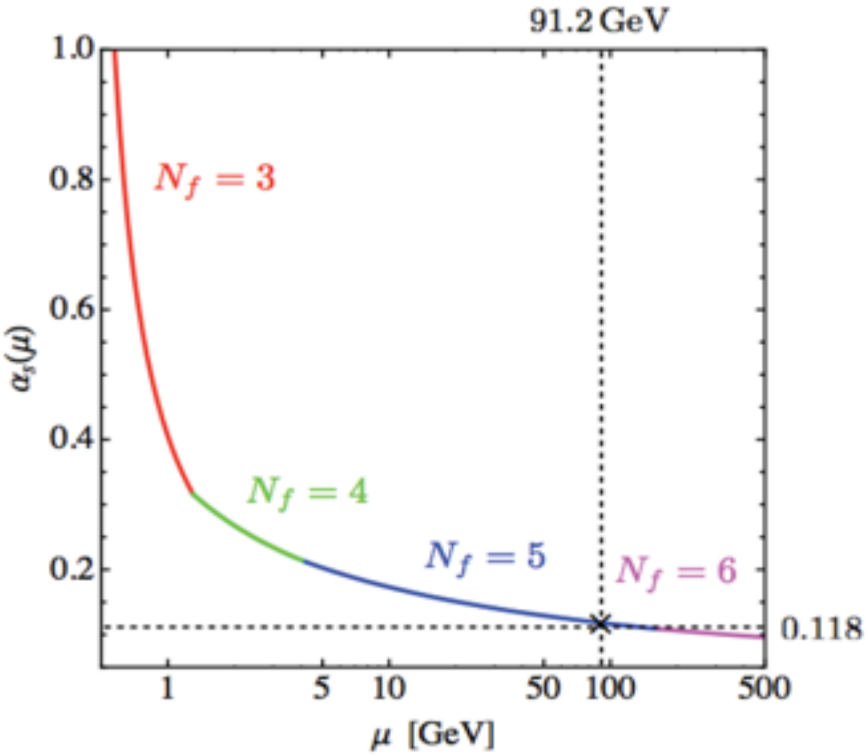

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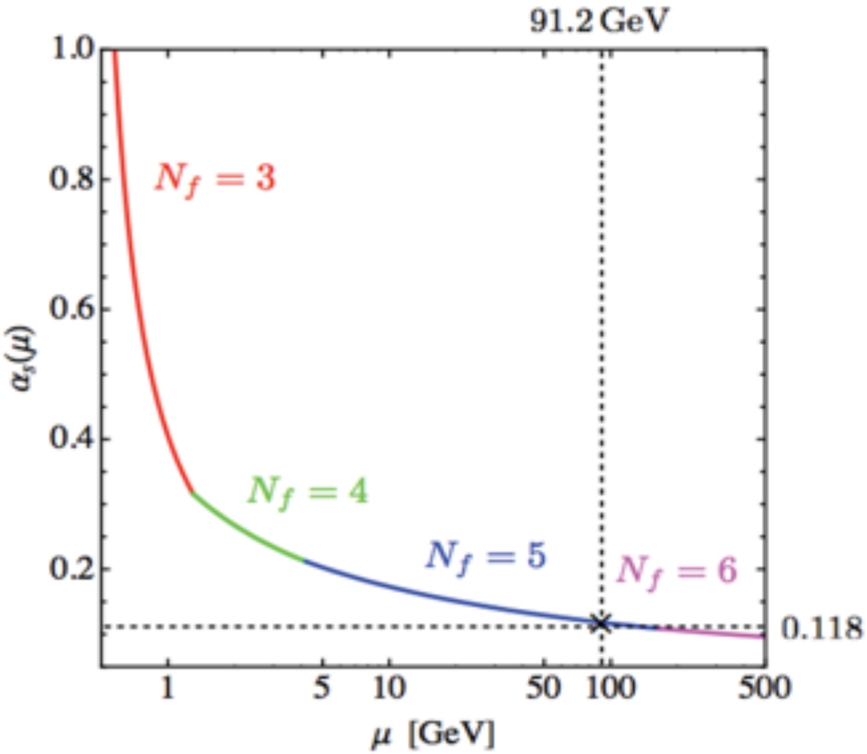


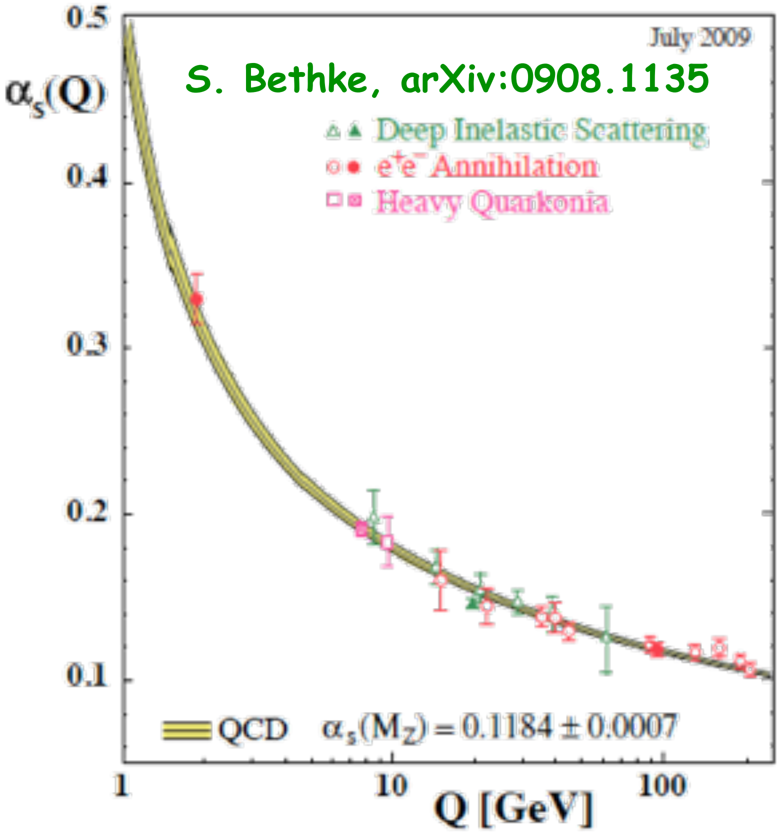
- coupling depends on number of active flavors
  (need matching a thresholds)

- can read off QED beta function (T_R coefficient)
  (only one flavor)

$$\beta_{\text{QED}} = \frac{1}{3\pi}\alpha^2 + \ldots$$

b_0 negative —> overall: positive beta fct.

consistent picture from many observables

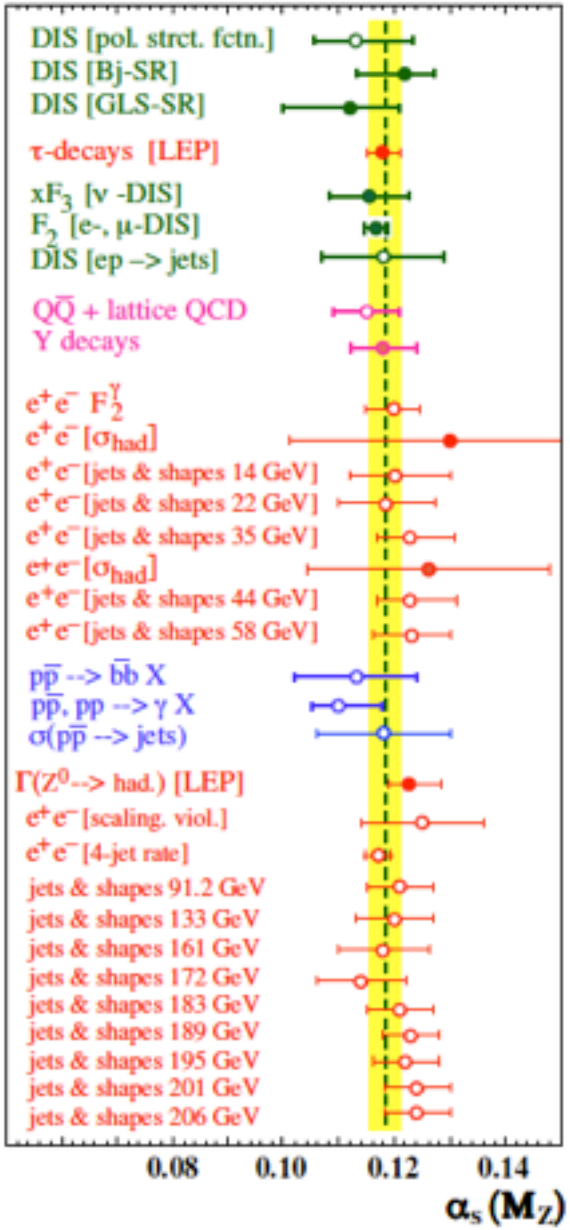


S. Bethke, arXiv:0908.1135

confinement

asymp. freedom

exp. evidence for log(Q^2) fall-off is persuasive



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- detectors are a long-distance away
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- **infrared safety**
- **factorization**

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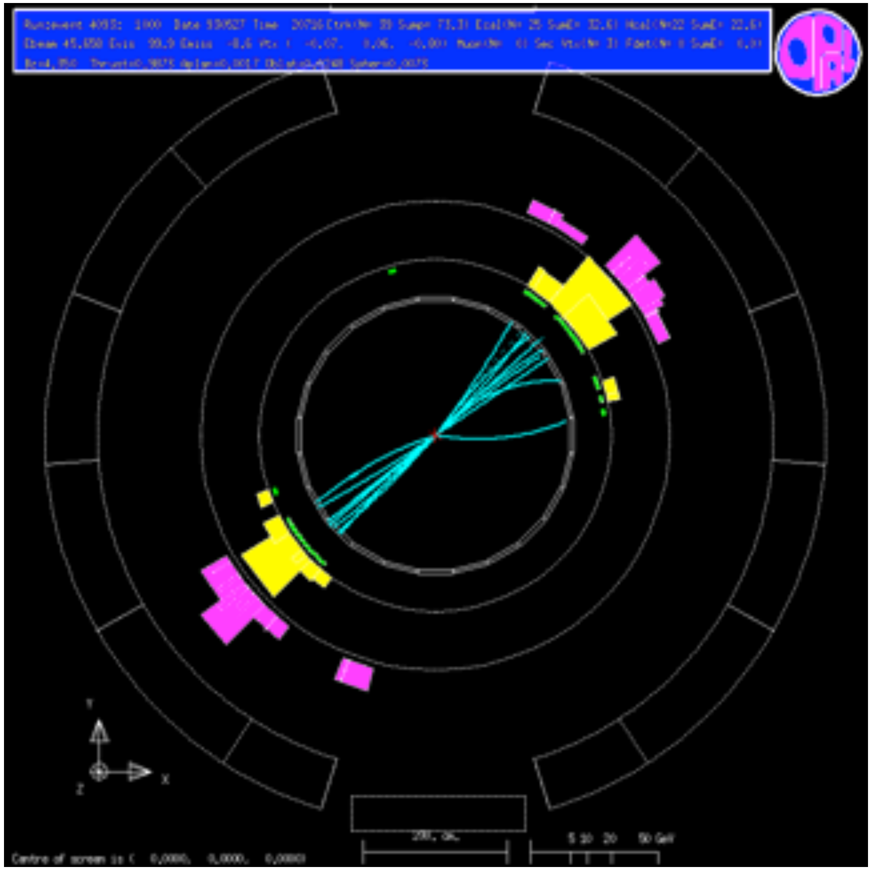
let's study electron-positron annihilation to see what this is all about ...

e^+e^- annihilation: the QCD guinea pig

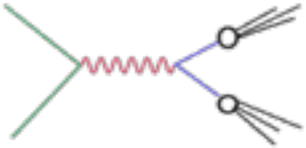
1989-2000

most of the hadronic events at CERN-LEP had **two back-to-back jets**

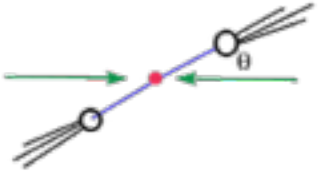


jet: pencil-like collection of hadrons

- jets resemble features of underlying 2->2 hard process $e^+e^- \rightarrow q\bar{q}$



- angular distribution of jet axis w.r.t. beam axis as predicted for **spin-½ quarks**

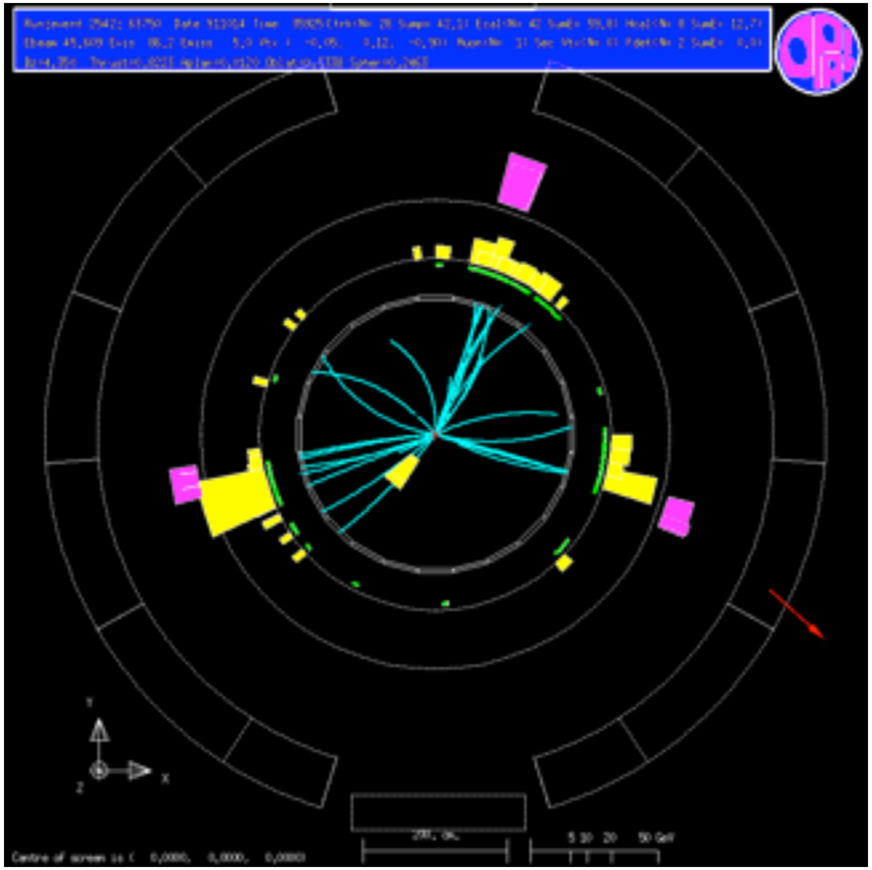


jets play major role in hadron-hadron collisions at TeVatron, RHIC, LHC

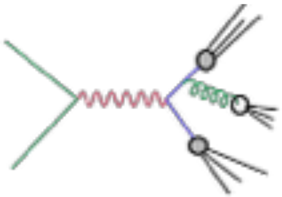
e^+e^- annihilation: three-jet events

about 10% of the events had a **third jet**

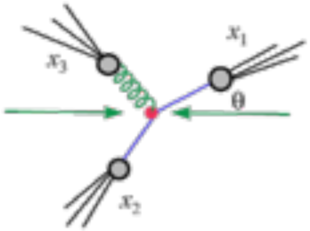
first discovered at
DESY-PETRA in 1979



- jets resemble features of underlying 2->3 hard process $e^+e^- \rightarrow q\bar{q}g$



- 10% rate consistent with $\alpha_s \simeq 0.1$ (**determination of α_s**)
- angular distribution of jets w.r.t. beam axis as expected for **spin-1 gluons**



e^+e^- annihilation: four-jet events

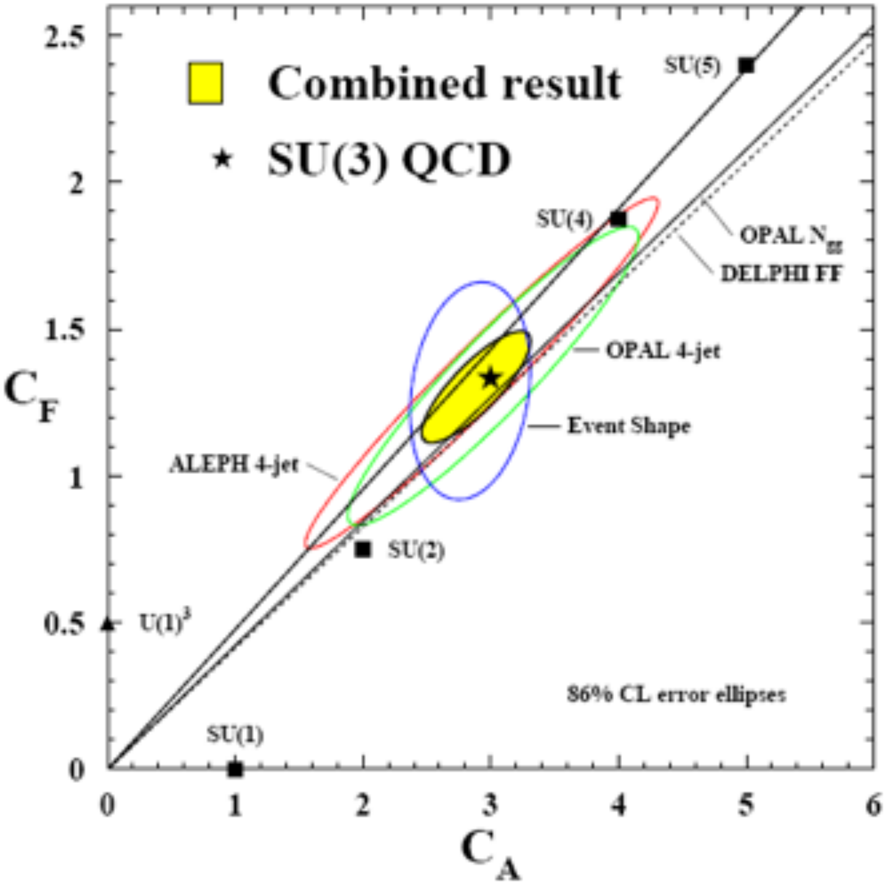
some events even had a **fourth jet**

extensively studied at LEP

- angular correlations between four jets depend on C_A/C_F and T_F/C_F
- sensitivity to non-Abelian **three-gluon-vertex**

  LO: Ellis, Ross, Terrano

- crucial test of QCD when combined with results for event shapes (thrust, etc.)



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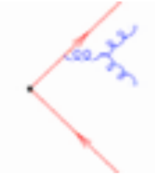

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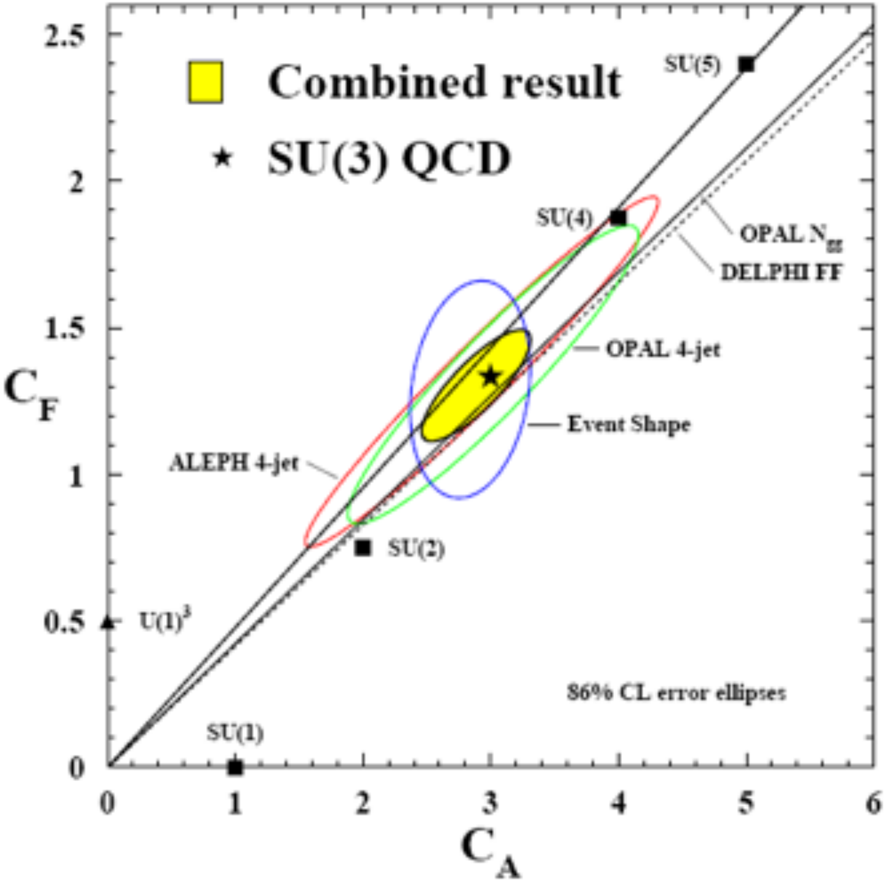

e^+e^- experiments played a vital role in establishing QCD as the correct theory of strong interactions and SU(3) as the underlying gauge group

recipe for quantitative calculations

(1) identify the final-state of interest and draw all relevant Feynman diagrams

(2) use SU(3) algebra to take care of QCD color factors

(3) compute the rest of the diagram using "Diracology"

traces of gamma matrices, spinors, …

(4) to turn squared matrix elements into a cross section we need to

- account for the available phase space (momentum d.o.f. in final-state)
- integrate out not observed d.o.f.
- normalize by incoming flux

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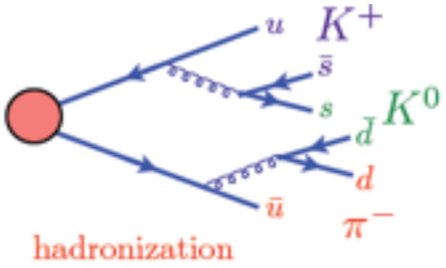

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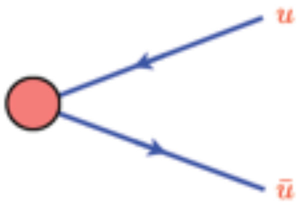
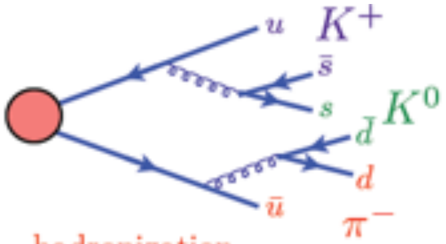
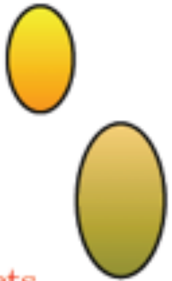


energetic partons



hadronization



jets

cleanest observables in QCD

will find that most "stuff"
is observed in the directions
of produced quarks & gluons
parton-hadron duality

bunch of automated LO tools

- LO estimates of cross sections are practically a solved problem
- many useful fully automated tools available (limitations for high multiplicities)

ALPGEN	M. L. Mangano et al. http://alpgen.web.cern.ch/alpgen/
AMEGIC++	F. Krauss et al. http://projects.hepforge.org/sherpa/dokuwiki/doku.php
CompHEP	E. Boos et al. http://comphep.sinp.msu.ru/
HELAC	C. Papadopoulos, M. Worek http://helac-phegas.web.cern.ch/helac-phegas/helac-phegas.html
Madgraph	F. Maltoni, T. Stelzer http://madgraph.hep.uiuc.edu/

...

let's have a closer look at the R-ratio already encountered in Part I

$$R \equiv \frac{e^+e^- \to \text{hadrons}}{e^+e^- \to \mu^+\mu^-} \propto N_c \sum_f Q_f^2$$

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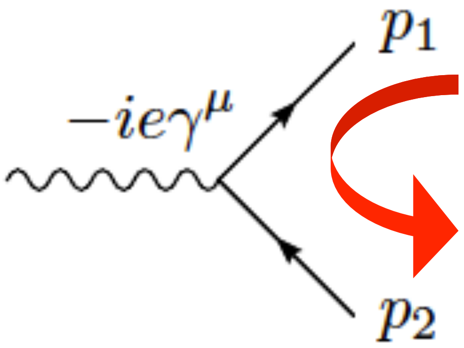
at LO described by:

vertex

$$M_0^\mu = \bar{u}(p_1)(-ie\gamma^\mu)v(p_2)$$

spinors for external lines

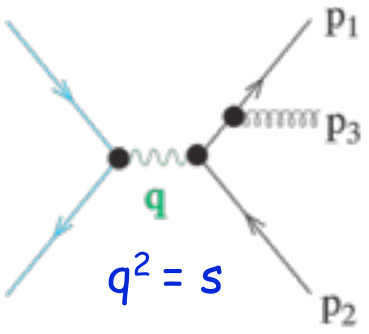


"read against the arrow"

exploring the QCD final-state: $e^+e^- \rightarrow$ 3 partons

simplest process in pQCD: $e^+e^- \rightarrow q\bar{q}g$

(all partons massless)



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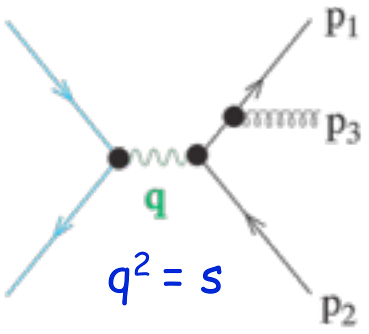

some kinematics first:

- energy fractions & conservation: $x_i \equiv \frac{2p_i \cdot q}{s} = \frac{E_i}{\sqrt{s}/2}$ $\sum x_i = \frac{2(\sum p_i) \cdot q}{s} = 2$

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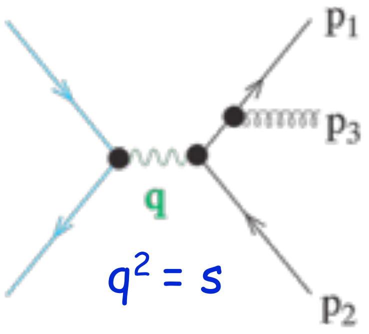

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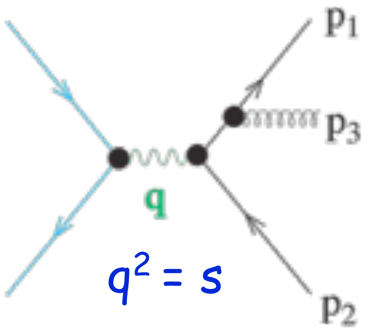

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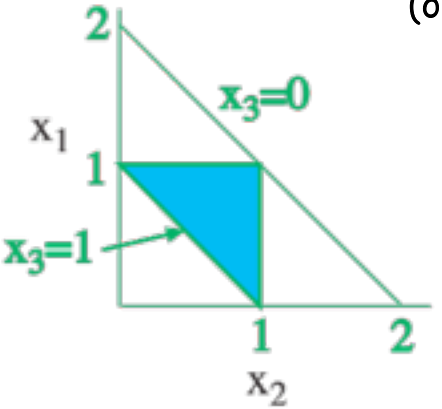
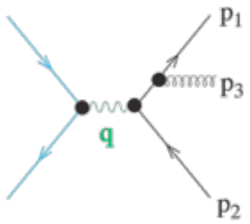
$$\Rightarrow \boxed{0 \leq x_i \leq 1}$$

allowed values for x_i lie within a triangle

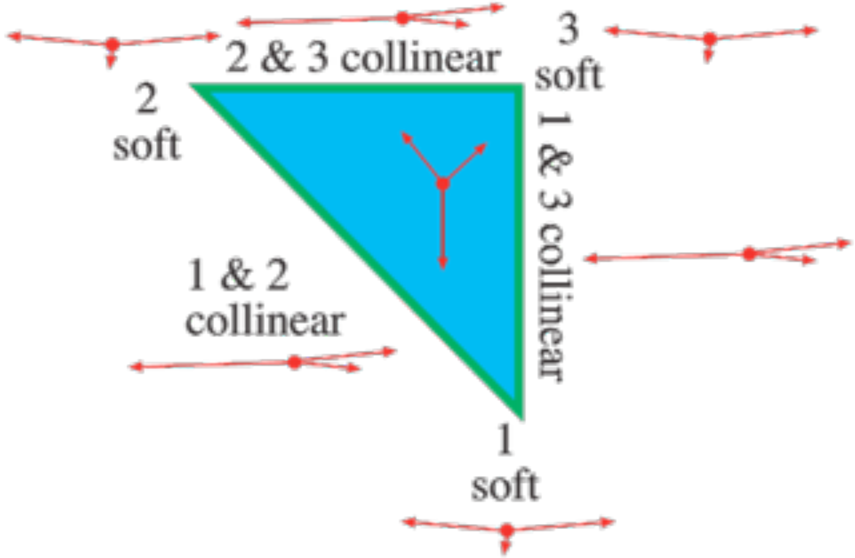


massless "Dalitz plot"

collinear and soft configurations



at the boundaries of phase space we encounter **special kinematic configurations**:



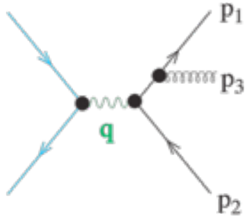
- “edges”: **two partons collinear**

$$\text{e.g. } \theta_{13} \to 0 \Leftrightarrow x_2 \to 1$$

- “corners”: **one parton soft**

$$p_i^\mu \to 0 \Leftrightarrow x_i \to 0$$

collinear and soft configurations



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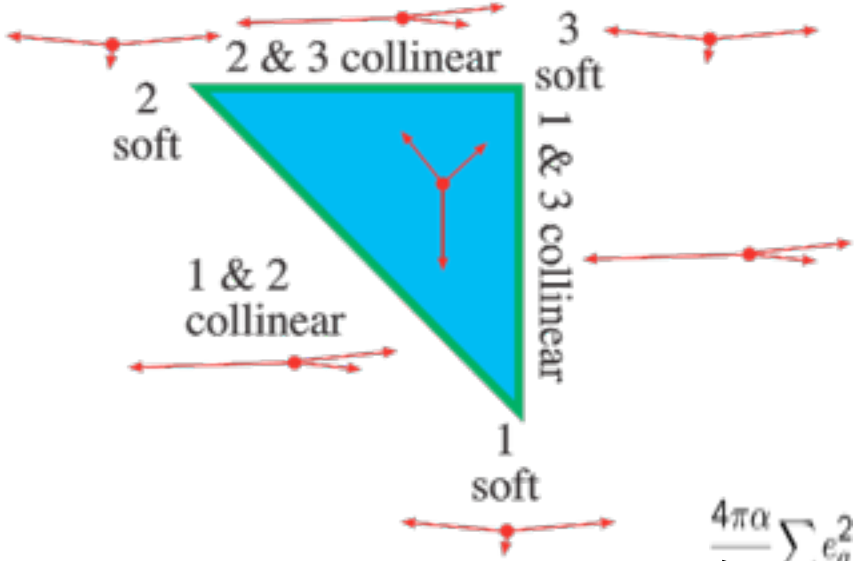

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$$\frac{4\pi\alpha}{s}\sum e_q^2$$

structure reflected
in the **cross section**:

$$\frac{1}{\sigma_0}\frac{d\sigma}{dx_1 dx_2} = \frac{\alpha_s}{2\pi} C_F \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

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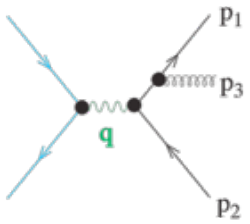

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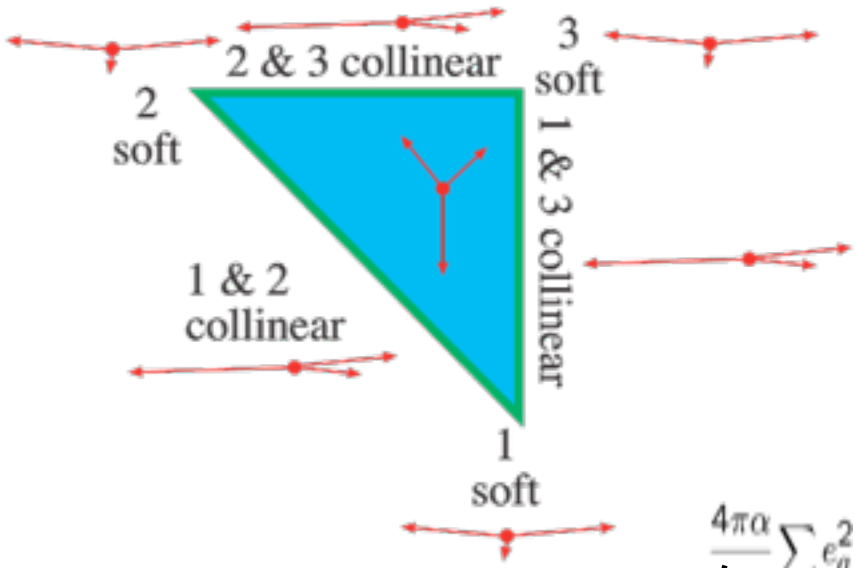

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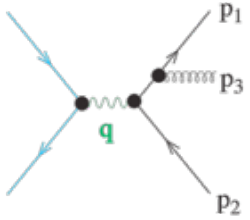
collinear singularities:

$x_1 \to 1$: gluon ∥ antiquark

$x_2 \to 1$: gluon ∥ quark

collinear and soft configurations



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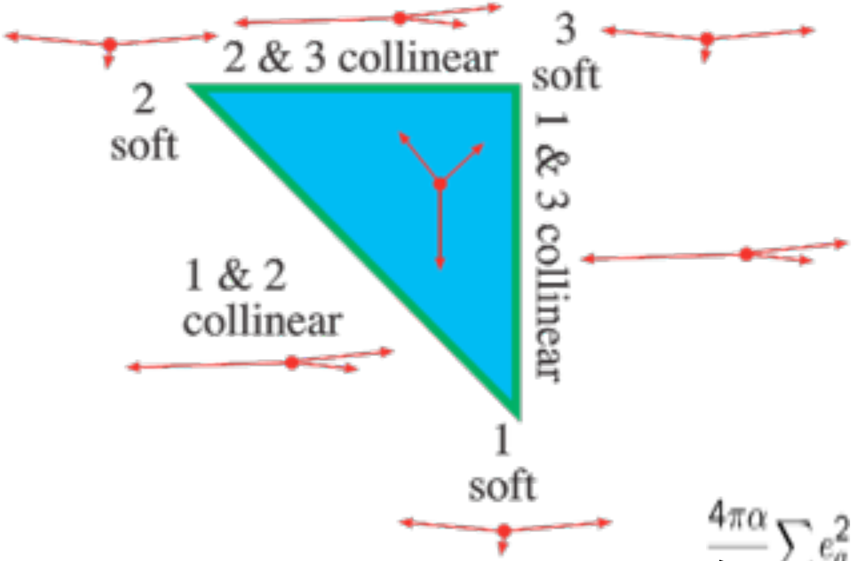

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soft gluon singularity:

$x_3 \to 0$: $p_3 \to 0$

$\leftrightarrow x_1 \to 1$ & $x_2 \to 1$

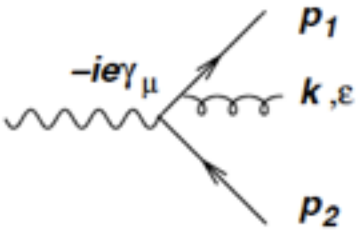
collinear singularities:

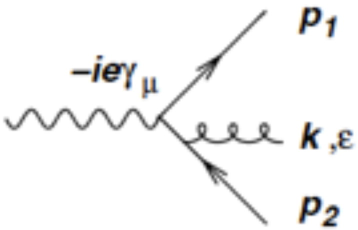
$x_1 \to 1$: gluon $\parallel$ antiquark

$x_2 \to 1$: gluon $\parallel$ quark

aside: some steps of the actual calculation

$$
\begin{aligned}
\mathcal{M}_{q\bar{q}g} = & \ \bar{u}(p_1) i g_s \not{\epsilon} t^A \frac{i}{\not{p}_1 + \not{k}} i e_q \gamma_\mu v(p_2) \\
& - \bar{u}(p_1) i e_q \gamma_\mu \frac{i}{\not{p}_2 + \not{k}} i g_s \not{\epsilon} t^A v(p_2)
\end{aligned}
$$





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propagator

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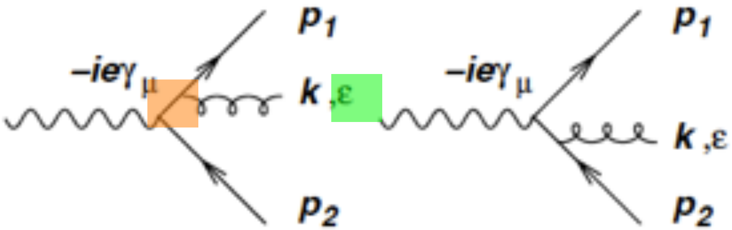
color

$$- \bar{u}(p_1) i e_q \gamma_\mu \frac{i}{\not{p}_2 + \not{k}} i g_s \not{\epsilon} t^A v(p_2)$$

polarization



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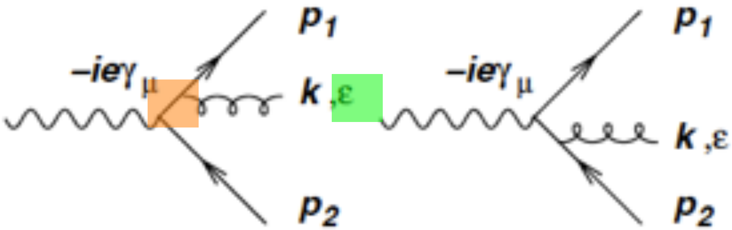

make gloun soft $k \ll p_{1,2}$ and square the amplitude

$$|M^2_{q\bar{q}g}| \simeq \sum_{A,\mathrm{pol}} \left| \bar{u}(p_1) i e_q \gamma_\mu t^A v(p_2)\, g_s \left(\frac{p_1.\epsilon}{p_1.k} - \frac{p_2.\epsilon}{p_2.k} \right) \right|^2$$

color factor

$$= -|M^2_{q\bar{q}}| C_F g_s^2 \left(\frac{p_1}{p_1.k} - \frac{p_2}{p_2.k} \right)^2 = |M^2_{q\bar{q}}| C_F g_s^2 \frac{2p_1.p_2}{(p_1.k)(p_2.k)}$$

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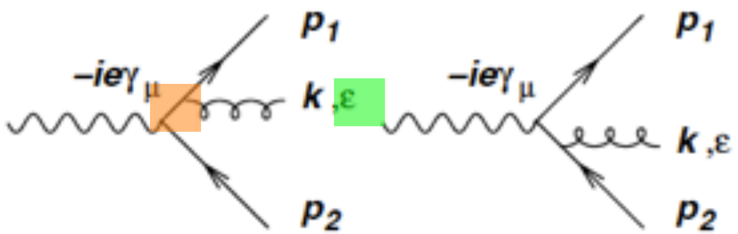

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sum over gluon polarizations

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Eikonal factor

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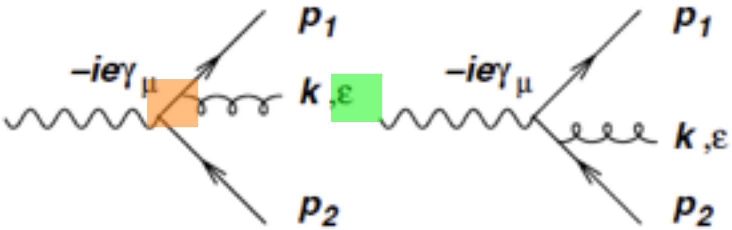

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Eikonal factor

include phase space for gluon

$$d\Phi_{q\bar{q}g} |M^2_{q\bar{q}g}| \simeq (d\Phi_{q\bar{q}} |M^2_{q\bar{q}}|) \frac{d^3\vec{k}}{2E(2\pi)^3} C_F g_s^2 \frac{2p_1.p_2}{(p_1.k)(p_2.k)}$$

factorize LO phase space

note: color will in general not factorize in soft limit

aside: some steps of the actual calculation - cont'd

$$d\Phi_{q\bar{q}g}|M^2_{q\bar{q}g}| \simeq (d\Phi_{q\bar{q}}|M^2_{q\bar{q}}|)\,\frac{d^3\vec{k}}{2E(2\pi)^3}C_F g_s^2\frac{2p_1.p_2}{(p_1.k)(p_2.k)}$$

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soft emission factor dS

$$EdE\, d\cos\theta\, \frac{d\phi}{2\pi} \cdot \frac{2\alpha_s C_F}{\pi} \frac{2p_1.p_2}{(2p_1.k)(2p_2.k)} \quad \begin{matrix} \theta \equiv \theta_{p_1 k} \\ \phi = \text{azimuth} \end{matrix}$$

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$\phi = \text{azimuth}$

express in terms of E,θ

$$= \frac{1}{E^2(1-\cos^2\theta)}$$

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express in terms of E,θ

$$= \frac{1}{E^2(1-\cos^2\theta)}$$

end up with

$$dS = \frac{2\alpha_s C_F}{\pi}\frac{dE}{E}\frac{d\theta}{\sin\theta}\frac{d\phi}{2\pi}$$

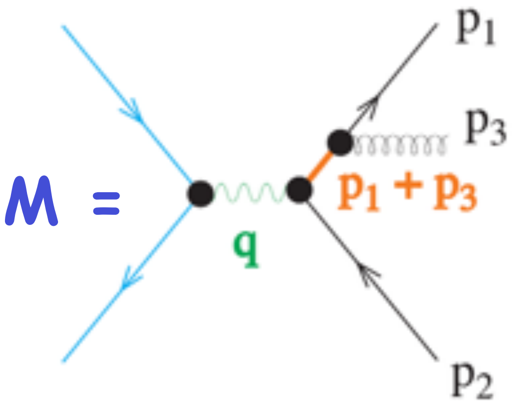
- It *diverges* for $E \to 0$ — *infrared (or soft) divergence*
- It *diverges* for $\theta \to 0$ and $\theta \to \pi$ — *collinear divergence*

general nature of these singularities

soft/collinear limit:

internal propagator goes on-shell

here: $$\frac{1}{(p_1+p_3)^2} = \frac{1}{2E_1E_3(1-\cos\theta_{13})}$$



note: "soft quarks" (here $E_1 \to 0$) never lead to singularities (canceled by numerator)

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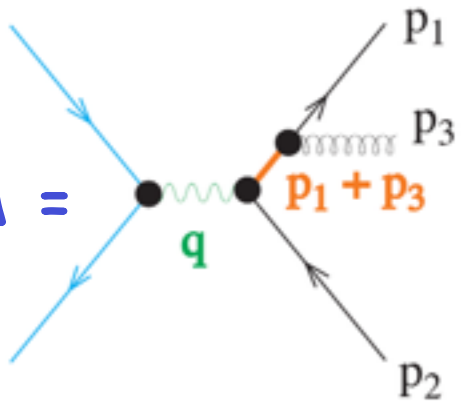

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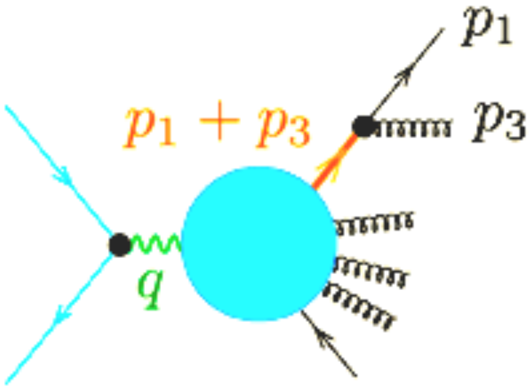
this structure is generic for QCD tree graphs:

$$\mathcal{M}_{n+1} \sim [\mathcal{M}_n]\Big|_{1,3\,\mathrm{on-shell}} \frac{\mathrm{spinors}}{(p1+p3)^2}$$

basis for parton-shower MC codes
like PYTHIA, HERWIG, SHERPA, …



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the lesson is:

whenever the 2->(n+1) kinematics collapses to an effective 2->n parton kinematics due to

- the emission of a soft gluon
- a collinear splitting of a parton into two partons

we have to be much more careful and work a bit harder!

this applies to all pQCD calculations

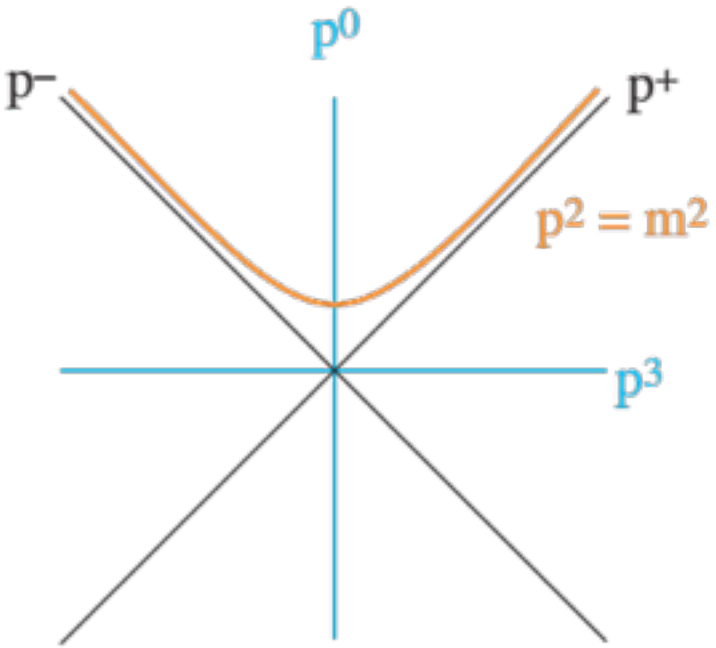
towards a space-time picture of the singularities

interlude: light-cone coordinates

$$p^{\pm} \equiv (p^0 \pm p^3)/\sqrt{2}$$

$$p^2 = 2p^+p^- - \vec{p}_T^{\,2}$$

$$p^- = (p_T^2 + m^2)/2p^+$$



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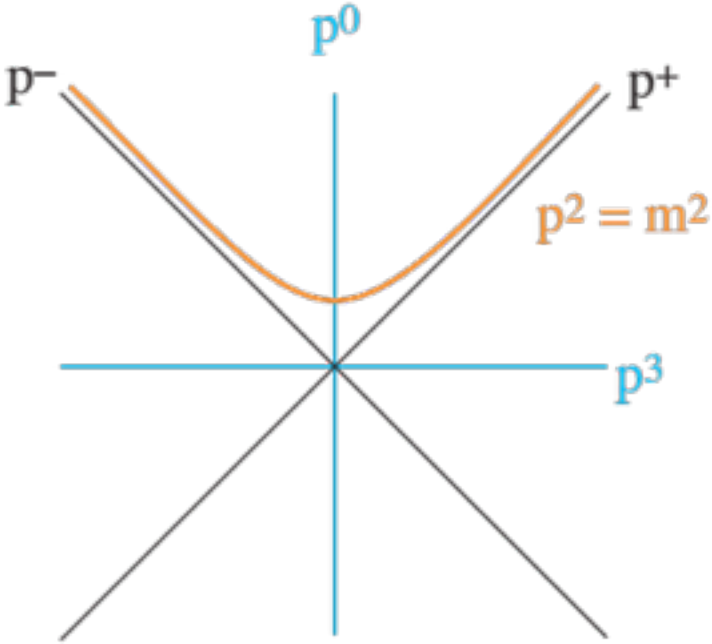

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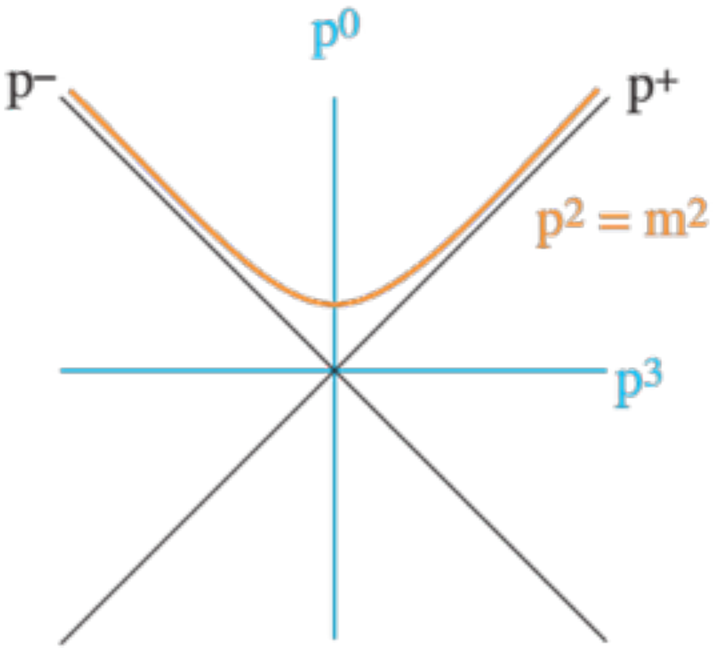

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Fourier transform

momentum space $\longleftrightarrow$ coordinate space

$$e^{ip\cdot x}$$

$$p \cdot x = p^+x^- + p^-x^+ - \vec{p}_T \cdot \vec{x}_T$$

--> x^- is conjugate to p^+ and x^+ is conjugate to p^-

space-time picture of the singularities

What does this imply for our propagator going on-shell?

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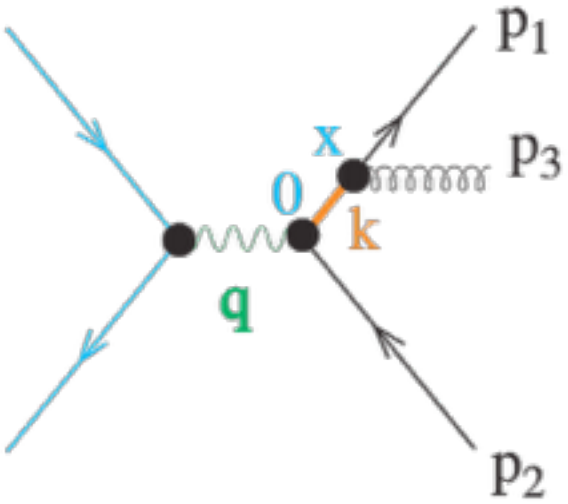

- define $k \equiv p_1 + p_3$
- use coordinates with k^+ large and $k_T = 0$
- $k^2 = 2\ k^+k^- \simeq 0$ corresponds to soft/collinear limit → k^- small

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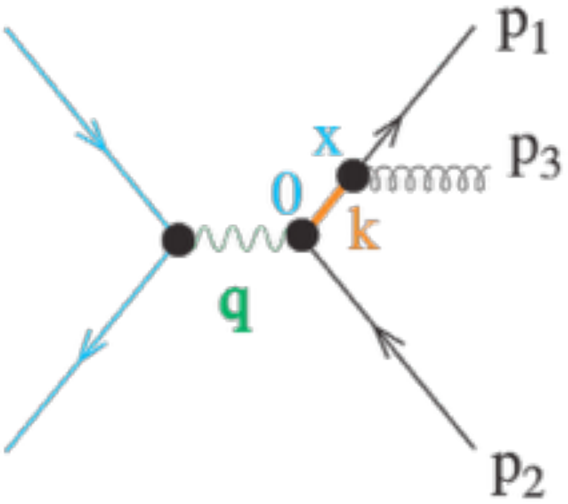

How far does the internal on-shell parton travel in space-time?

space-time picture of the singularities

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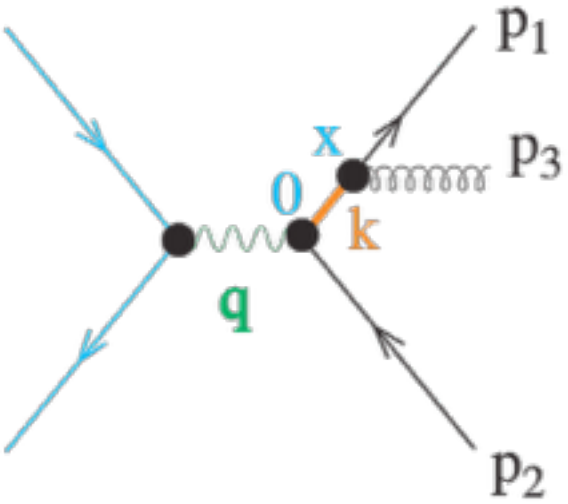

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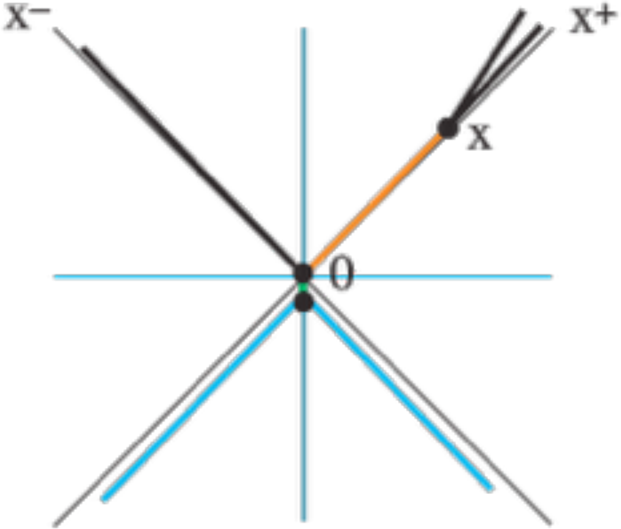
$$k^- \simeq (\vec{k}_T^2 + k^2)/\sqrt{s}$$ small

$\downarrow$ Fourier

$$x^+ \simeq 1/k^-$$ large

$$x^- \simeq 1/k^+$$ small



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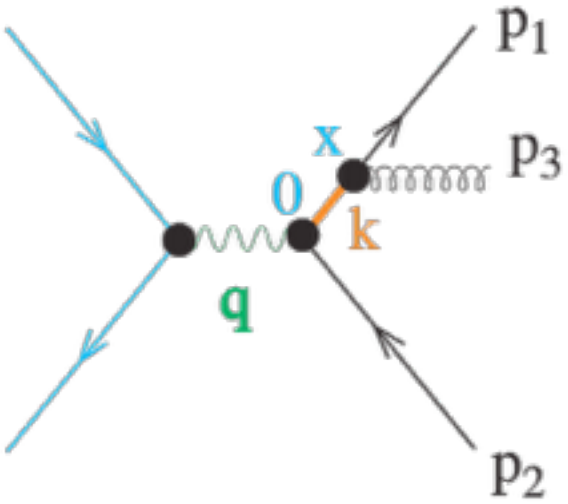

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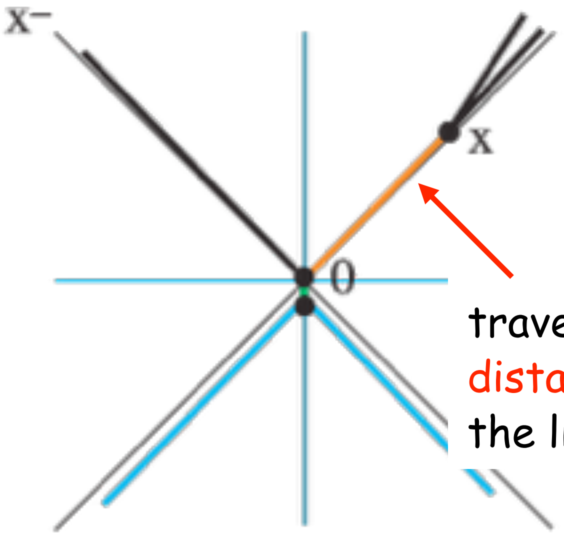

travels a long distance along the light-cone

upshot: soft/collinear singularities arise from interactions that happen a long time after the creation of the quark/antiquark pair

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upshot: soft/collinear singularities arise from interactions that happen a long time after the creation of the quark/antiquark pair

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so ...... What to do with the long-distance physics associated with these soft/collinear singularities? Is there any hope that we can predict some reliable numbers to compare with experiment?

to answer this, we have to formulate the

concept of infrared safety

infrared-safe observables

formal definition of infrared safety:

Kunszt, Soper; ...

study inclusive observables which do not distinguish between (n+1) partons and n partons in the soft/collinear (=degenerate) limit, i.e., are insensitive to what happens at long-distance

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$$\mathcal{I} = \frac{1}{2!}\int d\Omega_2 \frac{d\sigma[2]}{d\Omega_2} \mathcal{S}_2(p_1, p_2) + \frac{1}{3!}\int d\omega_2 dE_3 d\Omega_3 \frac{d\sigma[3]}{d\Omega_2 dE_3 d\Omega_3} \mathcal{S}_3(p_1, p_2, p_3) + \ldots$$

measurement fcts.
(define your observable)

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infrared safe **iff** [for $\lambda=0$ (soft) and $0 < \lambda < 1$ (collinear)]

$$\mathcal{S}_{n+1}(p_1,\ldots,(1-\lambda)p_n,\lambda p_n) = \mathcal{S}_n(p_1,\ldots,p_n)$$

physics behind formal IR safety requirement

cannot resolve soft and collinear partons experimentally

→ intuitively reasonable that a theoretical calculation can be infrared safe as long as it is insensitive to long-distance physics (not a priori guaranteed though)

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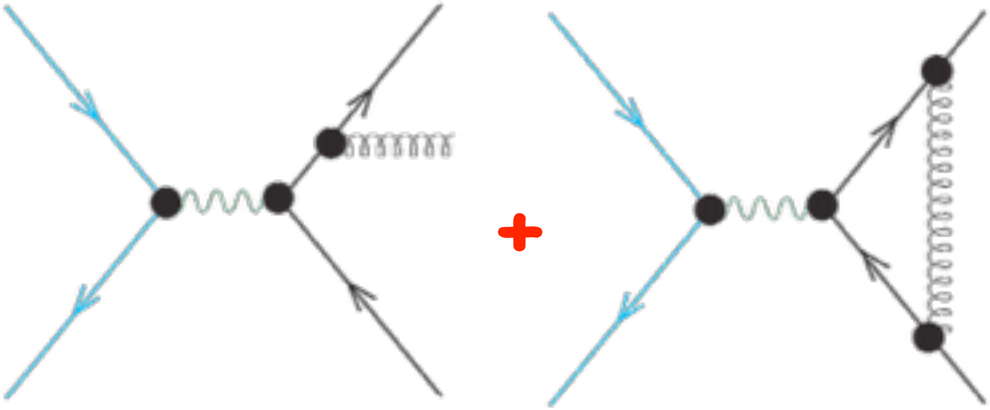
at a level of a pQCD calculation (e.g. e^+e^- at $O(\alpha_s)$, i.e., n=2)

$$S_{n+1}(p_1,\ldots,(1-\lambda)p_n,\lambda p_n)=S_n(p_1,\ldots,p_n)$$

→ singularities of real gluon emission and virtual corrections cancel in the sum



extension of famous theorems by Kinoshita-Lee-Nauenberg and Bloch-Nordsieck

example I: total cross section $e^+e^- \to$ hadrons

simplest case:

$$\boxed{S_n(p_1, \ldots, p_n) = 1}$$

fully inclusive quantity ⟷ we don't care what happens at long-distance

- the produced partons will all hadronize with probability one
- we do not observe a specific type of hadron
  (i.e. sum over a complete set of states)
- we sum over all degenerate kinematic regions

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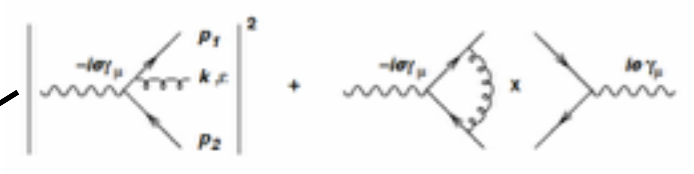
infrared safe by definition

R ratio:

$$R = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = N_c \sum e_q^2 \, (1 + \Delta_{\text{QCD}})$$



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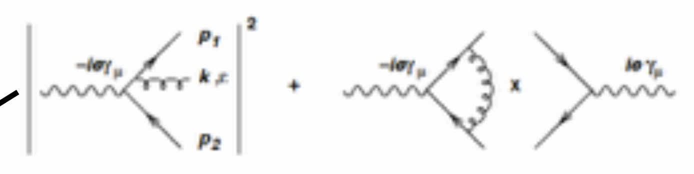

need to add up real and
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not IR safe:

- energy of hardest gluon in event
- multiplicity of gluons or 1-gluon cross section

example II: thrust distribution

somewhat less trivial: $d\sigma/dT$ (measure of the "event shape")

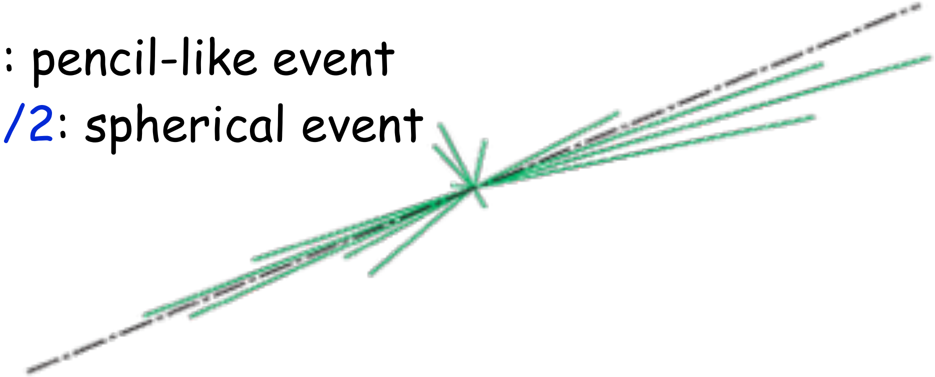
$$S_n(p_1,\ldots,p_n) \;=\; \delta(T - T_n(p_1,\ldots,p_n))$$
$$T_n(p_1,\ldots,p_n) \;\equiv\; \max|_{\vec{n}} \frac{\sum_{i=1}^{n} |\vec{p}_i \cdot \vec{n}|}{\sum_{i=1}^{n} |\vec{p}_i|}$$

<u>procedure</u>:

vary unit vector n to maximize the sum of the projections of p_i on n

T=1: pencil-like event

T=1/2: spherical event



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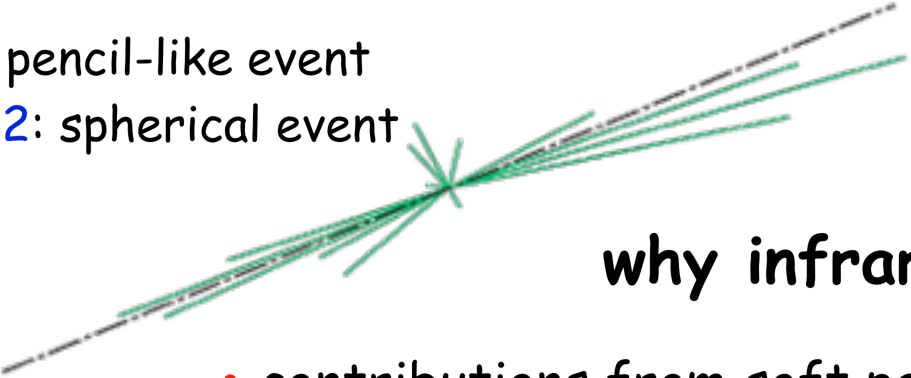


why infrared safe?

- contributions from soft particles with $\vec{p}_i \to 0$ drop out
- a collinear splitting does not change the thrust:

$$|(1-\lambda)\vec{p}_i \cdot \vec{n}| \;+\; |\lambda\vec{p}_i \cdot \vec{n}| = |\vec{p}_i \cdot \vec{n}|$$
$$|(1-\lambda)\vec{p}_i| \;+\; |\lambda\vec{p}_i| = |\vec{p}_i|$$

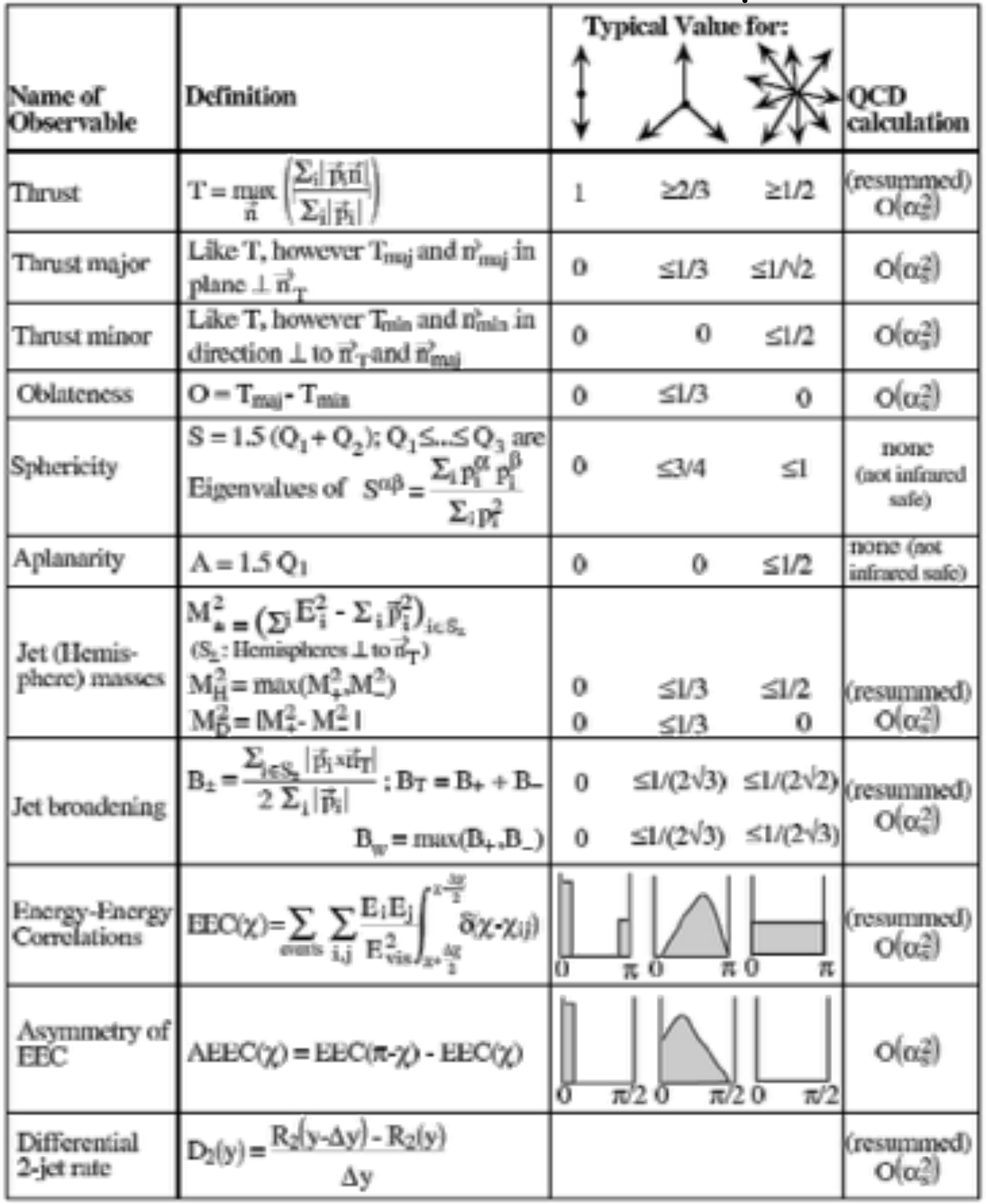
example III: event shape variables

there is a long list of similar infrared safe observables:

event-shapes: fertile ground for comparison between theory and experiment

- validity of pQCD calculations
- many ways to test SU(3) (color factors)
- spin of quarks and gluons
- measurements of α_s

taken from S. Bethke, hep-ex/0001023

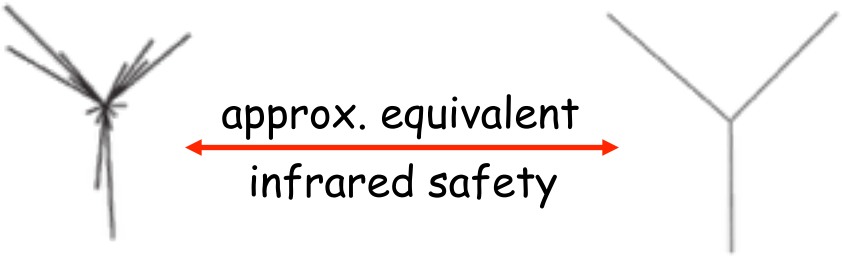


Name of Observable	Definition	Typical Value for:			QCD calculation
Thrust	$T = \max_{\vec{n}} \left(\frac{\Sigma_i \lvert \vec{p}_i \vec{n} \rvert}{\Sigma_i \lvert \vec{p}_i \rvert} \right)$	1	$\geq 2/3$	$\geq 1/2$	(resummed) $O(\alpha_s^2)$
Thrust major	Like T, however T_{maj} and $\vec{n}_{maj}$ in plane $\perp \vec{n}_T$	0	$\leq 1/3$	$\leq 1/\sqrt{2}$	$O(\alpha_s^2)$
Thrust minor	Like T, however T_{min} and $\vec{n}_{min}$ in direction $\perp$ to $\vec{n}_T$ and $\vec{n}_{maj}$	0	0	$\leq 1/2$	$O(\alpha_s^2)$
Oblateness	$O = T_{maj} - T_{min}$	0	$\leq 1/3$	0	$O(\alpha_s^2)$
Sphericity	$S = 1.5\,(Q_1 + Q_2)$; $Q_1 \leq \ldots \leq Q_3$ are Eigenvalues of $S^{\alpha\beta} = \frac{\Sigma_i p_i^\alpha p_i^\beta}{\Sigma_i p_i^2}$	0	$\leq 3/4$	≤ 1	none (not infrared safe)
Aplanarity	$A = 1.5\,Q_1$	0	0	$\leq 1/2$	none (not infrared safe)
Jet (Hemisphere) masses	$M_\pm^2 = (\Sigma_i E_i^2 - \Sigma_i \vec{p}_i^{\,2})_{i \in S_\pm}$ $(S_\pm$: Hemispheres $\perp$ to $\vec{n}_T)$ $M_H^2 = \max(M_+^2, M_-^2)$ $M_D^2 = \lvert M_+^2 - M_-^2 \rvert$	0 0	$\leq 1/3$ $\leq 1/3$	$\leq 1/2$ 0	(resummed) $O(\alpha_s^2)$
Jet broadening	$B_\pm = \frac{\Sigma_{i \in S_\pm} \lvert \vec{p}_i \times \vec{n}_T \rvert}{2\, \Sigma_i \lvert \vec{p}_i \rvert}$; $B_T = B_+ + B_-$ $B_W = \max(B_+, B_-)$	0 0	$\leq 1/(2\sqrt{3})$ $\leq 1/(2\sqrt{3})$	$\leq 1/(2\sqrt{2})$ $\leq 1/(2\sqrt{3})$	(resummed) $O(\alpha_s^2)$
Energy-Energy Correlations	$EEC(\chi) = \sum_{events} \sum_{i,j} \frac{E_i E_j}{E_{vis}^2} \int_{\chi + \frac{\Delta\chi}{2}}^{\chi - \frac{\Delta\chi}{2}} \delta(\chi - \chi_{ij})$	0 π	0 π	0 π	(resummed) $O(\alpha_s^2)$
Asymmetry of EEC	$AEEC(\chi) = EEC(\pi - \chi) - EEC(\chi)$	0 π/2	0 π/2	0 π/2	$O(\alpha_s^2)$
Differential 2-jet rate	$D_2(y) = \frac{R_2(y - \Delta y) - R_2(y)}{\Delta y}$				(resummed) $O(\alpha_s^2)$

most important example : n-jet cross section

experiment

QCD theory



approx. equivalent

infrared safety

real physical event
with 3 hadron-jets

theor. jet event
with 3 parton-jets

jets are the central link between theory and experiment

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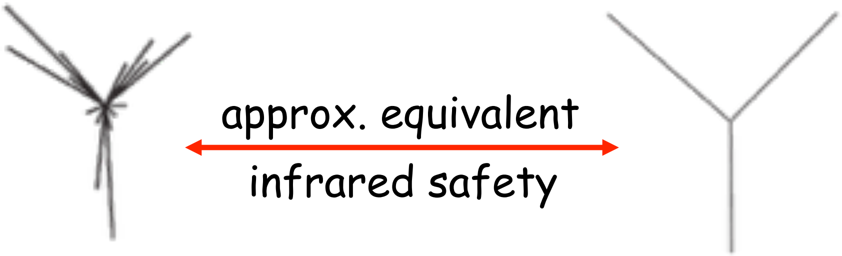

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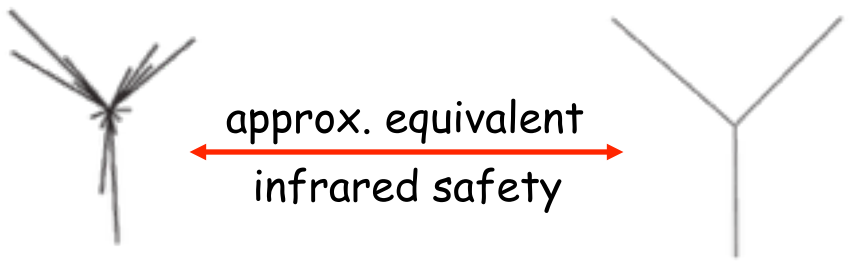

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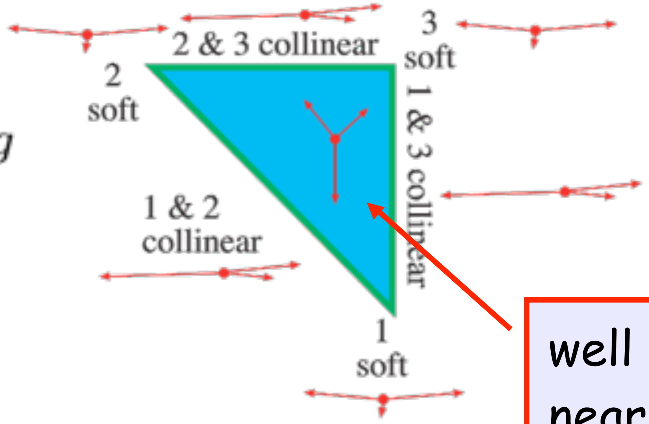
$e^+e^- \to q\bar{q}g$

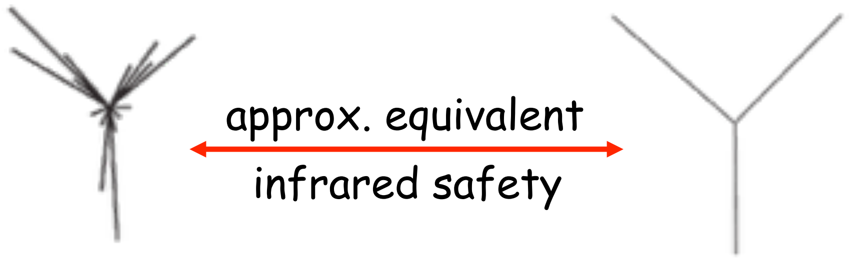


well inside: 3-jets
near edges: 2-jets

jet "measure"/"algorithm":
classify the final-state of hadrons (exp.) or partons (th.) according to the number of jets

most important example : n-jet cross section

experiment **QCD theory**



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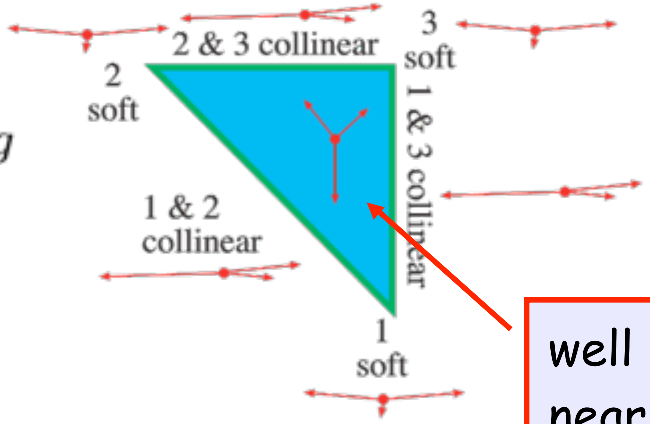

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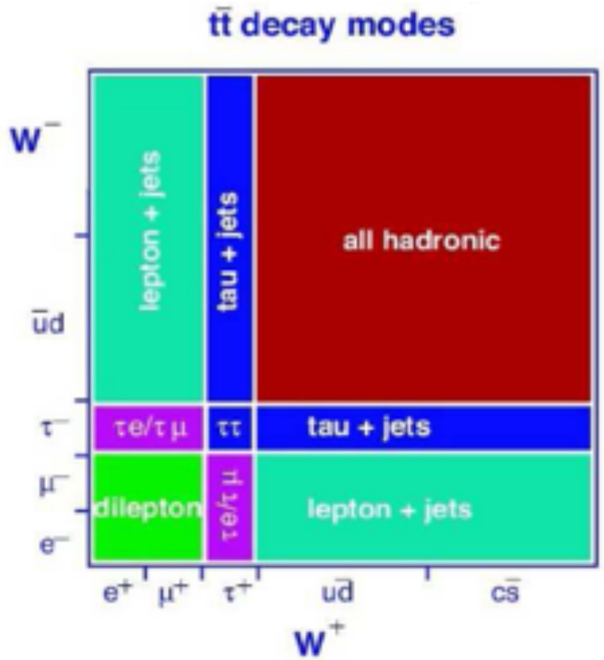
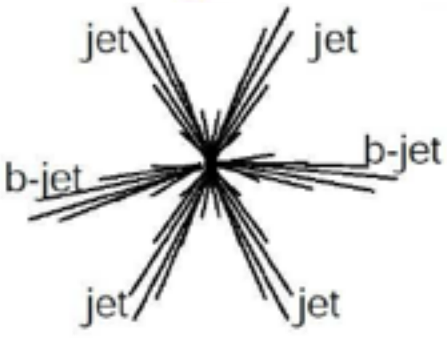
"2 or 3" depends on algorithm

jets – the central link between theory and experiment

input to almost all analyses at the LHC:

BSM & Higgs searches, top physics, PDF analyses, MC validation, …





All-hadronic
(BR~46%, huge bckg)

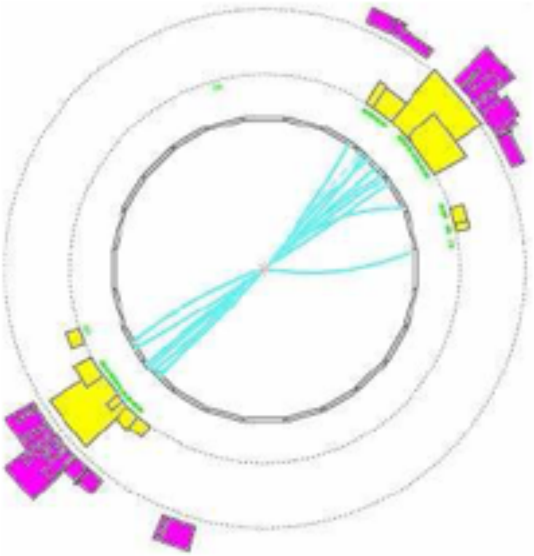
heavy objects have multi-jet final-states

- 10^7 top-antitop pairs for 10 fb^{-1} !
- vast number of QCD multi-jets:

# jets	# events for 10 fb^{-1}
3	$9 \cdot 10^8$
4	$7 \cdot 10^7$
5	$6 \cdot 10^6$
6	$3 \cdot 10^5$
7	$2 \cdot 10^4$
8	$2 \cdot 10^3$

tree level estimates:
Draggiotis, Kleiss, Papadopoulos
p_T(jet) > 60 GeV, θ_{ij} > 30deg., $|y_{ij}|$ < 3

seeing vs. defining jets



clearly (?) a 2-jet event

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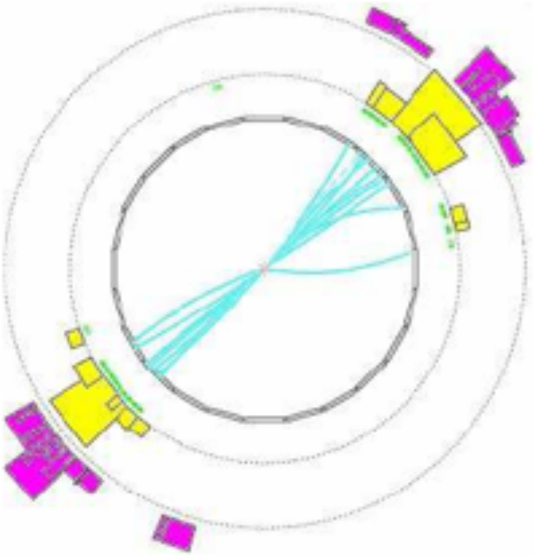

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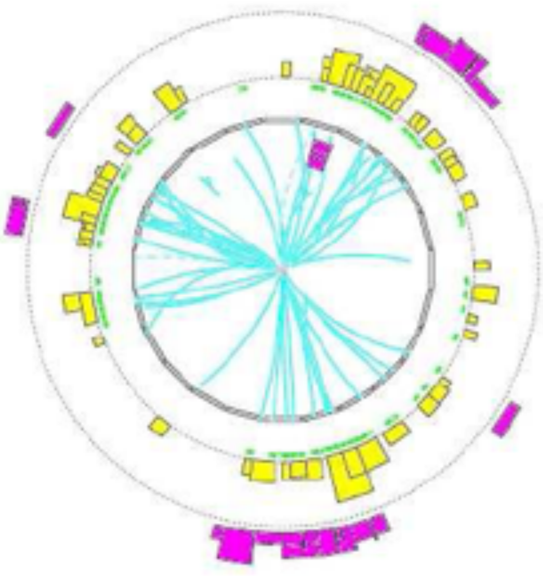

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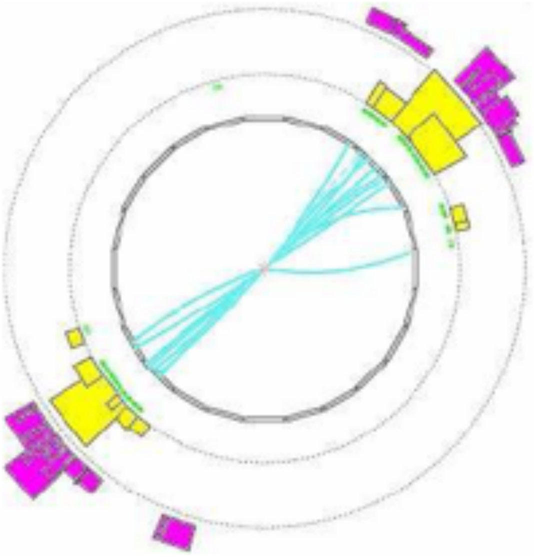

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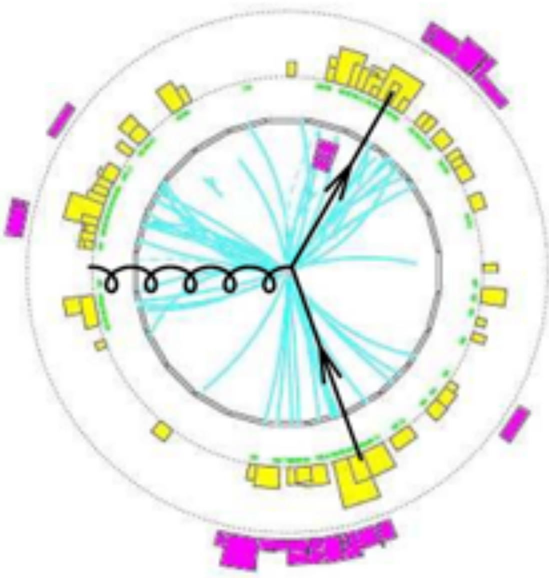

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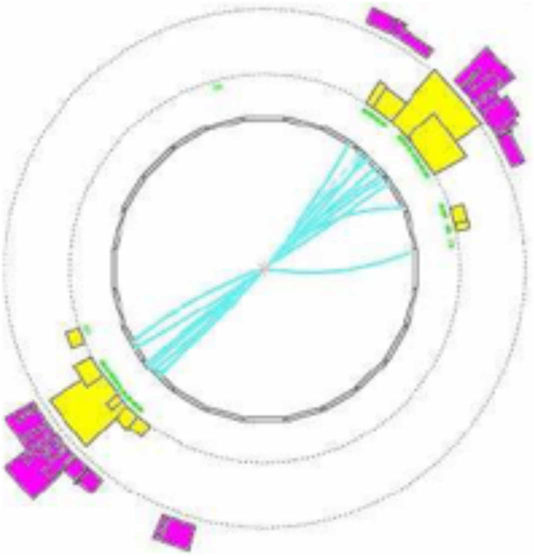

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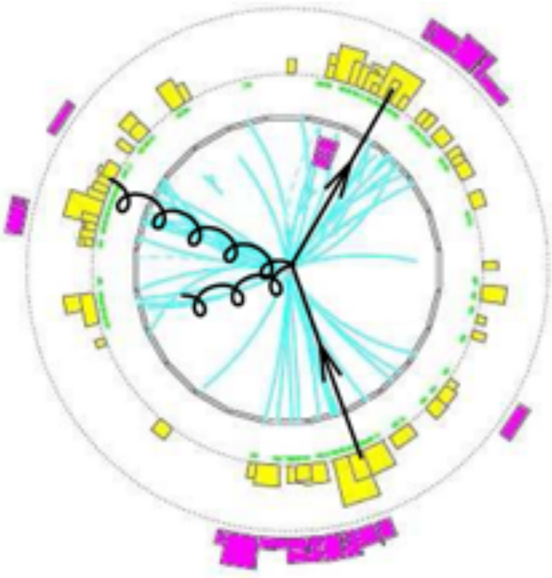

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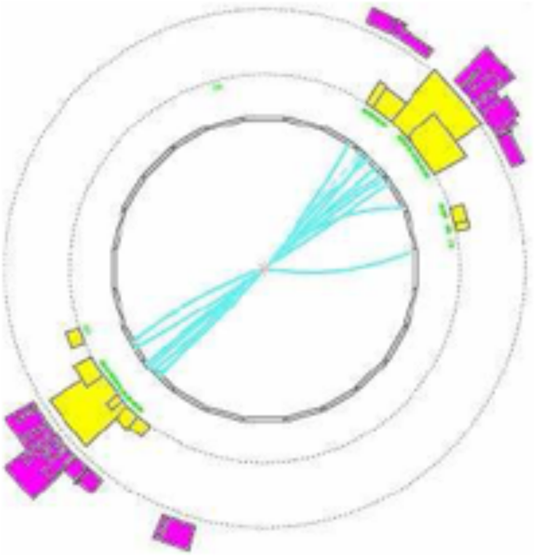
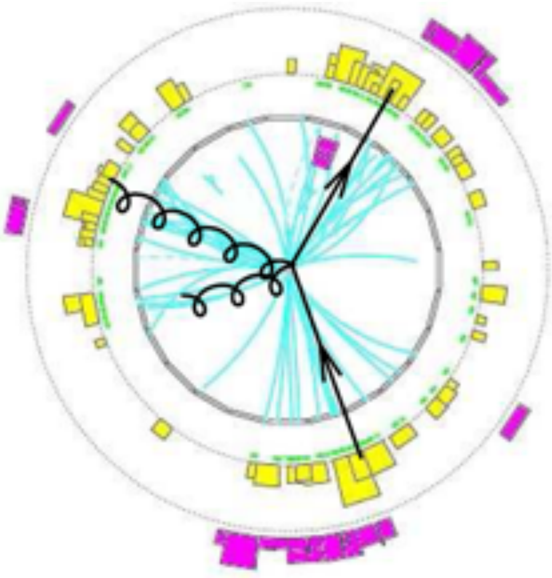
seeing vs. defining jets



clearly (?) a 2-jet event



how many jets do you count?

the "best" jet definition does not exist – construction is unavoidably ambiguous

basically two issues:

- which particles/partons get put together in a jet → **jet algorithm**
- how to combine their momenta → **recombination scheme**

basic requirements for a jet definition

projection to jets should be resilient to QCD & detector effects

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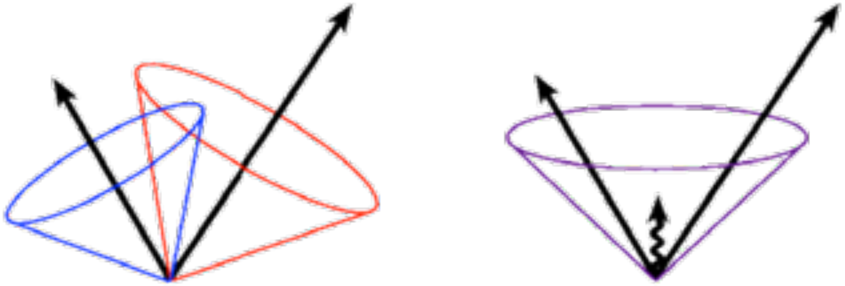

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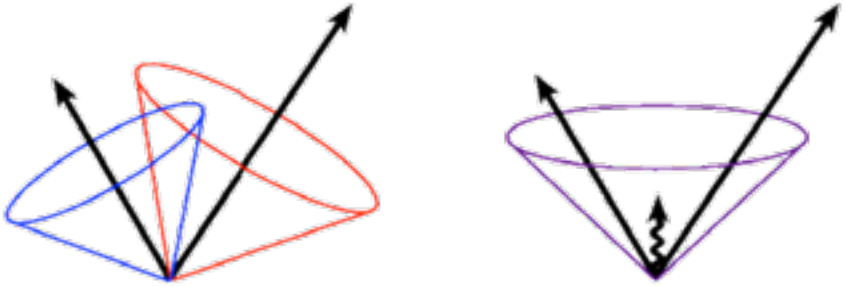

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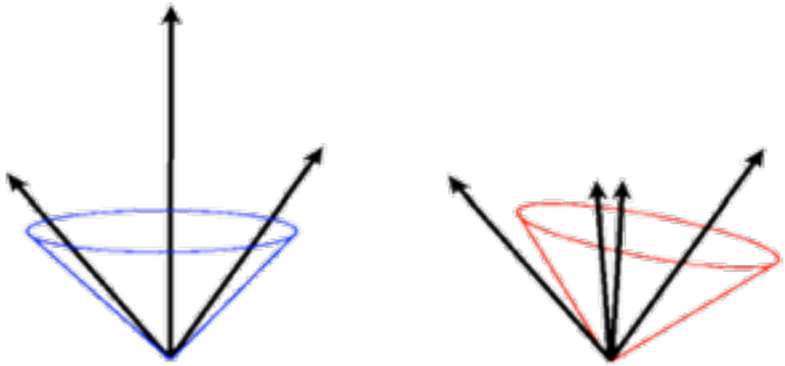

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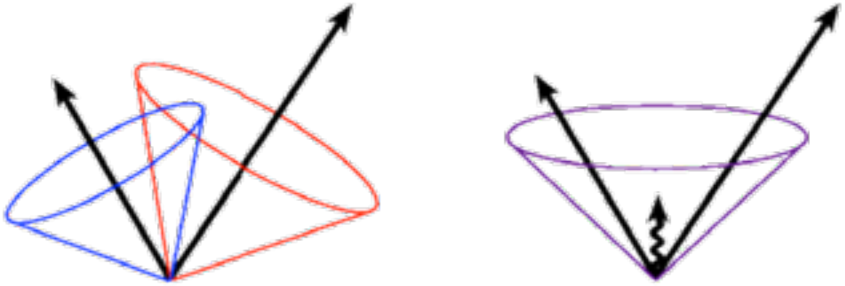

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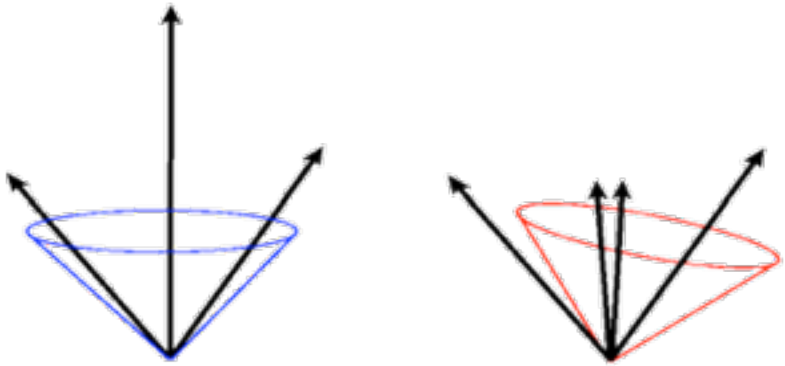

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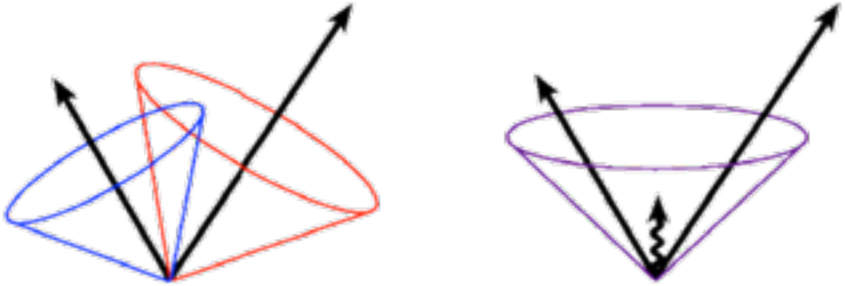

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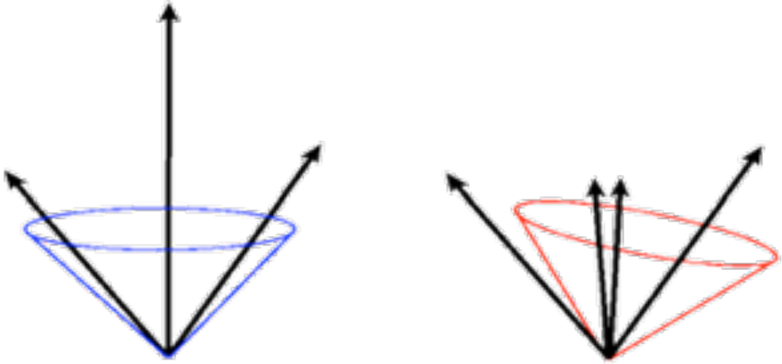
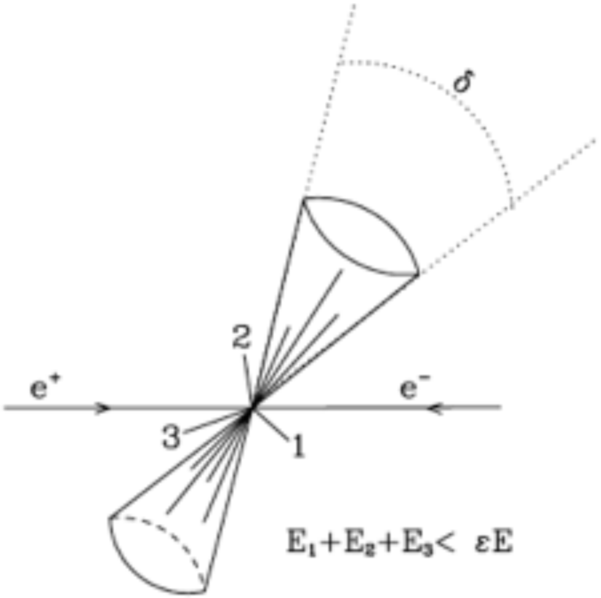


(anti-) k_T algorithms are the method of choice these days

Cacciari, Salam, Soyez (FastJet tool)

1st jet definition: Sterman and Weinberg



definition:

event has 2 jets if at least a fraction (1-ε) of the event energy in contained in two cones of opening angle δ

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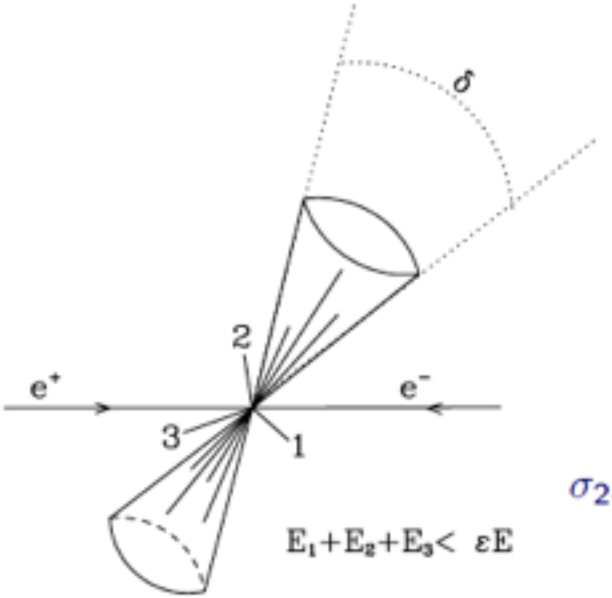

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real emission

$$\sigma_{2-jet} = \sigma_{q\bar{q}} \left(1 + \frac{2\alpha_s C_F}{\pi} \int \frac{dE}{E} \frac{d\theta}{\sin\theta} \left(R\left(\frac{E}{Q},\theta\right) \times \right.\right.$$
$$\left.\left. \times \left(1 - \Theta\left(\frac{E}{Q} - \epsilon\right)\Theta(\theta - \delta)\right) - V\left(\frac{E}{Q},\theta\right)\right)\right)$$

virtual

like the total cross section but emission with large E and θ is cut out

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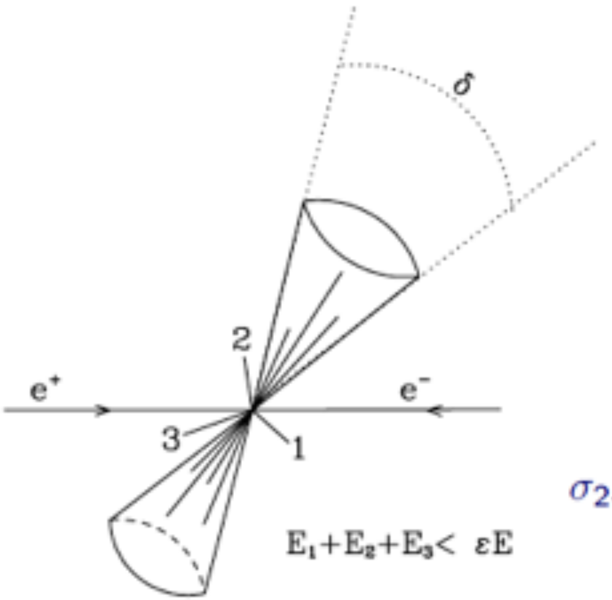

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virtual

like the total cross section but emission with large E and θ is cut out

find:

$$\sigma = \sigma_0\left(1+\frac{2\alpha_s C_F}{\pi}\ln\epsilon\ln\delta^2\right)$$

- if ε and/or δ become too small the results makes no sense (spoils KLN cancellation)

classes of jet algorithms

there are many algorithms to choose from!

basically two classes: "k_T-type" or "cone"

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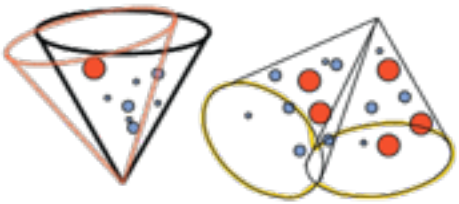

long. boost invariant cone size

$$R = \sqrt{\Delta\eta^2 + \Delta\phi^2}$$

top-down approach

cluster particles according to their distance in coordinate space

put cones along dominant direction of energy flow

potential problems with IR safety

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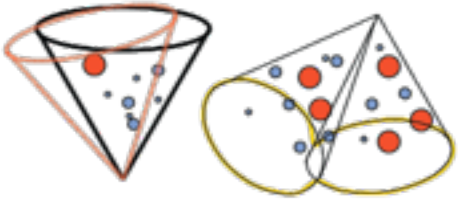
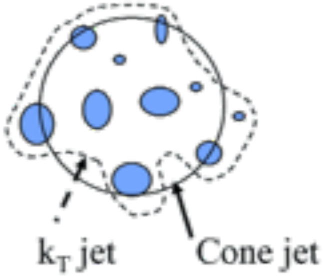


k_T type



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bottom-up approach

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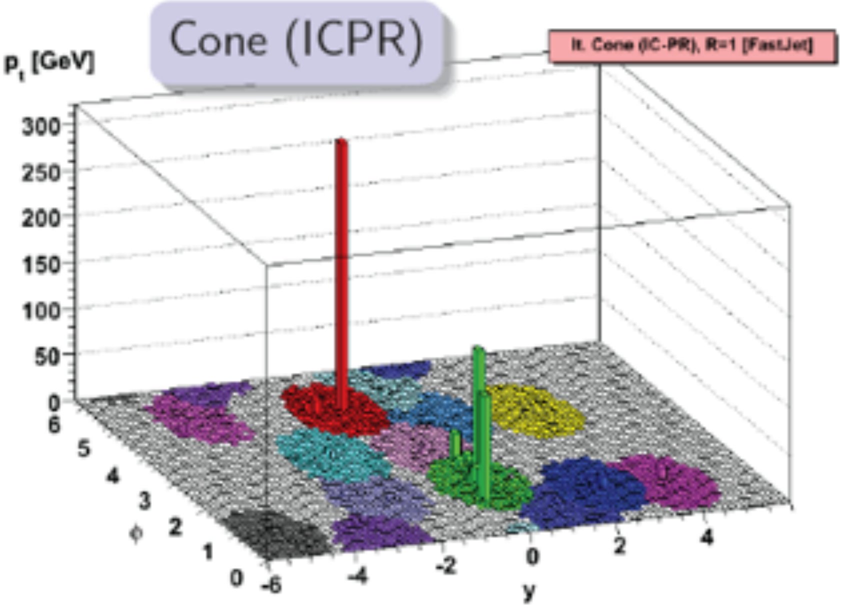
cluster particles according to their distance in momentum space

undo branchings occured in the perturbative QCD evolution:

e.g., pair particles with the smallest relative k_T

geometrical characteristics of jets

taken from
G. Salam



most cone algorithms produce
circular jets in y-Φ plane
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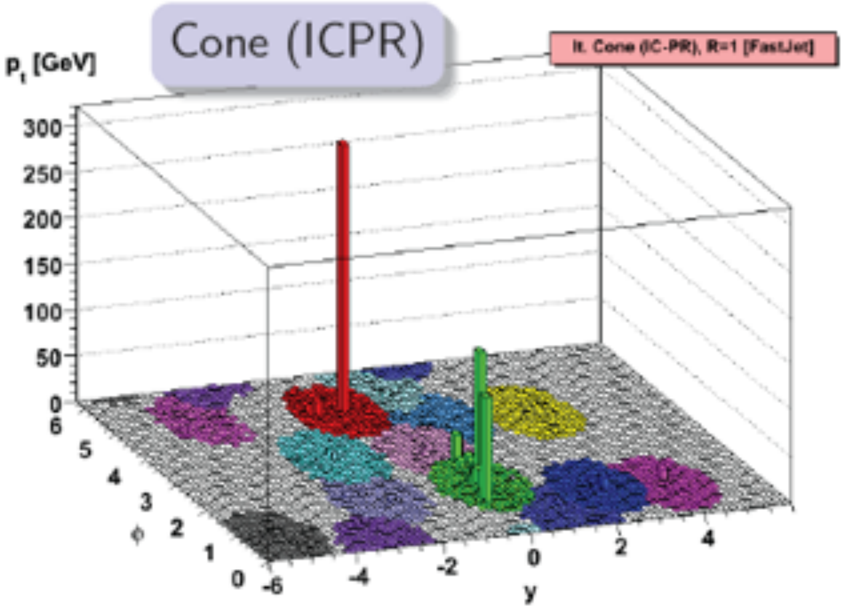

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k_T jets have irregular shape
because soft junk clusters first

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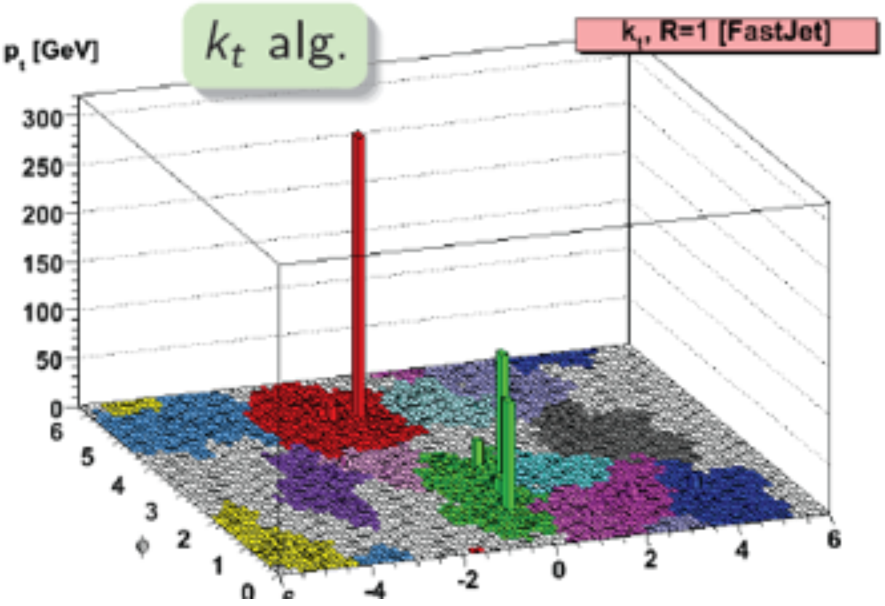

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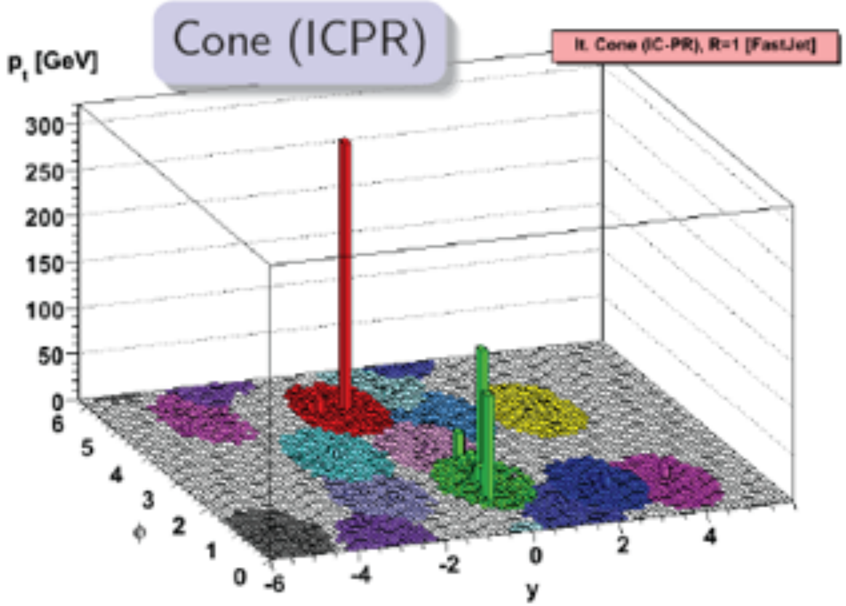

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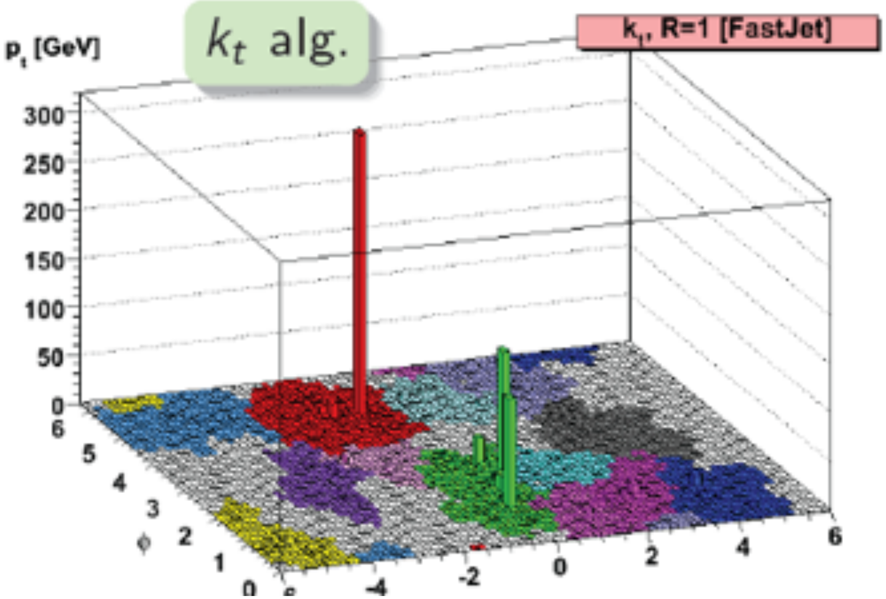

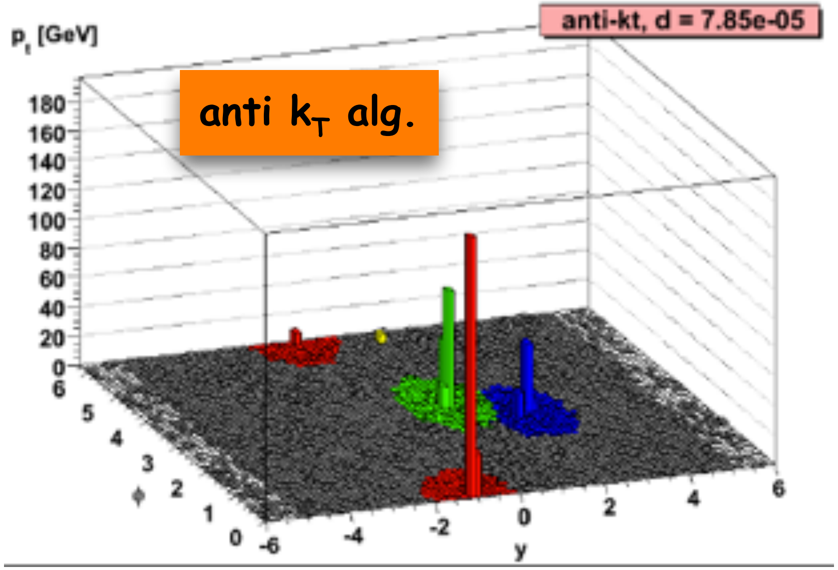

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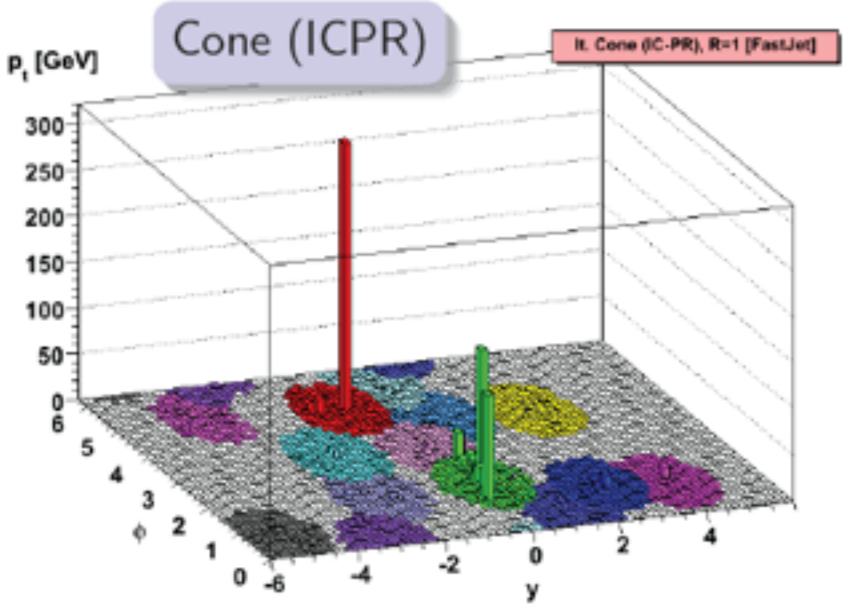

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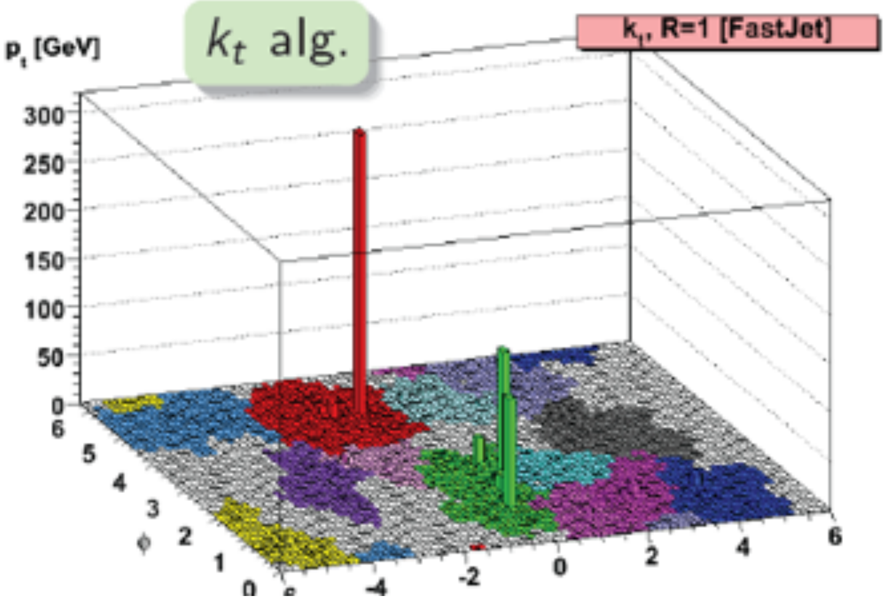

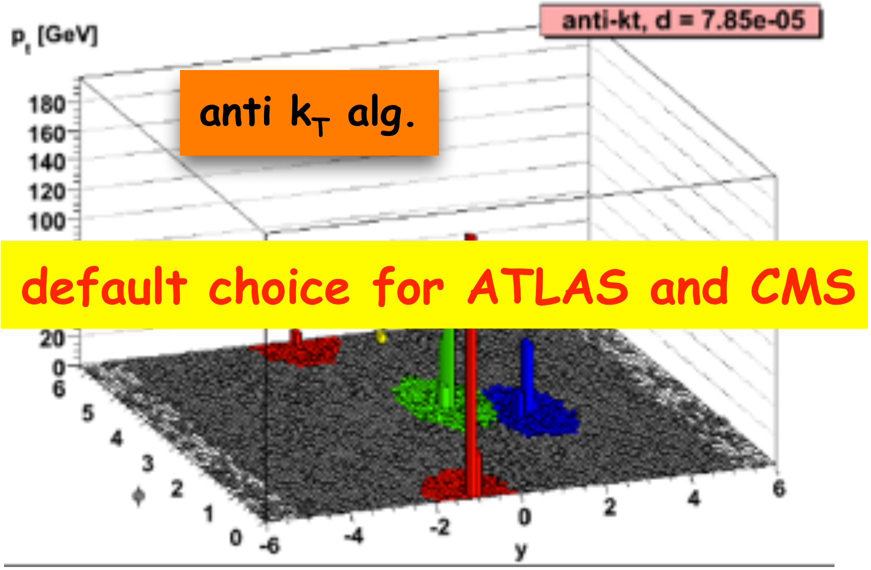

default choice for ATLAS and CMS

jets – final remarks

- n-jet vs. (n+1)-jet rate depends on algorithm
  → have to choose the same jet definition in exp. and theory

- have to be careful when comparing between different experiments or experiment and theory (often different jet algorithms!)

- many widely used jet definitions are **NOT IR safe!**
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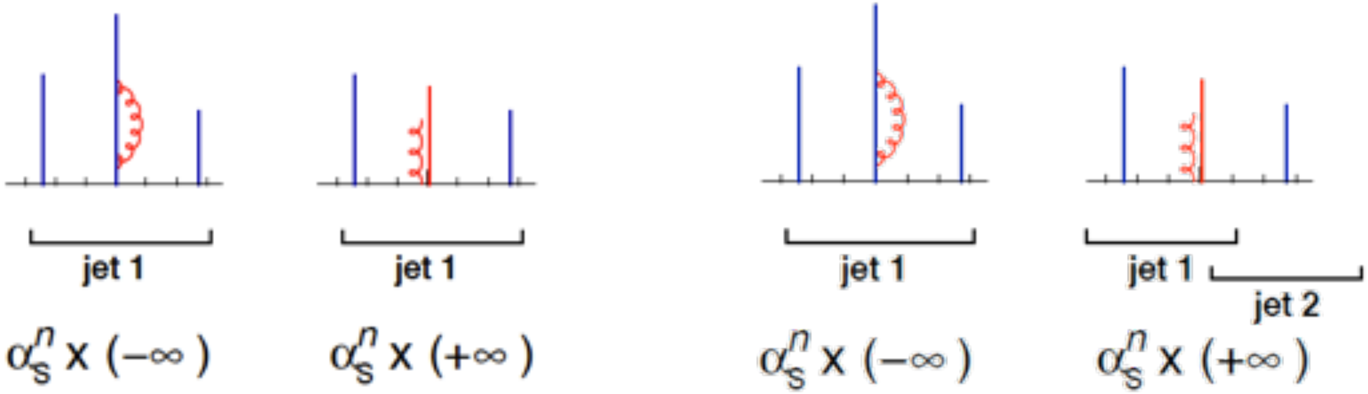

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divergencies cancel

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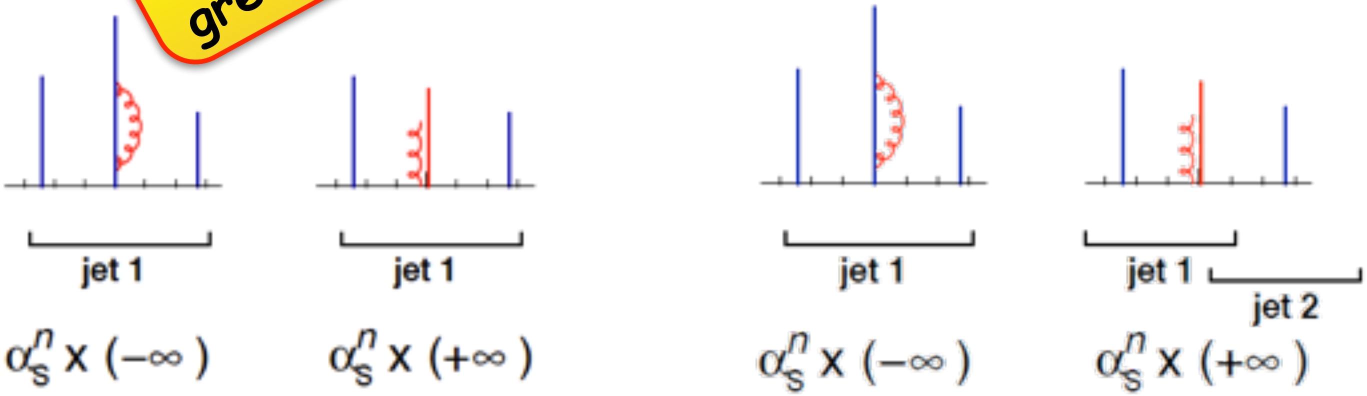
want to read more about jets?
great overview article: G. Salam, 0906.1833



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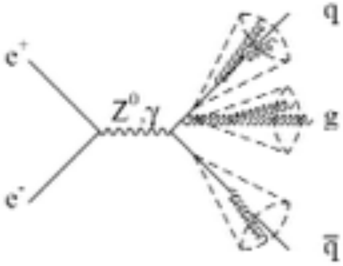
latest achievement: $e^+e^- \rightarrow$ 3 jets at NNLO

Gehrmann-De Ridder, Gehrmann, Glover, Heinrich; Weinzierl

up to 7 jets in NLO !!
leading color approx
Becker et al., 1111.1733



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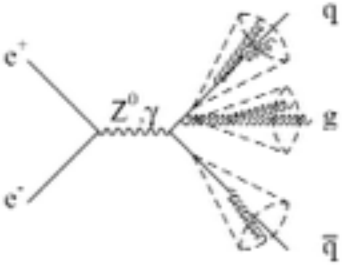

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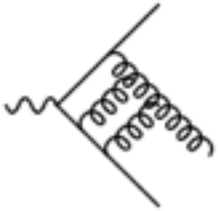

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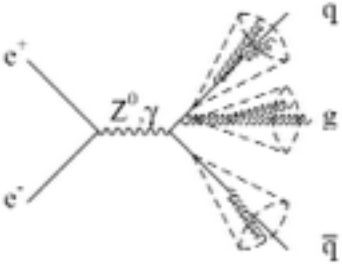

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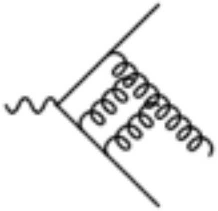

explicit IR poles from loop integrals

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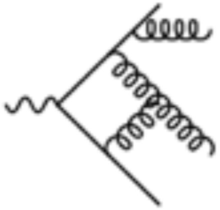

explicit IR poles from loop integrals

implicit IR poles from 1-unresolved radiation

soft, collinear

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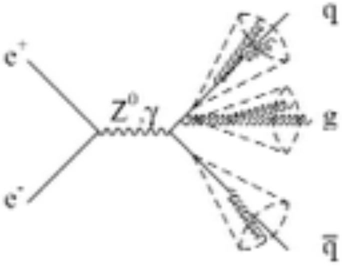

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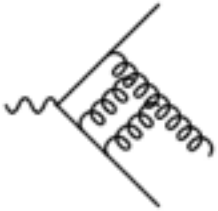

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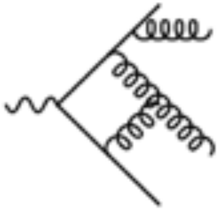
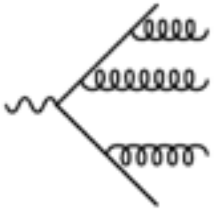


explicit IR poles from loop integrals

implicit IR poles from 1-unresolved radiation

soft, collinear

- tree level matrix elements (5 partons)



implicit IR poles from 2-unresolved radiation

double soft, soft/collinear,
double single collinear, triple collinear

structure of NNLO cross section

- complicated phase space (dΦ) integrations done with numerical (MC) methods
- different strategies for IR cancellations, most common: **subtraction method**

tricky issue: find NNLO subtraction functions which

- approximate cross section in all singular limits
- are sufficiently simple to be integrated analytically

$$\begin{aligned} d\sigma^{\text{n}-jets}_{\text{NNLO}} = & \int_{d\Phi_{n+2}} (d\sigma^R - d\sigma^S) \\ & + \int_{d\Phi_{n+1}} (d\sigma^{V,1} - d\sigma^{VS,1}) \\ & + \int_{d\Phi_n} d\sigma^{V,2} + \int_{d\Phi_{n+2}} d\sigma^S + \int_{d\Phi_{n+1}} d\sigma^{VS,1} \end{aligned}$$

analytically analytically

each line above is free of IR poles and numerically finite; implemented in EERAD3 code

1402.4140

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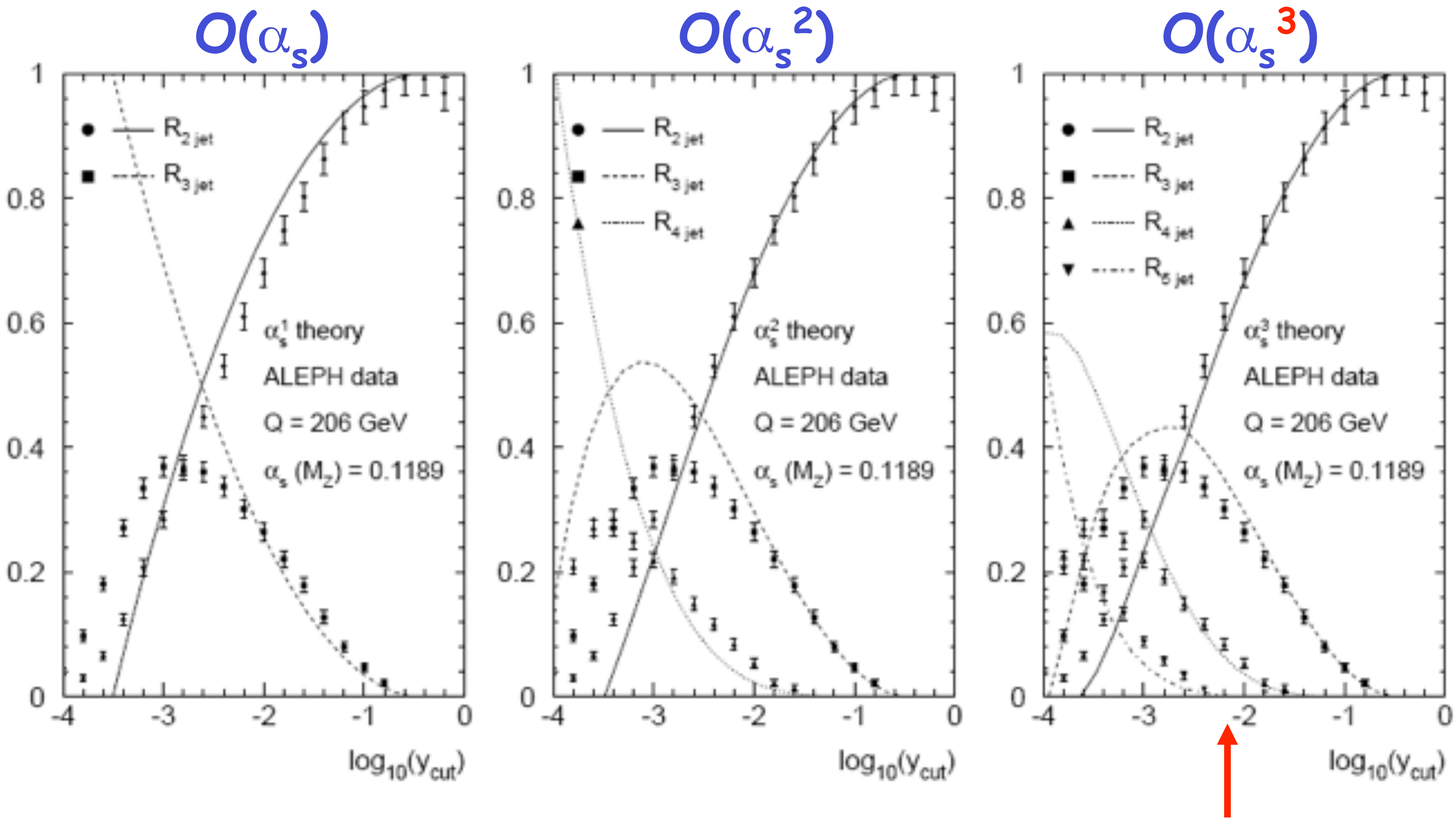
1402.4140

crucial step towards full NNLO corrections for 2 → 2 QCD processes

impact on e^+e^- jet rates

Gehrmann-De Ridder, Gehrmann, Glover, Heinrich; Weinzierl



$R_{n\ jet}$ normalized to σ_{tot} at given order

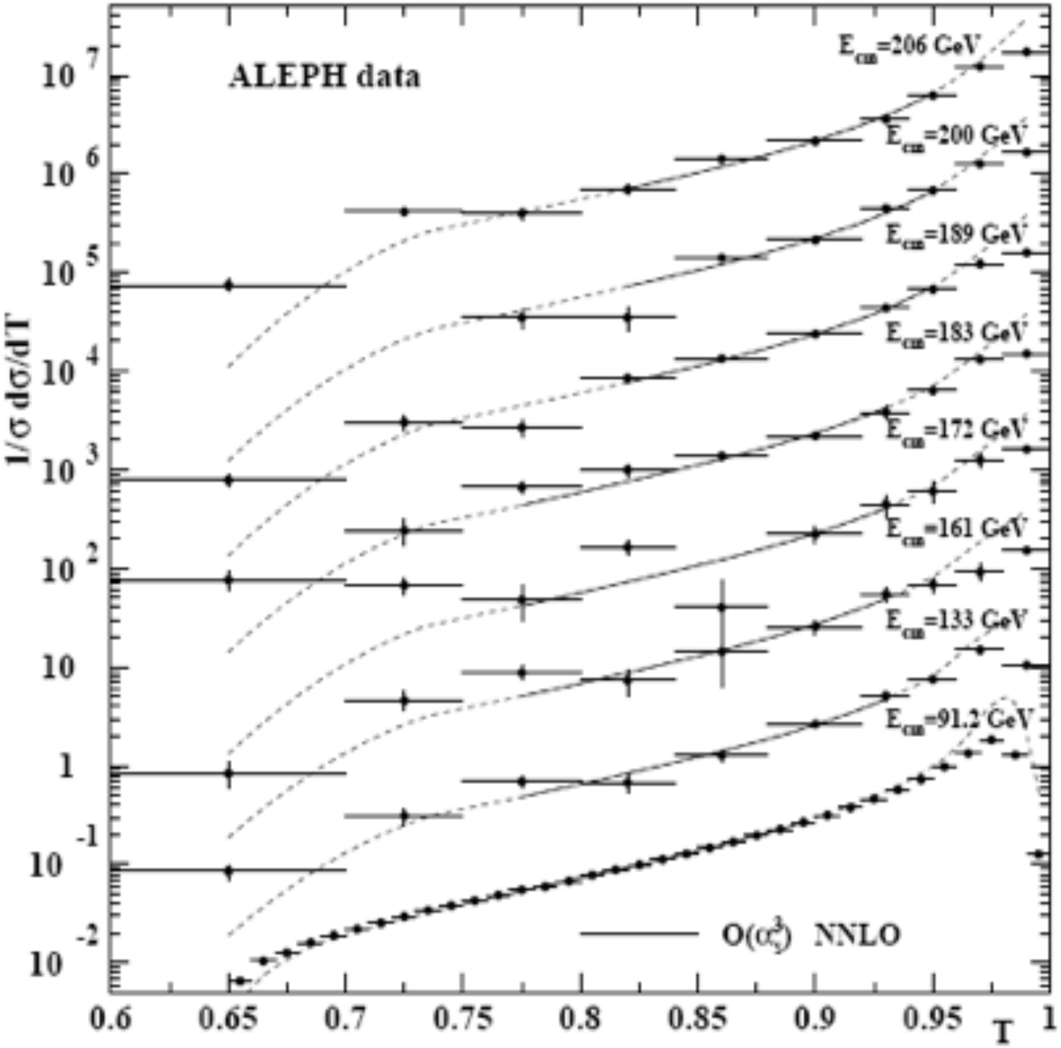
y_{cut}: jet resolution parameter
of Durham jet algorithm

$R_{2\ jet}$ at N^3LO
$R_{3\ jet}$ at NNLO
$R_{4\ jet}$ at NLO
$R_{5\ jet}$ at LO

inhibited radiation: all-order resummations

recall **thrust** variable: $T \equiv \max|_{\vec{n}} \frac{\sum_{i=1}^{n} |\vec{p}_i \cdot \vec{n}|}{\sum_{i=1}^{n} |\vec{p}_i|}$

T=1 : pencil-like event
T=1/2: spherical event



find: near perfect agreement with NNLO theory

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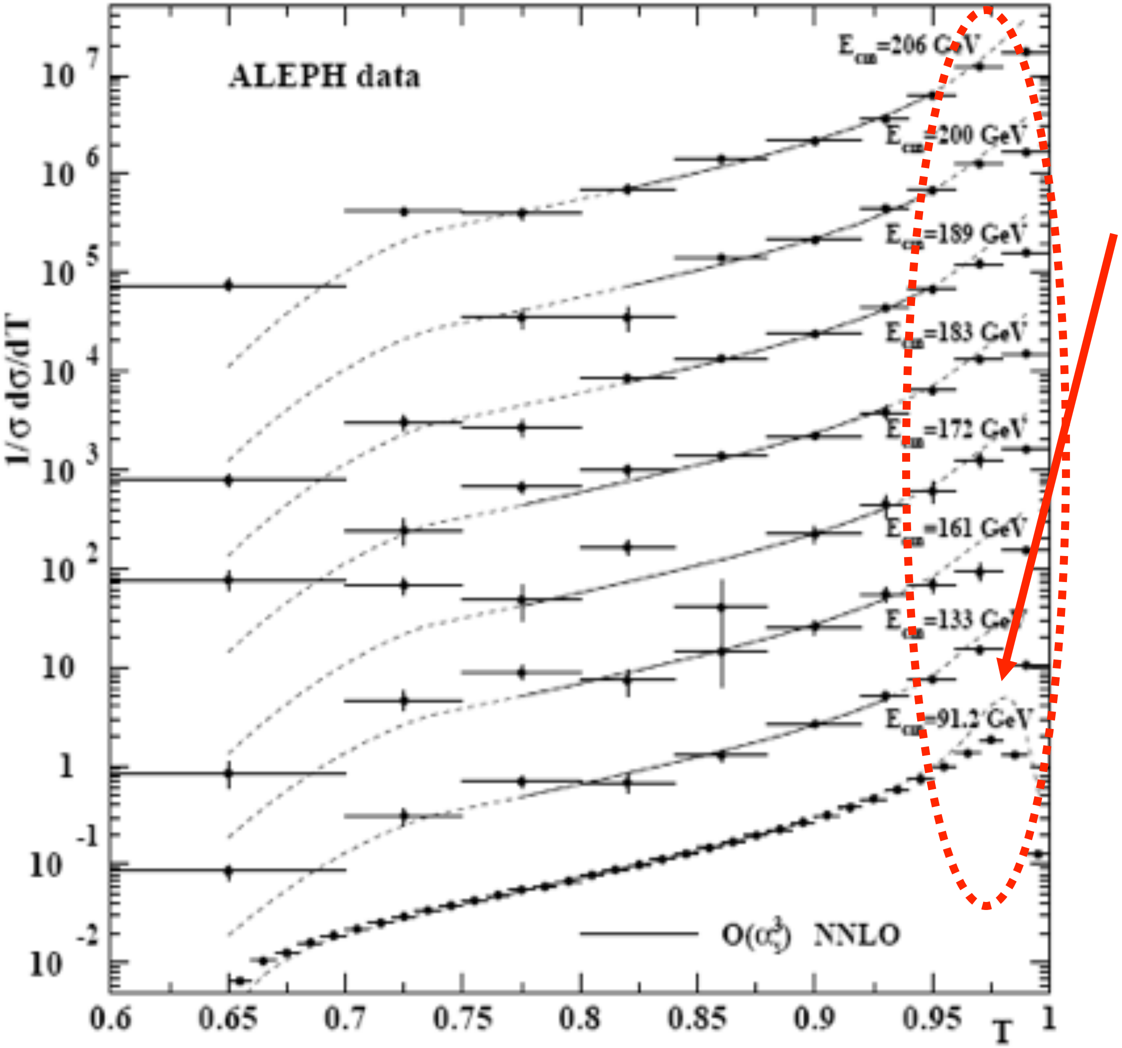

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closer look: trouble for T → 1

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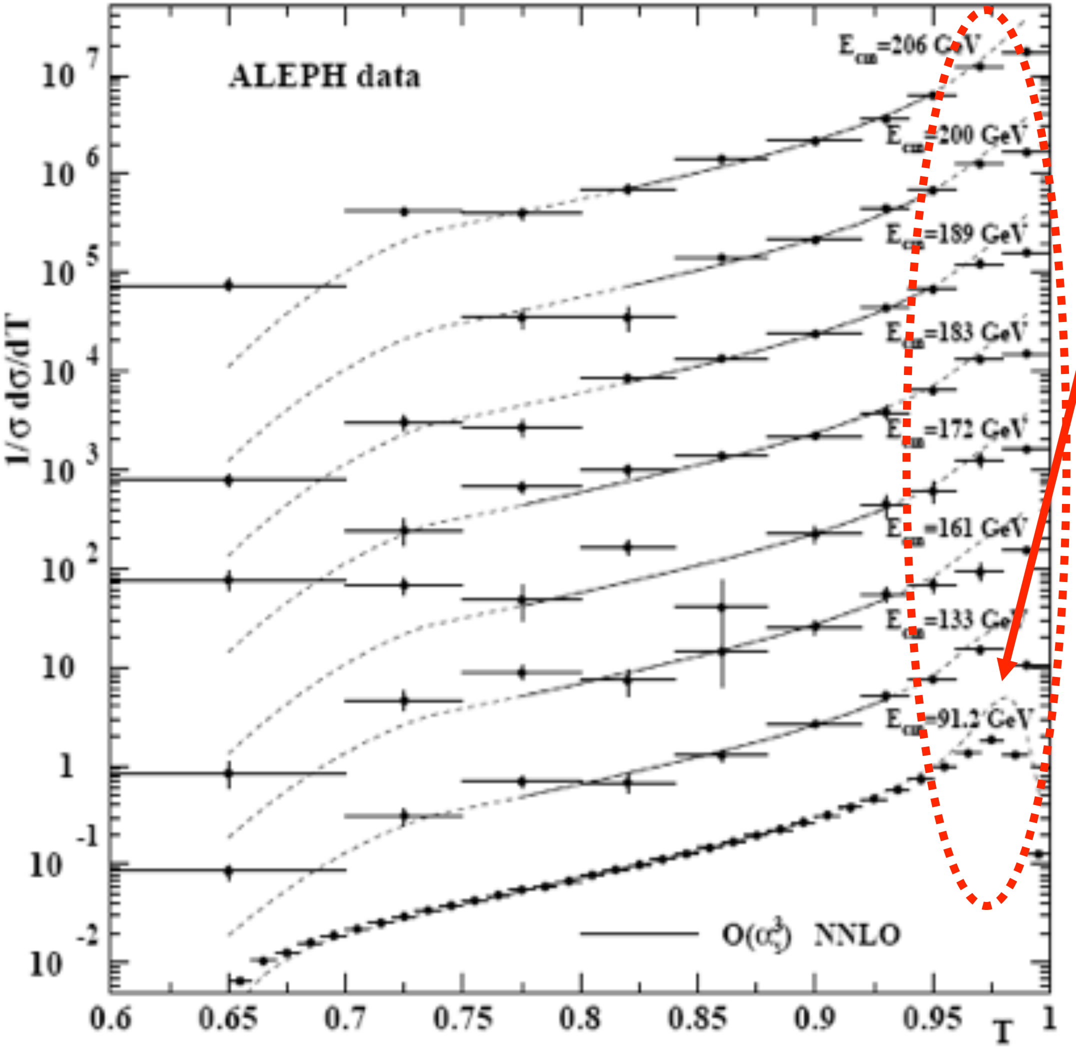

find: near perfect agreement with NNLO theory

closer look: trouble for T → 1

this is a general phenomenon for gauge theories !!

related to inhibited radiation near partonic threshold/excl. boundary

inhibited radiation: all-order resummations

What goes wrong for thrust?

- $T=1$ corresponds to 2-parton final state (just two back-to-back jets)
- if $T \to 1$ only soft/collinear gluons can be emitted (“**inhibited radiation**”) in events with an extra gluon
- IR singularities cancel between real emissions and loop corrections but leave **large logarithms** behind in each order of α_s
  here: $(\alpha_s \ln^2 [1-T])^n$ → spoil convergence of pQCD series even if a_s<<1

Can this be cured?

inhibited radiation: all-order resummations

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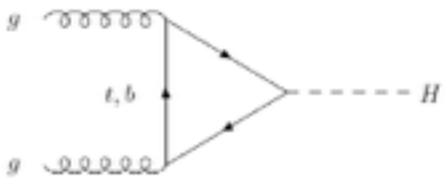
Can this be cured?

Yes! re-organize pQCD series to **resum large logs to all orders**

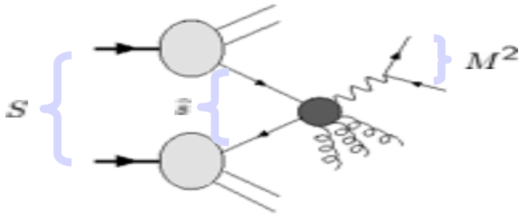
Sterman; Catani, Trentadue; Laenen, Oderda, Sterman; Catani et al.; Sterman, Vogelsang; Kidonakis, Owens; ...

of great phenomenological relevance in hadronic processes

examples:



high mass particles at the LHC

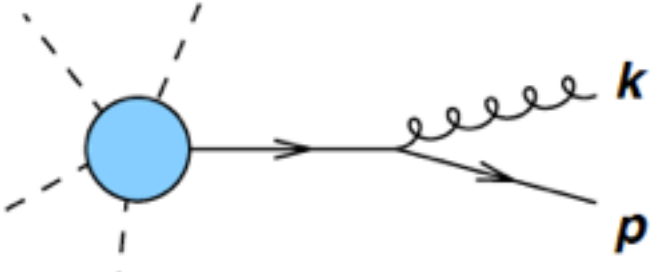


Drell-Yan pairs at fixed target exp.

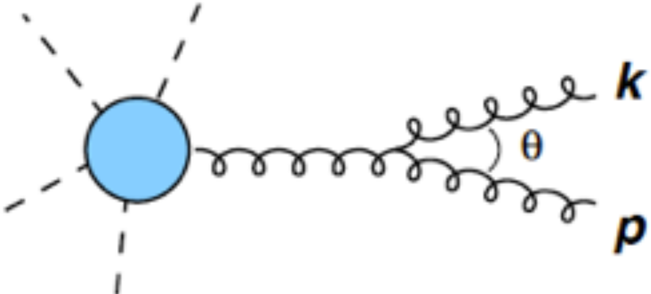
→ **more in Part IV**

recap: idea behind parton shower MC programs

- we have seen that emission of soft/collinear partons is favored
- we know exactly how and when it occurs (process-independent)



$$\frac{2\alpha_s C_F}{\pi}\frac{dE}{E}\frac{d\theta}{\theta}$$



$$\frac{2\alpha_s C_A}{\pi}\frac{dE}{E}\frac{d\theta}{\theta}$$

valid in
soft-collinear limit

this will provide the basis for a "parton shower"

- **main idea:** seek for an approx. result such that soft/collinear
  enhanced terms are included to all orders
  emissions are probabilistic (as needed to set up an event generator)

role of the Sudakov exponent

- the possible way to proceed is to ask

  “what is the probability of NOT radiating a gluon above a certain scale k_T?”

$$P(\text{no emission above } k_t) \sim 1 - \frac{2\alpha_s C_F}{\pi} \int \frac{dE}{E} \int \frac{d\theta}{\theta} \Theta(E\theta - k_t)$$

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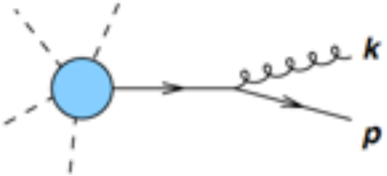

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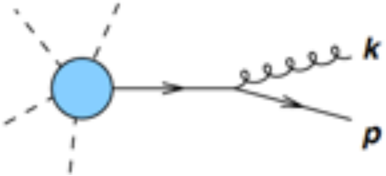

- generalized to all orders by exponentiation (**Sudakov exponent**)

$$\Delta(k_t, Q) \simeq \exp\left[-\frac{2\alpha_s C_F}{\pi}\int^{Q}\frac{dE}{E}\int^{\pi/2}\frac{d\theta}{\theta}\Theta(E\theta - k_t)\right]$$

bounded between 0 and 1 (probability)

(here: some simplifying assumptions !!)

- probability distribution for gluon emission given by $\frac{dP}{dk_{t1}} = \frac{d\Delta(k_{t1}, Q)}{dk_{t1}}$

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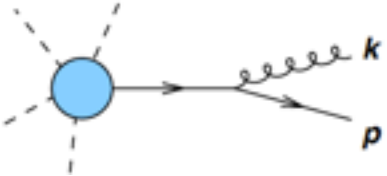

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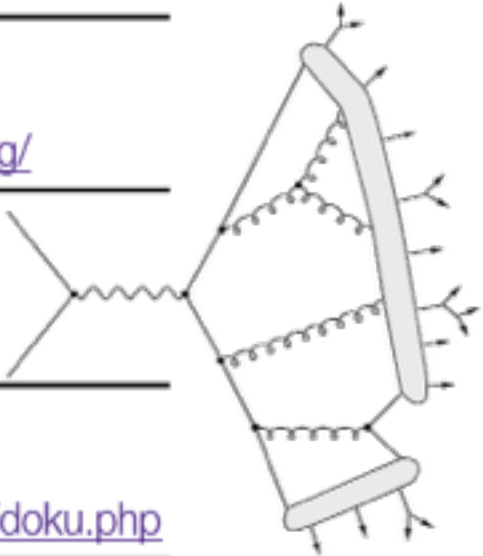
(here: some simplifying assumptions !!)

- probability distribution for gluon emission given by $\frac{dP}{dk_{t1}} = \frac{d\Delta(k_{t1}, Q)}{dk_{t1}}$
- used in MC to generate subsequent ordered branchings, e.g., $k_{T1} > k_{T2} > \ldots$
- stop at some small cut-off scale and then use some model to produce hadrons

some popular parton shower programs

PYTHIA	T. Sjöstrand et al. http://home.thep.lu.se/~torbjorn/Pythia.html
HERWIG	G. Corcella et al. http://hepwww.rl.ac.uk/theory/seymour/herwig/
HERWIG++	S. Gieseke et al. http://projects.hepforge.org/herwig/
SHERPA	F. Krauss et al. http://projects.hepforge.org/sherpa/dokuwiki/doku.php
ISAJET	H. Baer et al. http://www.nhn.ou.edu/~isajet/



- can fail in high-multiplicity events or when large-angle emissions are relevant
- do better than fixed order calculations at lowish scales
- matching with NLO matrix elements well advanced: MC@NLO, POWHEG, …

summary so far

pQCD cannot give all the answers
but it does cover a lot of ground
despite the “long-distance problem”

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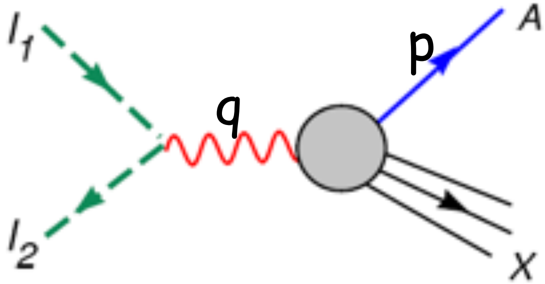
despite the "long-distance problem"

the **concept of factorization** will allow us to compute cross sections for a much wider class of processes than considered so far (involving hadrons in the initial and/or final state)

LHC, RHIC, COMPASS, ..., EIC, ...

identified hadrons: a new “long distance problem”

consider the one-particle inclusive cross section:



identified hadron

e.g. ($A = \pi$)

not measured

$$\boxed{\frac{d\sigma(e^+e^- \to \pi + X)}{dE_\pi}}$$

not infrared safe by itself!

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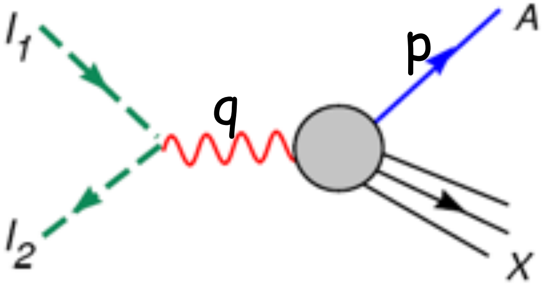

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(leads to uncanceled singularities -> meaningless)

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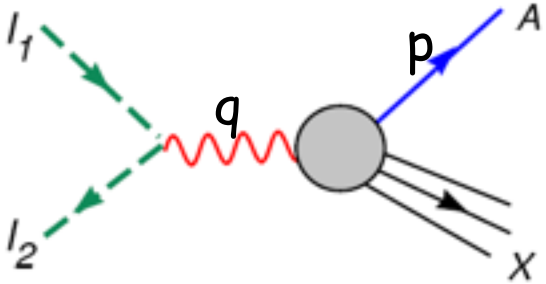

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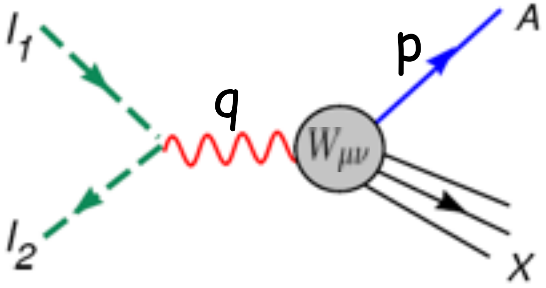
general feature of QCD processes with observed (=identified) hadrons in the initial and/or final state

factorization

strategy: try to factorize the physical observable into a calculable infrared safe and a non-calculable but universal piece

how does it work?



leptonic tensor

$$d\sigma = \frac{4\alpha^2}{sQ^2}\frac{d^3\vec{p}}{2|\vec{p}|}L^{\mu\nu}W_{\mu\nu}$$

hadronic tensor

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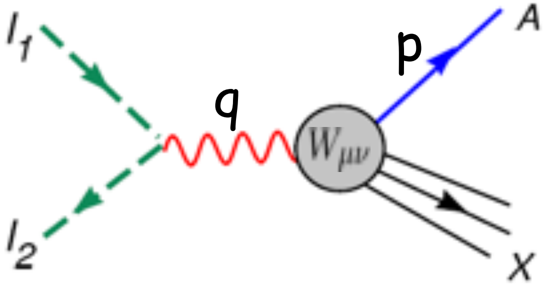

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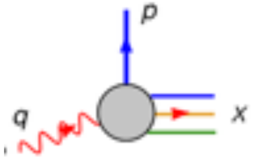
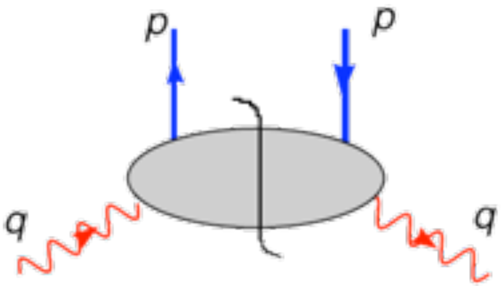


square of the hadronic scattering amplitude summed over all final-states X except A(p)



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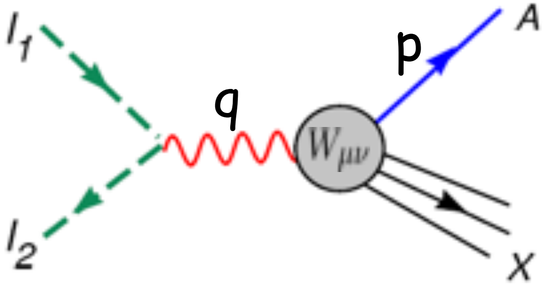

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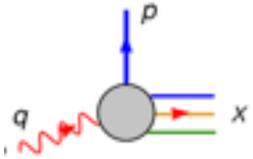

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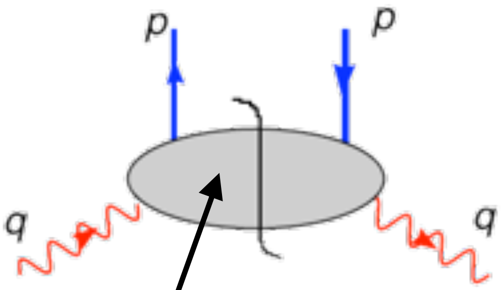

needed to factorize long-distance physics

concept of factorization - pictorial sketch

factorization = isolating and absorbing infrared singularities accompanying observed hadrons

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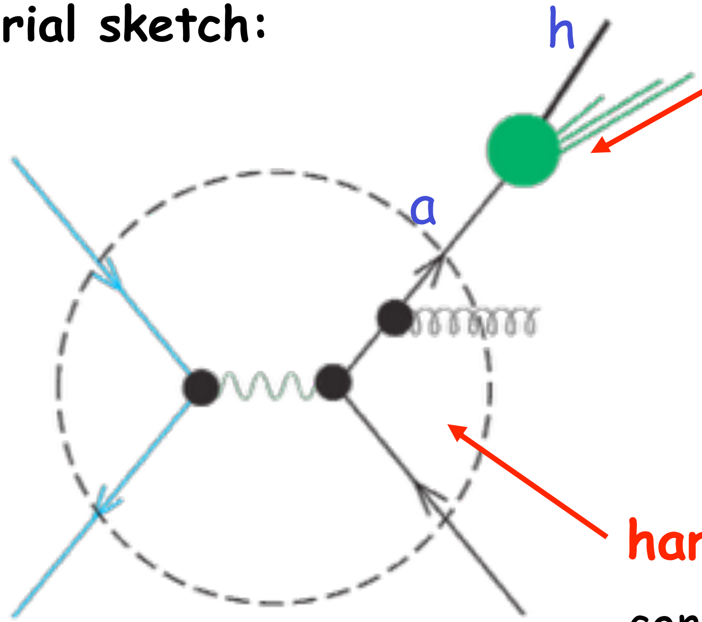
pictorial sketch:



fragmentation functions D_a^h

contains all **long-distance** interactions
hence not calculable but universal

physical interpretation:

probability to find a hadron carrying
a certain momentum of parent parton

hard scattering $\widehat{F}_a$

contains only **short-distance** physics
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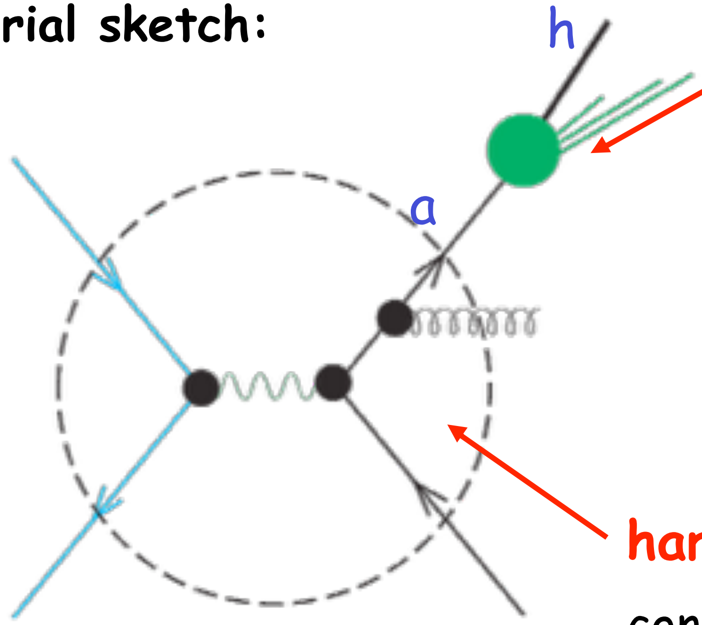

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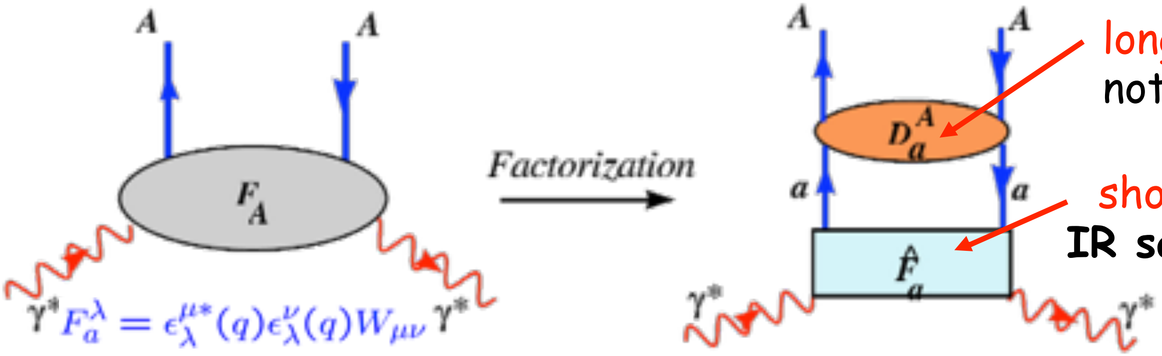
aside: fragmentation fcts. play an important role in learning about nucleon (spin) structure from semi-inclusive DIS data by COMPASS & HERMES or from hadron production at RHIC

factorization - detailed picture

more explicitly



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not calculable

short distance
IR safe, calculable

$\gamma^* F_a^\lambda = \epsilon_\lambda^{\mu*}(q)\epsilon_\lambda^{\nu}(q)W_{\mu\nu}\,\gamma^*$

λ=L,T (pol. of γ^*)

$$\frac{d\sigma}{dz\,d\cos\theta} = \frac{\pi\alpha^2}{2s}\left[F_A^T(z,Q)(1+\cos^2\theta) + F_A^L(z,Q)\sin^2\theta\right]$$

where

$$\boxed{F_A^{T,L}(z,Q) = \sum_a \hat{F}_a^{T,L}(z,\frac{Q}{\mu_f}) \otimes D_a^h(z,\mu_f)}$$

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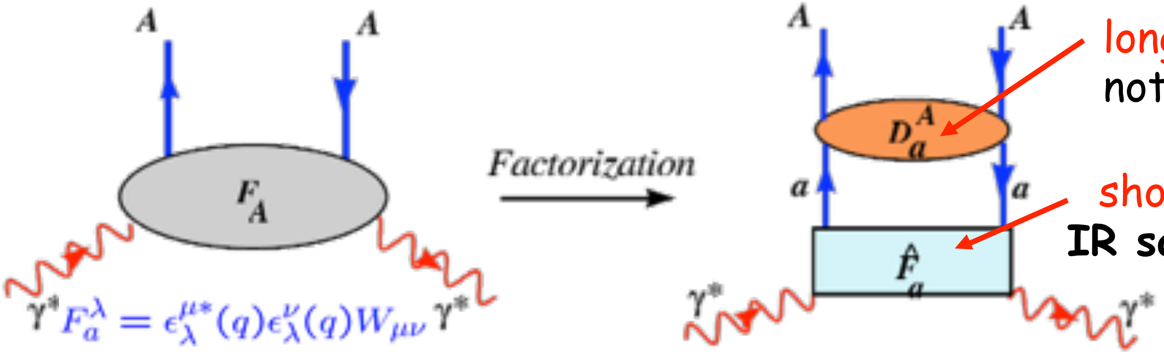

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factorization scale (arbitrary!)

characterizes the boundary between
short and long-distance physics

physics indep. of μ_f → renormalization group

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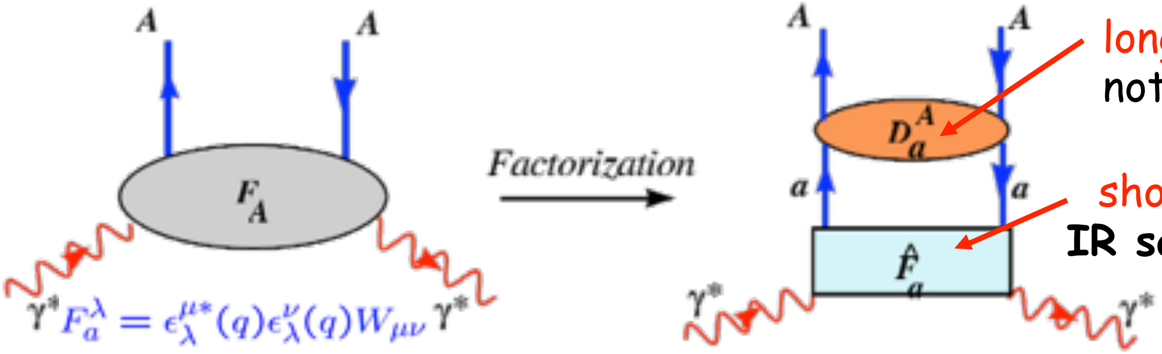

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"convolution"

$$f(x)\otimes g(x)\equiv\int_x^1\frac{dy}{y}f\left(\frac{x}{y}\right)g(y)$$

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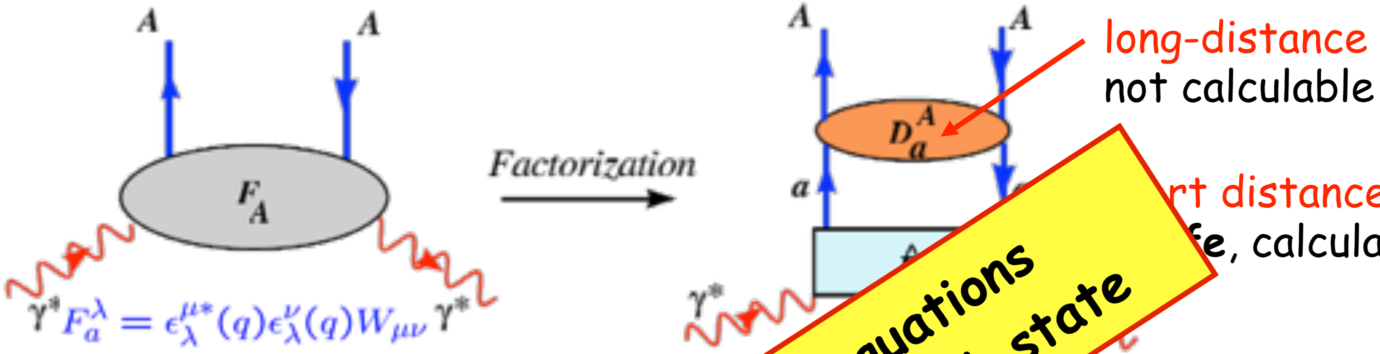

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before studying renormalization group equations
let's first introduce hadrons also in the initial-state

where

$$F_A^{T,L}(\quad \sum_a F_a^{T,L}\left(z,\frac{Q}{\mu_f}\right) \otimes D_a^h(z,\mu_f)$$

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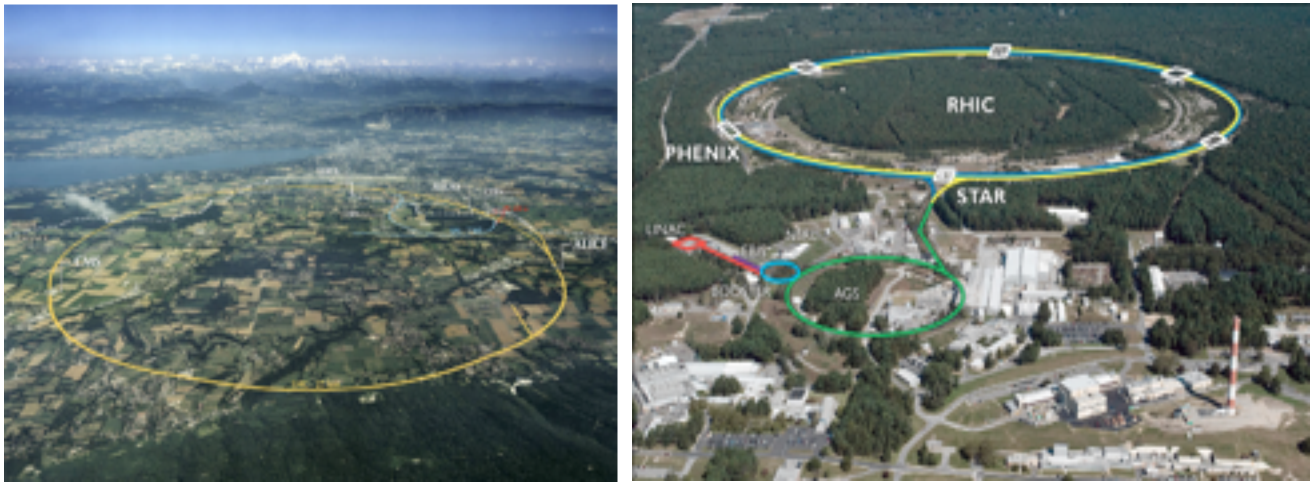
take home message for part II

THE QCD TOOLBOX

- QCD is a non-Abelian gauge theory: gluons are self-interacting —> asymptotic freedom (large Q), confinement (small Q)
- QCD calculations are singular when any two partons become collinear or a gluon becomes soft; basis for parton shower MCs
- choose infrared/collinear safe observables for comparison between experiment and perturbative QCD
- jets (= cluster of partons): best link between theory and exp.; needs a proper IR safe jet definition in theory and experiment
- infrared cancellation leaves large logarithms behind which become important in certain regions of phase-space —> all-order resummations
- factorization allows to deal with hadronic processes introduces arbitrary scale —> leads to RGEs

early microscopes



the World's most powerful microscopes

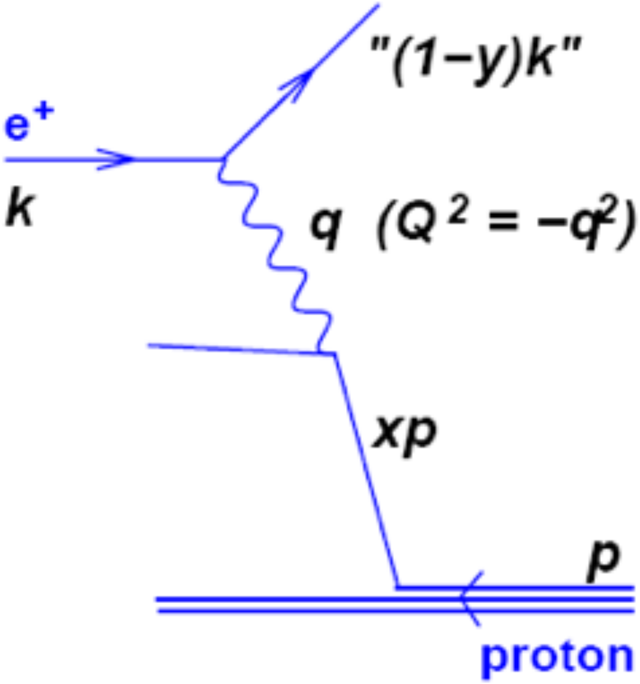
Part III

inward bound: "femto-spectroscopy"

QCD initial state, partons, DIS, factorization, renormalization group, hadron-hadron collisions

partons in the initial state: the DIS process

start with the simplest process: **deep-inelastic scattering**



relevant kinematics:

$$x = \frac{Q^2}{2p \cdot q} \qquad y = \frac{p \cdot q}{p \cdot k} \qquad Q^2 = xys$$

- Q^2: photon virtuality ↔ resolution r~1/Q at which the proton is probed
- x: long. momentum fraction of struck parton in the proton
- y: momentum fraction lost by electron in the proton rest frame

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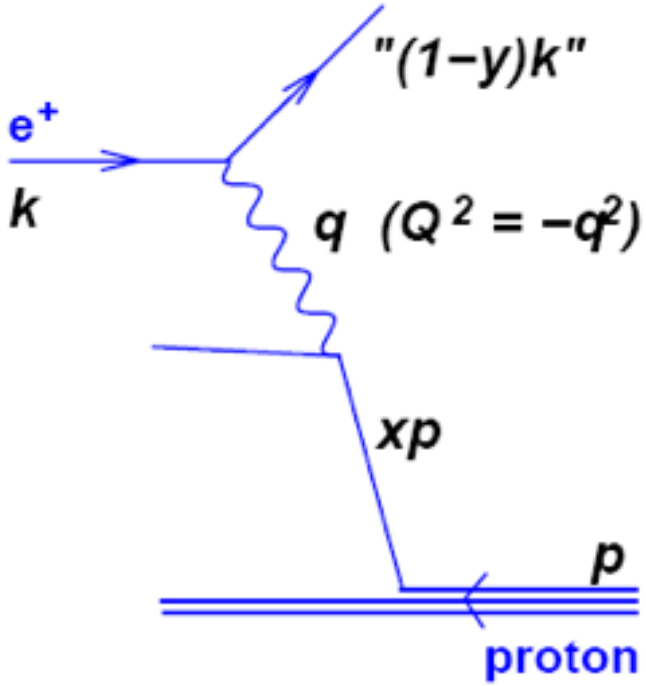

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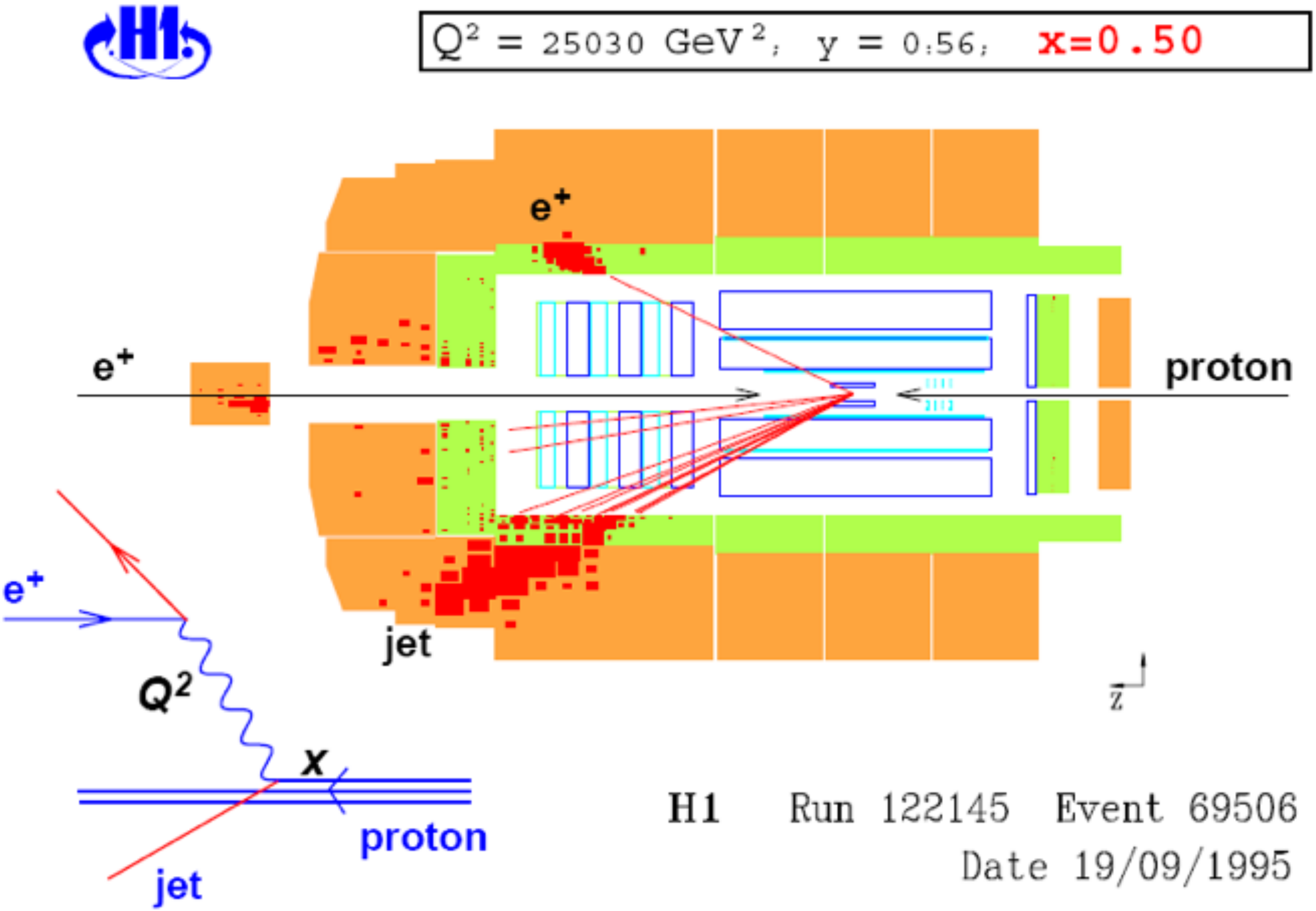
"deep-inelastic": $Q^2 \gg 1$ GeV^2

"scaling limit": $Q^2 \to \infty$, x fixed

resolution: $\frac{\hbar}{Q} \approx \frac{2 \times 10^{-16}\mathrm{m}}{Q[\mathrm{GeV}]}$

$r \sim 1/Q$

a typical DIS event

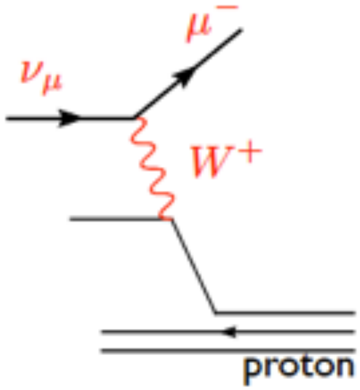


Q^2 = 25030 GeV^2; y = 0.56; x=0.50

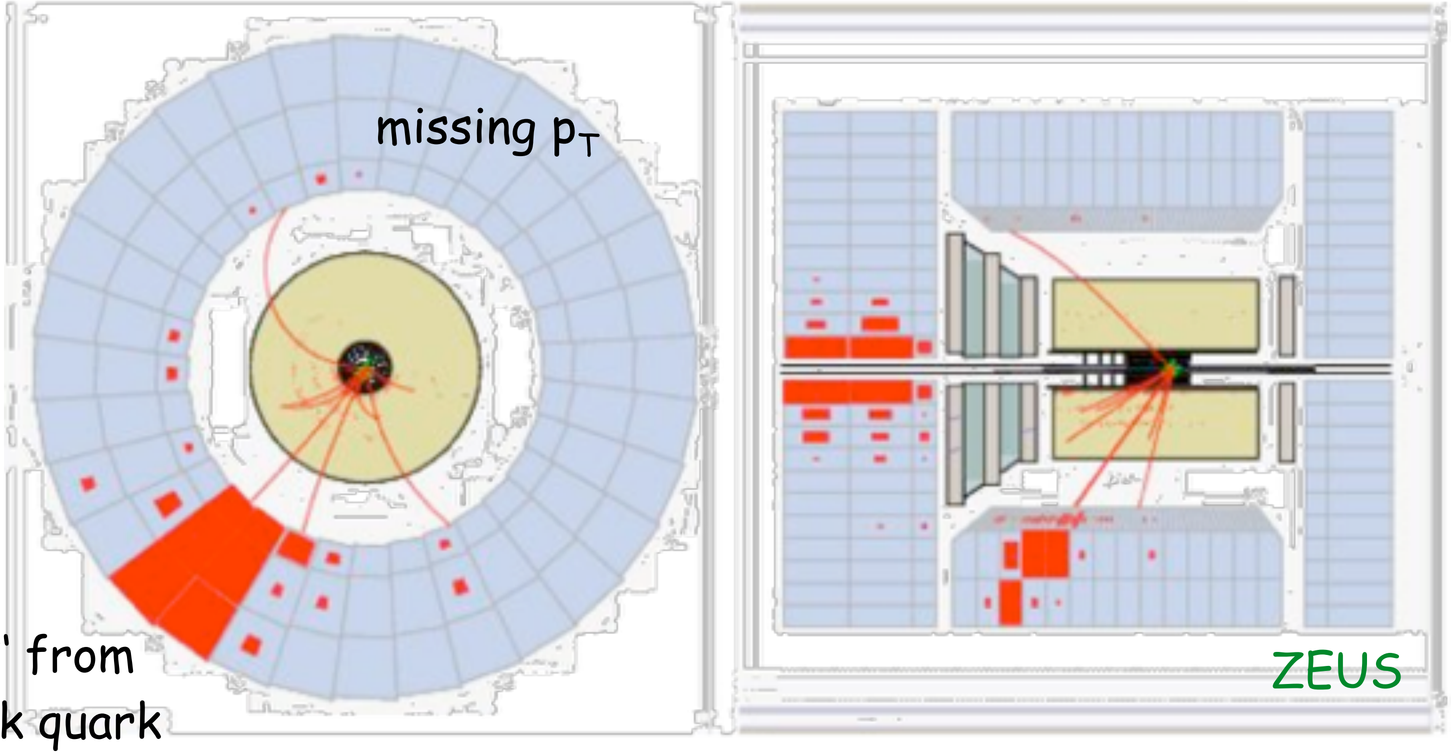
H1 Run 122145 Event 69506

Date 19/09/1995

a charged current DIS event



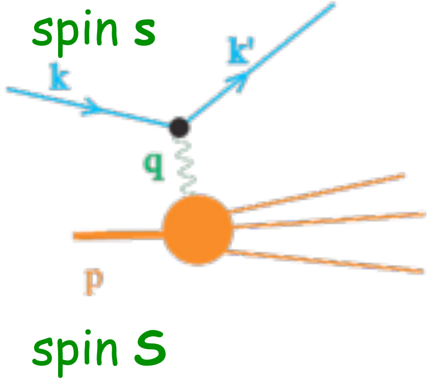
a charged current event with W-boson-exchange
(the electron turns into a neutrino which is "invisible")



for simplicity we will restrict ourselves to photon exchange though

analysis of DIS: 1st steps

electroweak theory tells us how the virtual vector boson (here γ^*) couples:



$$d\sigma = \frac{4\alpha^2}{s}\frac{d^3\vec{k}'}{2|\vec{k}'|}\frac{1}{Q^4}L^{\mu\nu}(k,q,s)\boxed{W_{\mu\nu}(p,q,S)}$$

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contains information
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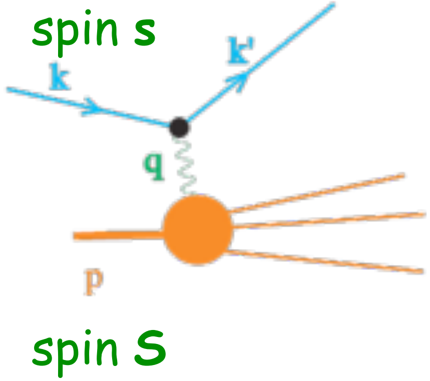

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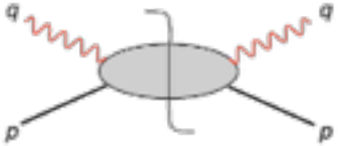
leptonic tensor from QED

hadronic tensor contains information about hadronic structure

parity & Lorentz inv., hermiticity $W^{\nu\mu}=W^{\mu\nu*}$, current conservation $q_\mu W^{\mu\nu}=0$ **dictate**:

$$W^{\mu\nu}(P,q,S) = \frac{1}{4\pi}\int d^4z\, e^{iq\cdot z}\,\langle P,S|\,J_\mu(z)\,J_\nu(0)\,|P,S\rangle$$



$$= \left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)F_1(x,Q^2) + \left(P^\mu - \frac{P\cdot q}{q^2}q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2}q^\nu\right)F_2(x,Q^2)$$

$$+\,i\,M\,\varepsilon^{\mu\nu\rho\sigma}q_\rho\left[\frac{S_\sigma}{P\cdot q}\,g_1(x,Q^2) + \frac{S_\sigma(P\cdot q)-P_\sigma(S\cdot q)}{(P\cdot q)^2}\,g_2(x,Q^2)\right]$$

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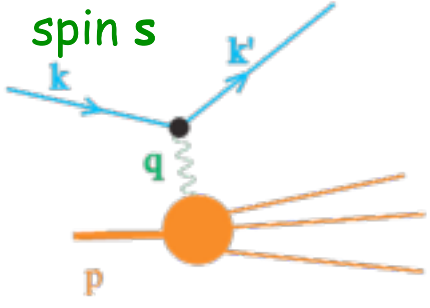

spin s

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$$d\sigma = \frac{4\alpha^2}{s}\frac{d^3\vec{k}'}{2|\vec{k}'|}\frac{1}{Q^4}L^{\mu\nu}(k,q,s)\boxed{W_{\mu\nu}(p,q,S)}$$

leptonic tensor from QED

hadronic tensor contains information about hadronic structure

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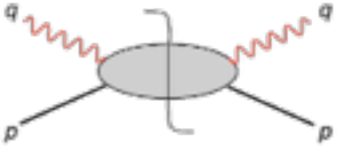

$$W^{\mu\nu}(P,q,S) = \frac{1}{4\pi}\int d^4z\, e^{iq\cdot z}\,\langle P,S|\,J_\mu(z)\,J_\nu(0)\,|P,S\rangle$$

$$= \left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)F_1(x,Q^2) + \left(P^\mu - \frac{P\cdot q}{q^2}q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2}q^\nu\right)F_2(x,Q^2)$$

unpol.structure fcts.$F_{1,2}$

$$+\,i\,M\,\varepsilon^{\mu\nu\rho\sigma}q_\rho\left[\frac{S_\sigma}{P\cdot q}\,g_1(x,Q^2)+\frac{S_\sigma(P\cdot q)-P_\sigma(S\cdot q)}{(P\cdot q)^2}\,g_2(x,Q^2)\right]$$

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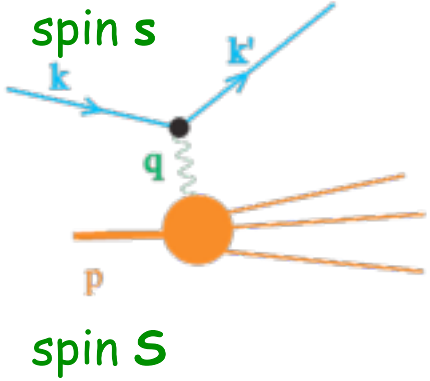

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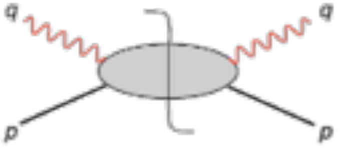

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pol. structure fcts. $g_{1,2}$ **– measure** $W(P,q,S)$ – $W(P,q,-S)$ **!**

SLAC-MIT experiment of 1969

1990

two unexpected results:

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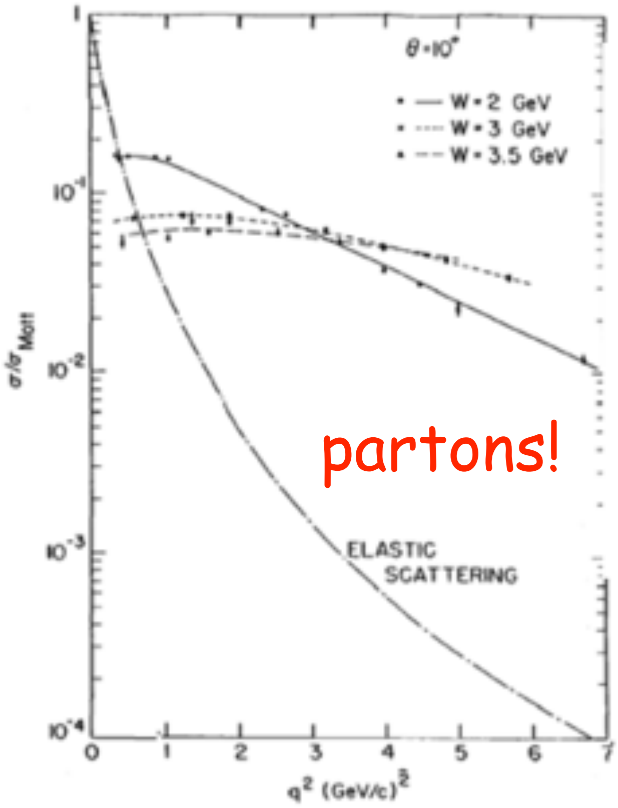

partons!

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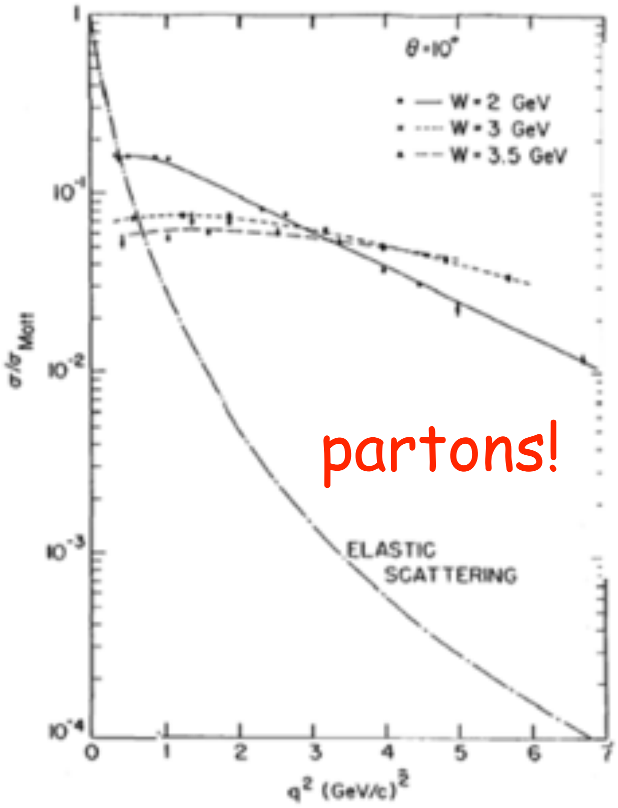

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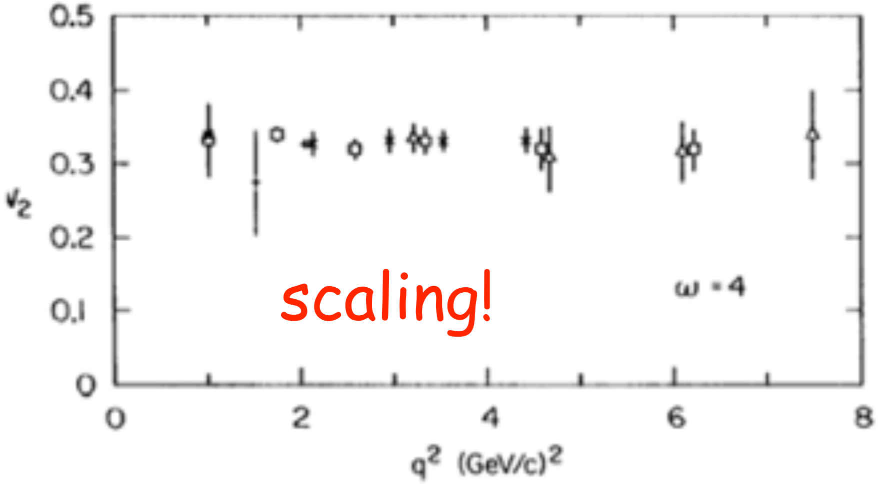
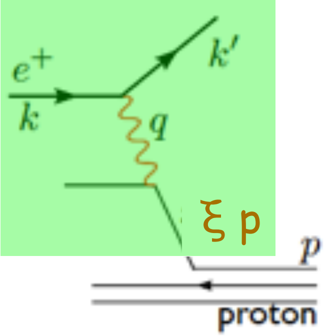


scaling!

birth of the pre-QCD parton model

DIS in the naïve parton model

let's do a quick calculation: consider **electron-quark scattering**



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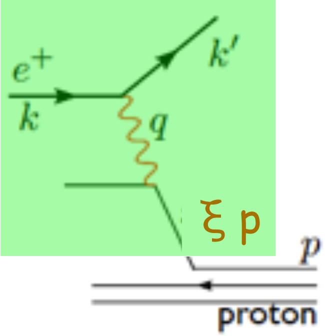

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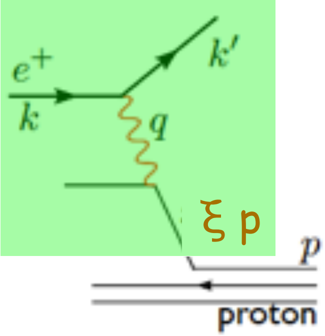

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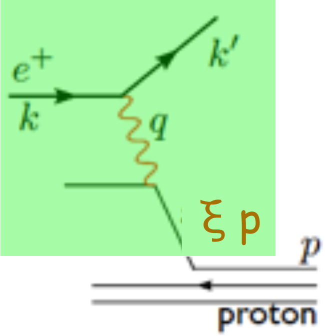

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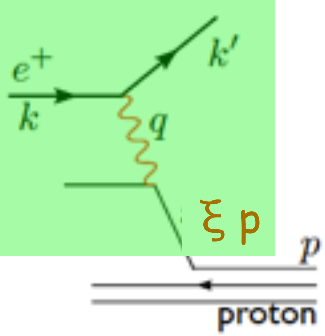

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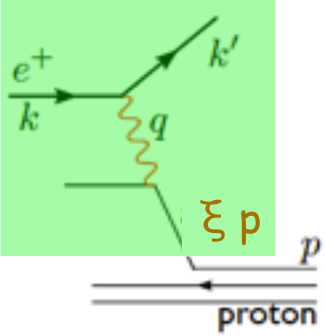

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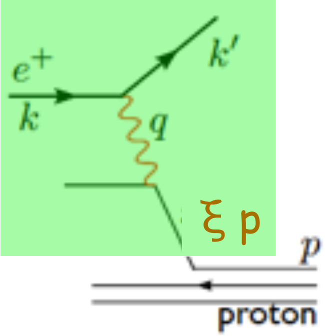

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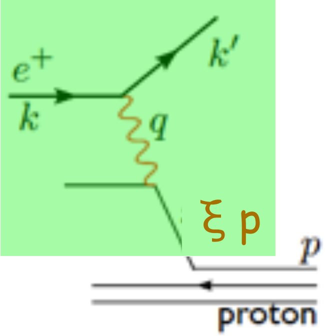

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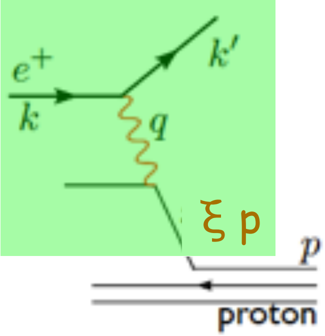

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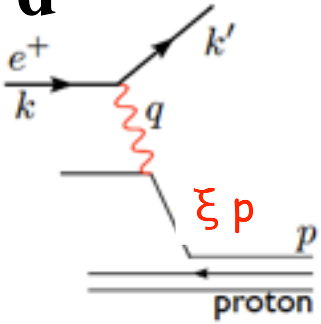

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to what one obtains with the hadronic tensor (on the quark level)

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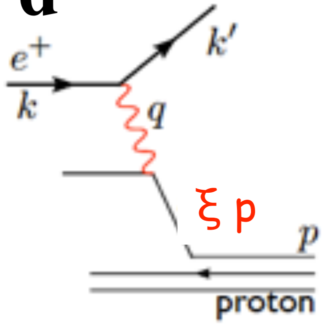
DIS in the naïve parton model cont'd



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and read off

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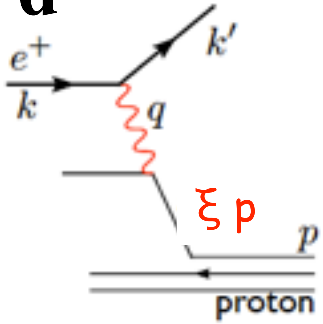
Callan Gross relation

reflects spin 1/2 nature of quarks

DIS in the naïve parton model cont’d



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proton structure functions then obtained by weighting the quark str. fct. with the **parton distribution functions** (probability to find a quark with momentum ξ)

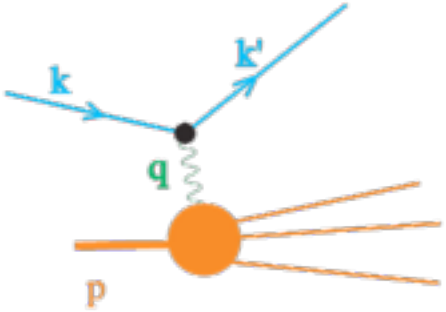
$$\mathbf{F_2} = \mathbf{2xF_1} = \sum_{q,q'}\int_0^1 d\xi\,\mathbf{q}(\xi)\,\mathbf{xe_q^2}\,\delta(\mathbf{x}-\xi)$$

$$= \sum_{q,q'} \mathbf{e_q^2}\,\mathbf{x}\,\mathbf{q(x)}$$

DIS measures the charged-weighted sum of quarks and antiquarks

“scaling” - no dependence on scale Q

space-time picture of DIS



this can be best understood in a reference frame where the proton moves very fast and Q>>m_h is big

(recall light-cone kinematics from part II)

4-vector	hadron rest frame	Breit frame
$(p^+, p^-, \vec{p}_T)$	$\frac{1}{\sqrt{2}}(m_h, m_h, \vec{0})$	$\frac{1}{\sqrt{2}}(\frac{Q}{x}, \frac{x m_h^2}{Q}, \vec{0})$
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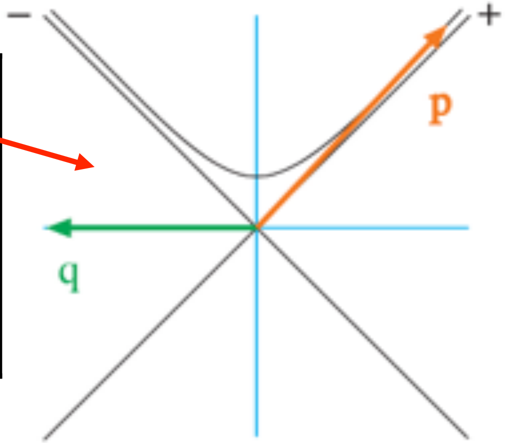

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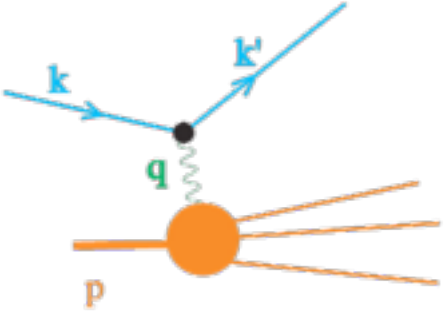

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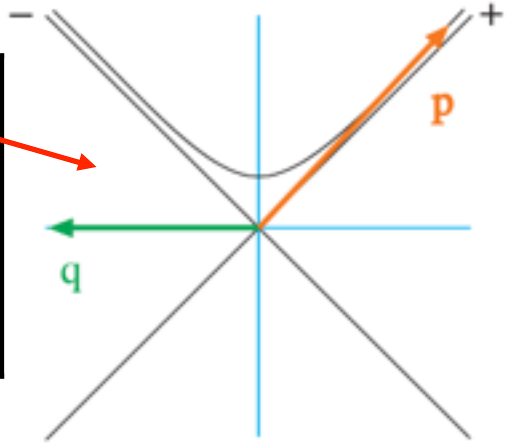


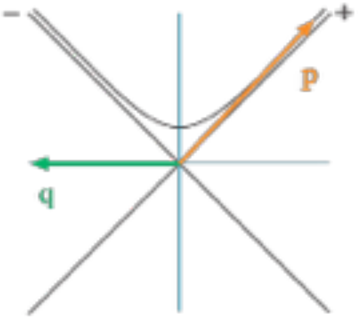
Lorentz boost

in general $(a^+, a^-, \vec{a}_T) \rightarrow (e^{\omega} a^+, e^{-\omega} a^-, \vec{a}_T) = (a'^+, a'^-, \vec{a}')$

here: $e^{\omega} = Q/(x m_h)$

space-time picture of DIS – cont’d

simple estimate for typical time-scale of interactions among the partons inside a fast-moving hadron:



rest frame: $\Delta x^{+} \sim \Delta x^{-} \sim \frac{1}{m}$

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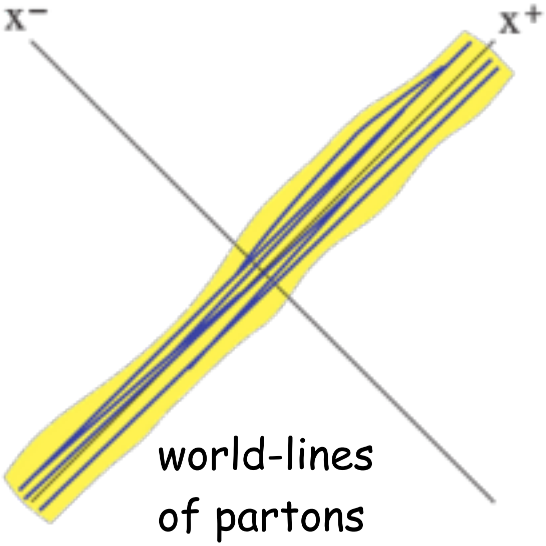

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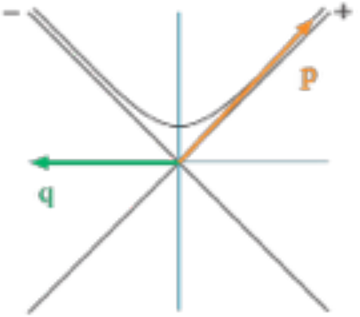

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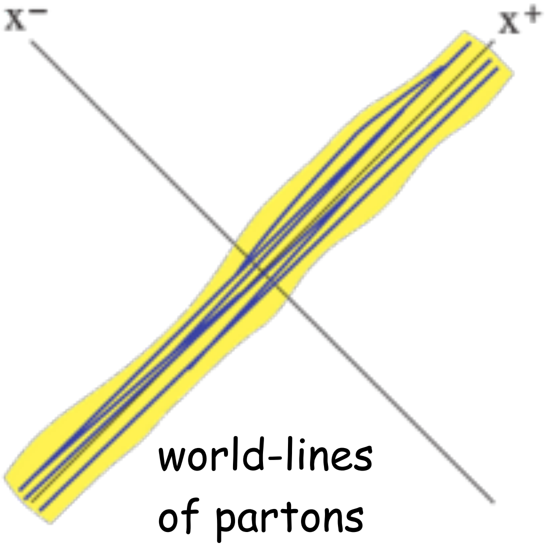
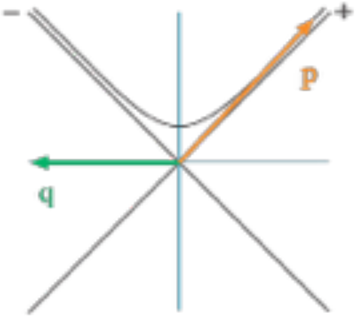


space-time picture of DIS – cont'd



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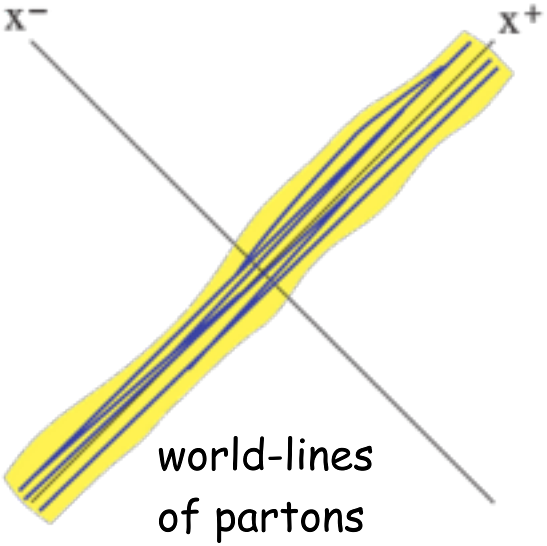

world-lines of partons

interactions between partons are spread out inside a fast moving hadron

How does this compare with the time-scale of the hard scattering?

foundation of naïve Parton Model

Feynman;
Bjorken, Paschos

Breit frame:

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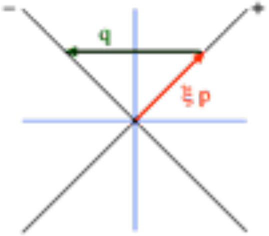
struck quark
on-shell



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foundation of naïve Parton Model

Feynman;
Bjorken, Paschos

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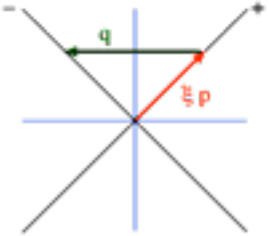

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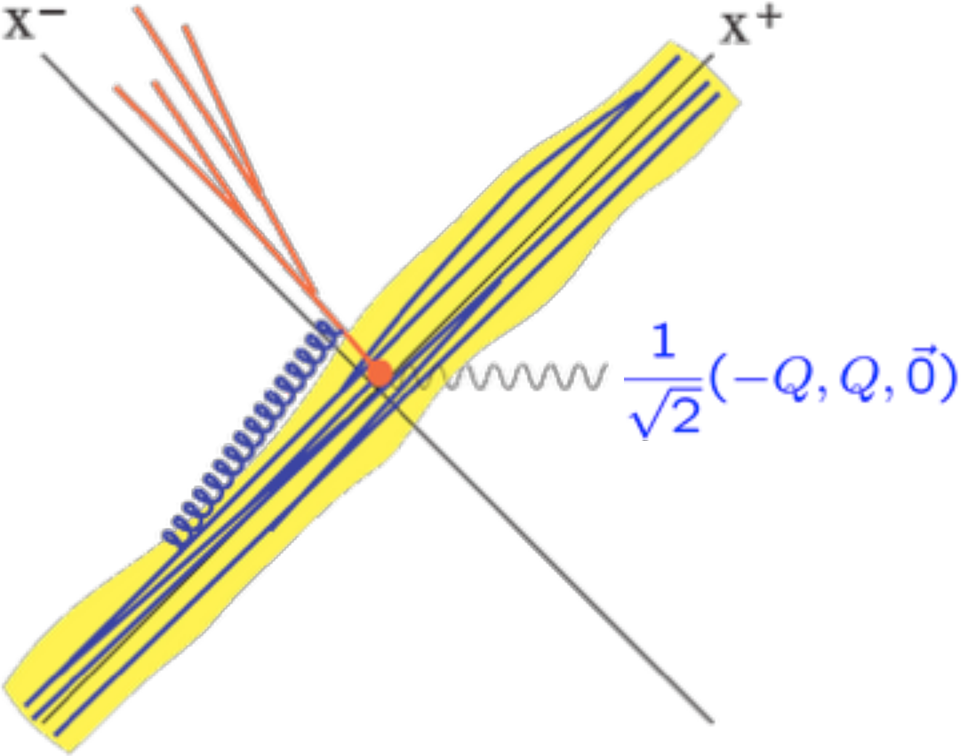

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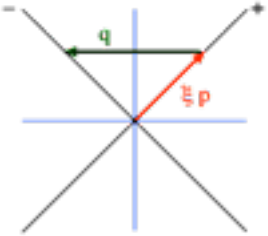

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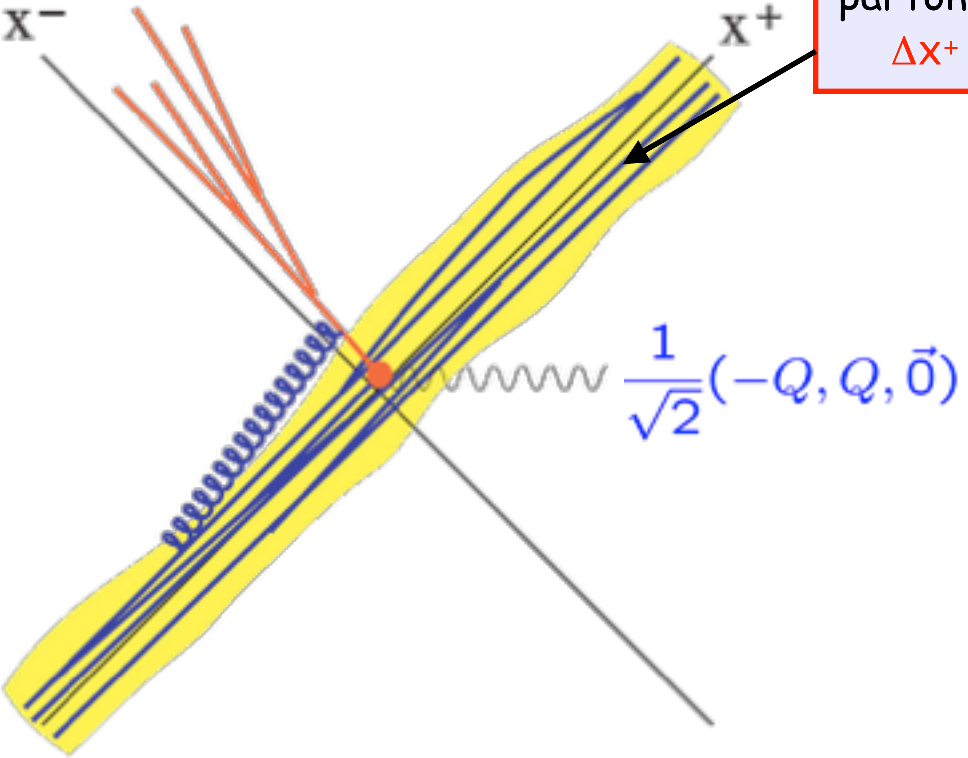

interactions of partons dilated
$\Delta x^+ \approx Q/m^2$

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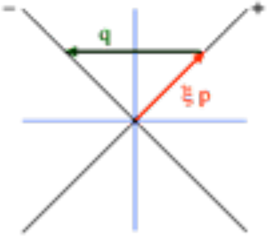

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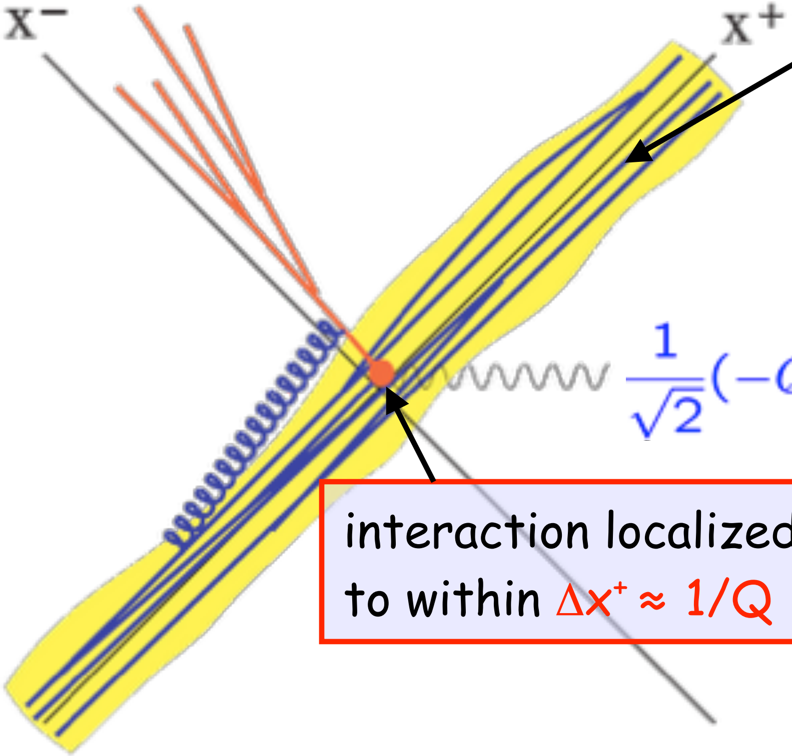

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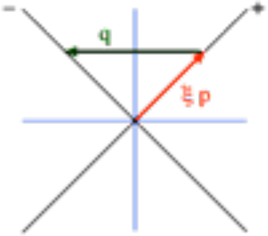

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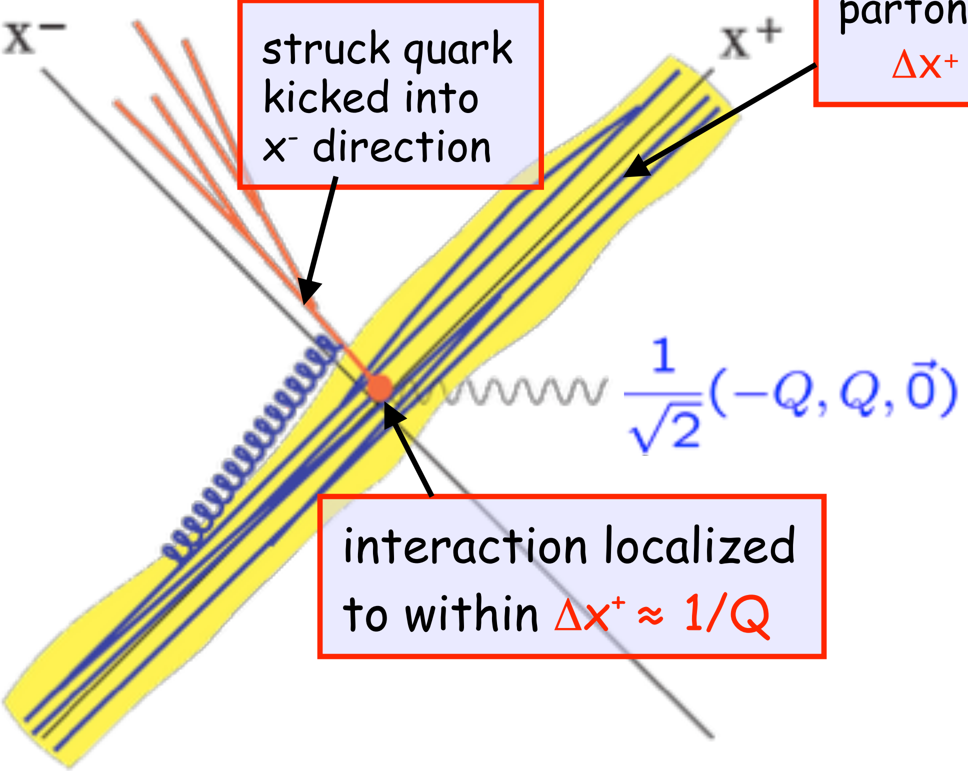

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struck quark kicked into x^- direction

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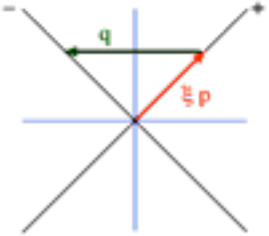

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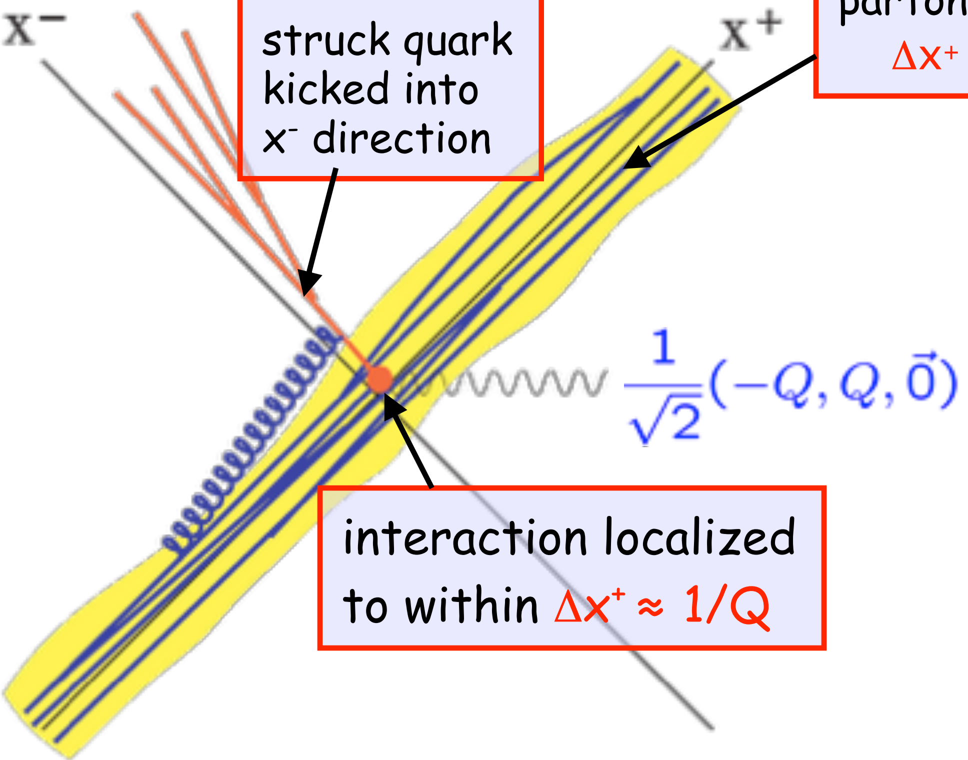

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interactions of partons dilated $\Delta x^+ \approx Q/m^2$

interaction localized to within $\Delta x^+ \approx 1/Q$

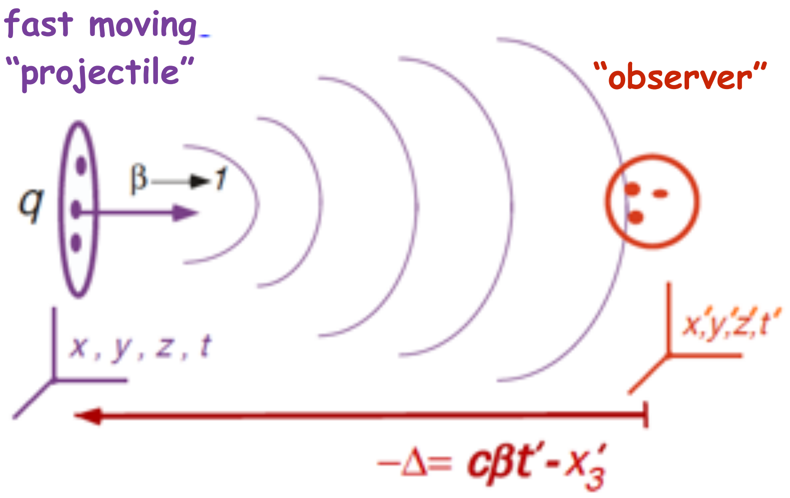
upshot:

- partons are free during the hard interaction
- lepton scatters off free partons incoherently
- convenient to introduce momentum fractions

$0 < \xi_i \equiv p_i^+/p^+ < 1$

a “classical” view of factorization

adapted from G. Sterman’s lectures



accelerated charges produce classical radiation

QFT assembles field from infinite # of soft quanta

Lorentz transformation $\mathbf{x_3} = \gamma(\beta \mathbf{ct}' - \mathbf{x}'_3) \equiv -\gamma\boldsymbol{\Delta}$

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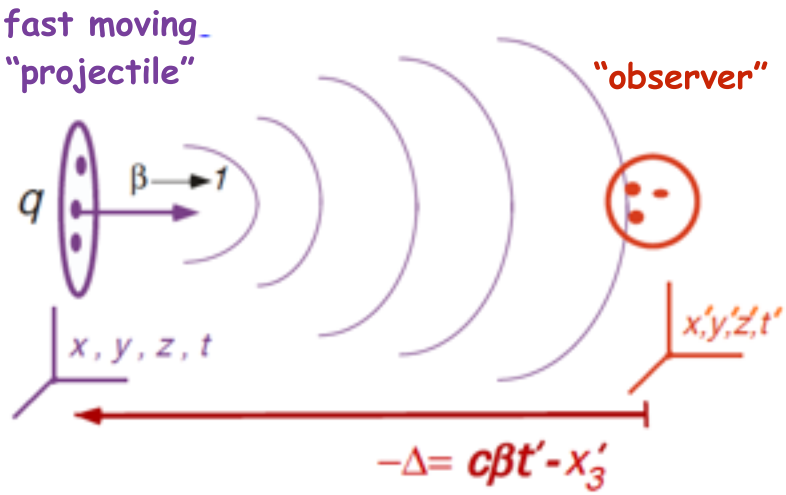

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field	x-frame	x′ frame	Lorentz factor
scalar field $\phi(\mathbf{x})$	$\frac{\mathbf{q}}{\mathbf{|\tilde{x}|}}$	$\frac{\mathbf{q}}{\mathbf{(x_T^2 + \gamma^2\Delta^2)^{1/2}}}$	$\mathbf{1/\gamma}$

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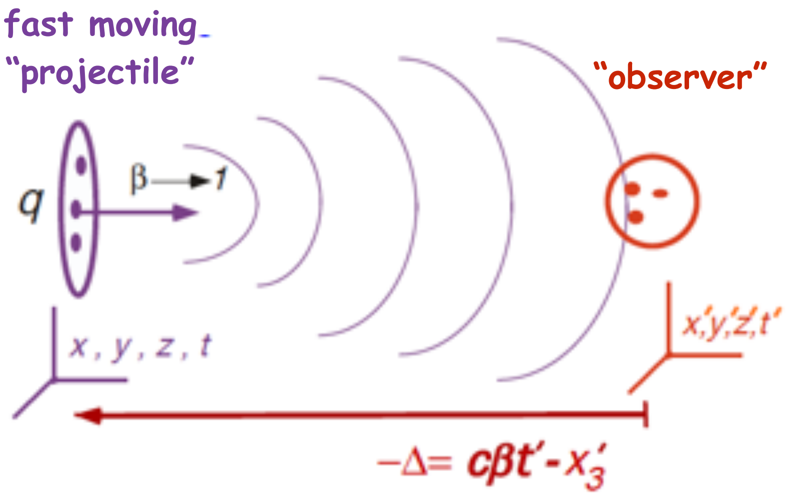

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gauge field $\mathbf{A_0(x)}$	$\frac{\mathbf{q}}{\lvert\tilde{\mathbf{x}}\rvert}$	$\frac{-\mathbf{q}\gamma}{(\mathbf{x_T^2} + \gamma^2\mathbf{\Delta}^2)^{1/2}}$	$\gamma^{\mathbf{0}}$

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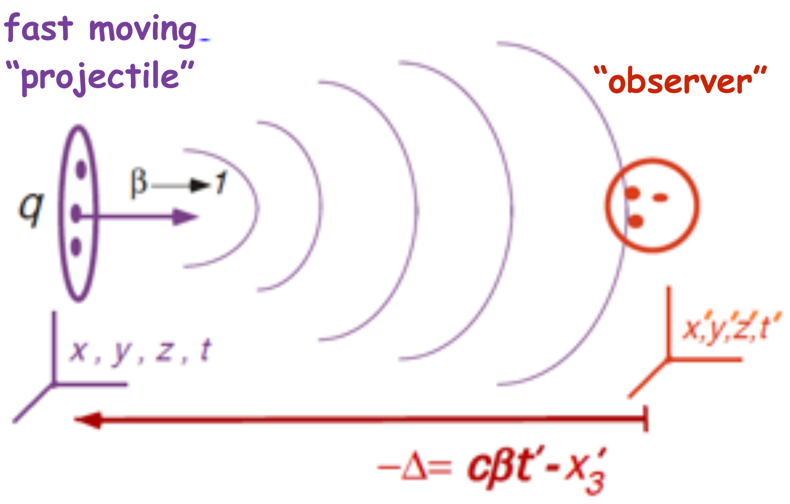

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“field strength” $\mathbf{E_3(x)}$	$\frac{\mathbf{q}}{\lvert\tilde{\mathbf{x}}\rvert^2}$	$\frac{-\mathbf{q}\gamma\mathbf{\Delta}}{(\mathbf{x_T^2} + \gamma^2\mathbf{\Delta}^2)^{3/2}}$	$1/\gamma^2$

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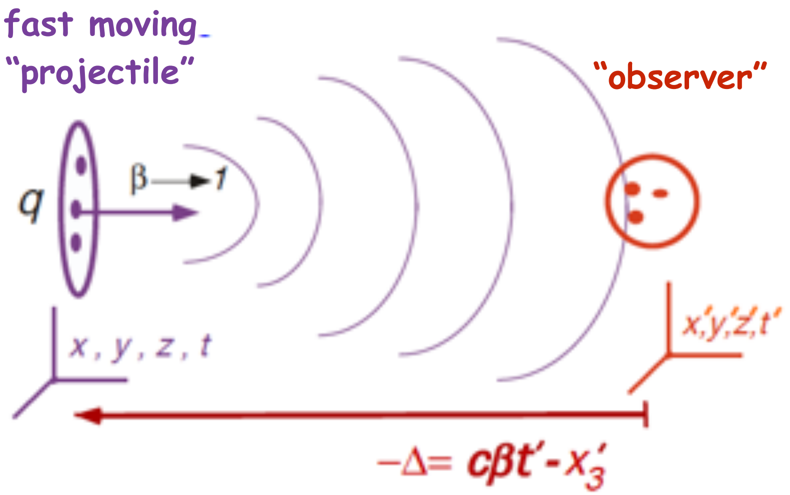

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upshot

☑ physical fields are Lorentz contracted

fast moving "projectile" sees much shorter distance x_3 than "observer"

☑ physical field does not overlap with observer until moment of "scattering"

☑ corrections (= "advanced effects") power suppressed $\propto (\mathbf{1} - \beta)$

☑ much the same reasoning for final-state

sum rules and isospin

for the quark distributions in a proton there are several sum rules to obey

$$\int_0^1 dx \sum_i x f_i^{(p)}(x) = 1$$

momentum sum rule

quarks share proton momentum

$$\int_0^1 dx \left(f_u^{(p)}(x) - f_{\bar{u}}^{(p)}(x) \right) = 2$$

$$\int_0^1 dx \left(f_d^{(p)}(x) - f_{\bar{d}}^{(p)}(x) \right) = 1$$

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flavor sum rules

conservation of quantum numbers

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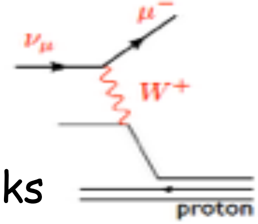
isospin symmetry relates a neutron to a proton (just u and d interchanged)

$$F_2^n(x) = x\left(\frac{1}{9}d_n(x) + \frac{4}{9}u_n(x)\right) = x\left(\frac{4}{9}d_p(x) + \frac{1}{9}u_p(x)\right)$$

- measuring both allows to determine u^p and d^p separately
- note: CC DIS couples to weak charges and separates quarks and antiquarks

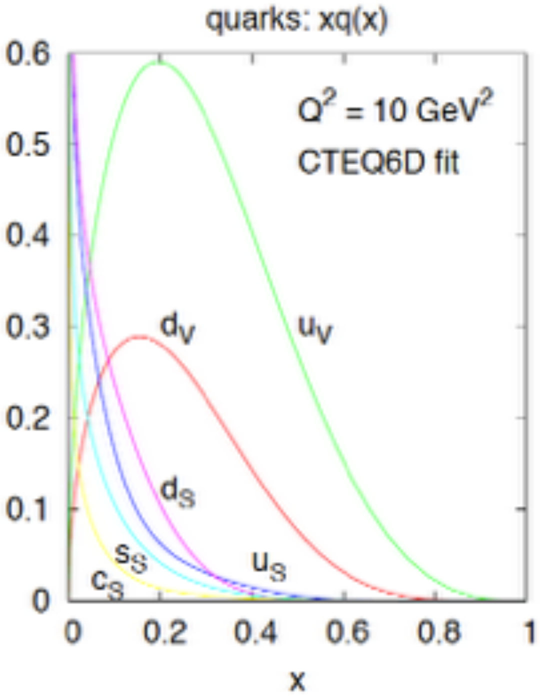


momentum sum rule in the naïve parton model

$$\int_0^1 dx \sum_i x f_i^{(p)}(x) = 1$$

u_v	0.267
d_v	0.111
u_s	0.066
d_s	0.053
s_s	0.033
c_c	0.016
total	0.546

half of the momentum is missing



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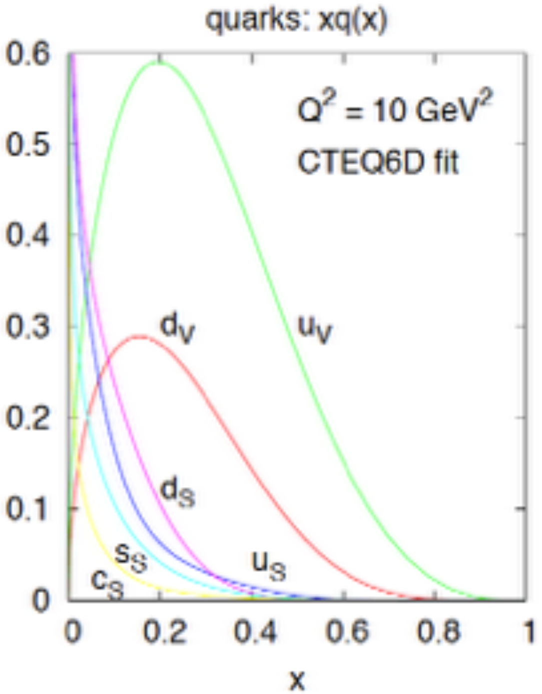

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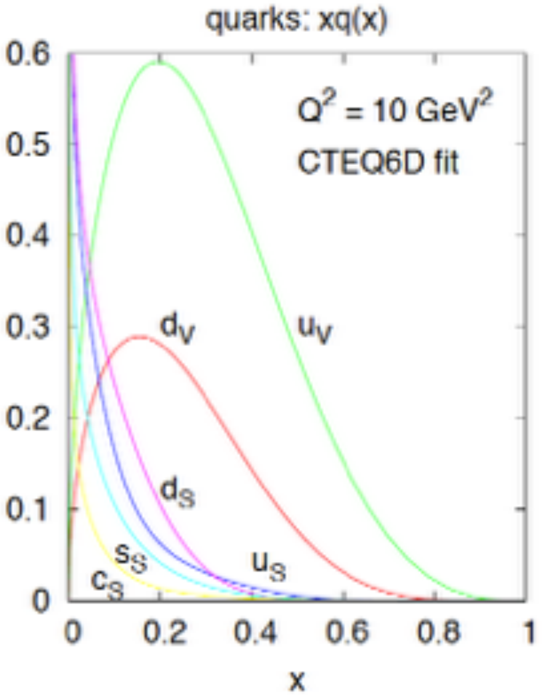
how can they couple?



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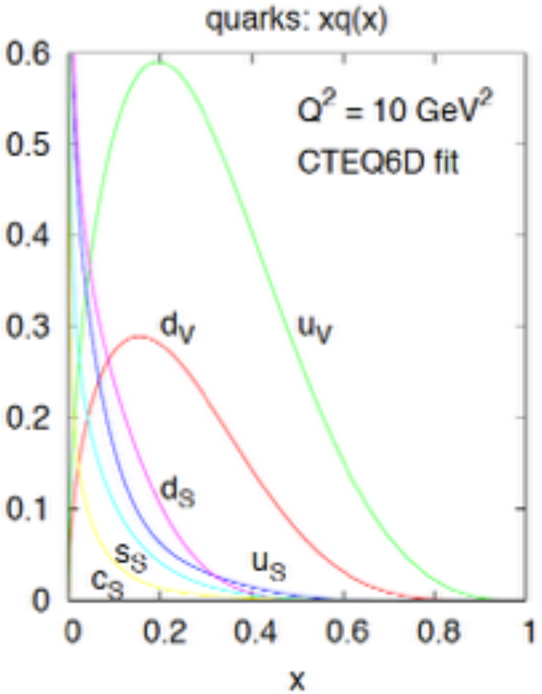

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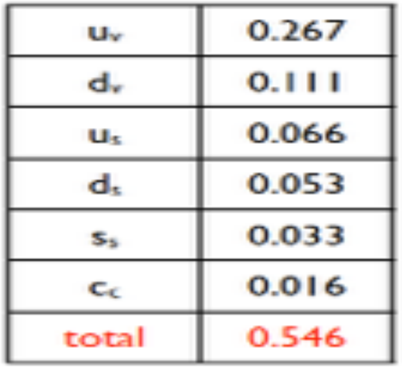

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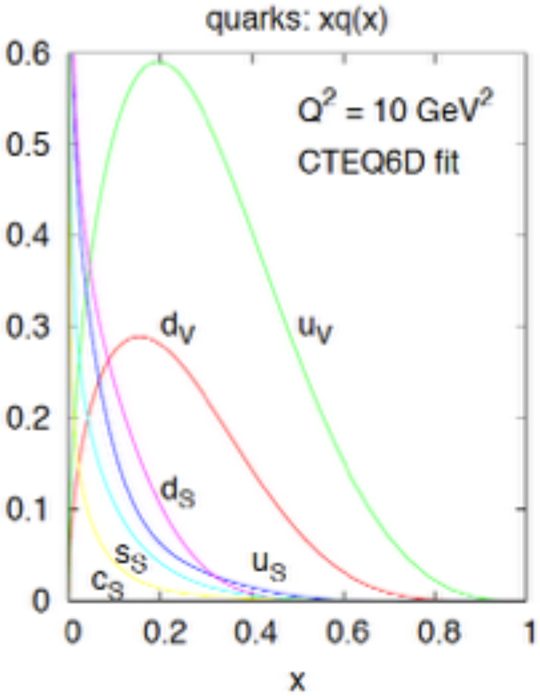

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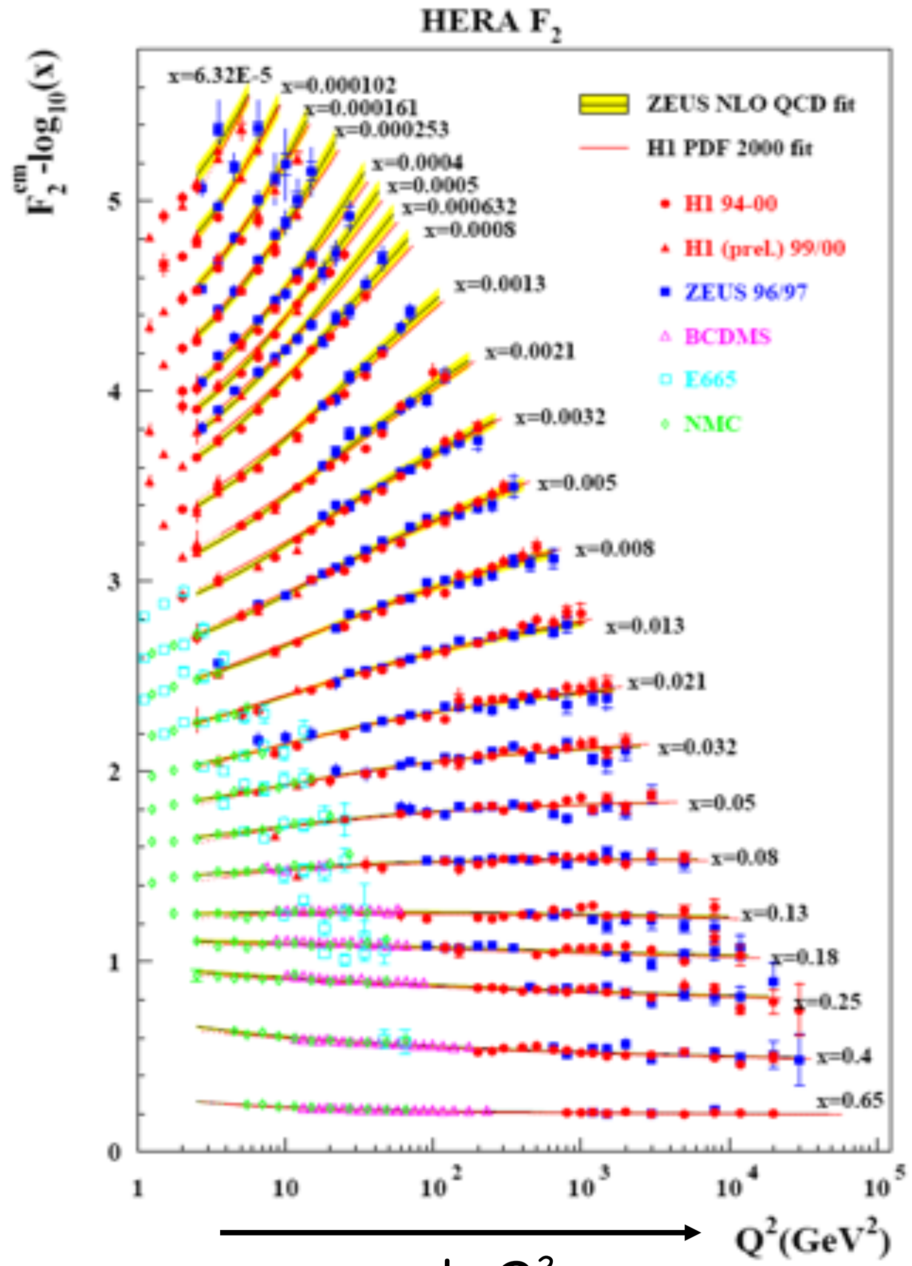
gluons will enter the game and everything will become scale dependent

Naïve parton model vs. experiment



scale Q^2

find strong scaling violations

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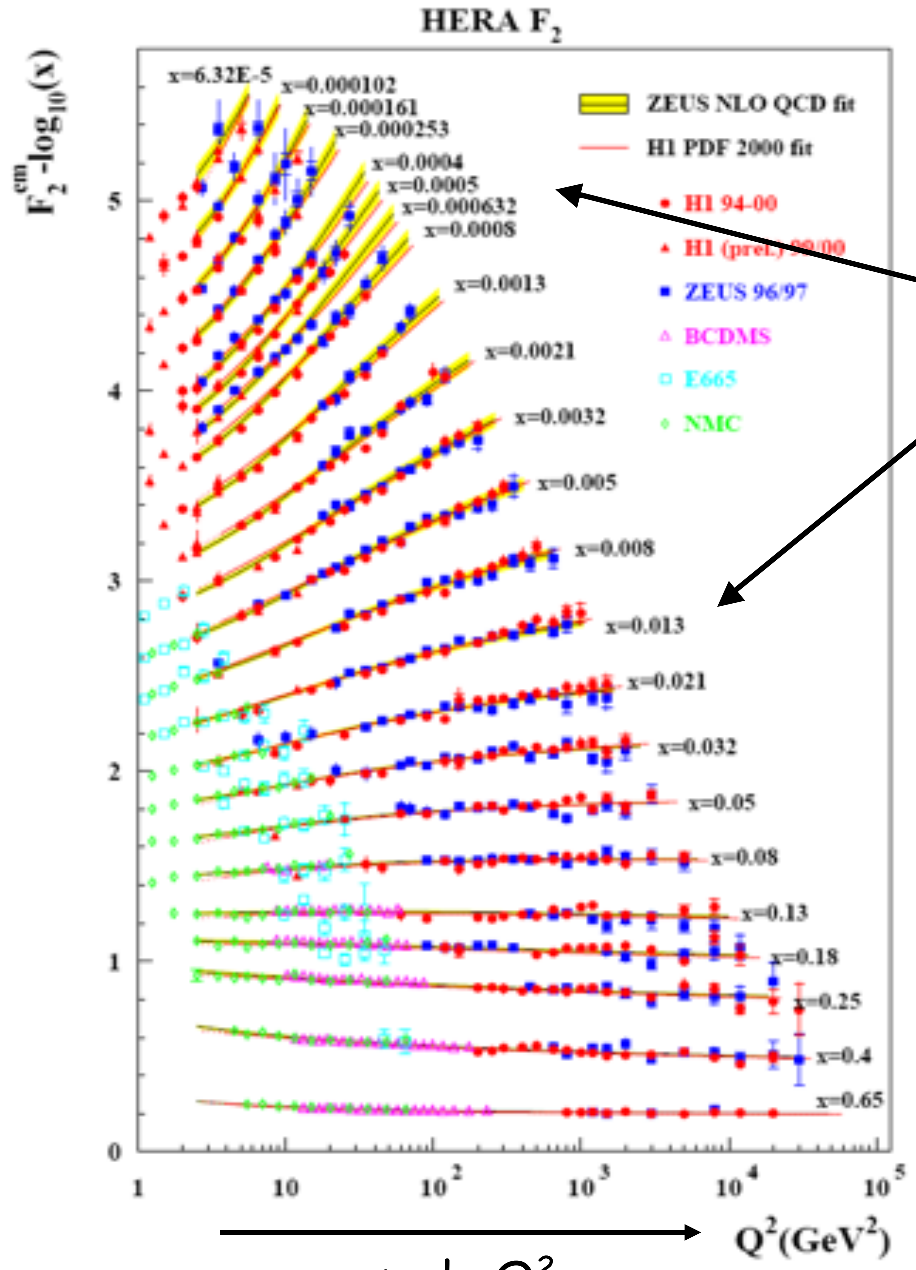

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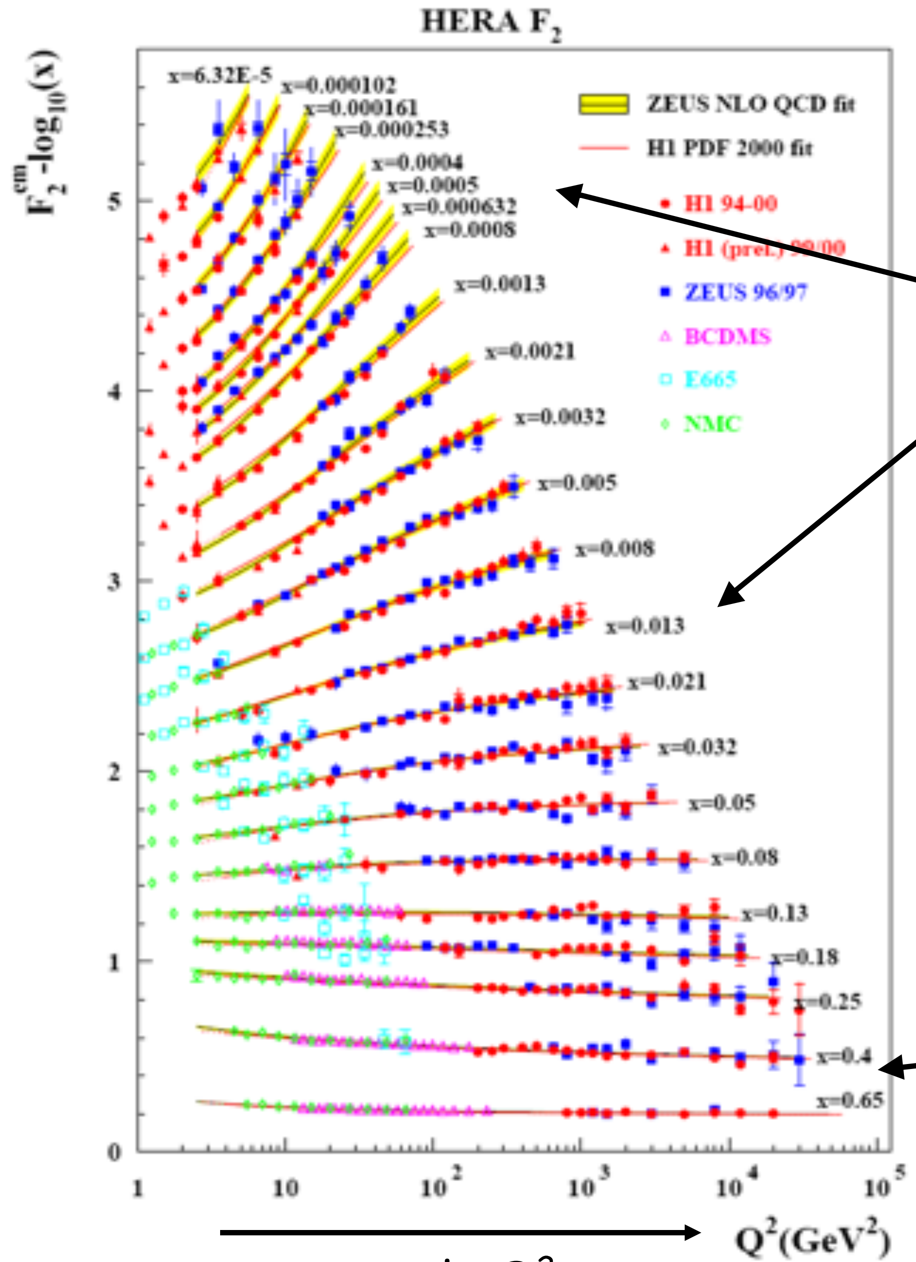

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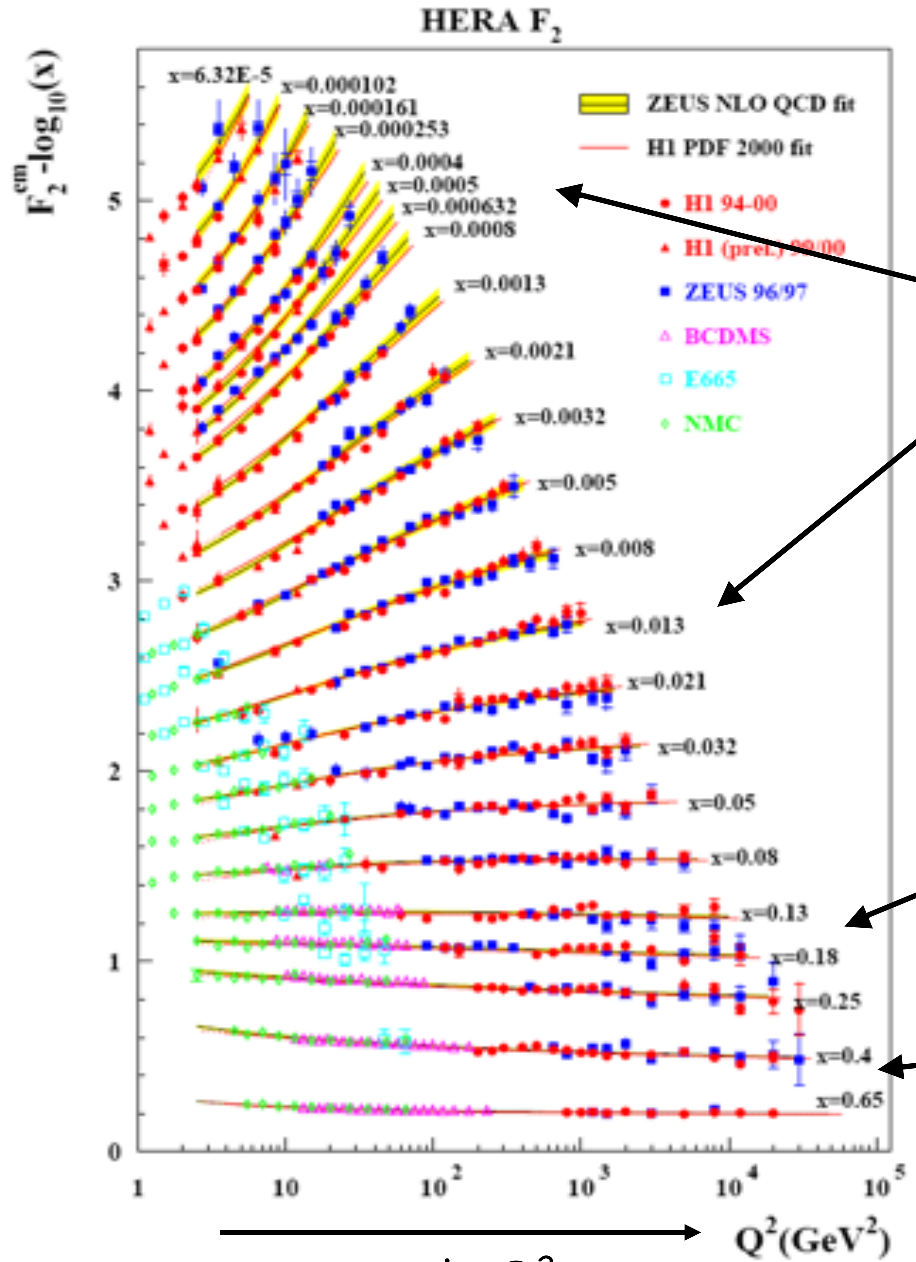


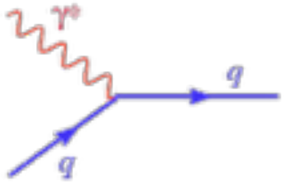
significant rise at small x

approximate scaling only
around x ≃ 0.15

decrease at high x

scale Q^2

DIS in the QCD improved parton model



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now we have to study **QCD dynamics in DIS**

- this leads to similar problems already encountered in e^+e^-

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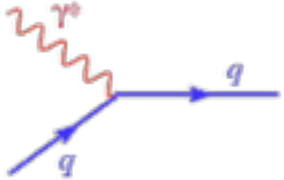

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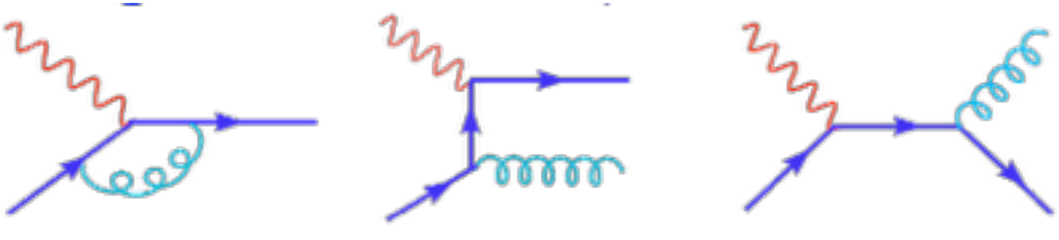
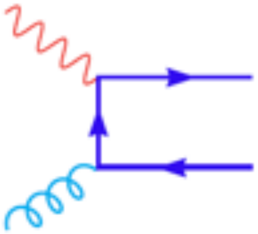


α_S corrections to the LO process



photon-gluon fusion

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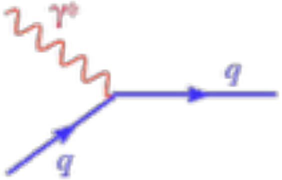

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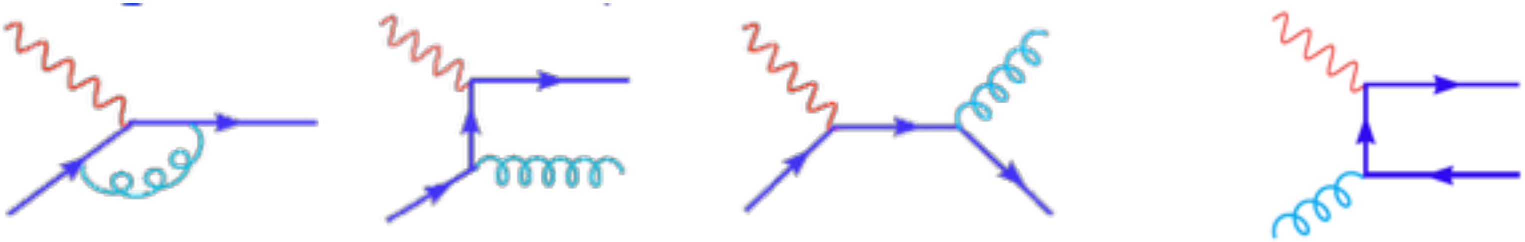

α_S corrections to the LO process photon-gluon fusion

caveat: have to expect divergencies (recall 2nd part)
related to soft/collinear emission or from loops

we cannot calculate with infinities → introduce a "**regulator**"
and remove it in the end

regularization methods

regulating divergencies is the 1st step in higher order calculations

standard regulators in QCD calculations:

- **dimensional regularization**

  change dimension of space-time to 4-2ε

  → calculations (integrals) rather involved;

  works in general, i.e., to all orders

  issues: γ_5 (spin, e.-w. couplings), SUSY, helicity violation

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  intuitive and transparent; stick to four dimensions

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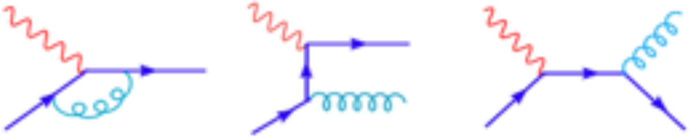
only if we have done everything consistently, including factorization, we can safely remove the regulator and can compare to experiment

general structure of the O(α_s) corrections

using small (artificial) quark/gluon masses as regulator we obtain:

$$\left.\frac{d^2\hat{\sigma}}{dx\,dQ^2}\right|_{F_2} \equiv \hat{F}_2^q = e_q^2 x \left[\delta(1-x) + \frac{\alpha_s(\mu_r)}{4\pi}\left[P_{qq}(x)\ln\frac{Q^2}{m_g^2} + C_2^q(x)\right]\right]$$



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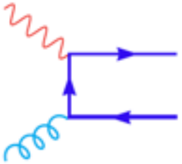
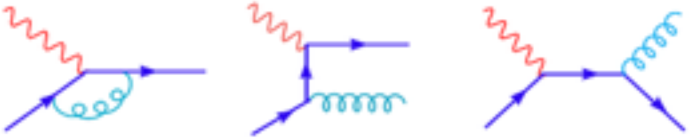


general structure of the O(α_s) corrections

using small (artificial) quark/gluon masses as regulator we obtain:



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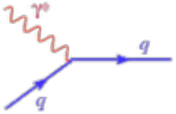

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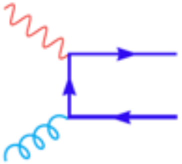

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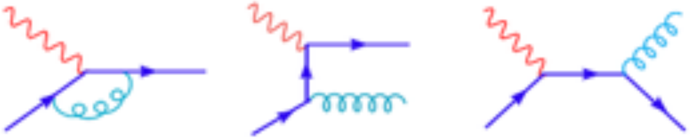

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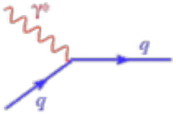

large logarithms
(collinear emission)

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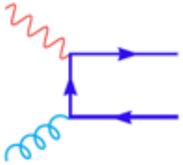

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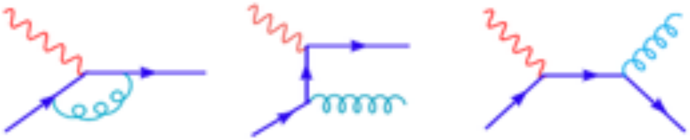

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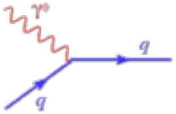

large logarithms
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finite
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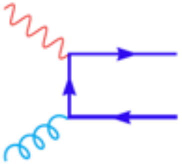

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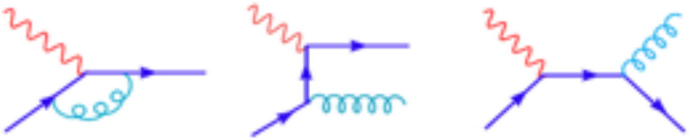

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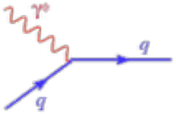

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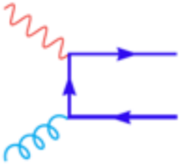
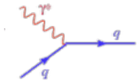


to see what happens to the logs we have to convolute our results with the PDFs

factorization of collinear singularities

for the quark part we obtain:



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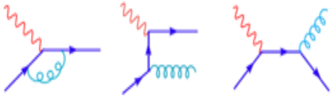
similarly for the gluonic part

from 

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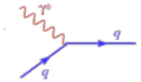

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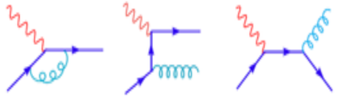
from 

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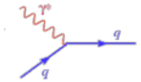

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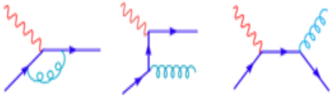

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at order α_s: (can be generalized to all orders)

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factorization of collinear singularities

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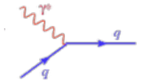

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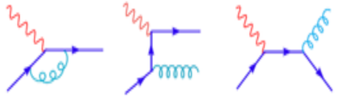

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absorbs all long-distance singularities at a **factorization scale** μ_f into $f_{a,0}$

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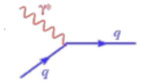

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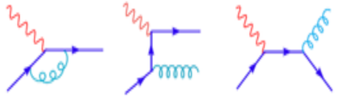

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physical/renormalized densities: not calculable in pQCD but **universal**

general structure of a factorized cross section

putting everything together, keeping only terms up to α_s:

$$
\begin{aligned}
F_2(x,Q^2) \;&=\; x \sum_{a=q,\bar{q}} e_q^2 \int_x^1 \frac{d\xi}{\xi} f_a(\xi,\mu_f^2) \\
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both, pdf's and the short-dist. coefficient depend on μ_f

(choice of μ_f: shifting terms between long- and short-distance parts)

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short-distance "Wilson coefficient"

choice of the factorization scheme

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yet another scale: μ_r

due to the renormalization
of ultraviolet divergencies

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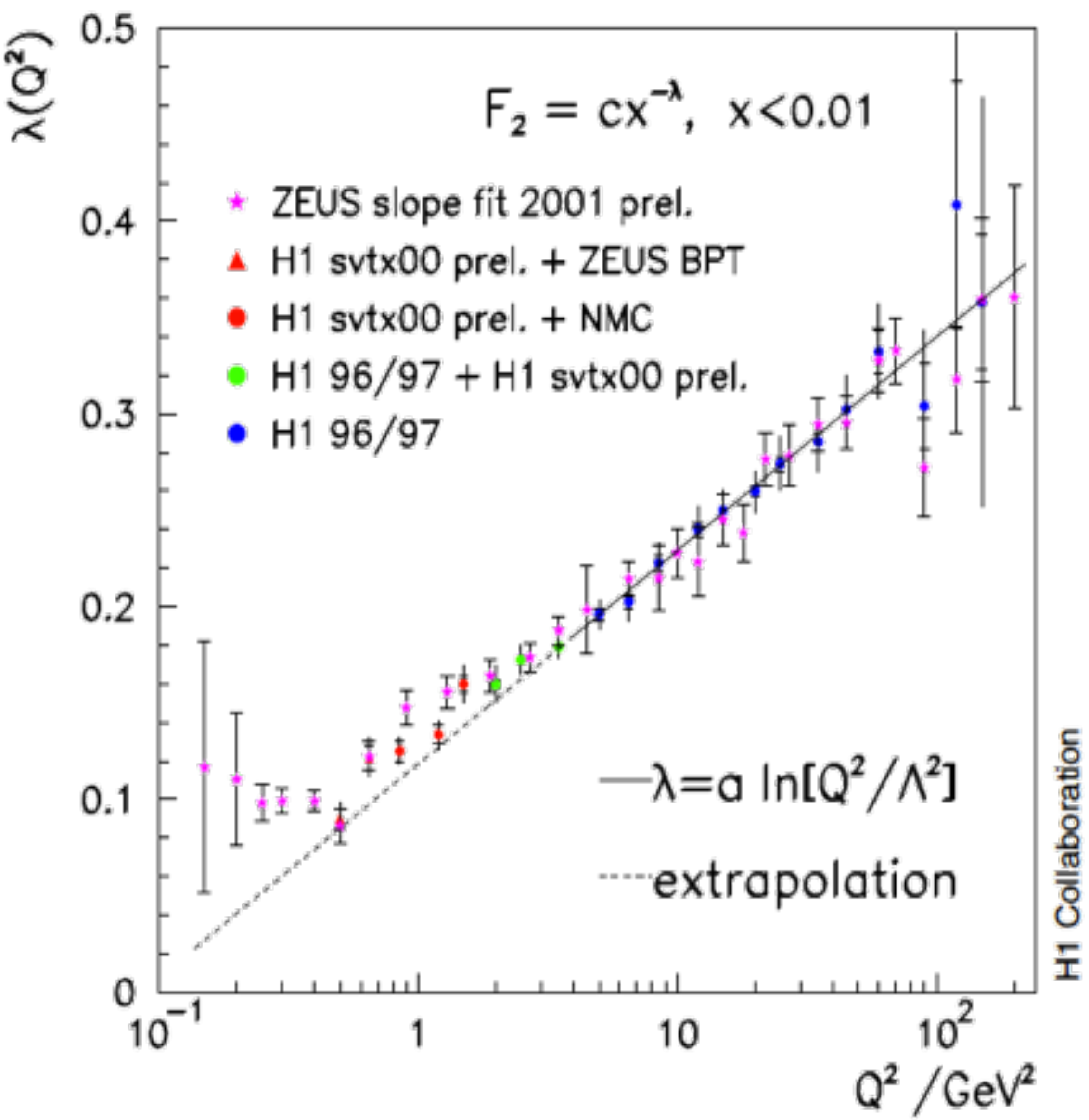
choice of the factorization scheme

this result is readily extended to hadron-hadron collisions

lesson: theorists are not afraid of infinities

ALRIGHT RUTH, I ABOUT GOT THIS ONE RENORMALIZED.

HERA's legacy: rise of F_2 vs Q^2



rise of F_2 can be expressed as

$$F_2 = cx^{-\lambda(Q^2)},\ x < 0.01$$

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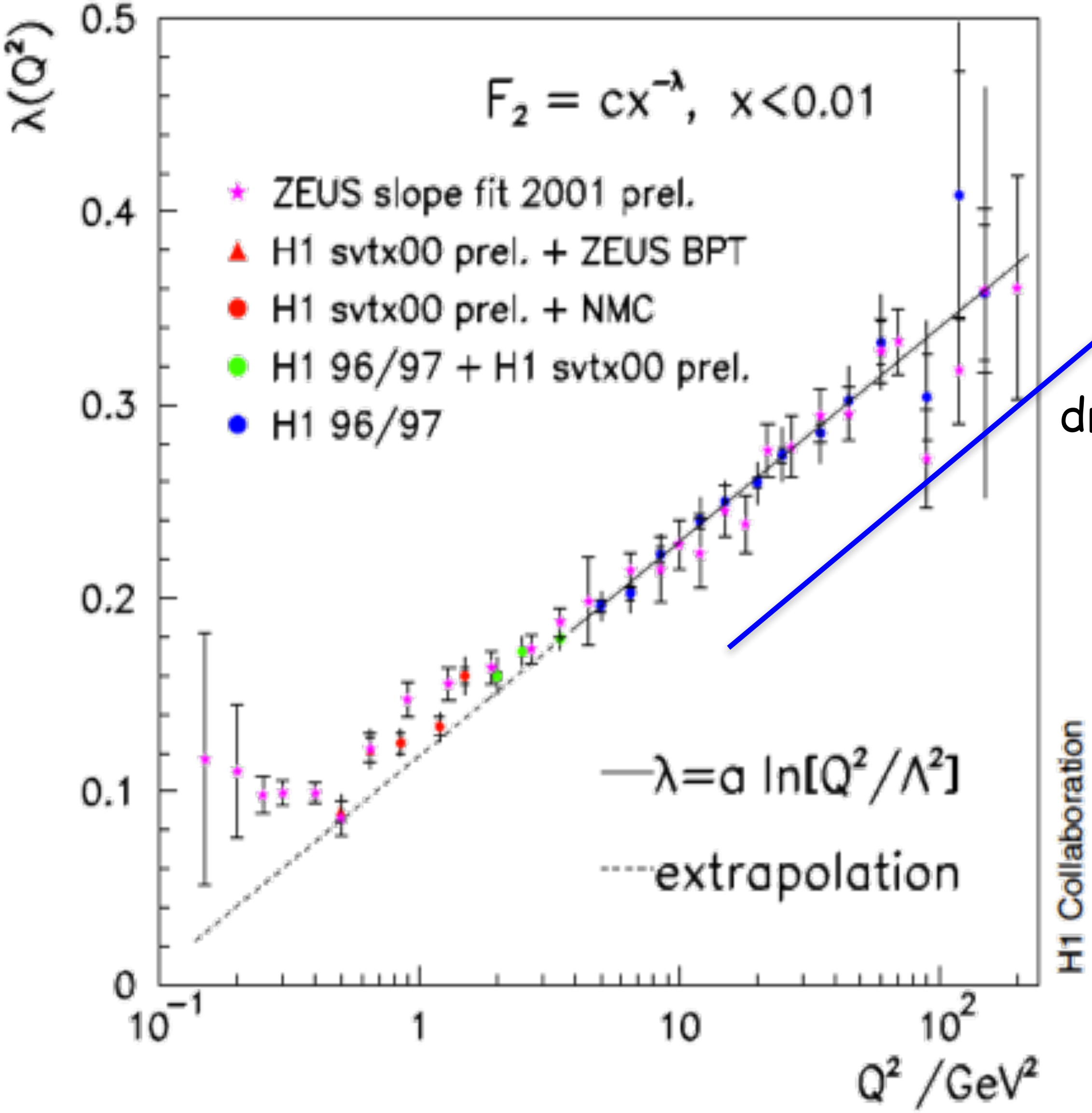

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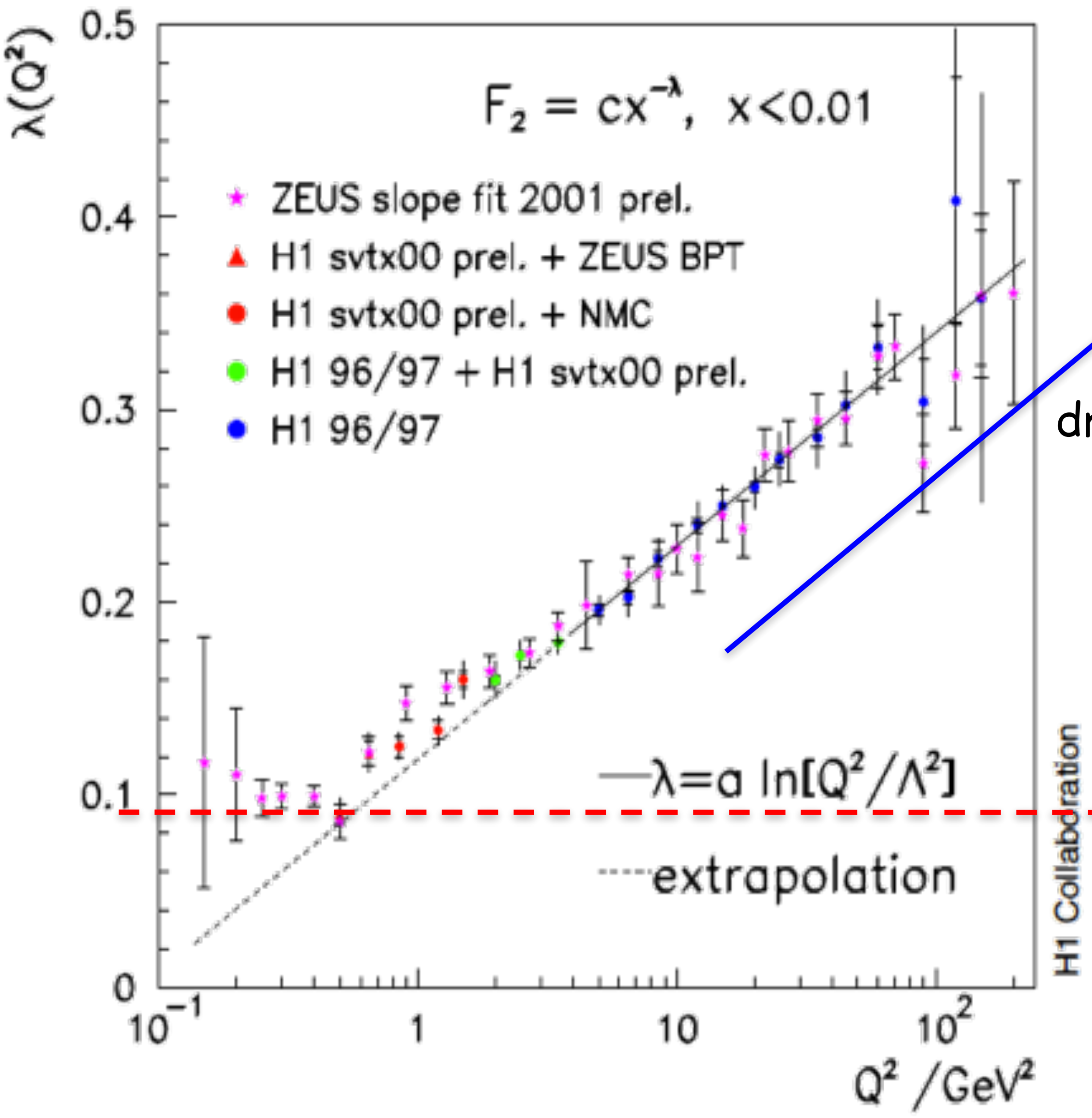

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$$F_2 = cx^{-\lambda(Q^2)},\ x < 0.01$$

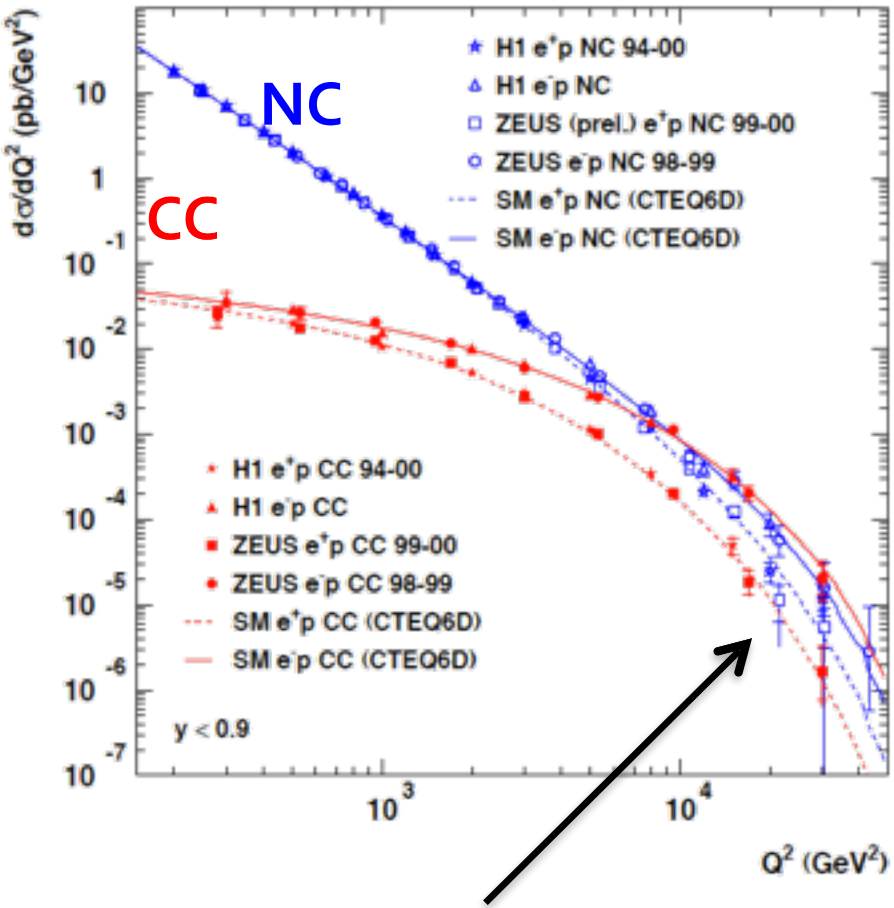
driven by evolution of gluon distribution

F_2 flattens around $Q^2 = 1\ \mathrm{GeV}^2$

change from partonic to hadronic behavior

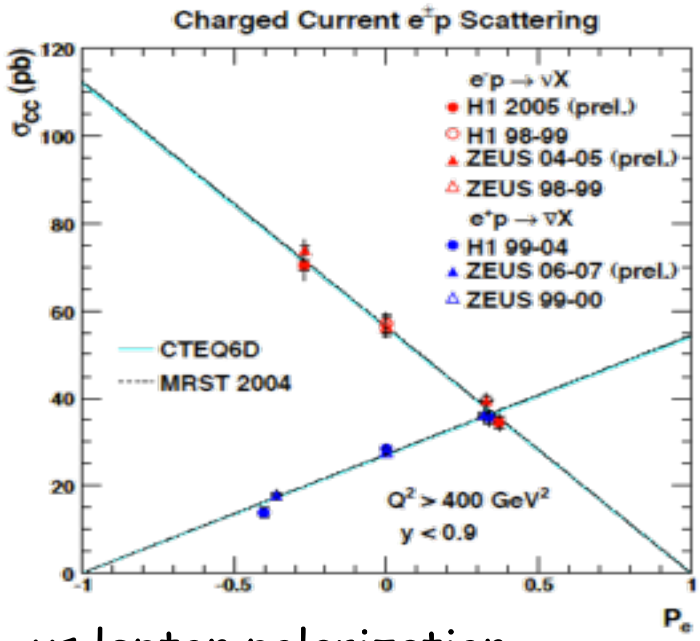
transition can be described in the **"color dipole model"**

NC & CC DIS: test of e-w theory !

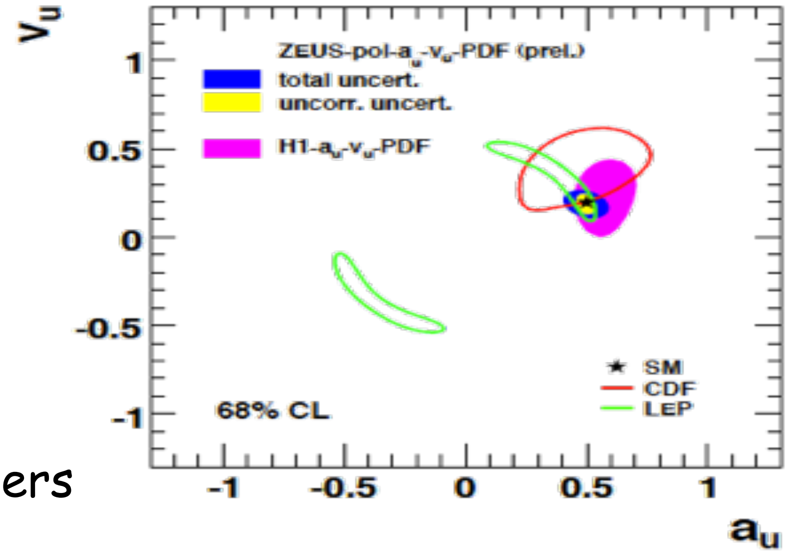


✓ e-w unification at high Q^2

✓ $\sigma(e^-p) \simeq u > \sigma(e^+p) \simeq d$



✓ σ_{CC} vs lepton polarization



✓ extraction of e-w parameters

universal PDFs → key to predictive power of pQCD

once PDFs are extracted from one set of experiments, e.g. DIS, we can use them to **predict cross sections** in, say, hadron-hadron collisions

parton densities are **universal**

→ there must be a process-independent **precise definition**

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small print: we need to specify a common factorization scheme for short- and long-distance physics (= choice of z_{ij} in our result for F_2)

standard choice: **modified minimal subtraction ($\overline{\text{MS}}$) scheme**
(closely linked to dim. regularization; used in all PDF fits)

less often used: **DIS scheme** = "maximal" subtraction where all $O(\alpha_s)$ corrections in DIS are absorbed into PDFs
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classic (but old-fashioned) definition of PDFs through their Mellin moments in Wilson-Zimmermann's operator product expansion (OPE)

Bardeen, Buras, Duke, Muta

PDFs as bi-local operators

Curci, Furmanski, Petronzio; Collins, Soper
see, e.g., D. Soper, hep-lat/9609018

more physical formulation in Bjorken-x space:
matrix elements of bi-local operators on the light-cone

for quarks: (similar for gluons; easy to include spin $\gamma^+ \rightarrow \gamma^+\gamma_5$)

$$f_a(\xi, \mu_f) = \frac{1}{2}\int \frac{dy^-}{2\pi} e^{-i\xi p^+ y^-} \langle p|\overline{\Psi}_a(0, y^-, \vec{0})\gamma^+ \mathcal{F}\Psi_a(0)|p\rangle_{\overline{\text{MS}}}$$

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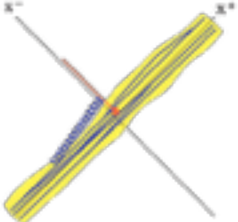
- in general we need a "**gauge link**" for a gauge invariant definition:

Foundations of
Perturbative QCD

JOHN COLLINS

$$\mathcal{F} = \mathcal{P}\exp\left(-ig\int_0^{y^-} dz^- A_c^+(0,z^-,\vec{0})T_c\right)$$



crucial role for a special class of "**transverse-momentum dep. PDFs**"
describing phenomena with transverse polarization **("Sivers function",** …)

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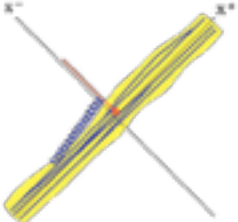

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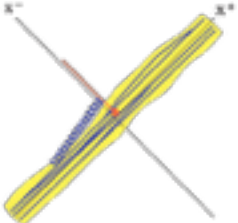

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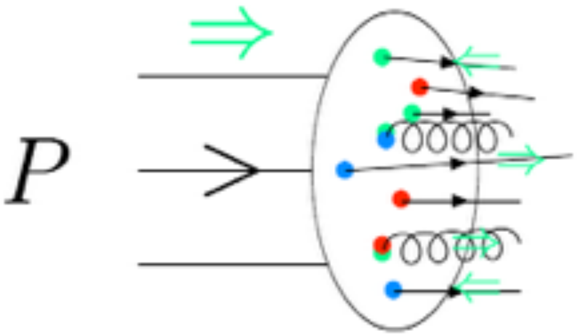
- interpretation as "number operator" only in "**$A^+ = 0$ gauge**"
- turn into local operators (→ lattice QCD) if taking moments $\int_0^1 d\xi\, \xi^n$

pictorial representation of PDFs

suppose we could take a snapshot of a nucleon with positive helicity



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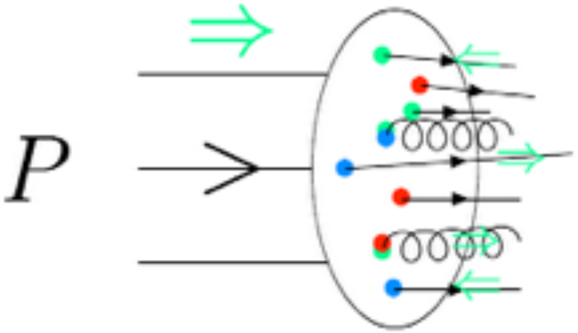

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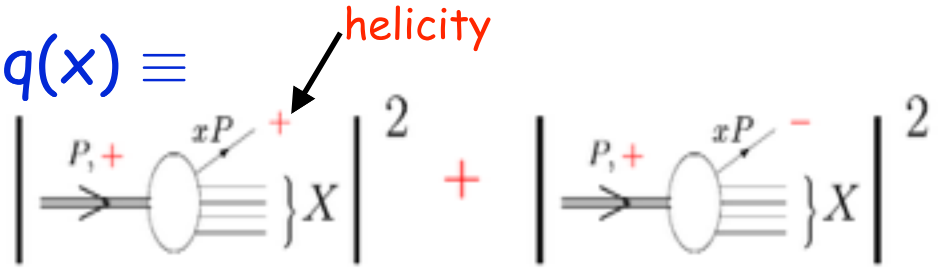

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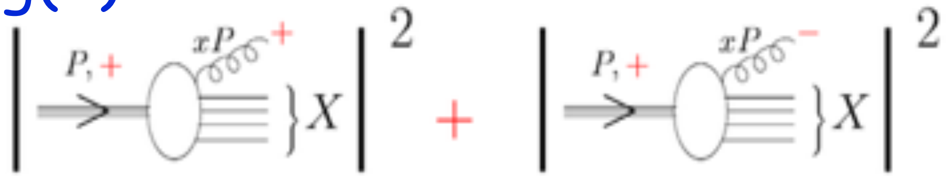

unpolarized PDFs

→ LHC phenomenology, etc.

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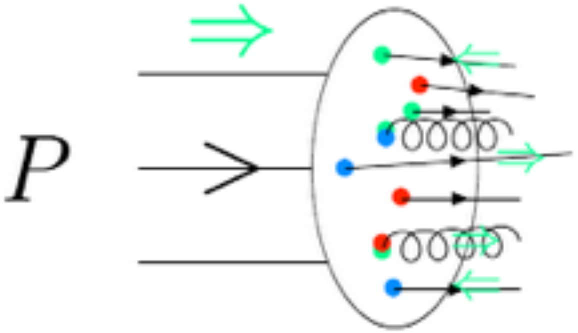

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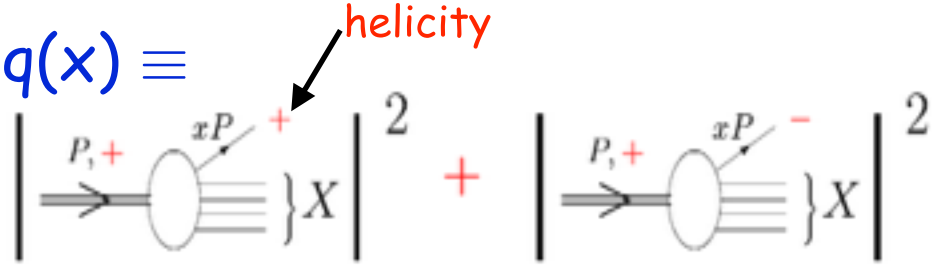

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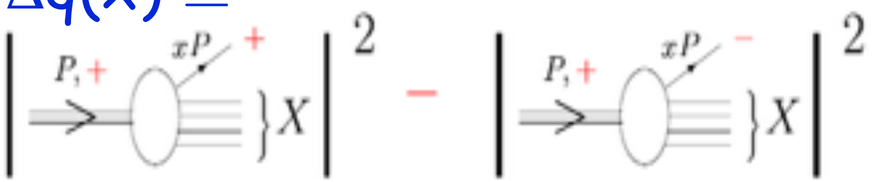

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unpolarized PDFs

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helicity-dep. PDFs

→ spin of the nucleon

towards renormalization group equations

so far: infinities related to long-time/distance physics (soft/collinear emissions)

these singularities cancel for infrared safe observables
or can be systematically removed (factorization) by “hiding” them
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loop momenta can be very large (=infinite)
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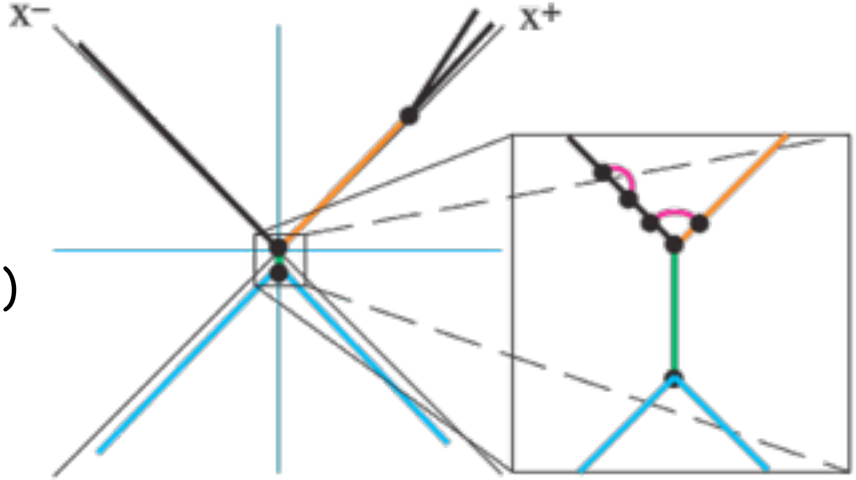

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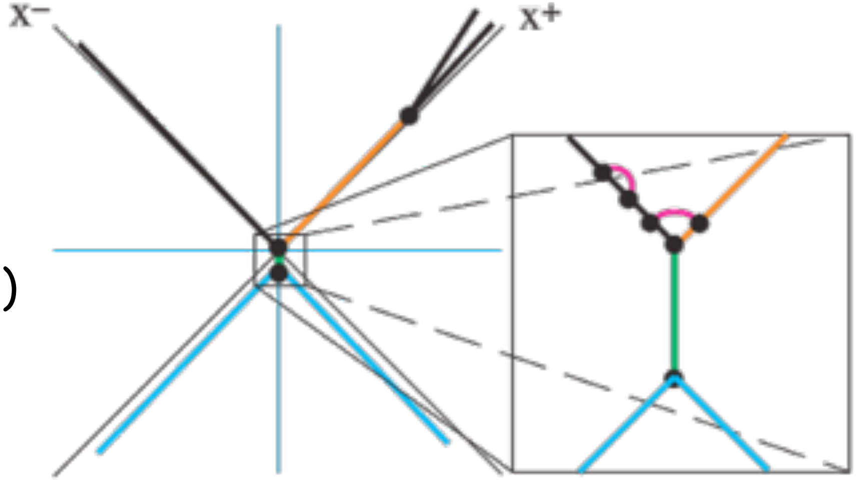

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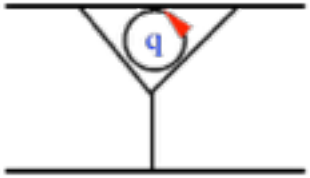
again, we need a suitable regulator for
divergent loop integrations:

$$\int_0^\infty d^4q$$



UV cut-off vs. **dim. regularization**

intuitive; not beyond NLO	involved; works to all orders

the importance of scales

factorization and renormalization play similar roles at opposite ends of the energy range of pQCD

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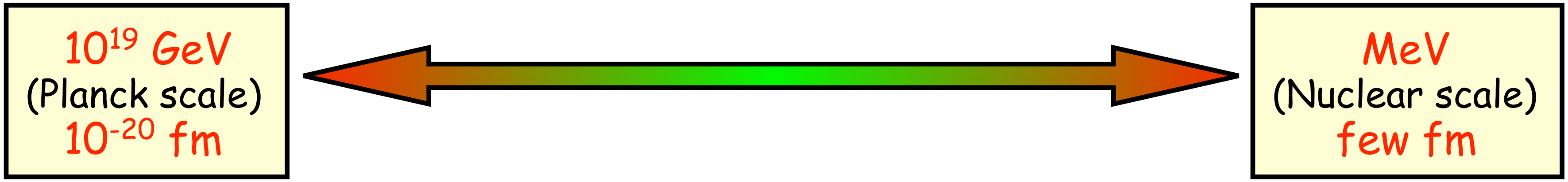

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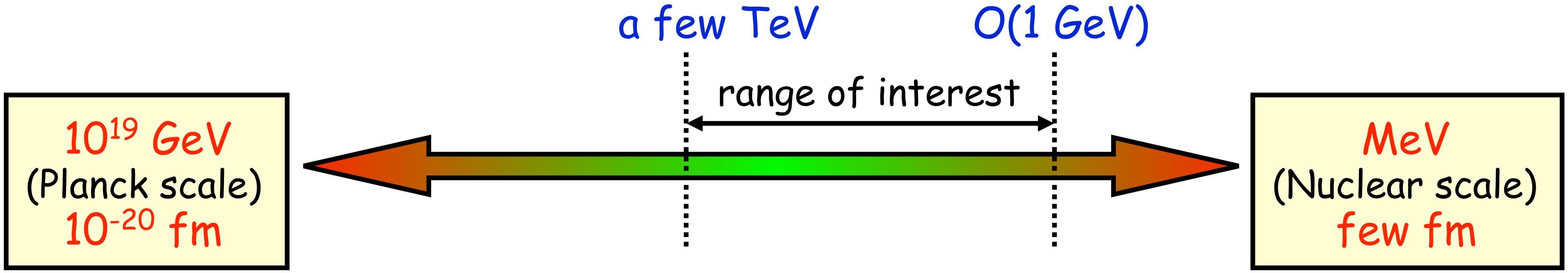

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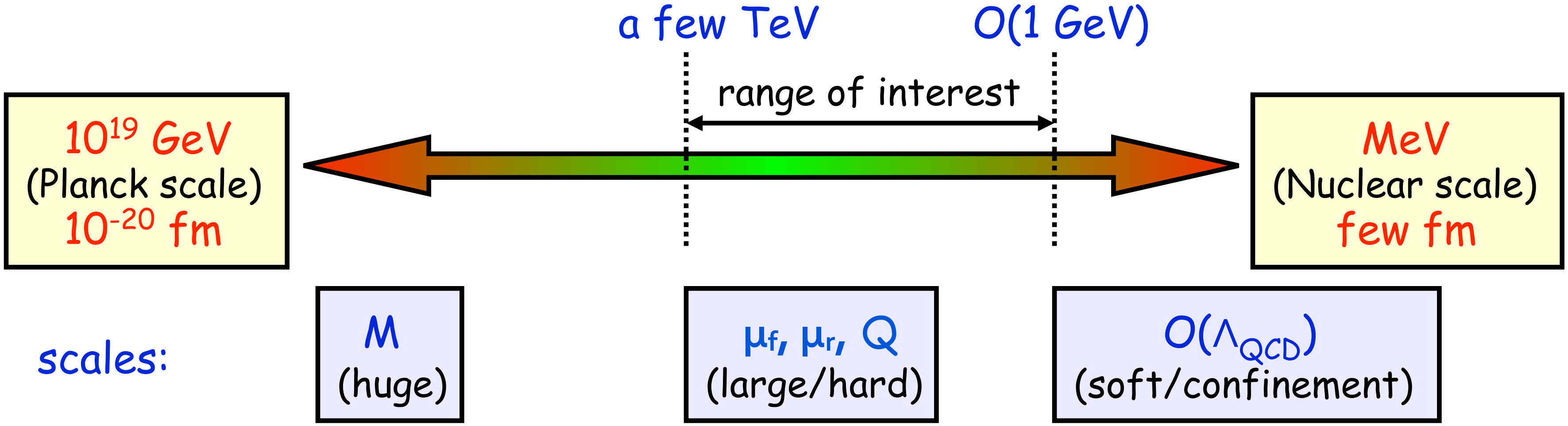

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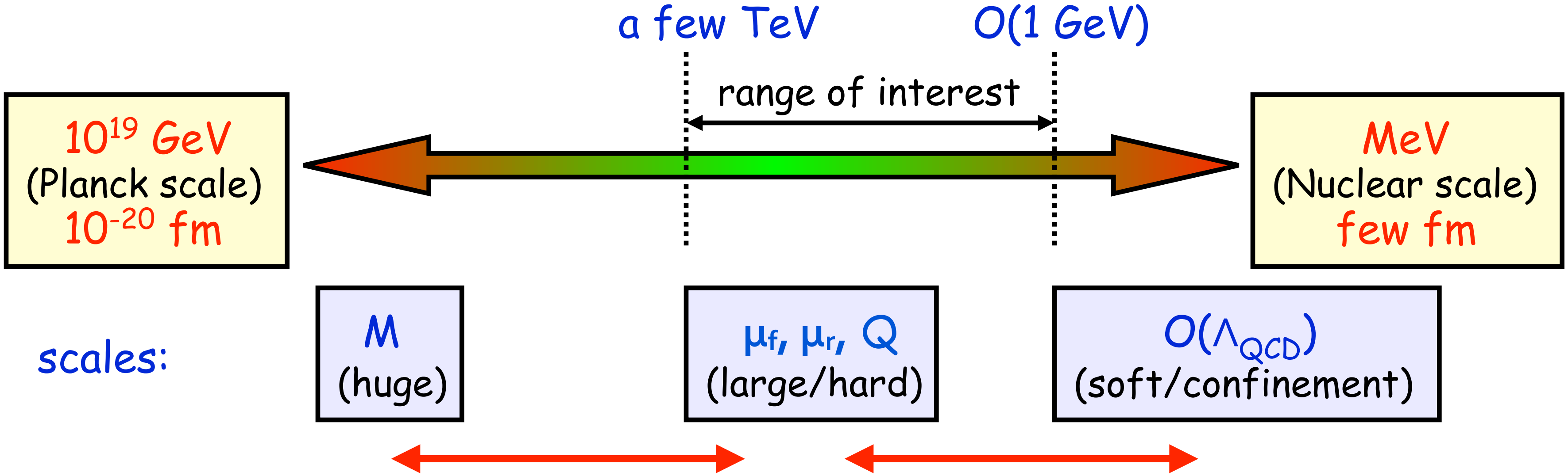

renormalization group equations (RGE) relate physics at diff. scales

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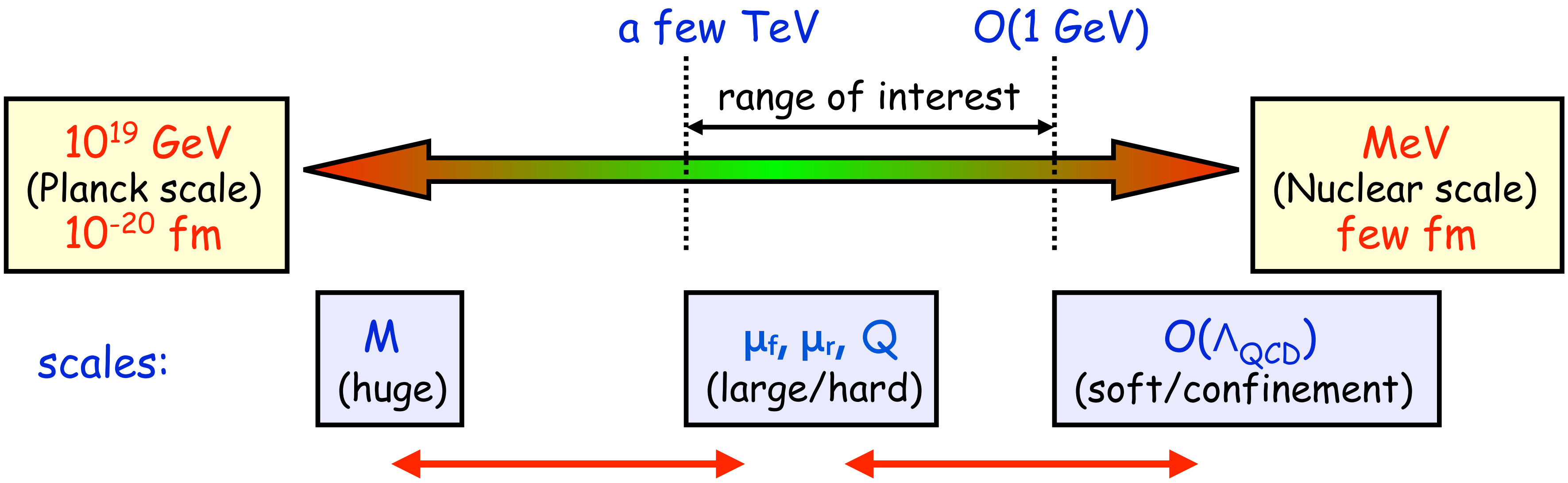

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hides our ignorance of
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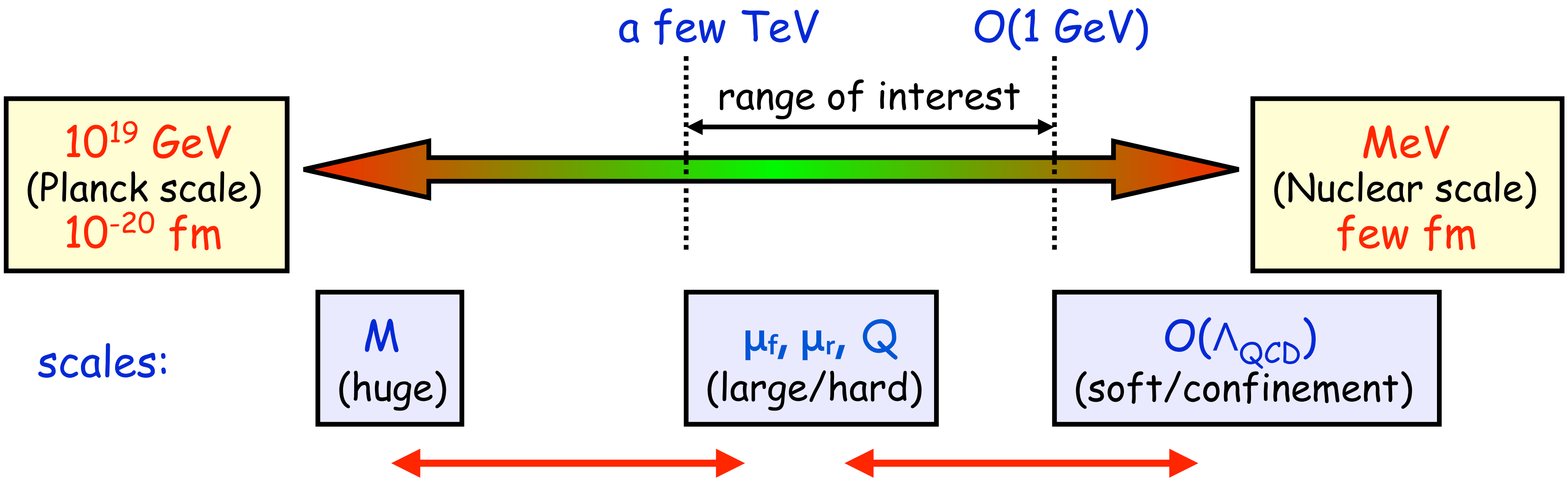

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IR/collinear factorization

hides non-perturbative QCD
at confinement scale in
$f_a(x,\mu_f), \Delta f_a(x,\mu_f), D_a^H(z,\mu_f), \ldots$

RGE: the swiss army knife of pQCD

we use α_s (and f_a, D_c^H) to absorb UV (IR) divergencies

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the physical idea behind this is beautiful & simple:

both scale parameters μ_f and μ_r are not intrinsic to QCD

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$$\mu_{r,f}\frac{d\sigma}{d\mu_{r,f}} = \frac{d\sigma}{d\ln\mu_{r,f}} = 0$$ ➡ **renormalization group equations**

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renormalization group equations

all we need is a reference measurement at some scale μ_0

scale evolution of α_s and parton densities

simplest example of RGE: running coupling α_s derived from $\frac{d\sigma}{d \ln \mu_r} = 0$

→ recall part II $$\frac{da_s}{d \ln \mu^2} = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - \beta_3 a_s^5 + \ldots \qquad a_s \equiv \frac{\alpha_s}{4\pi}$$

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scale dependence of PDFs: more complicated

simplified example:
F_2 for one quark flavor

$$F_2(x, Q^2) = q(x, \mu_f) \otimes \hat{F}_2(x, \frac{Q}{\mu_f})$$

physical quark pdf hard cross section

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$$\int_0^1 dx x^{n-1}\left[\int_x^1 \frac{dy}{y} f(y) g\left(\frac{x}{y}\right)\right] =$$

$$\int_0^1 dx x^{n-1} \int_0^1 dy \int_0^1 dz\, \delta(x-zy) f(y) g(z) = f(n)\, g(n)$$

simplest example of DGLAP evolution

Dokshitzer; Gribov, Lipatov; Altarelli, Parisi

now we can compute $\frac{dF_2(x,Q^2)}{d\ln\mu_f} = 0$

$$\longleftrightarrow \qquad \frac{dq(n,\mu_f)}{d\ln\mu_f}\hat{F}_2(n,\frac{Q}{\mu_f}) + q(n,\mu_f)\frac{d\hat{F}_2(n,\frac{Q}{\mu_f})}{d\ln\mu_f} = 0$$

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→ **once we know the PDFs at a scale μ_0 we can predict them at $\mu > \mu_0$**

factorization → evolution → resummation

physical interpretation of the evolution eqs.:

RGE resums collinear emissions to all orders

factorization → evolution → resummation

physical interpretation of the evolution eqs.:

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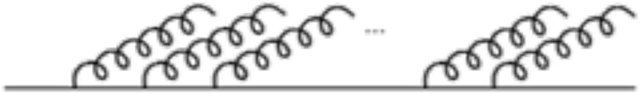
- to see this expand the solution in α_s:



$$\exp[\ldots] = 1 + \frac{\alpha_s}{2\pi} P_{qq}(n) \ln \frac{\mu_f}{\mu_0} + \frac{1}{2}\left[\frac{\alpha_s}{2\pi} P_{qq}(n) \ln \frac{\mu_f}{\mu_0}\right]^2 + \ldots$$

factorization $\to$ evolution $\to$ resummation

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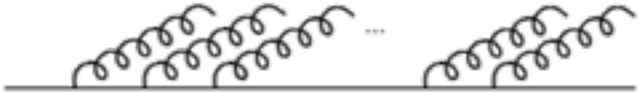

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- the **splitting functions** $P_{ij}(n)$ **or** $P_{ij}(x)$ multiplying the log's are universal and calculable in pQCD order by order in α_s

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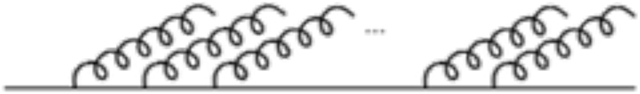

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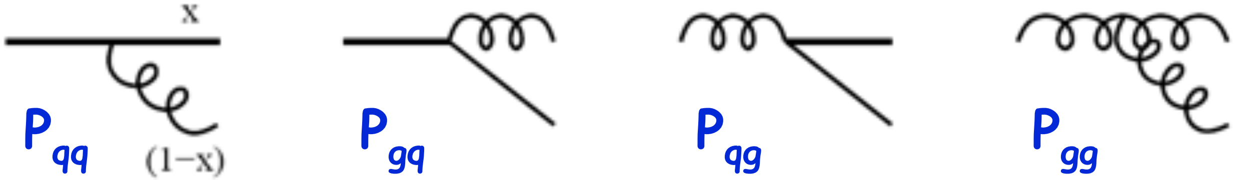
- the **splitting functions** $P_{ij}(n)$ **or** $P_{ij}(x)$ multiplying the log's are universal and calculable in pQCD order by order in α_s
- the physical meaning of the splitting functions is easy:

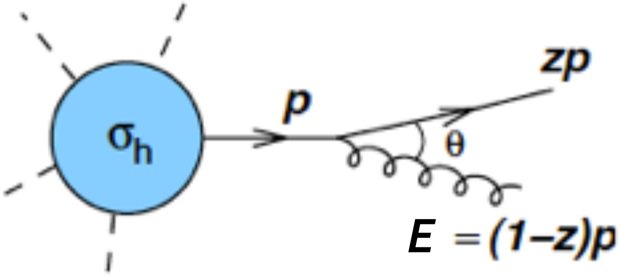
$P_{ij}(x)$: probability that a parton j splits collinearly into a parton i (and something) carrying a momentum fraction x



factorization recap: final-state vs initial-state

recall what we learned for **final-state radiation**

$$\sigma_{h+g} \simeq \sigma_h \frac{\alpha_s C_F}{\pi} \frac{dE}{E} \frac{d\theta^2}{\theta^2}$$



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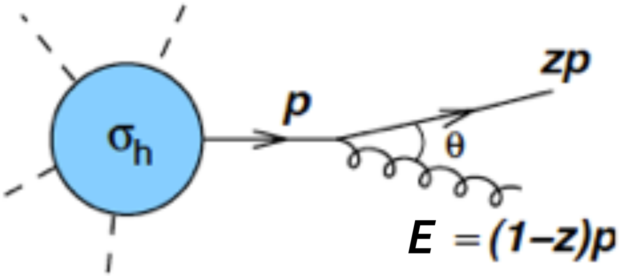

$$\sigma_{h+g} \simeq \sigma_h \frac{\alpha_s C_F}{\pi} \frac{dE}{E} \frac{d\theta^2}{\theta^2}$$

and rewrite in terms of new variable k_T

$$\sigma_{h+g} \simeq \sigma_h \frac{\alpha_s C_F}{\pi} \frac{dz}{1-z} \frac{dk_t^2}{k_t^2}$$

where we have used

$$\mathbf{E} = (\mathbf{1} - \mathbf{z})\mathbf{p}$$
$$\mathbf{k_T} = \mathbf{E} \sin\theta \simeq \mathbf{E}\theta$$

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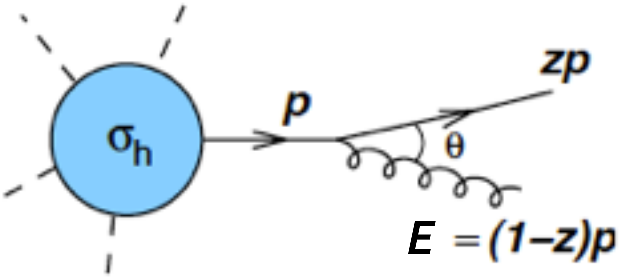

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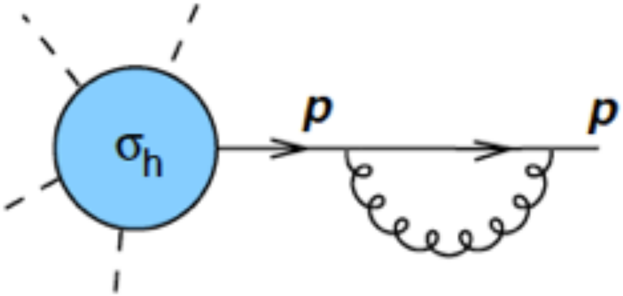
KLN: if we avoid distinguishing quark and collinear quark-gluon final-states (like for **jets**) divergencies cancel against virtual corrections

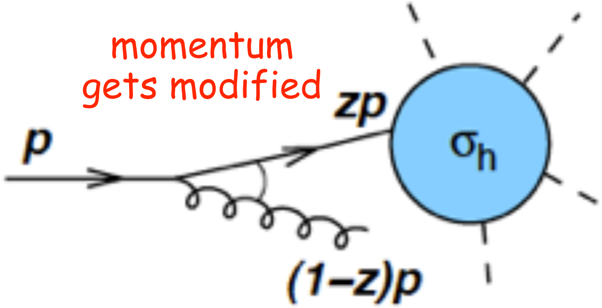


$$\sigma_{h+V} \simeq -\sigma_h \frac{\alpha_s C_F}{\pi} \frac{dz}{1-z} \frac{dk_t^2}{k_t^2}$$

factorization recap: initial-state peculiarities

initial-state radiation: crucial difference – hard scattering happens after splitting

$$\sigma_{g+h}(p) \simeq \sigma_h(zp) \frac{\alpha_s C_F}{\pi} \frac{dz}{1-z} \frac{dk_t^2}{k_t^2}$$


momentum gets modified

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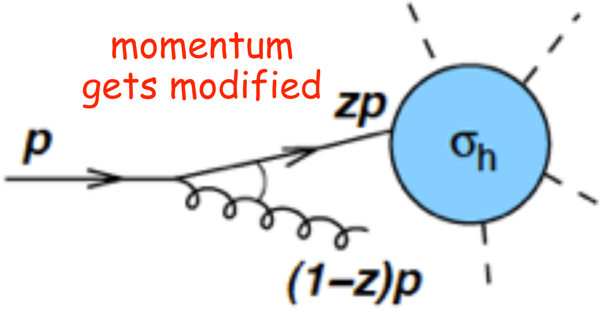
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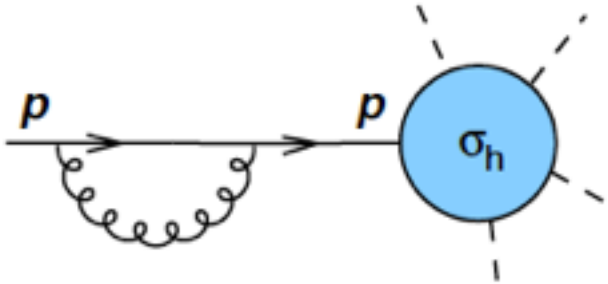
but for the virtual piece the momentum is unchanged



$$\sigma_{V+h}(p) \simeq -\sigma_h(p)\frac{\alpha_s C_F}{\pi}\frac{dz}{1-z}\frac{dk_t^2}{k_t^2}$$

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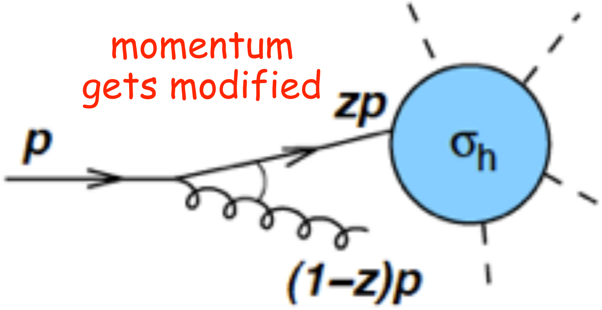

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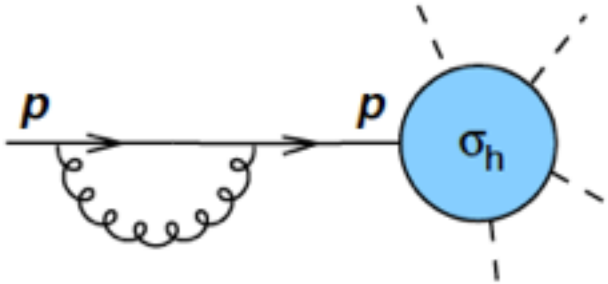

$$\sigma_{V+h}(p) \simeq -\sigma_h(p)\frac{\alpha_s C_F}{\pi}\frac{dz}{1-z}\frac{dk_t^2}{k_t^2}$$

hence, the sum receives two contributions with different momenta

$$\sigma_{g+h} + \sigma_{V+h} \simeq \frac{\alpha_s C_F}{\pi}\int\frac{dk_t^2}{k_t^2}\frac{dz}{1-z}[\sigma_h(zp) - \sigma_h(p)]$$

disclaimer: we assume that $k_T \ll Q$ (large) to ignore other transverse momenta

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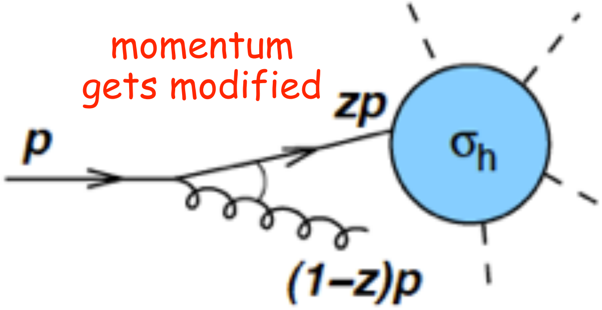

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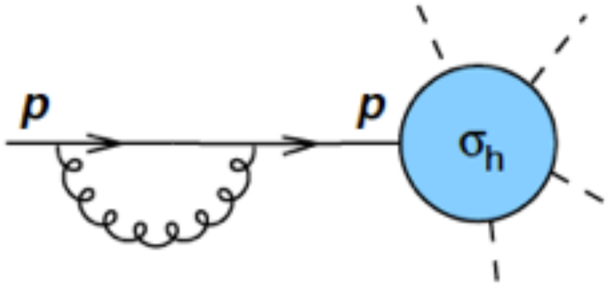

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leads to uncanceled collinear singularity

disclaimer: we assume that $k_T \ll Q$ (large) to ignore other transverse momenta

factorization revisited: collinear singularity

$$\sigma_{g+h} + \sigma_{V+h} \simeq \frac{\alpha_s C_F}{\pi} \underbrace{\int_0^{Q^2} \frac{dk_t^2}{k_t^2}}_{\text{infinite}} \underbrace{\int \frac{dz}{1-z} [\sigma_h(zp) - \sigma_h(p)]}_{\text{finite}}$$

- z=1: soft divergence cancels (KLN) as $\sigma_h(zp) - \sigma_h(p) \to 0$
- arbitrary z: $\sigma_h(zp) - \sigma_h(p) \neq 0$ but z integration is finite
- but k_T integration always diverges (at lower limit)

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reflects collinear singularity

cross sections with incoming partons not collinear safe

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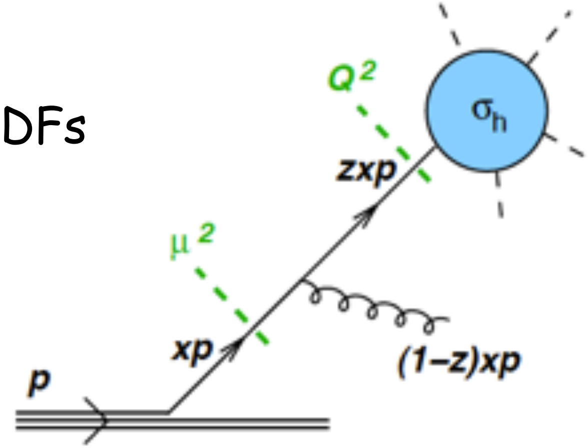
reflects collinear singularity

cross sections with incoming partons not collinear safe

factorization = collinear "cut-off"

- absorb divergent small k_T region in non-perturbative PDFs

$$\sigma_1 \simeq \frac{\alpha_s C_F}{\pi} \underbrace{\int_{\mu^2}^{Q^2} \frac{dk_t^2}{k_t^2}}_{\text{finite (large)}} \underbrace{\int \frac{dx\, dz}{1-z} [\sigma_h(zxp) - \sigma_h(xp)]\, q(x,\mu^2)}_{\text{finite}}$$



anatomy of splitting functions

splitting functions may receive two kinds of contributions:

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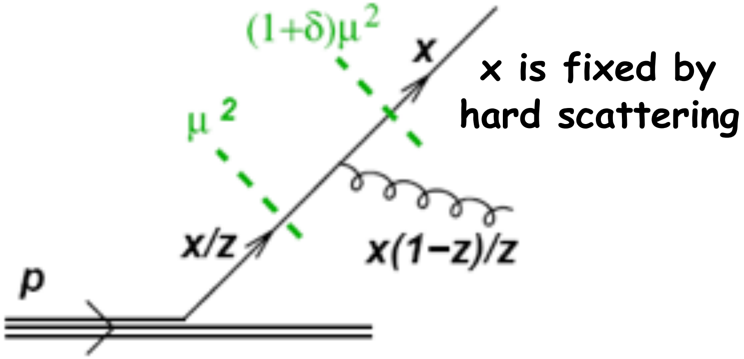

real emission
"something happens"

$$\frac{dq(x,\mu^2)}{d\ln\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 dz P_{qq}(z)\frac{q(x/z,\mu^2)}{z}$$

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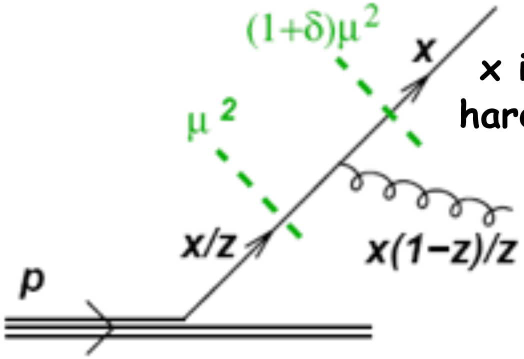

x is fixed by hard scattering

real emission
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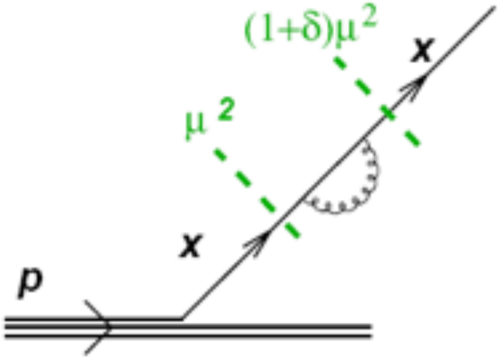
+



virtual emission
"nothing happens"

$$\frac{dq(x,\mu^2)}{d\ln\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 dz P_{qq}(z)\frac{q(x/z,\mu^2)}{z} - \frac{\alpha_s}{2\pi}\int_0^1 dz P_{qq}(z) q(x,\mu^2)$$

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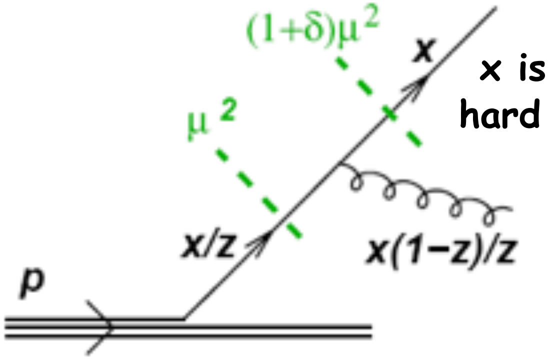

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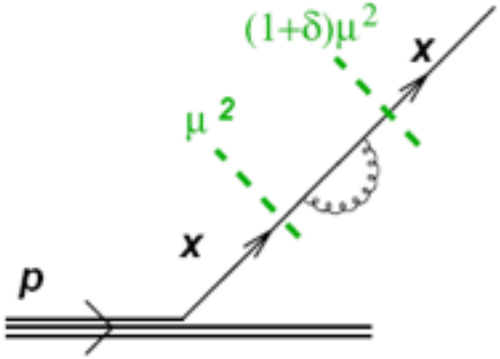

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combine !

$$\frac{dq(x,\mu^2)}{d\ln\mu^2} = \frac{\alpha_s}{2\pi}\underbrace{\int_x^1 dz\, P_{qq}(z)\frac{q(x/z,\mu^2)}{z}}_{P_{qq}\otimes q}$$

$$P_{qq}(z) = C_F\left(\frac{1+z^2}{1-z}\right)_+$$

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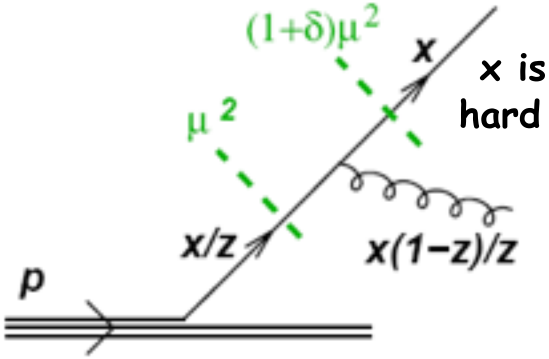

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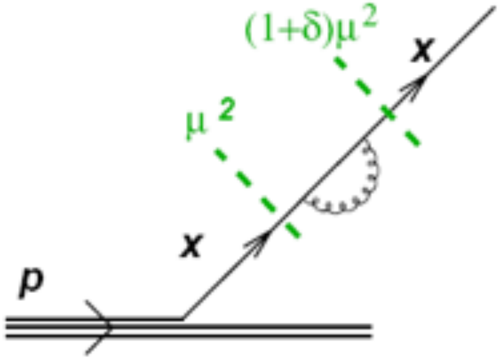

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involves "**plus distribution**"

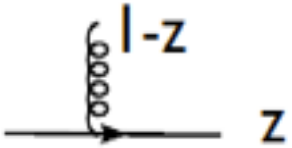
$$\int_0^1 dz\,[g(z)]_+ f(z) \equiv \int_0^1 dz\, g(z)\,[f(z) - f(1)]$$

condition: f(z) sufficiently smooth for z→ 1

properties of LO splitting functions

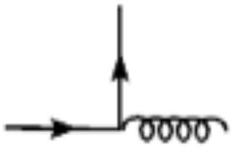
in general, quarks and gluons can split into quarks and gluons -> 4 functions

$$P_{qq}^{(0)} = P_{\bar{q}\bar{q}}^{(0)} = C_F \left[\frac{\mathbf{1+z^2}}{\mathbf{(1-z)}_+} + \frac{3}{2}\delta(1-z) \right]$$

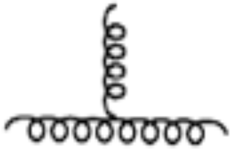


$$P_{qg}^{(0)} = P_{\bar{q}g}^{(0)} = T_R \left(z^2 + (1-z) \right)$$

$$P_{gq}^{(0)} = P_{g\bar{q}}^{(0)} = C_F \frac{1+(1-z)^2}{z}$$



$$P_{gg}^{(0)} = 2C_A \left[z \left(\frac{1}{1-z} \right)_+ + \frac{1-z}{z} + z(1-z) + b_0 \delta(1-z) \right]$$



in higher orders more complicated, as $\mathbf{P_{q_i q_j} \neq 0}$ arise

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soft gluon divergence (z=1)
regulated by plus distribution

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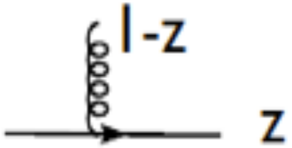

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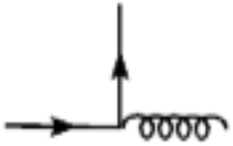

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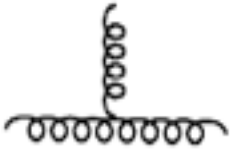

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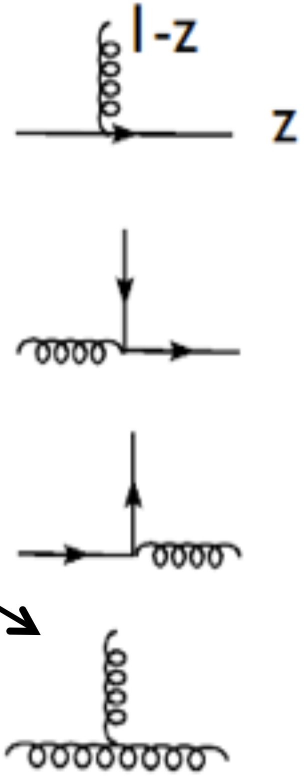
soft gluon divergence (z=1)
regulated by plus distribution

symmetric under
z -> (1-z)
except virtuals



in higher orders more complicated, as $\mathbf{P_{q_iq_j} \neq 0}$ arise

reaching for precision

$$\begin{aligned}
P^{(0)}_{\mathrm{ns}}(x) &= C_F\left(2p_{\mathrm{qq}}(x)+3\delta(1-x)\right) \\
P^{(0)}_{\mathrm{ps}}(x) &= 0 \\
P^{(0)}_{\mathrm{qg}}(x) &= 2n_f\, p_{\mathrm{qg}}(x) \\
P^{(0)}_{\mathrm{gq}}(x) &= 2C_F\, p_{\mathrm{gq}}(x) \\
P^{(0)}_{\mathrm{gg}}(x) &= C_A\left(4p_{\mathrm{gg}}(x)+\frac{11}{3}\delta(1-x)\right)-\frac{2}{3}n_f\,\delta(1-x)
\end{aligned}$$

LO: 1973

reaching for precision

$$
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\end{aligned}
$$

LO: 1973

$$
\begin{aligned}
P^{(1)+}_{\mathrm{ns}}(x) &= 4C_AC_F\Big(p_{\mathrm{qq}}(x)\Big[\frac{67}{18}-\zeta_2+\frac{11}{6}\mathrm{H}_0+\mathrm{H}_{0,0}\Big]+p_{\mathrm{qq}}(-x)\Big[\zeta_2+2\mathrm{H}_{-1,0}-\mathrm{H}_{0,0}\Big]\\
&\quad+\frac{14}{3}(1-x)+\delta(1-x)\Big[\frac{17}{24}+\frac{11}{3}\zeta_2-3\zeta_3\Big]\Big)-4C_Fn_f\Big(p_{\mathrm{qq}}(x)\Big[\frac{5}{9}+\frac{1}{3}\mathrm{H}_0\Big]+\frac{2}{3}(1-x)\\
&\quad+\delta(1-x)\Big[\frac{1}{12}+\frac{2}{3}\zeta_2\Big]\Big)+4C_F{}^2\Big(2p_{\mathrm{qq}}(x)\Big[\mathrm{H}_{1,0}-\frac{3}{4}\mathrm{H}_0+\mathrm{H}_2\Big]-2p_{\mathrm{qq}}(-x)\Big[\zeta_2+2\mathrm{H}_{-1,0}\\
&\quad-\mathrm{H}_{0,0}\Big]-(1-x)\Big[1-\frac{3}{2}\mathrm{H}_0\Big]-\mathrm{H}_0-(1+x)\mathrm{H}_{0,0}+\delta(1-x)\Big[\frac{3}{8}-3\zeta_2+6\zeta_3\Big]\Big)\\
P^{(1)-}_{\mathrm{ns}}(x) &= P^{(1)+}_{\mathrm{ns}}(x)+16C_F\Big(C_F-\frac{C_A}{2}\Big)\Big(p_{\mathrm{qq}}(-x)\Big[\zeta_2+2\mathrm{H}_{-1,0}-\mathrm{H}_{0,0}\Big]-2(1-x)\\
&\quad-(1+x)\mathrm{H}_0\Big)
\end{aligned}
$$

$$
P^{(1)}_{\mathrm{ps}}(x) = 4C_Fn_f\Big(\frac{20}{9}\frac{1}{x}-2+6x-4\mathrm{H}_0+x^2\Big[\frac{8}{3}\mathrm{H}_0-\frac{56}{9}\Big]+(1+x)\Big[5\mathrm{H}_0-2\mathrm{H}_{0,0}\Big]\Big)
$$

$$
\begin{aligned}
P^{(1)}_{\mathrm{qg}}(x) &= 4C_An_f\Big(\frac{20}{9}\frac{1}{x}-2+25x-2p_{\mathrm{qg}}(-x)\mathrm{H}_{-1,0}-2p_{\mathrm{qg}}(x)\mathrm{H}_{1,1}+x^2\Big[\frac{44}{3}\mathrm{H}_0-\frac{218}{9}\Big]\\
&\quad+4(1-x)\Big[\mathrm{H}_{0,0}-2\mathrm{H}_0+x\mathrm{H}_1\Big]-4\zeta_2x-6\mathrm{H}_{0,0}+9\mathrm{H}_0\Big)+4C_Fn_f\Big(2p_{\mathrm{qg}}(x)\Big[\mathrm{H}_{1,0}+\mathrm{H}_{1,1}+\mathrm{H}_2\\
&\quad-\zeta_2\Big]+4x^2\Big[\mathrm{H}_0+\mathrm{H}_{0,0}+\frac{5}{2}\Big]+2(1-x)\Big[\mathrm{H}_0+\mathrm{H}_{0,0}-2x\mathrm{H}_1+\frac{29}{4}\Big]-\frac{15}{2}-\mathrm{H}_{0,0}-\frac{1}{2}\mathrm{H}_0\Big)
\end{aligned}
$$

$$
\begin{aligned}
P^{(1)}_{\mathrm{gq}}(x) &= 4C_AC_F\Big(\frac{1}{x}+2p_{\mathrm{gq}}(x)\Big[\mathrm{H}_{1,0}+\mathrm{H}_{1,1}+\mathrm{H}_2-\frac{11}{6}\mathrm{H}_1\Big]-x^2\Big[\frac{8}{3}\mathrm{H}_0-\frac{44}{9}\Big]+4\zeta_2-2\\
&\quad-7\mathrm{H}_0+2\mathrm{H}_{0,0}-2\mathrm{H}_1x+(1+x)\Big[2\mathrm{H}_{0,0}-5\mathrm{H}_0+\frac{37}{9}\Big]-2p_{\mathrm{gq}}(-x)\mathrm{H}_{-1,0}\Big)-4C_Fn_f\Big(\frac{2}{3}x\\
&\quad-p_{\mathrm{gq}}(x)\Big[\frac{2}{3}\mathrm{H}_1-\frac{10}{9}\Big]\Big)+4C_F{}^2\Big(p_{\mathrm{gq}}(x)\Big[3\mathrm{H}_1-2\mathrm{H}_{1,1}\Big]+(1+x)\Big[\mathrm{H}_{0,0}-\frac{7}{2}+\frac{7}{2}\mathrm{H}_0\Big]-3\mathrm{H}_{0,0}\\
&\quad+1-\frac{3}{2}\mathrm{H}_0+2\mathrm{H}_1x\Big)
\end{aligned}
$$

$$
\begin{aligned}
P^{(1)}_{\mathrm{gg}}(x) &= 4C_An_f\Big(1-x-\frac{10}{9}p_{\mathrm{gg}}(x)-\frac{13}{9}\Big(\frac{1}{x}-x^2\Big)-\frac{2}{3}(1+x)\mathrm{H}_0-\frac{2}{3}\delta(1-x)\Big)+4C_A{}^2\Big(27\\
&\quad+(1+x)\Big[\frac{11}{3}\mathrm{H}_0+8\mathrm{H}_{0,0}-\frac{27}{2}\Big]+2p_{\mathrm{gg}}(-x)\Big[\mathrm{H}_{0,0}-2\mathrm{H}_{-1,0}-\zeta_2\Big]-\frac{67}{9}\Big(\frac{1}{x}-x^2\Big)-12\mathrm{H}_0\\
&\quad-\frac{44}{3}x^2\mathrm{H}_0+2p_{\mathrm{gg}}(x)\Big[\frac{67}{18}-\zeta_2+\mathrm{H}_{0,0}+2\mathrm{H}_{1,0}+2\mathrm{H}_2\Big]+\delta(1-x)\Big[\frac{8}{3}+3\zeta_3\Big]\Big)+4C_Fn_f\Big(2\mathrm{H}_0\\
&\quad+\frac{2}{3}\frac{1}{x}+\frac{10}{3}x^2-12+(1+x)\Big[4-5\mathrm{H}_0-2\mathrm{H}_{0,0}\Big]-\frac{1}{2}\delta(1-
\end{aligned}
$$

Curci, Furmanski, Petronzio;
Floratos et al., …

NLO: 1980

P_{ij} @ NNLO: a landmark calculation

10000 diagrams, 10^5 integrals, 10 man years, and several CPU years later:

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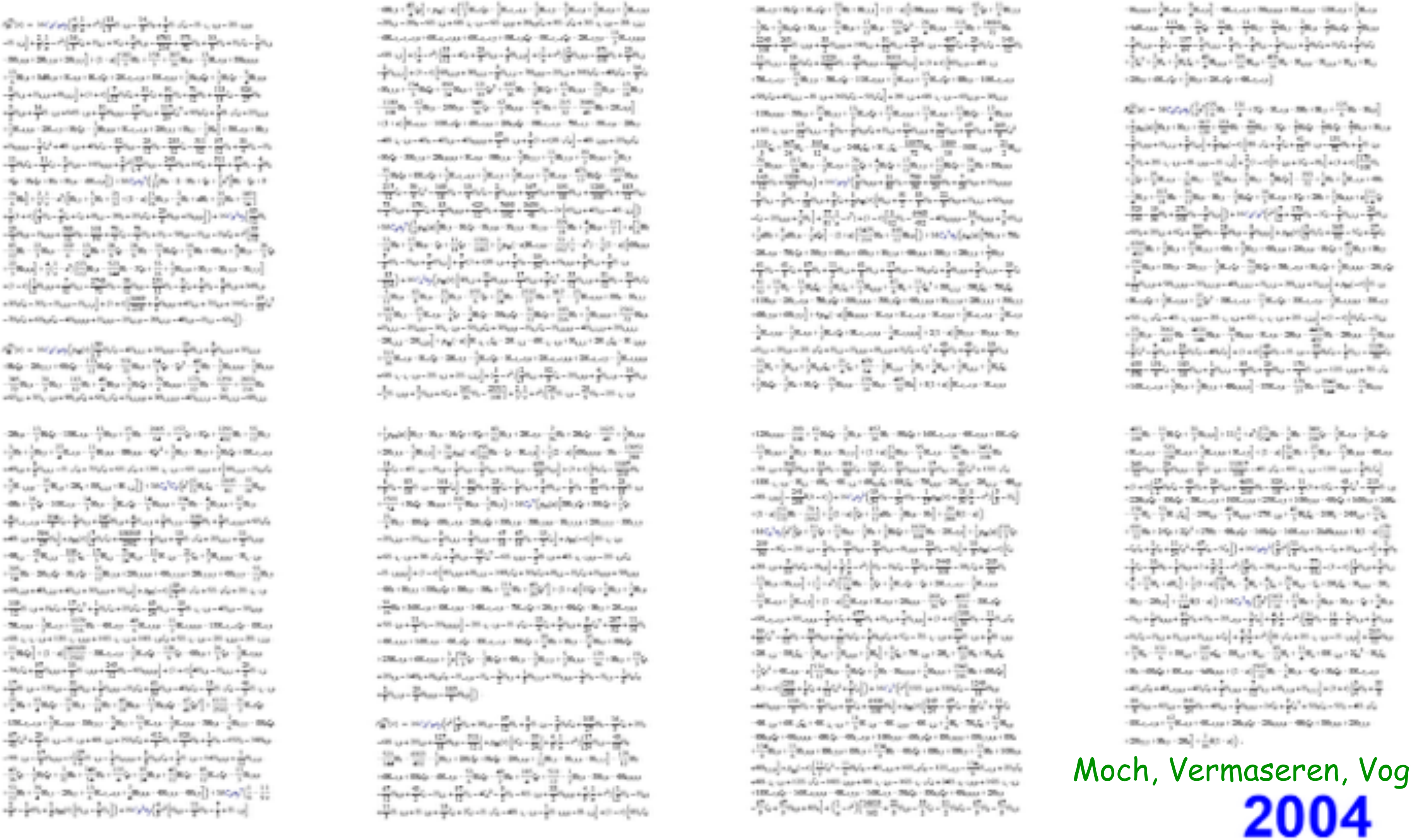

Moch, Vermaseren, Vogt

2004

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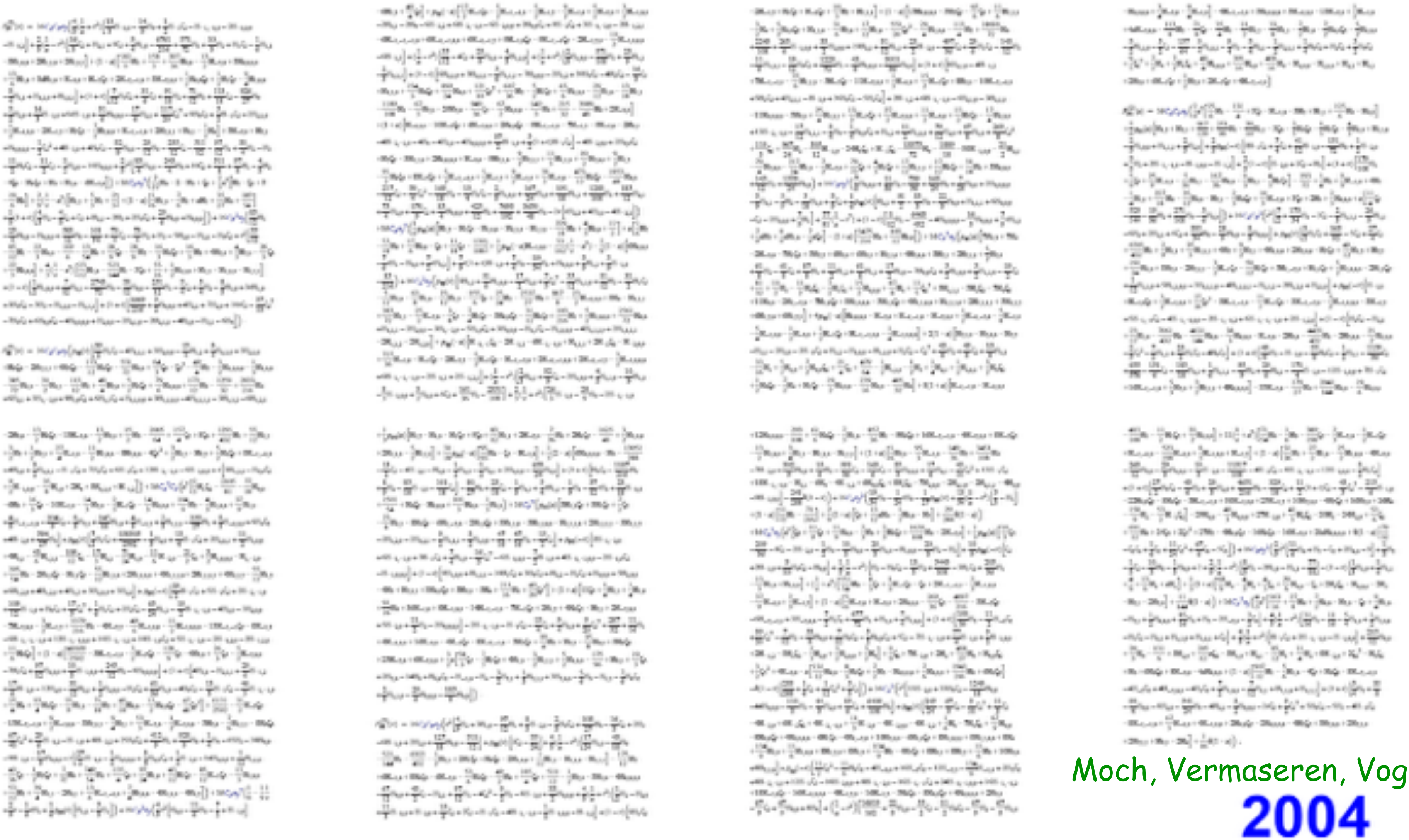

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NNLO the new emerging standard in QCD – essential for precision physics

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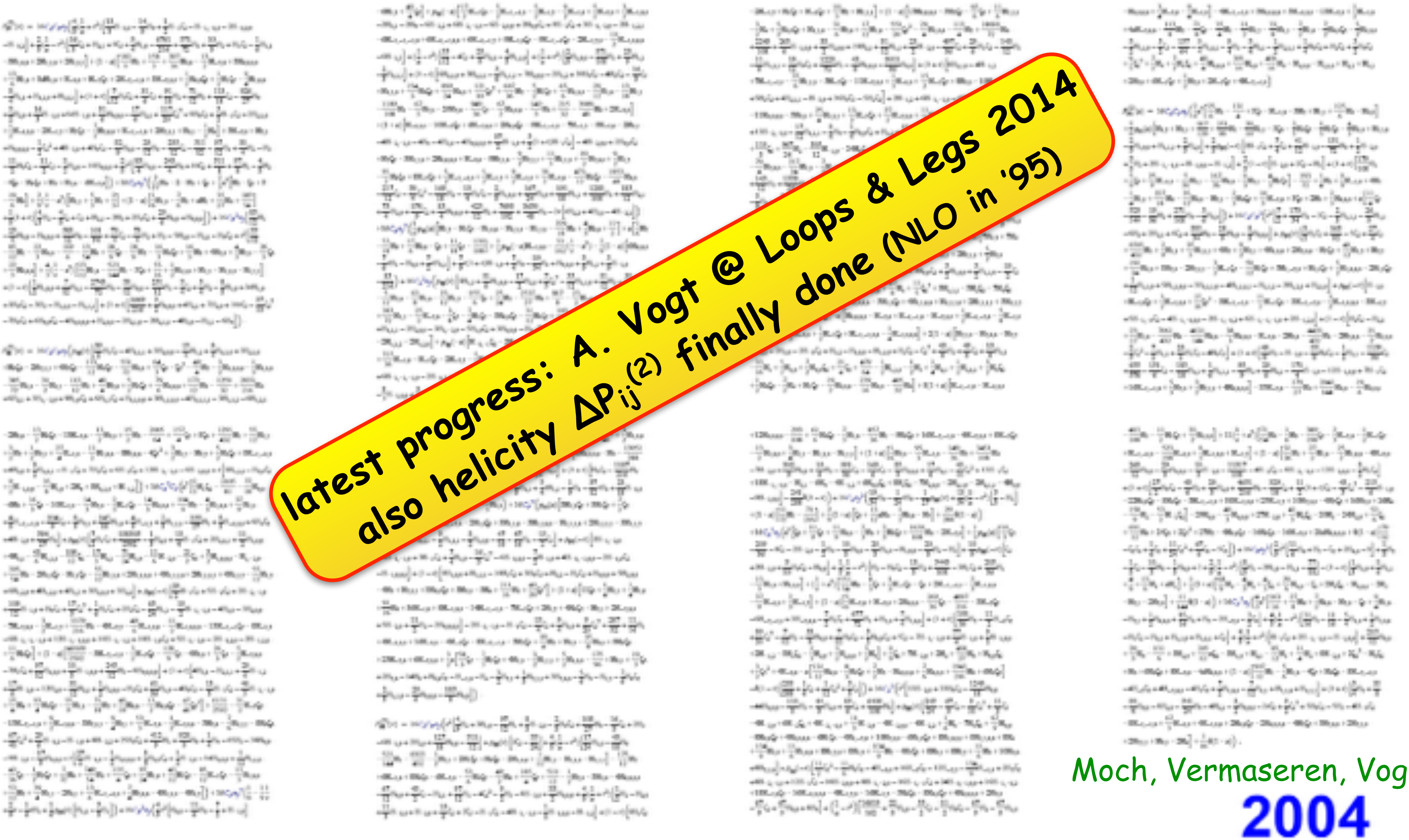


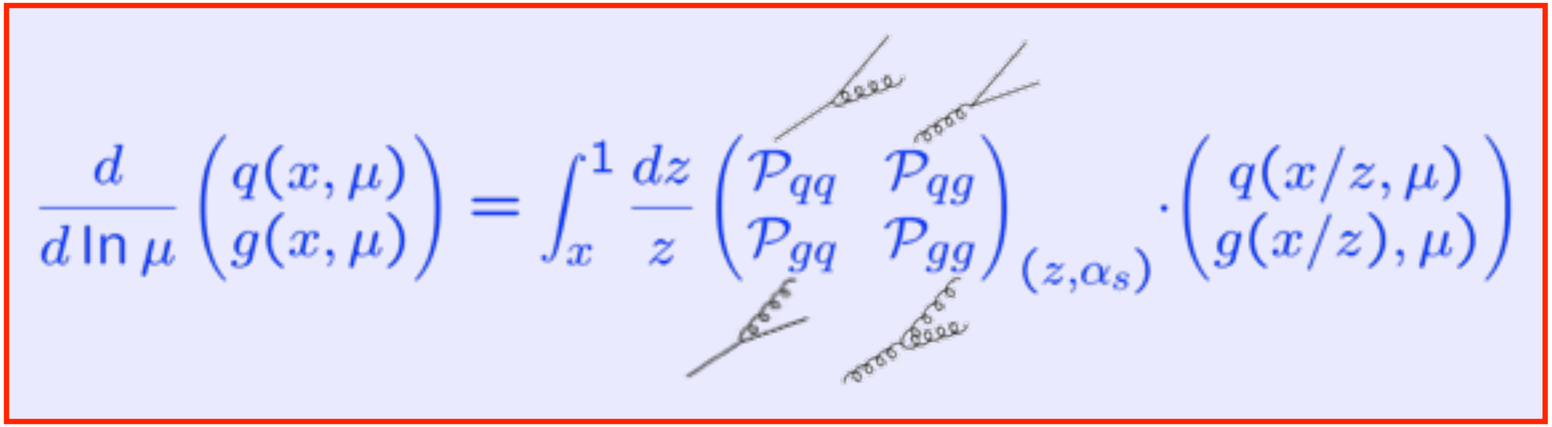
latest progress: A. Vogt @ Loops & Legs 2014
also helicity $\Delta P_{ij}^{(2)}$ finally done (NLO in '95)

Moch, Vermaseren, Vogt
2004

NNLO the new emerging standard in QCD – essential for precision physics

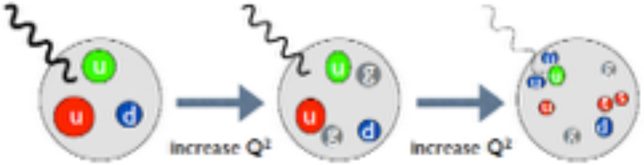
DGLAP evolution in full glory

taking quarks and gluons together: coupled integro-differential equations



$$\frac{d}{d\ln\mu}\begin{pmatrix} q(x,\mu)\\ g(x,\mu)\end{pmatrix} = \int_x^1 \frac{dz}{z}\begin{pmatrix}\mathcal{P}_{qq} & \mathcal{P}_{qg}\\ \mathcal{P}_{gq} & \mathcal{P}_{gg}\end{pmatrix}_{(z,\alpha_s)} \cdot \begin{pmatrix} q(x/z,\mu)\\ g(x/z),\mu)\end{pmatrix}$$

best solved in Mellin moment space: set of ordinary differential eqs.;
no closed solution in exp. form beyond LO (commutators of P matrices!)



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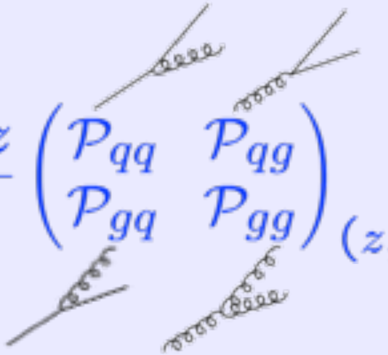

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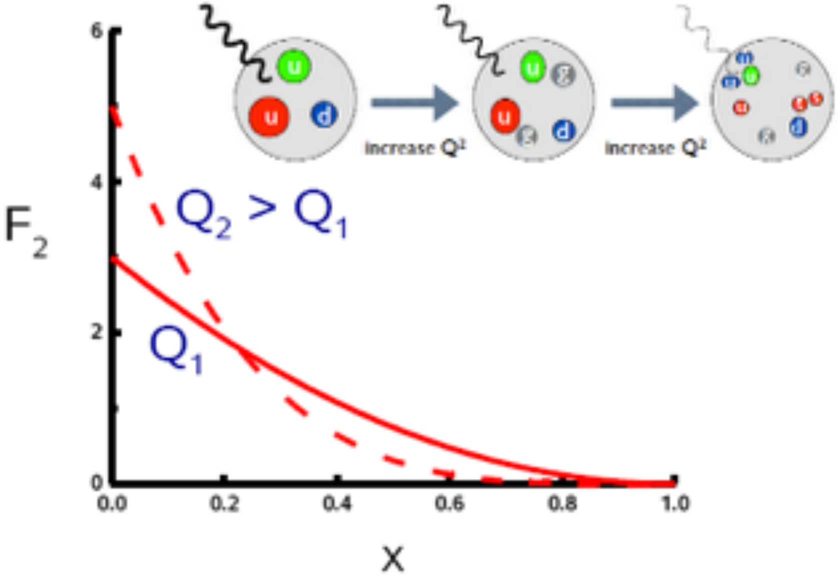
main effect/prediction of evolution:

partons loose energy by evolution!

- large x depletion
- small x increase



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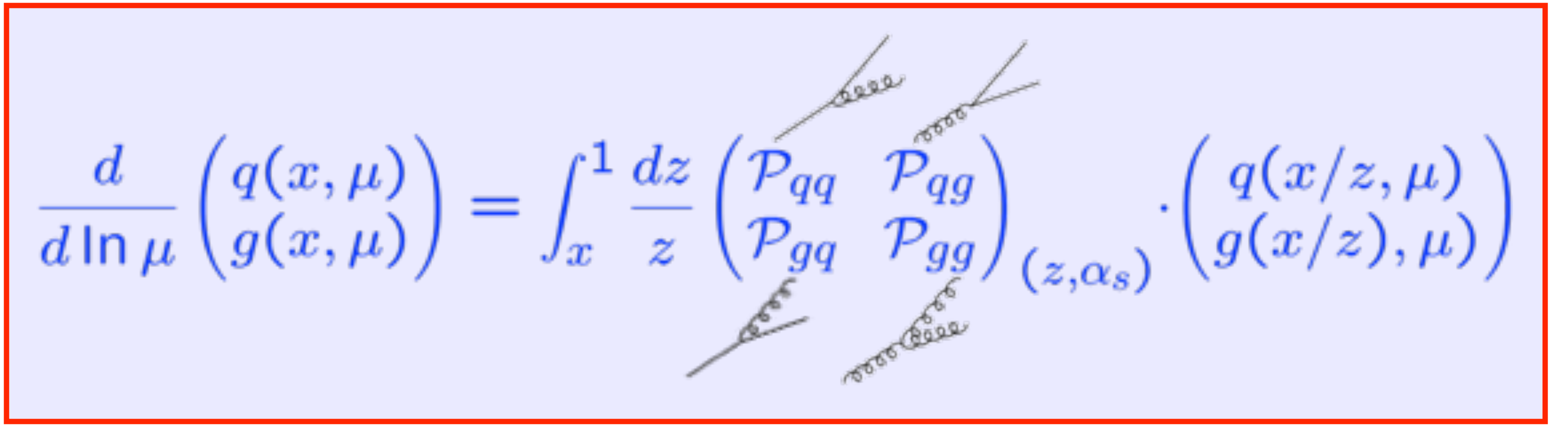

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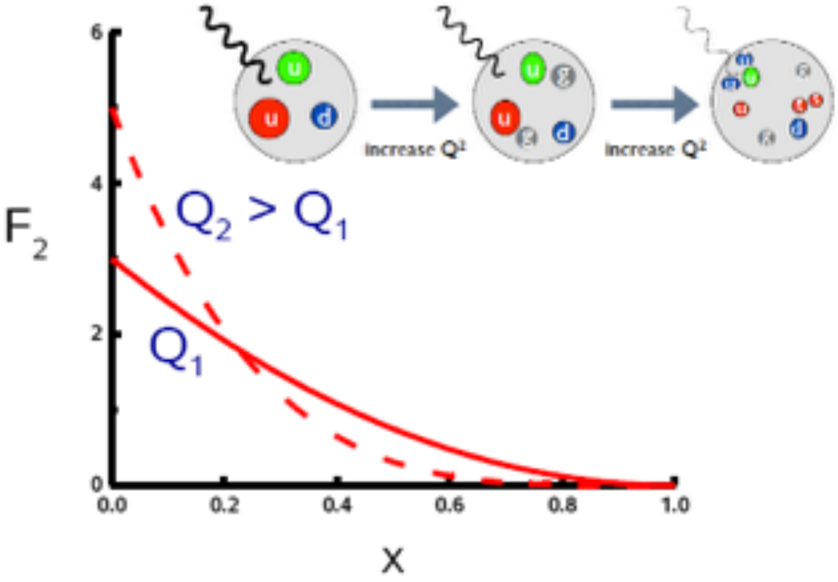
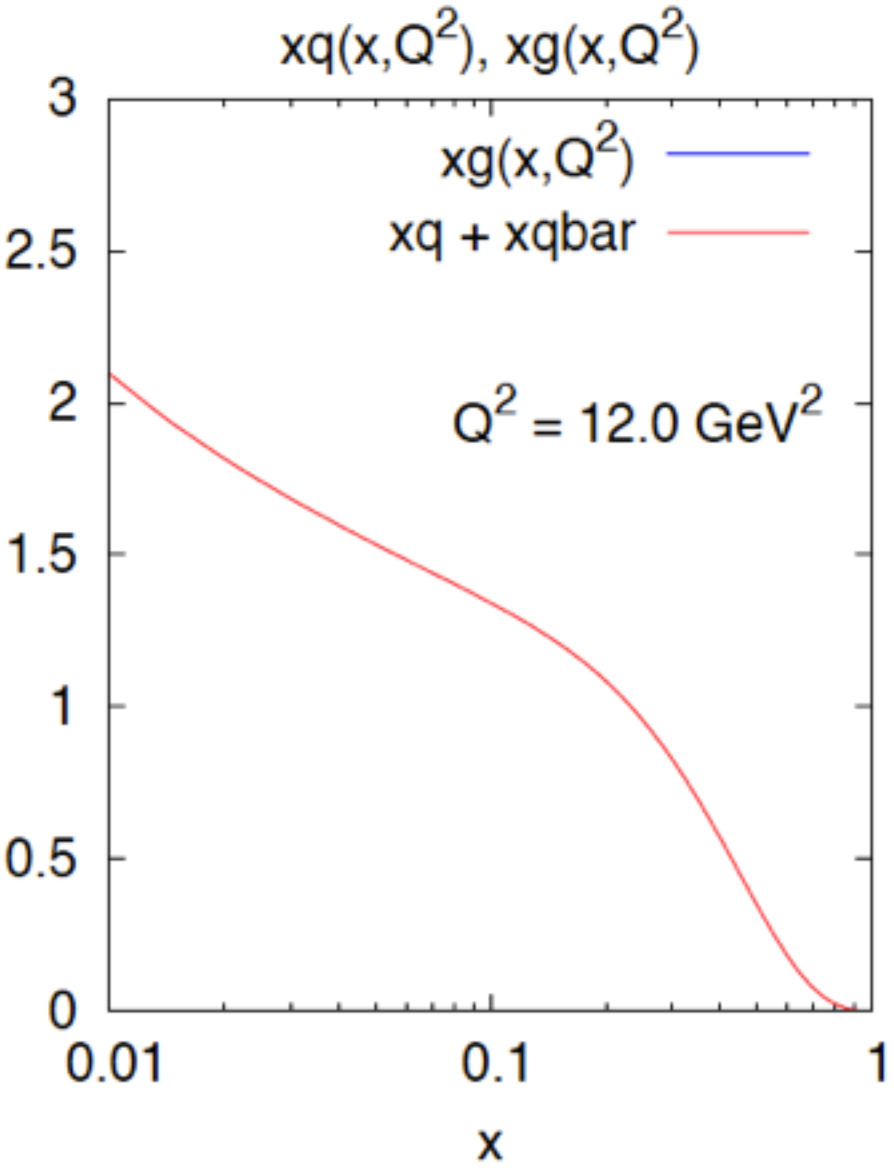
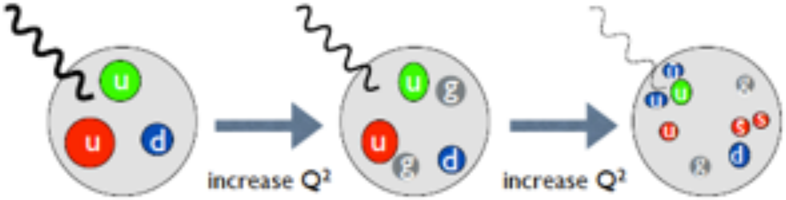
exactly as observed in experiment
huge success of pQCD



DGLAP evolution at work: toy example





start off from just quarks, no gluons

- quarks reduced at large x
- gluons rise quickly at small x
  (which, btw, also generates sea quarks)

taken from G. Salam

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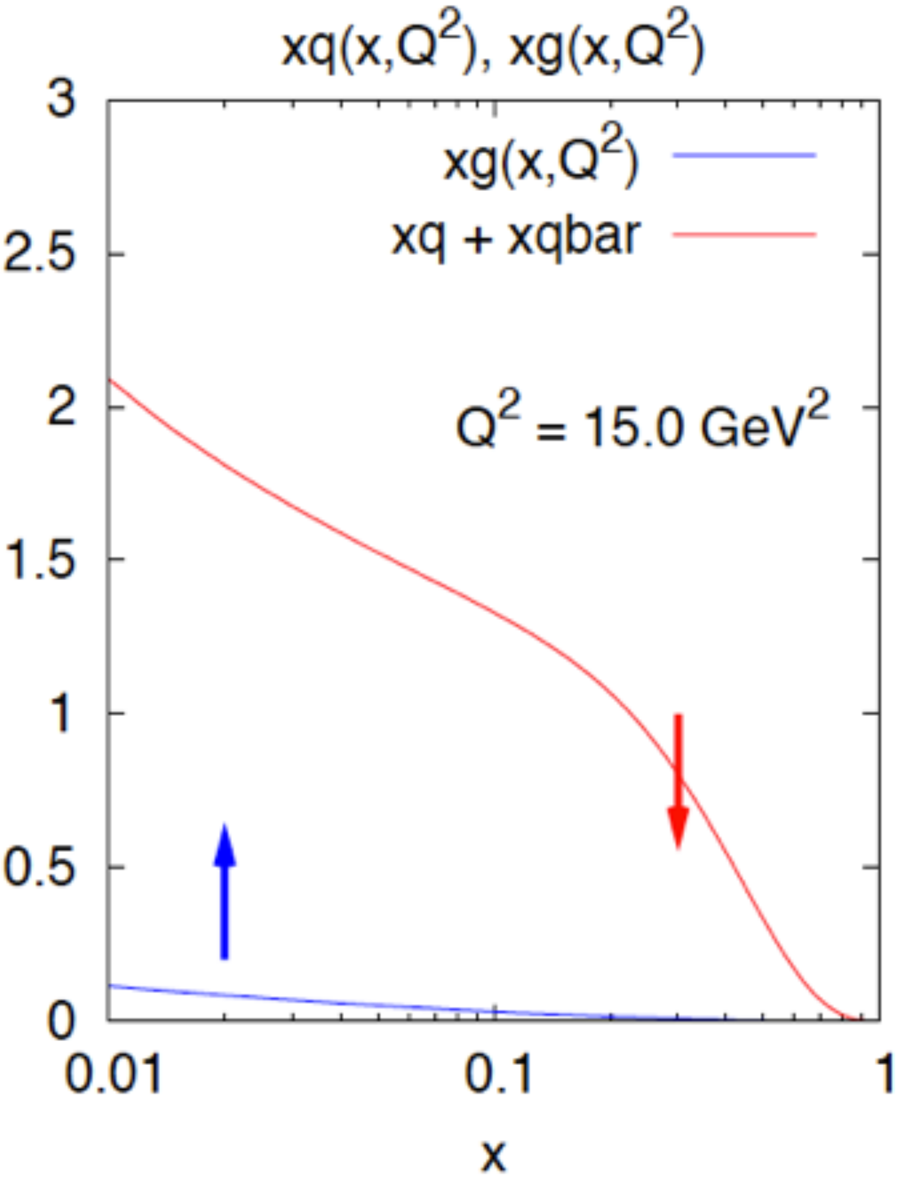

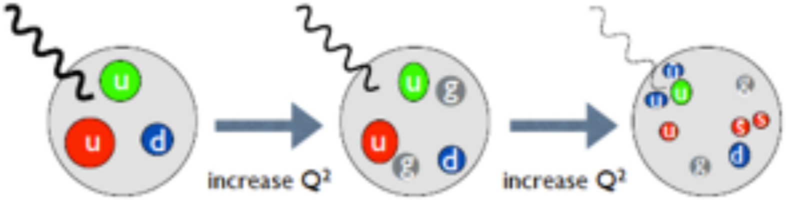

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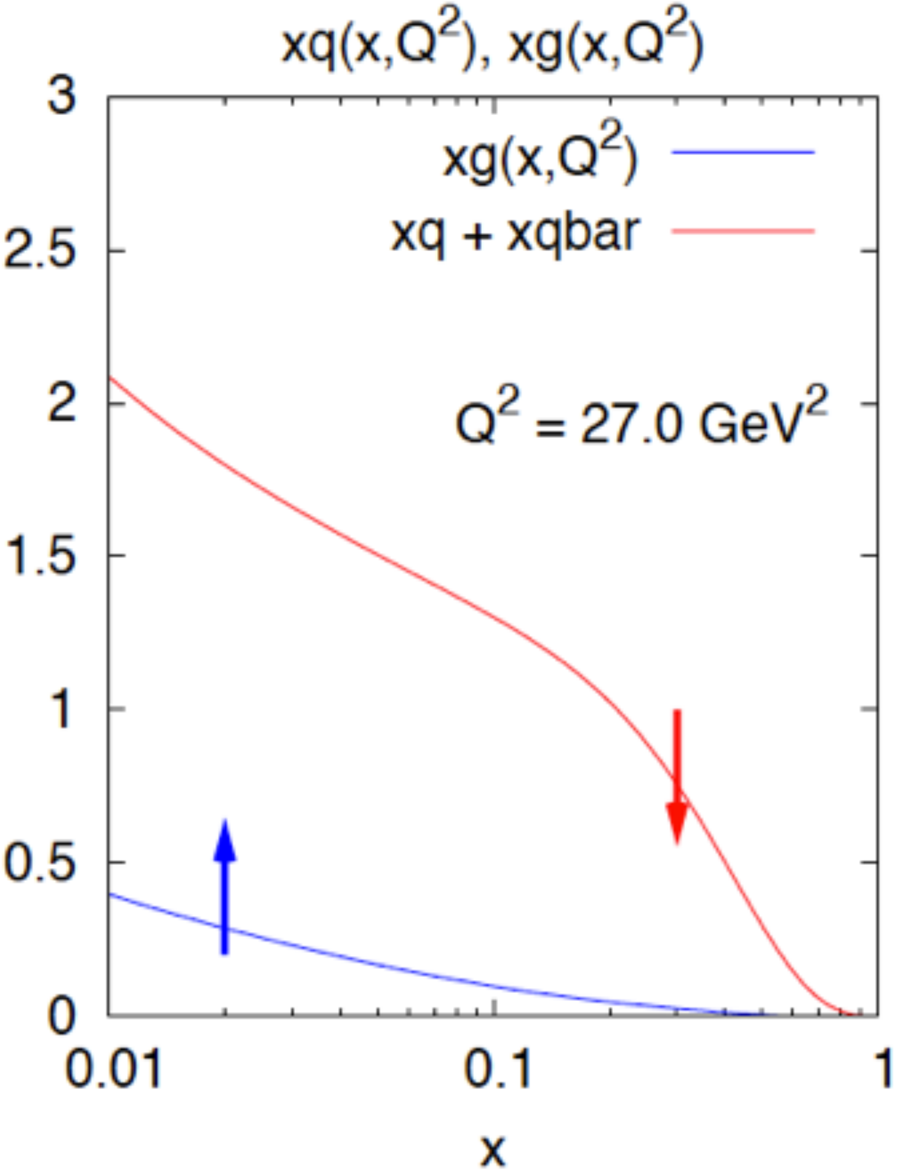

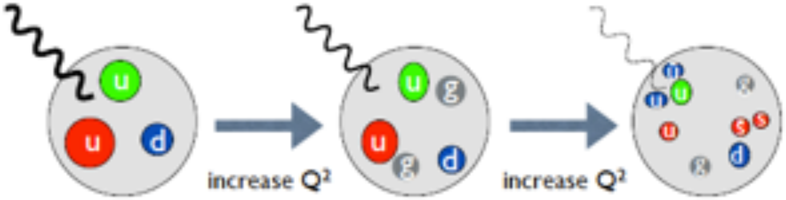

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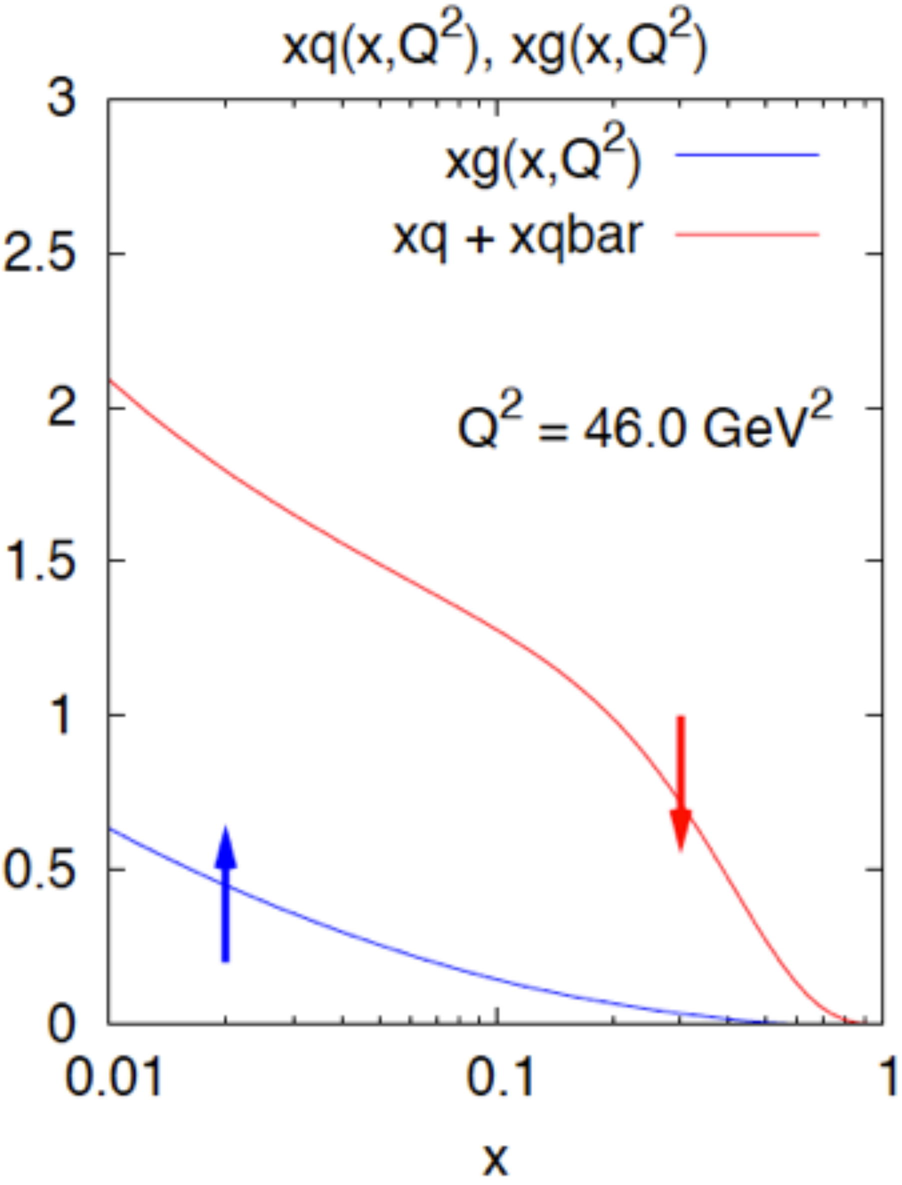

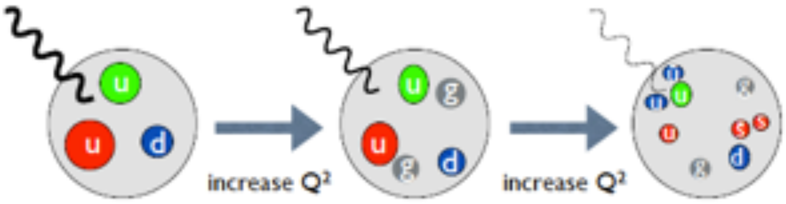

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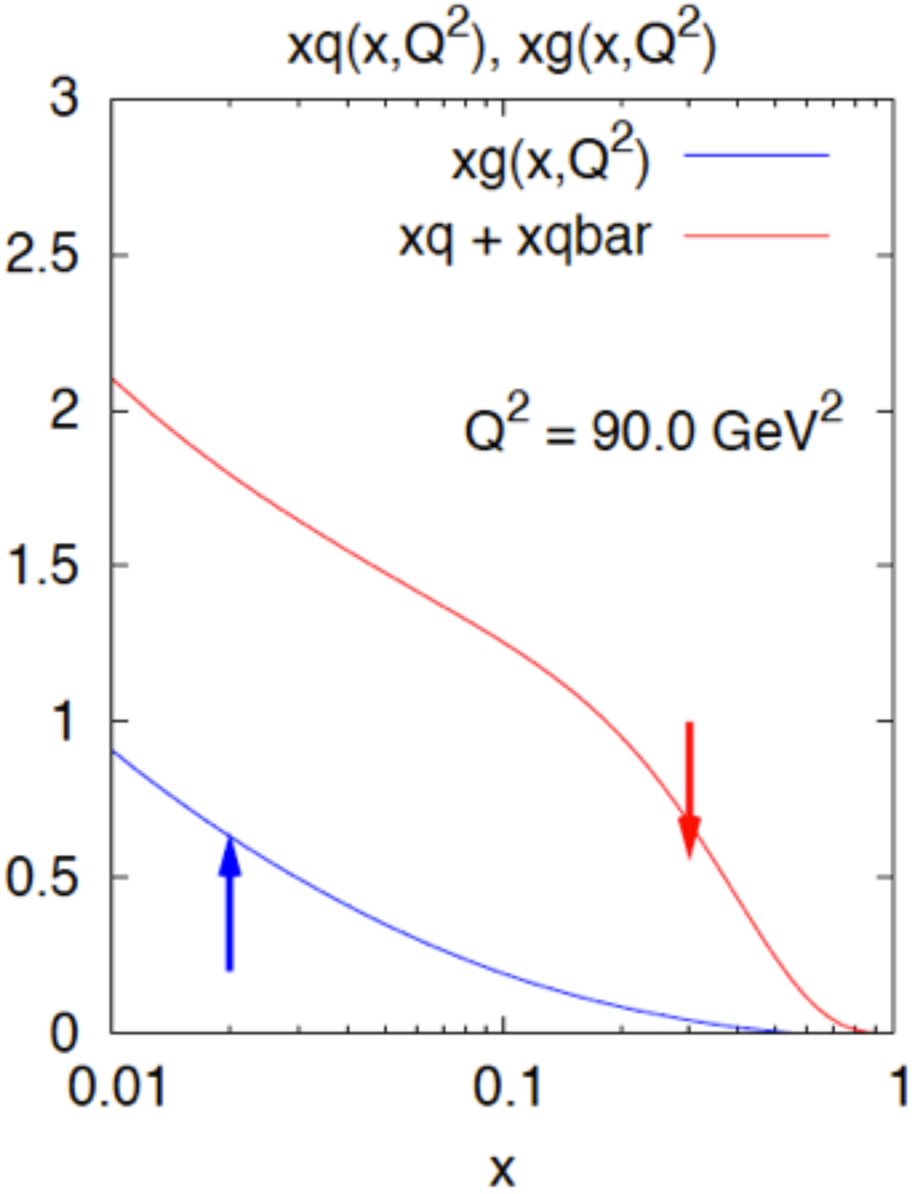

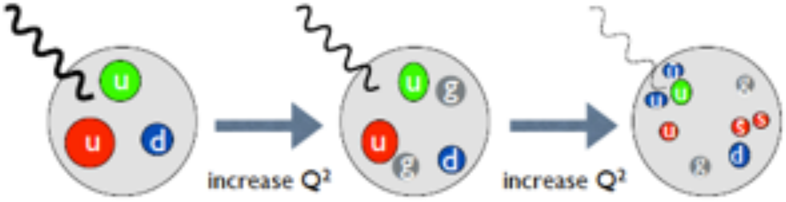

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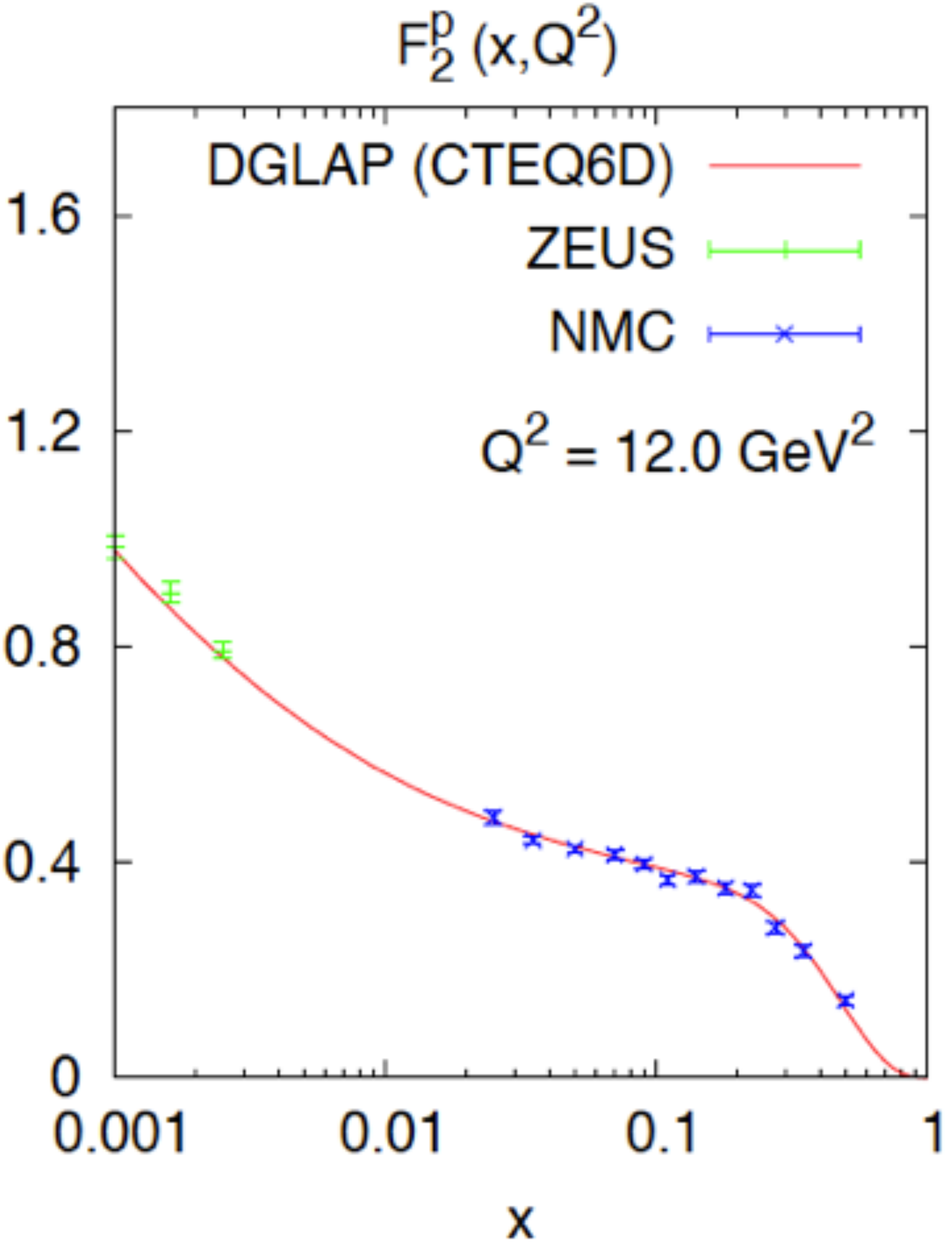
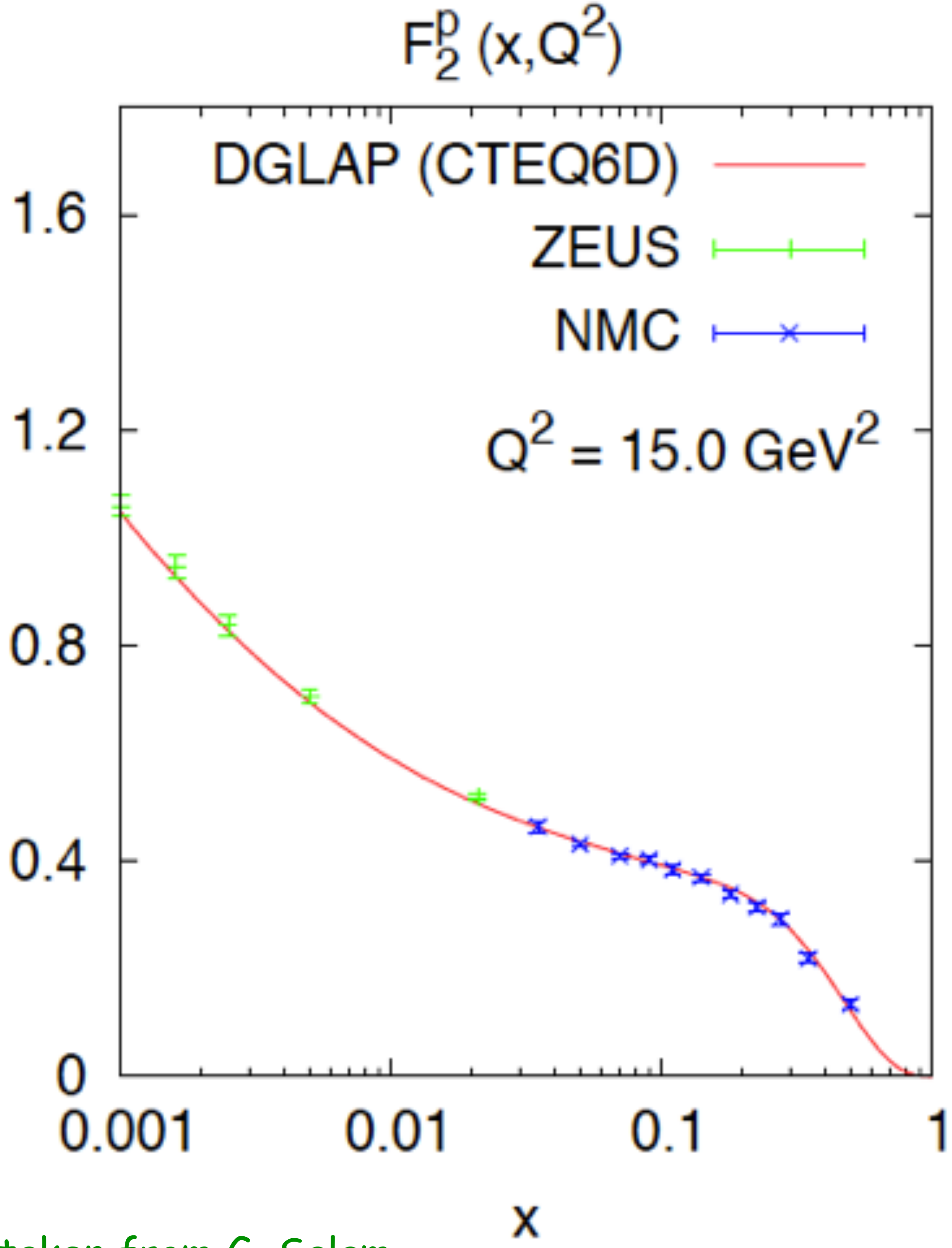
DGLAP evolution seen in DIS data



- use one of the global fits of PDFs to data by CTEQ
- steep rise of F_2 at small x (due to gluon evolution)

taken from G. Salam

DGLAP evolution seen in DIS data



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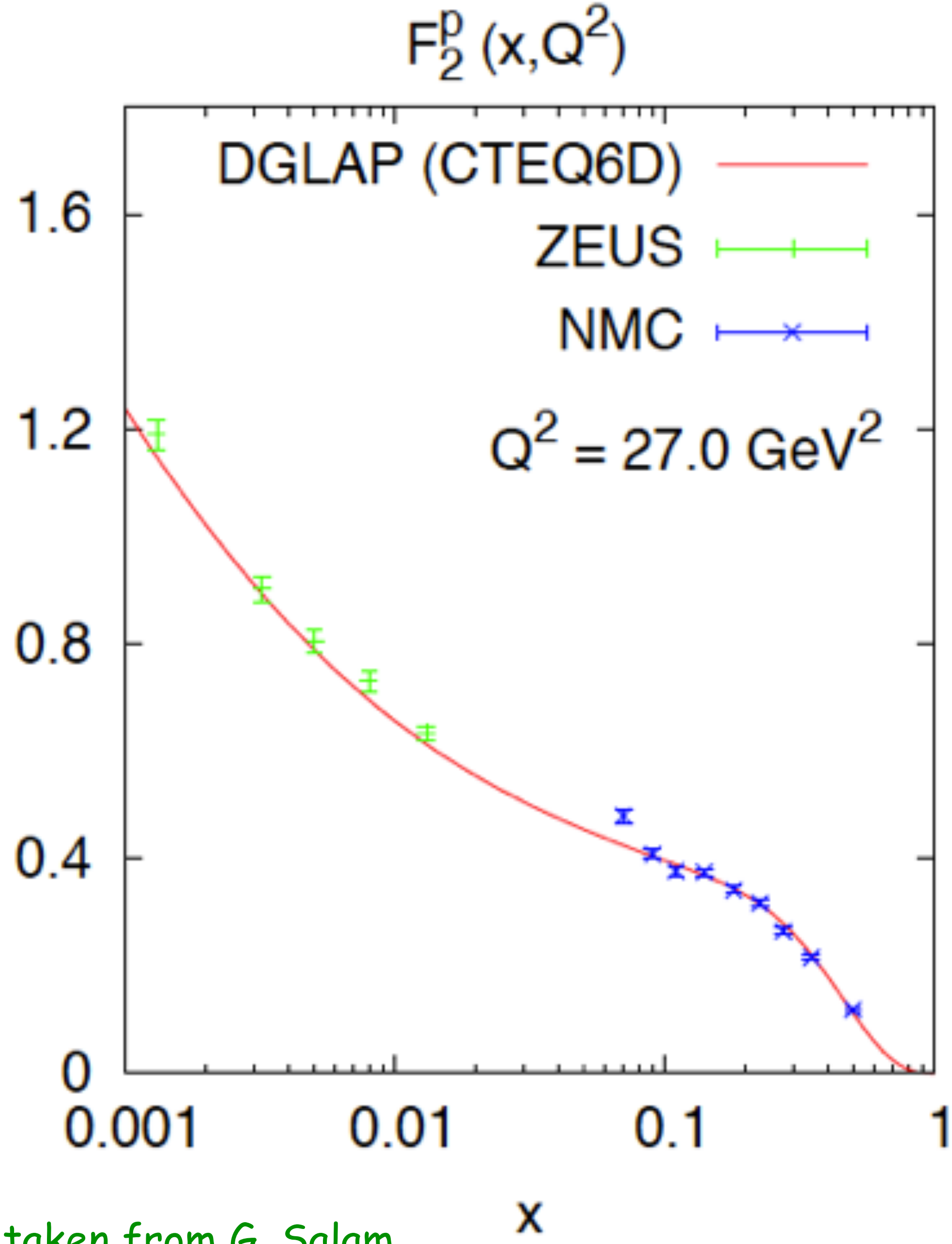

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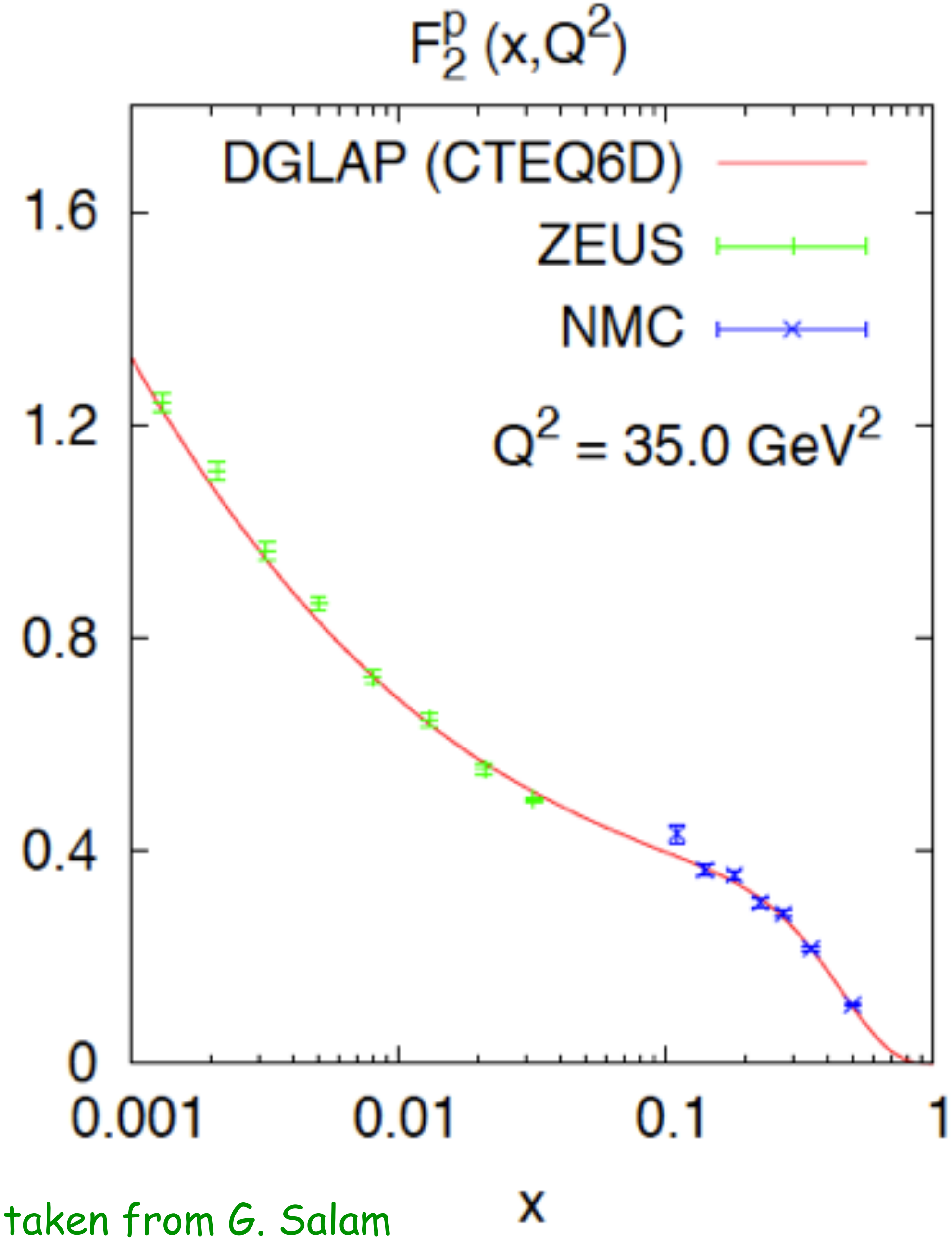

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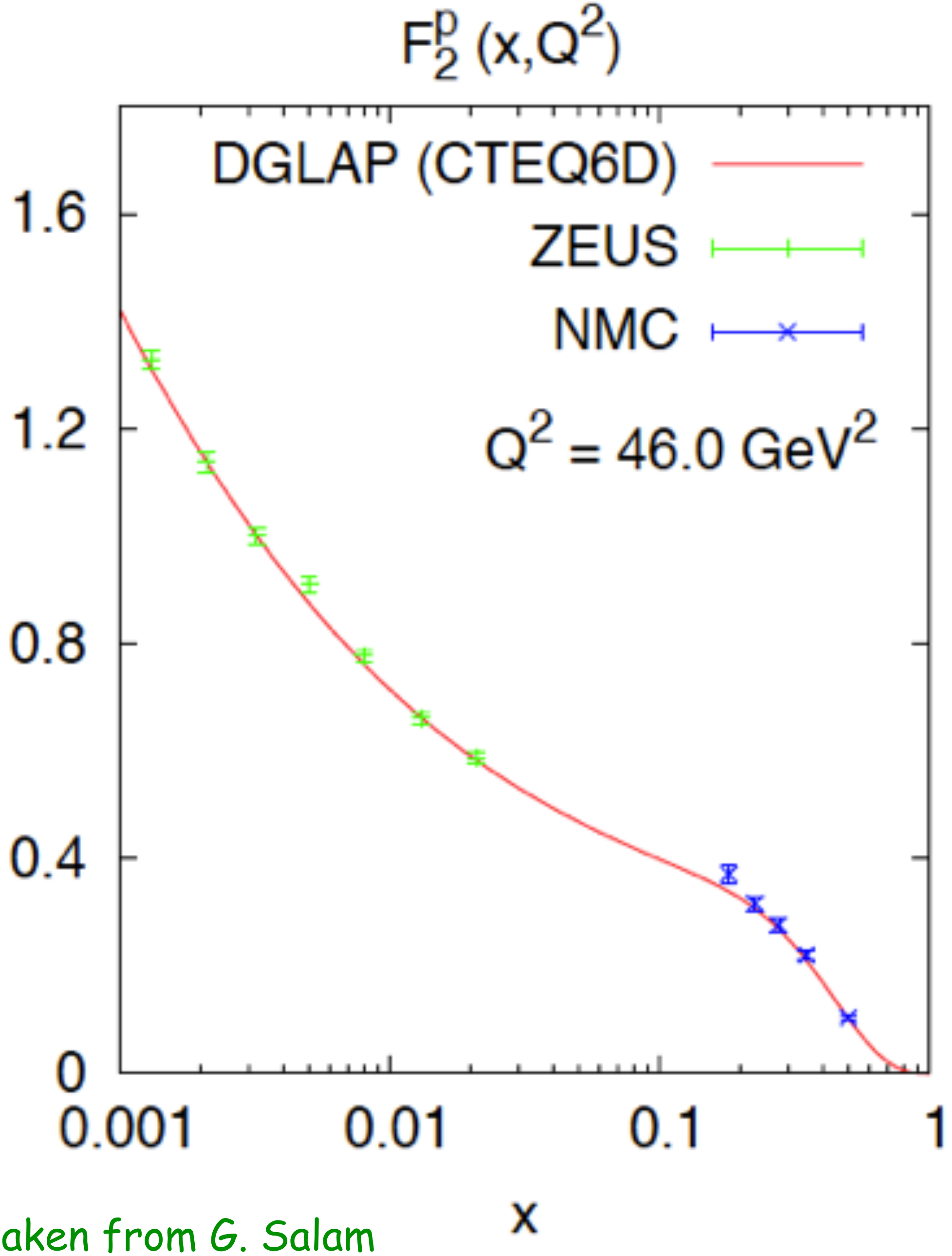

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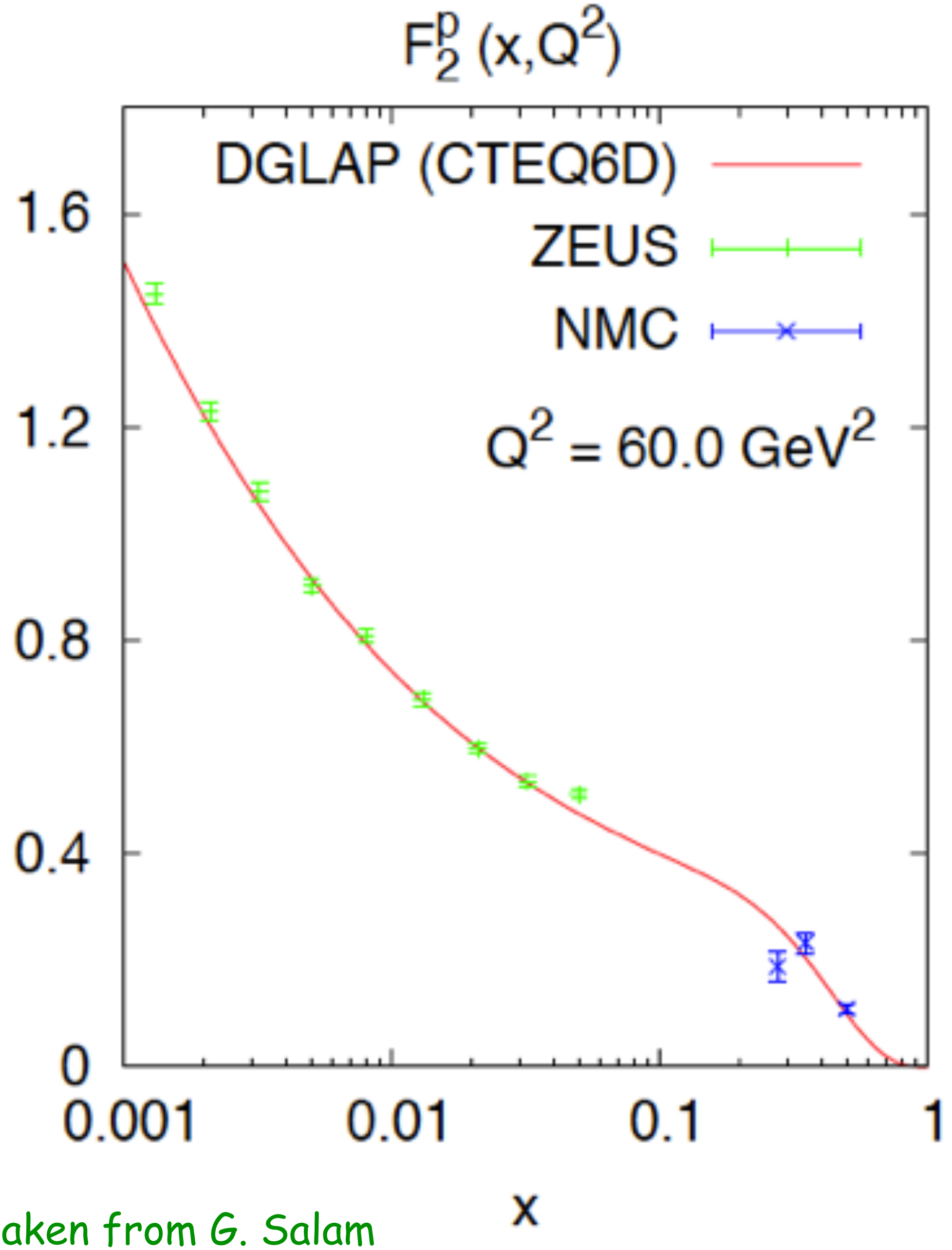

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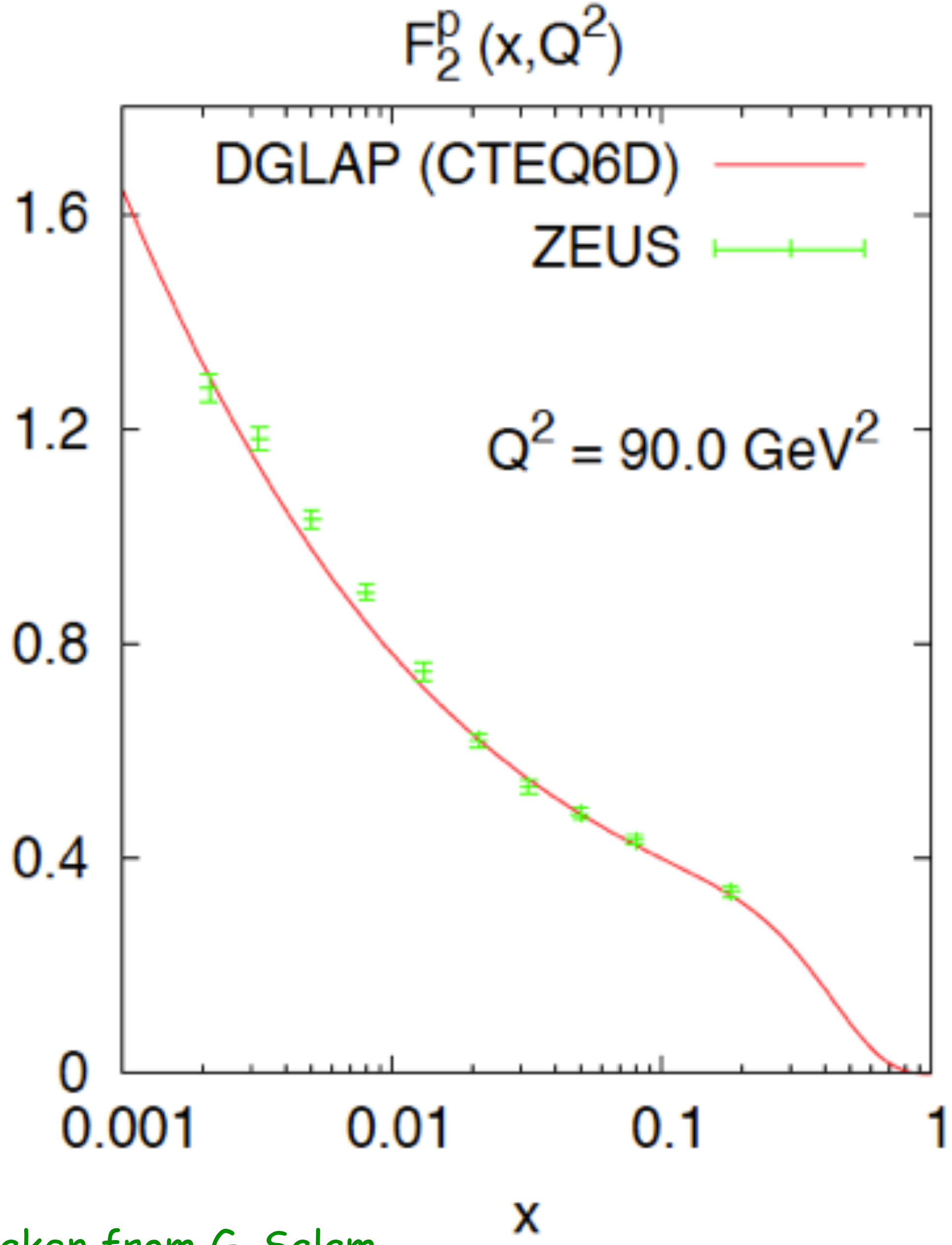

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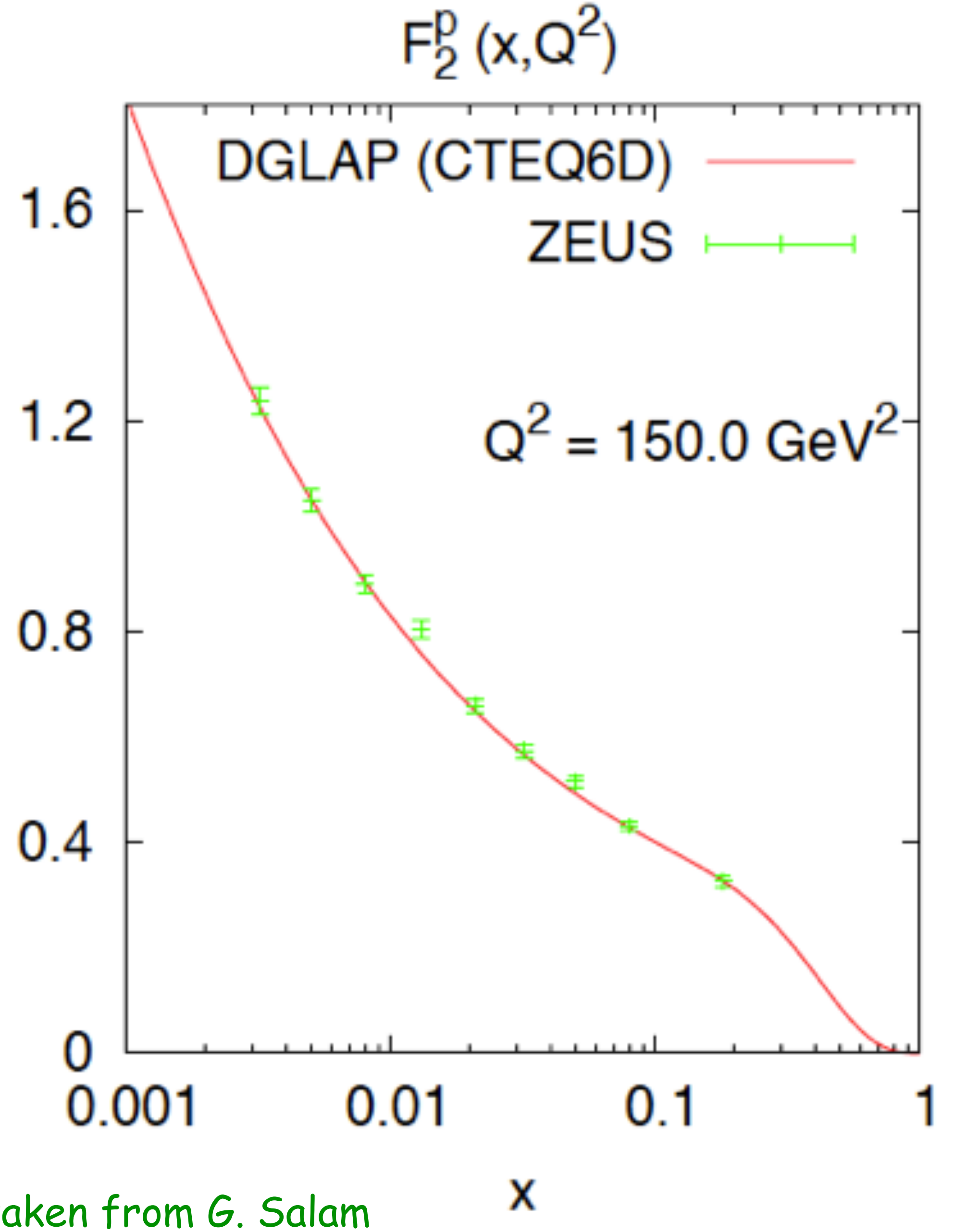

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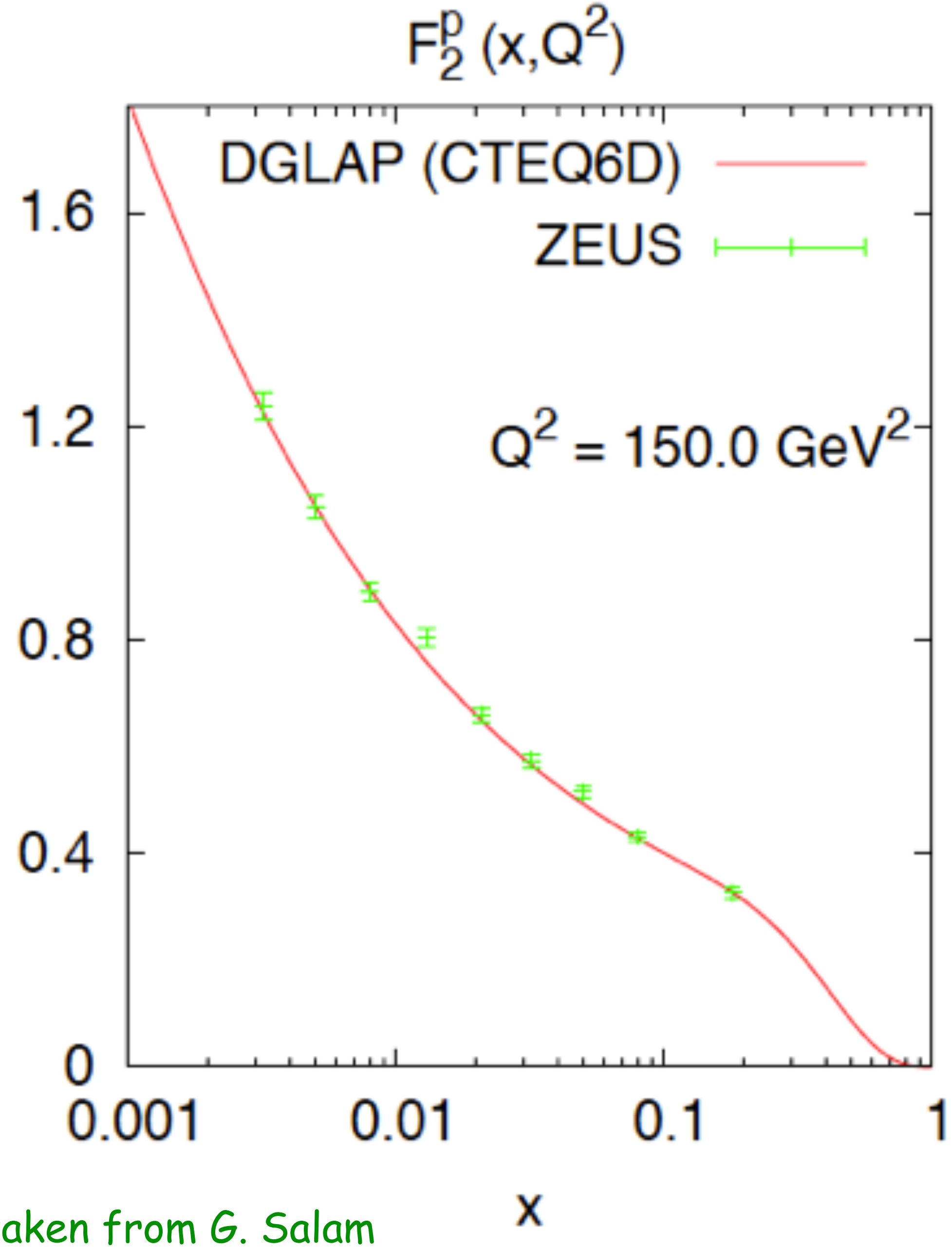

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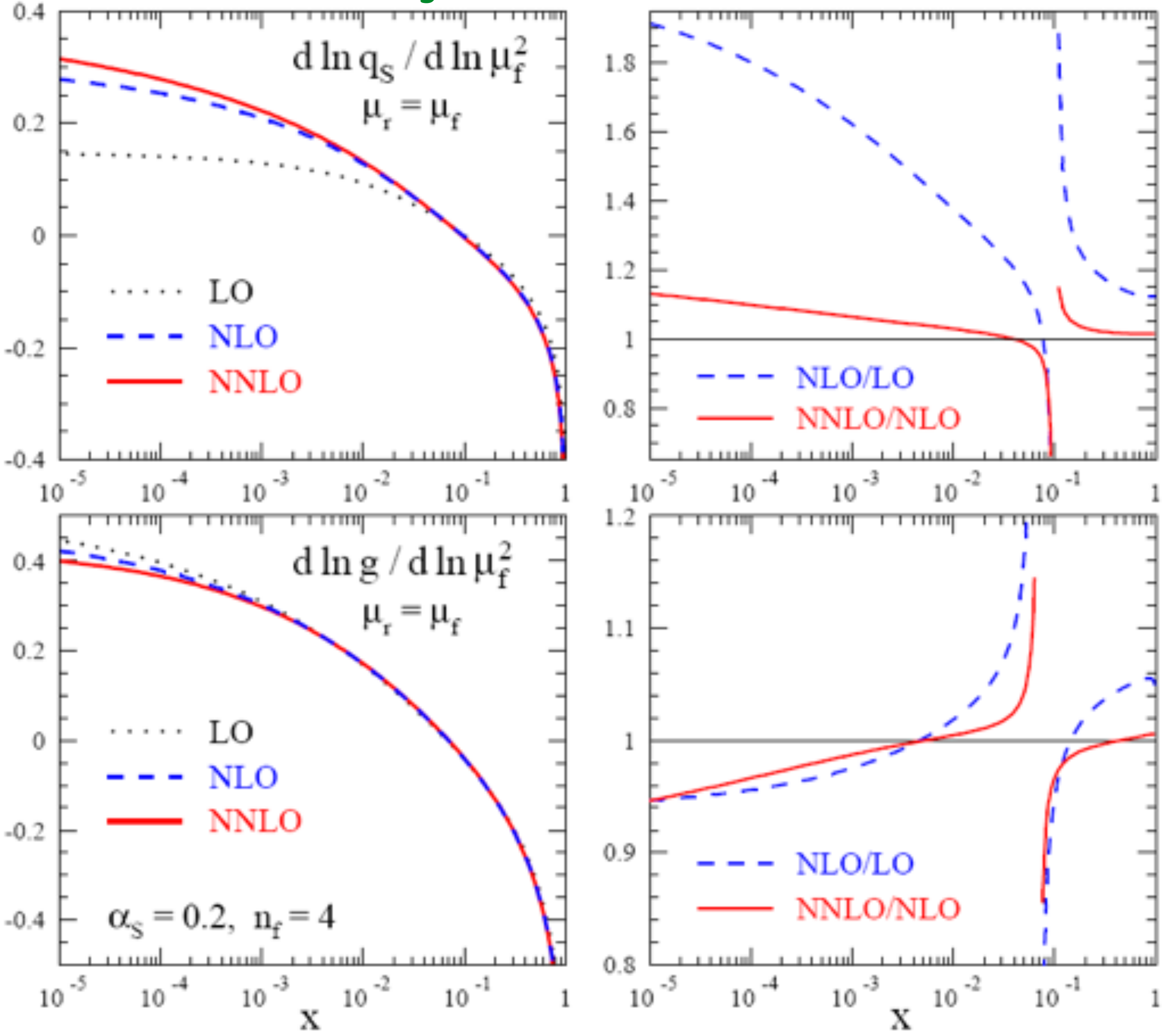
major success of pQCD and DGLAP evolution

taken from G. Salam

perturbative stability of evolution

Moch, Vermaseren, Vogt

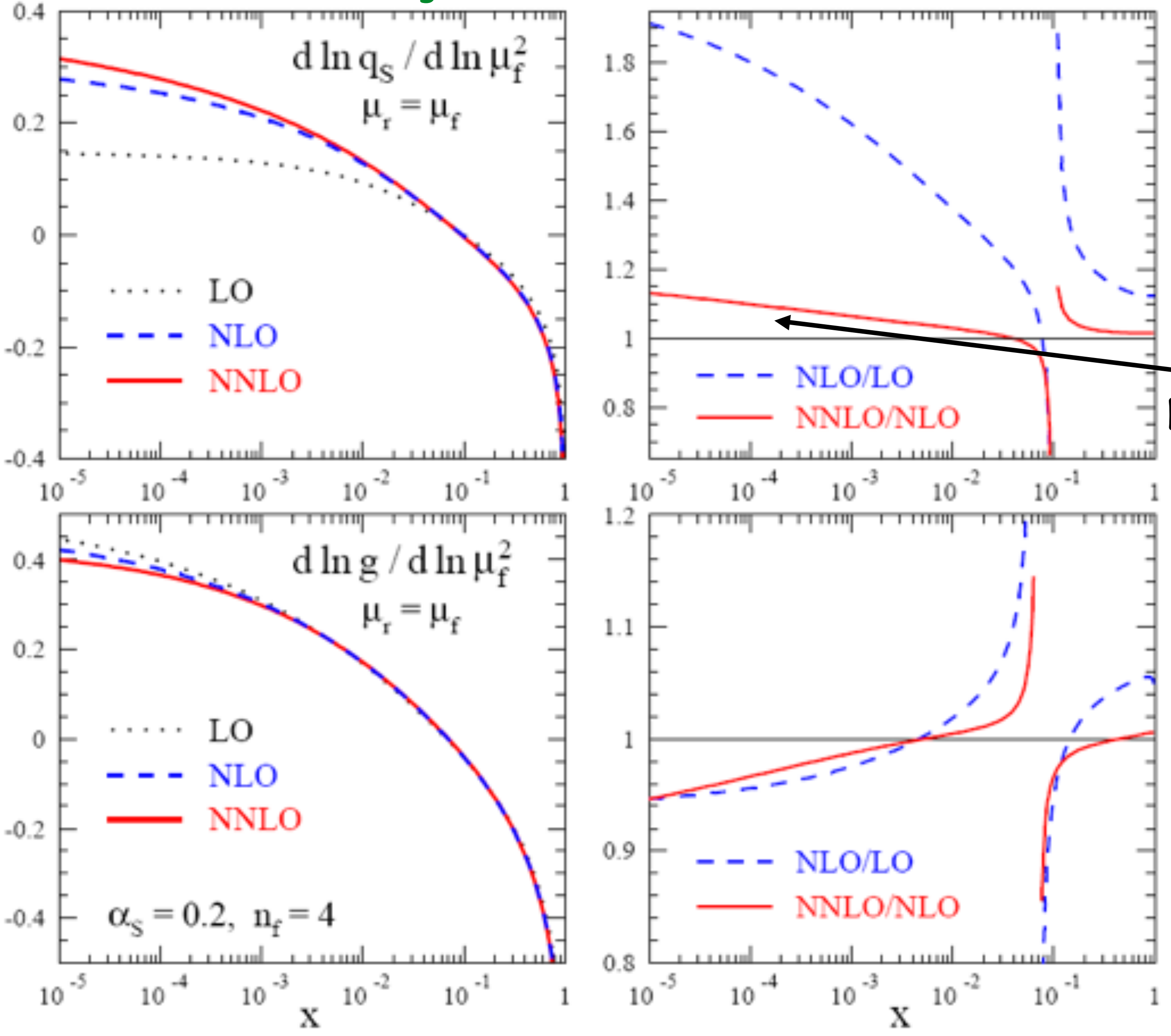


quarks

gluons

perturbative stability of evolution

Moch, Vermaseren, Vogt



quarks

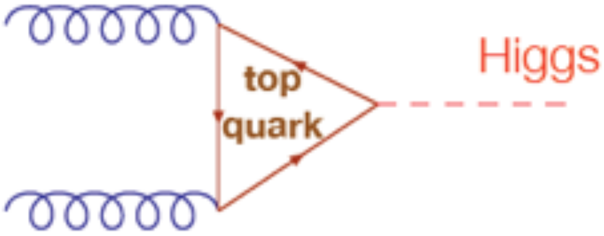
NNLO: >10% for $x<10^{-4}$

gluons

aside: universality of splitting fcts

example taken from J. Campbell

let's look at collinear singularities in a "QCD-ish" effective theory



can be simplified in the limit of infinite top mass

effective Lagrangian

$$\mathcal{L}_{\mathrm{ggH}} = \frac{\mathbf{C}}{\mathbf{2}}\,\mathbf{H}\,\mathbf{F}^{\mathbf{a}}_{\mu\nu}\mathbf{F}_{\mathbf{a}}^{\mu\nu}$$

coupling

$$\mathbf{C} = \frac{\alpha_{\mathbf{s}}}{\mathbf{6}\pi\,\mathbf{v}}$$

Higgs

usual gluon field strength

aside: universality of splitting fcts

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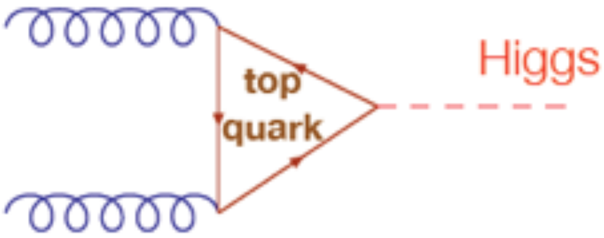

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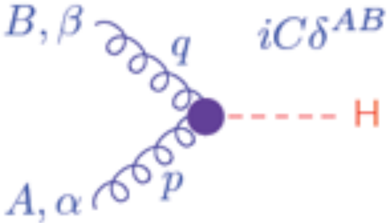
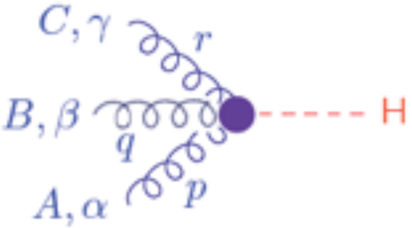
Higgs

usual gluon field strength

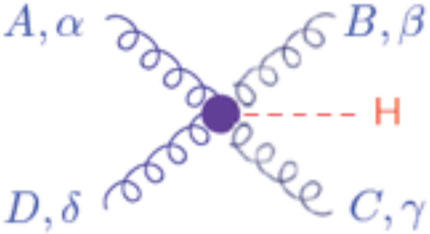
gives rise to new Feynman rules



$$iC\delta^{AB}\left(p\cdot q\, g^{\alpha\beta} - p^{\beta}q^{\alpha}\right)$$



$$-Cg_s f^{ABC}\Big[g^{\alpha\beta}(p^{\gamma}-q^{\gamma}) + g^{\beta\gamma}(q^{\alpha}-r^{\alpha}) + g^{\gamma\alpha}(r^{\beta}-p^{\beta})\Big]$$

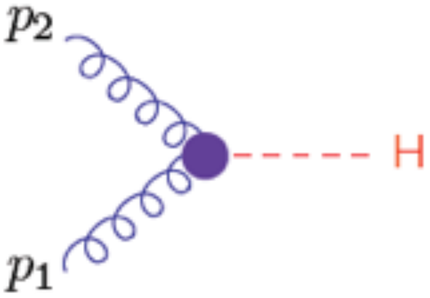


resembles all the features of QCD and reproduces full QCD calculation to within 10 – 20%

so, what do we encounter in an actual calculation?

sketch of a calculation in the effective Hgg theory

start with the tree-level diagram (recall: one-loop in full QCD)



$$|\mathcal{M}_{Hgg}|^2 = 2(N_c^2 - 1)C^2 m_H^4$$

sketch of a calculation in the effective Hgg theory

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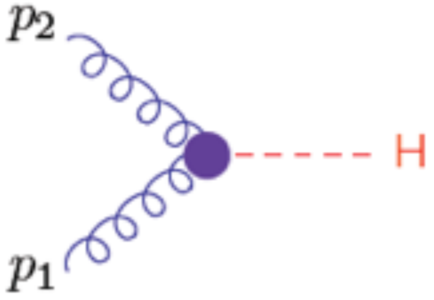
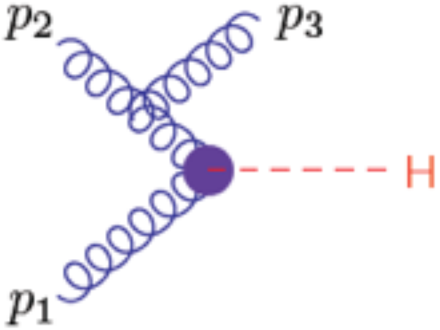


$$|\mathcal{M}_{Hgg}|^2 = 2(N_c^2 - 1)C^2 m_H^4$$

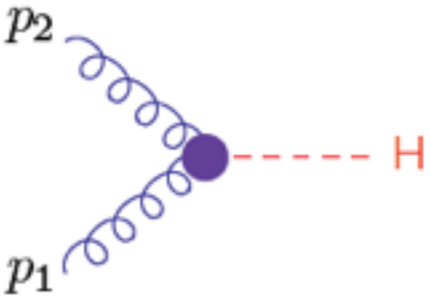
then add another gluon



$$|\mathcal{M}_{Hggg}|^2 = 4N_c(N_c^2 - 1)C^2 g_s^2 \times \left(\frac{m_H^8 + (2p_1.p_2)^4 + (2p_1.p_3)^4 + (2p_2.p_3)^4}{8p_1.p_2\, p_1.p_3\, p_2.p_3}\right)$$

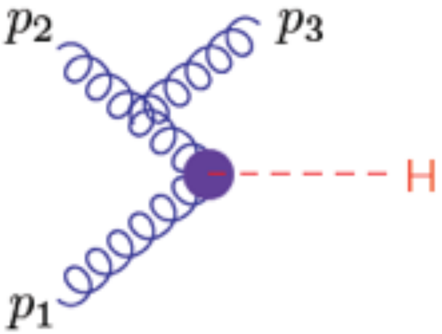
sketch of a calculation in the effective Hgg theory

start with the tree-level diagram (recall: one-loop in full QCD)



$$|\mathcal{M}_{Hgg}|^2 = 2(N_c^2-1)C^2 m_H^4$$

then add another gluon



$$|\mathcal{M}_{Hggg}|^2 = 4N_c(N_c^2-1)C^2 g_s^2 \times \left(\frac{m_H^8 + (2p_1.p_2)^4 + (2p_1.p_3)^4 + (2p_2.p_3)^4}{8p_1.p_2\, p_1.p_3\, p_2.p_3}\right)$$

plug in

and evaluate in the collinear limit for p_2 and p_3

use
$$\mathbf{2p_2 \cdot p_3 \rightarrow 0}$$
$$\mathbf{2p_1 \cdot p_2 \rightarrow zm_H^2}$$
$$\mathbf{2p_1 \cdot p_3 \rightarrow (1-z)m_H^2}$$

… arriving at a familiar function

find

$$|\mathcal{M}_{Hggg}|^2 \xrightarrow{\text{coll.}} 4N_c(N_c^2-1)C^2 g_s^2 m_H^4 \left(\frac{1+z^4+(1-z)^4}{2z(1-z)p_2.p_3}\right)$$

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collinear sing. associated with
familiar gluon-gluon splitting fct.

$$P_{gg}(z) = 2N_c\left(\frac{z^2+(1-z)^2+z^2(1-z)^2}{z(1-z)}\right)$$

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universal ✓

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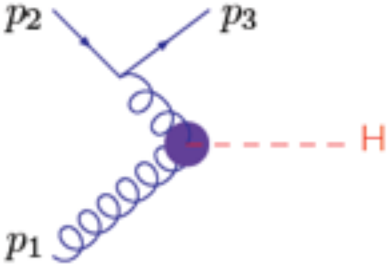
similarly, one can obtains P_{qg} from



factorization in hadron-hadron collisions

What happens when two hadrons collide ?

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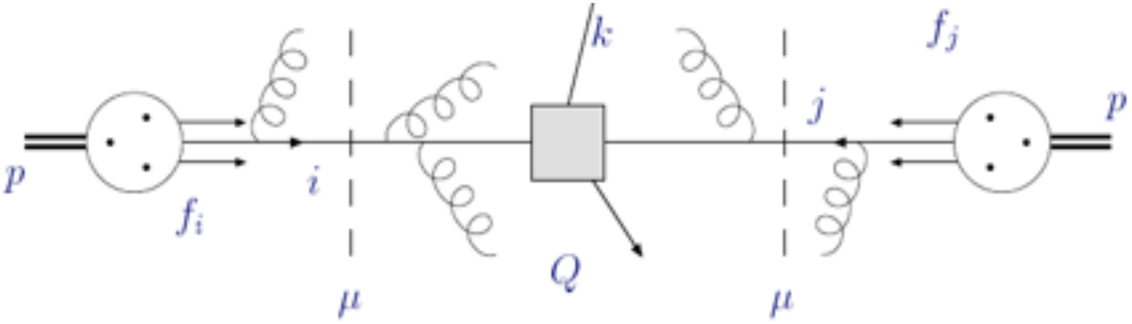
What happens when two hadrons collide ?

straightforward generalization of the concepts discussed so far:



jets, hadrons, heavy quarks, ...

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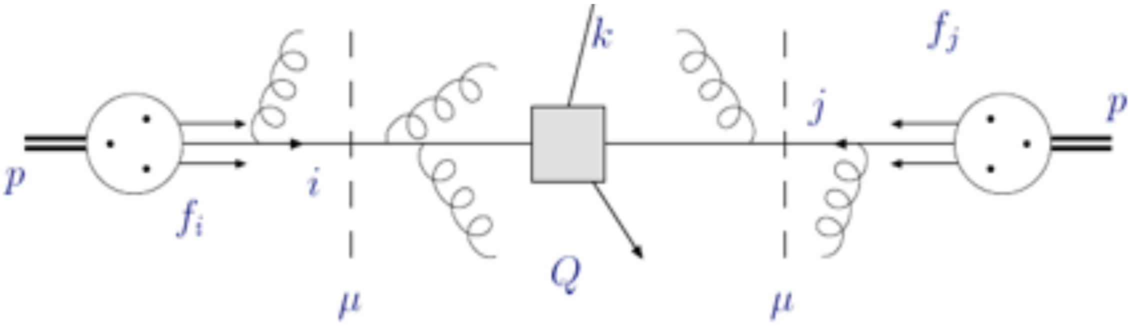


$$d\sigma = \sum_{ij}\int dx_i dx_j\, f_i(x_i,\mu^2)\, f_j(x_j,\mu^2)\, d\hat{\sigma}_{ij}(\alpha_s(\mu_r),Q^2,\mu^2,x_i,x_j)$$

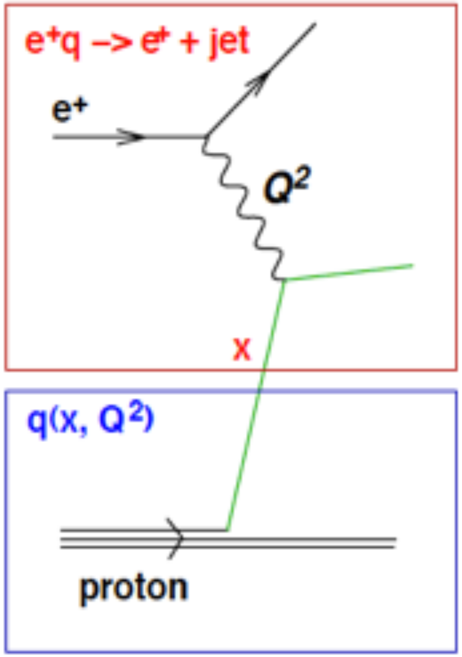
non-perturbative but universal PDFs $\xleftrightarrow[\text{by }\mu]{\text{linked}}$ **hard scattering of two partons → pQCD**

factorization at work

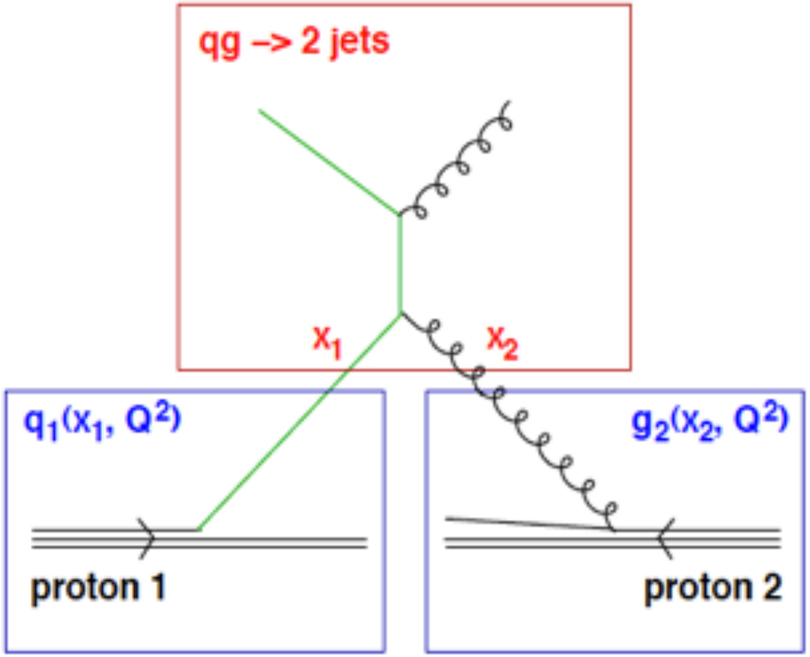
key assumption that a cross section factorizes into

- hard (perturbatively calculable) process-dep. partonic subprocesses
- non-perturbative but universal parton distribution functions

has great **predictive power** and can be challenged experimentally:



$$\sigma_{ep} = \sigma_{eq} \otimes q$$



$$\sigma_{pp\to 2\,jets} = \sigma_{qg\to 2\,jets} \otimes q_1 \otimes g_2 + \cdots$$

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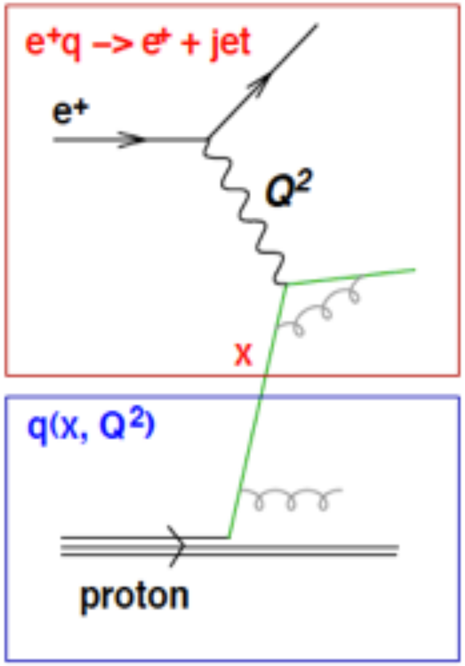

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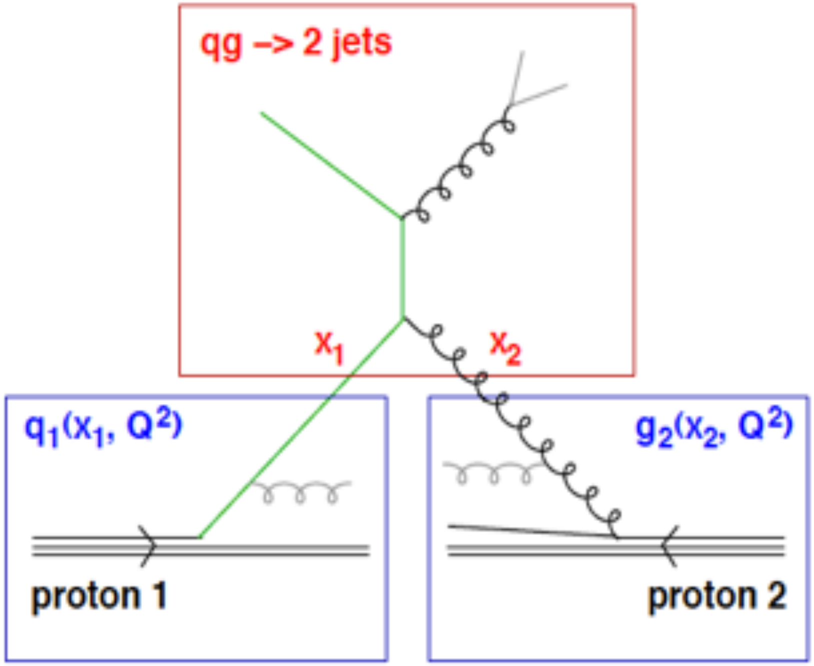

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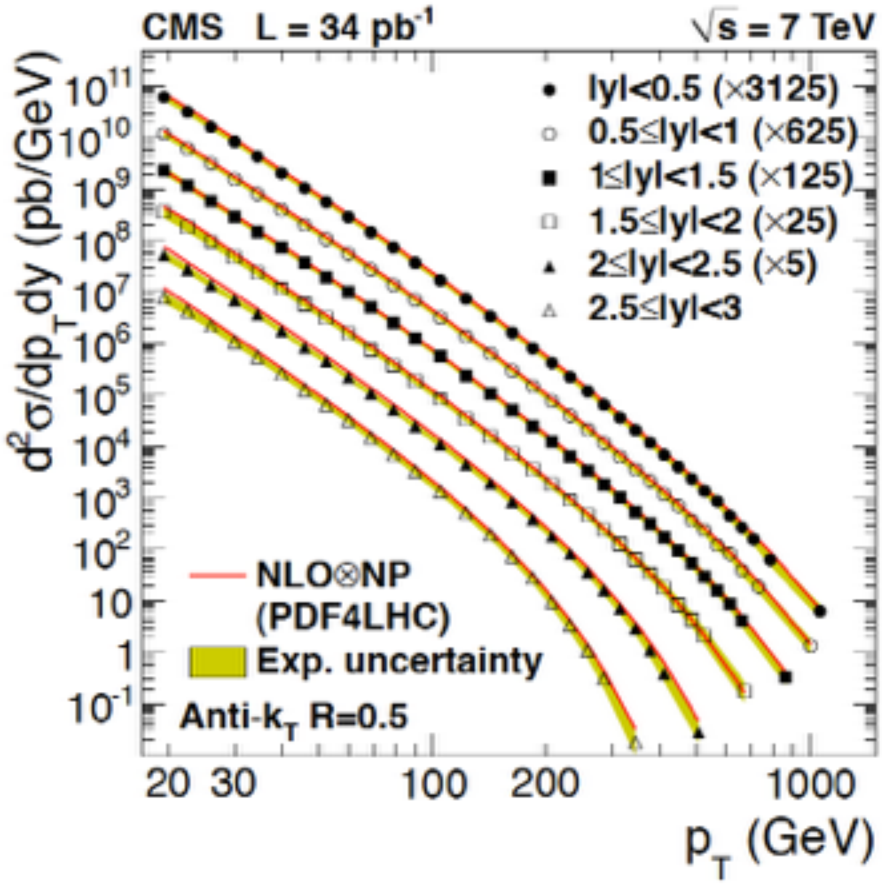
factorization: so far a success story



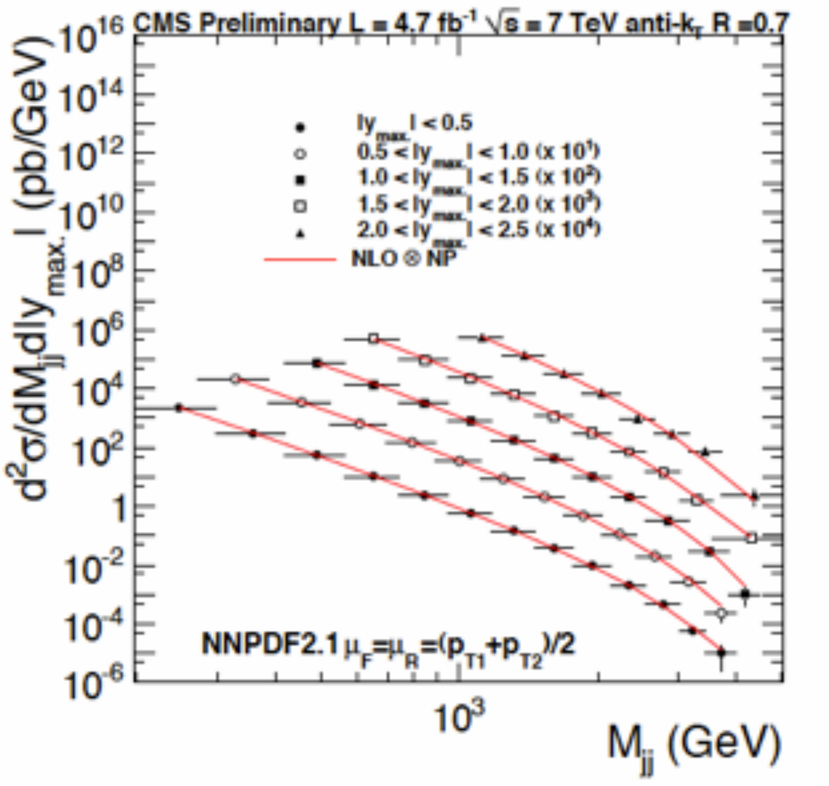
two recent examples from the LHC:

1-jet and di-jet cross sections

many other final-states available

$$y = \ln\tan\frac{\theta}{2} \sim \frac{1}{2}\ln\frac{x_1}{x_2} \qquad M = \sqrt{x_1 x_2 s}$$

$$x_1 = \frac{M}{\sqrt{s}}e^{+y} \qquad x_2 = \frac{M}{\sqrt{s}}e^{-y}$$



results now start to being used in global fits to constrain PDFs

particularly sensitive to gluons

$$gg \rightarrow gg \qquad gq \rightarrow gq$$

proofs of factorization

- to prove the validity of **factorization to all orders** of pQCD is a highly theoretical and technical matter

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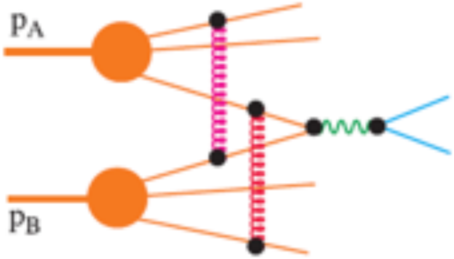

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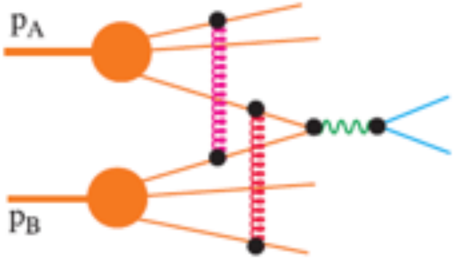

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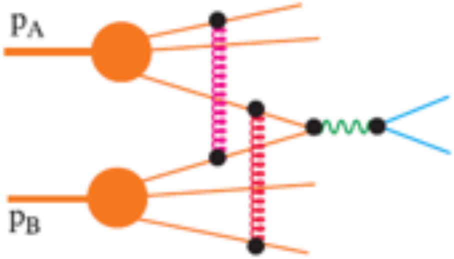

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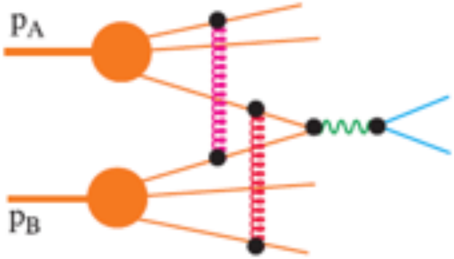

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---

recall: the renormalizibility of a non-abelian gauge theory like QCD was demonstrated by 't Hooft and Veltman

1999

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keys to resolve the apparent dilemma:

- asymptotic freedom
- infrared safety
- factorization theorems & renormalizibility

pQCD: a tool for the most violent collisions

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high-p_T jet: factorization!

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"soft stuff": difficult!

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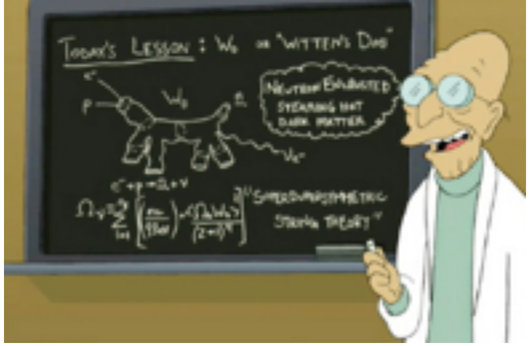
"underlying event": more than difficult

to take home from this part of the lectures

INWARD BOUND - FEMTO'SCOPY



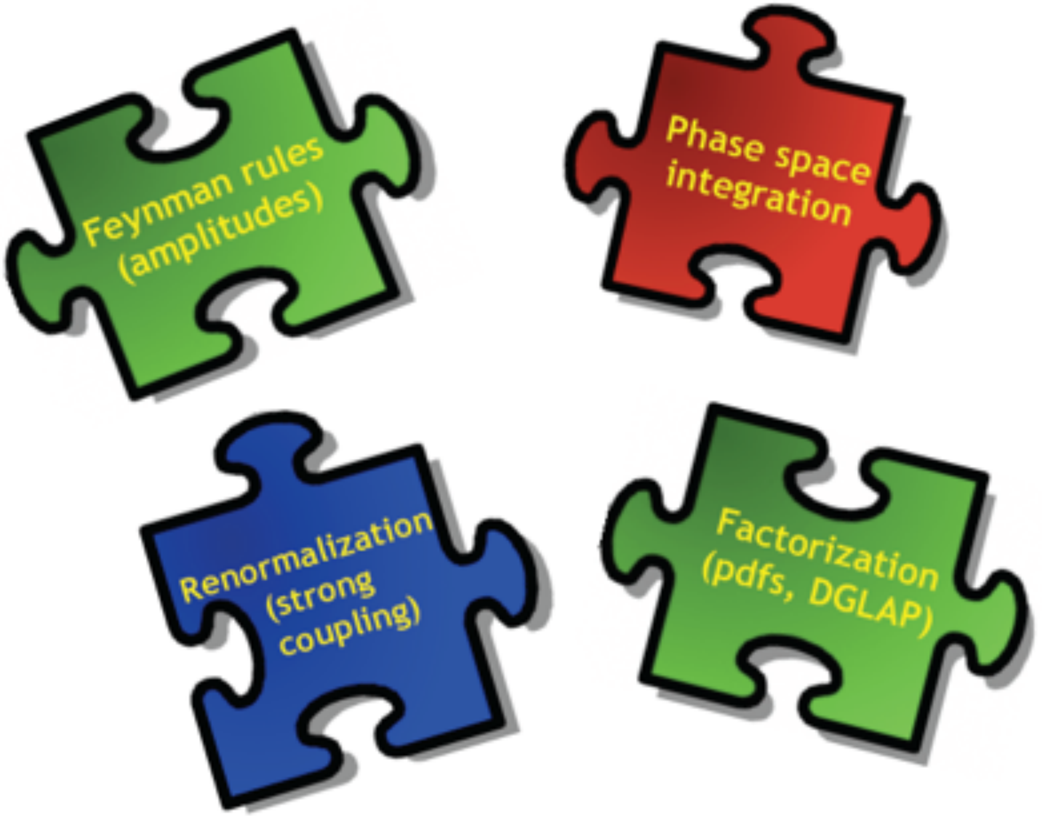
- factorization = isolating and absorbing long-distance singularities accompanying identified hadrons into parton densities (initial state) and fragmentation fcts. (final state)

- factorization and renormalization introduce arbitrary scales
  → powerful concept of renormalization group equations
  → α_s, PDFs, frag. fcts. depend on energy/resolution

- PDFs (and frag. fcts) have definitions as bilocal operators

- hard hadron-hadron interactions factorize as well: ffdσ

- strict proofs of factorization only for limited class of processes



Part IV

some applications & advanced topics

scales and theoretical uncertainties; Drell-Yan process
small-x physics; global QCD analysis; resummations

30+ years of hadron collider physics and counting

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CERN S$p\bar{p}$S [1981 → 1990]

$p\bar{p}$ collisions 540, 630 GeV

W,Z discovery, jets, …
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CERN LHC [operating]

pp collisions up to 14(?) TeV

a QCD machine, discoveries ?

also a PbPb and pPb program

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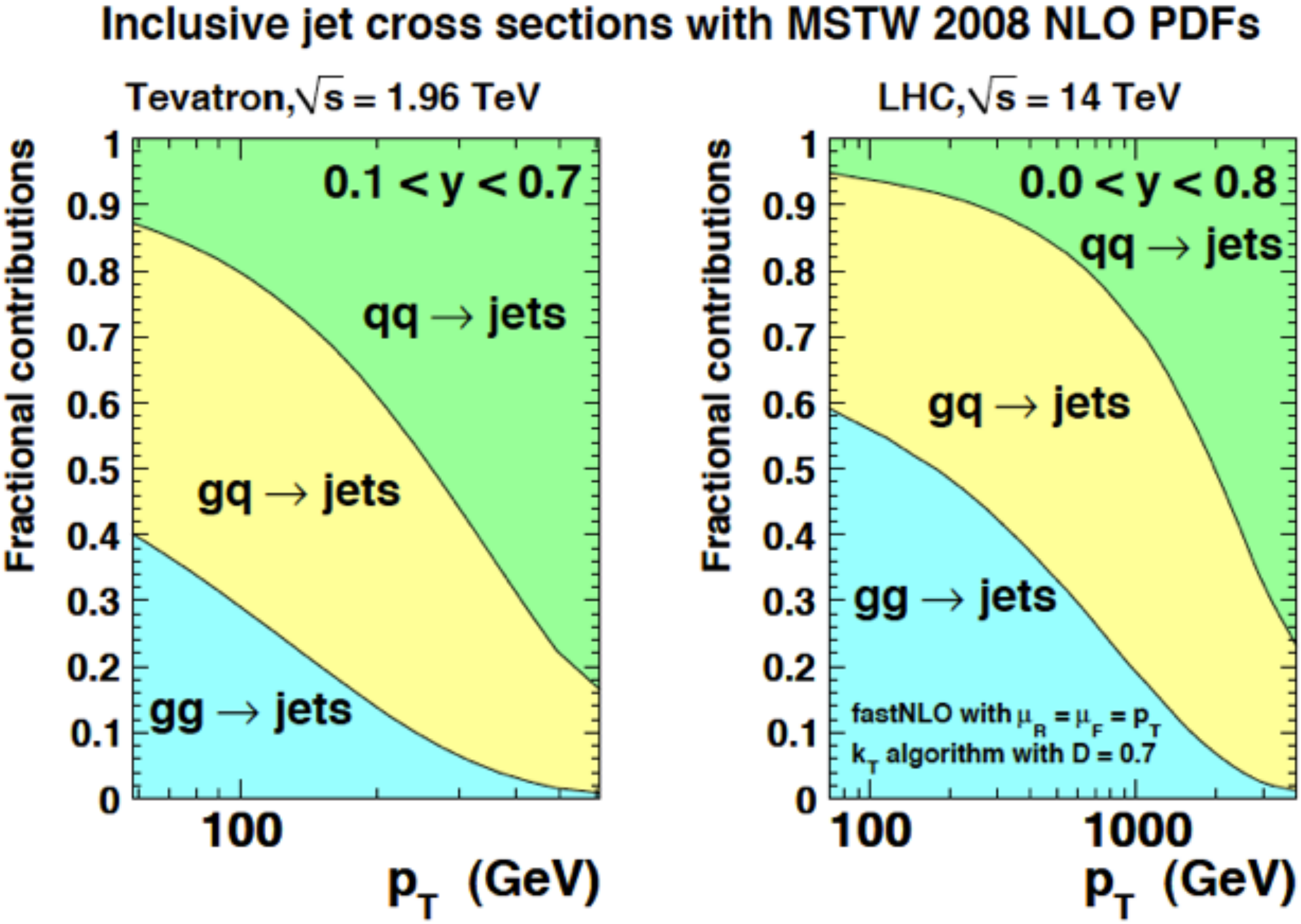
jets: which parton processes contribute



from G. Salam's lectures

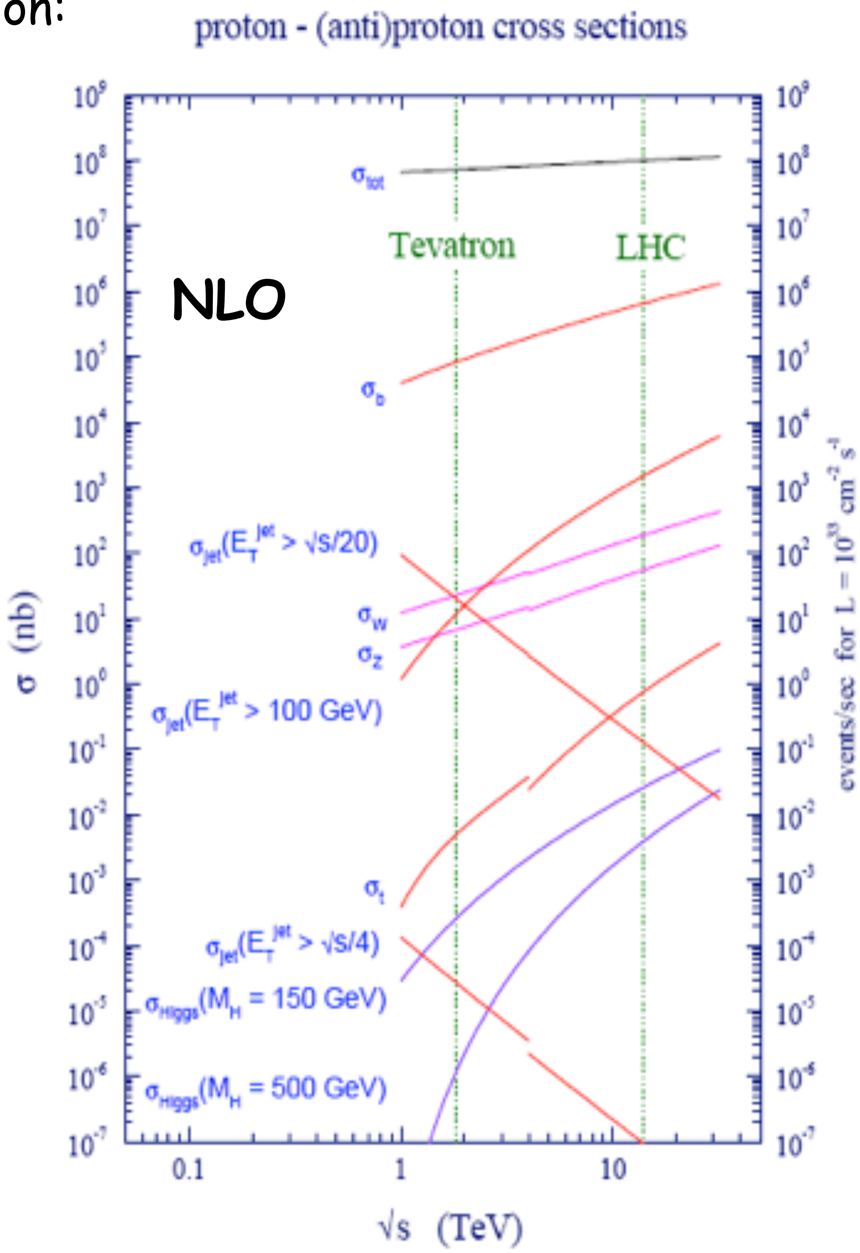
hadron colliders are gluon dominated up to rather large p_T

pQCD at the energy frontier

pQCD essential in solving the master equation:
"New Physics = data – Standard Model"

issues:

- **large rates for SM processes**
  e.g.: leptonic events for 10 fb^{-1} from
  W's (300M), Z's (33M), top (2.4M)
- **even lots of multi-particle states**
  —> background to "new physics"
- **QCD + e.w. effects mix**
  LHC well above e.w. scale M_Z
  —> e.w. bosons are "light"



Campbell, Huston, Stirling

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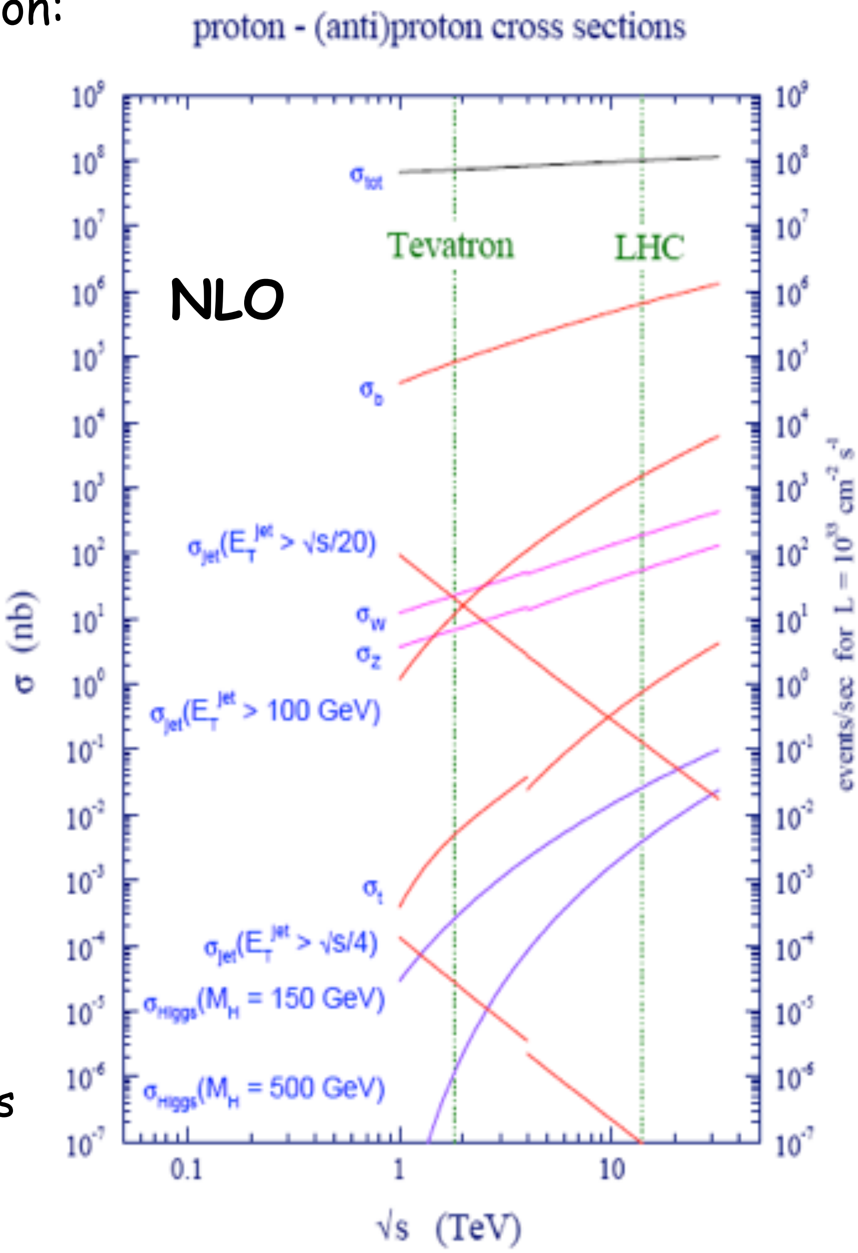
need:

precision, precision, precision …

for hard scattering, PDFs, theor. uncertainties
+ novel methods for processes with many legs

Campbell, Huston, Stirling



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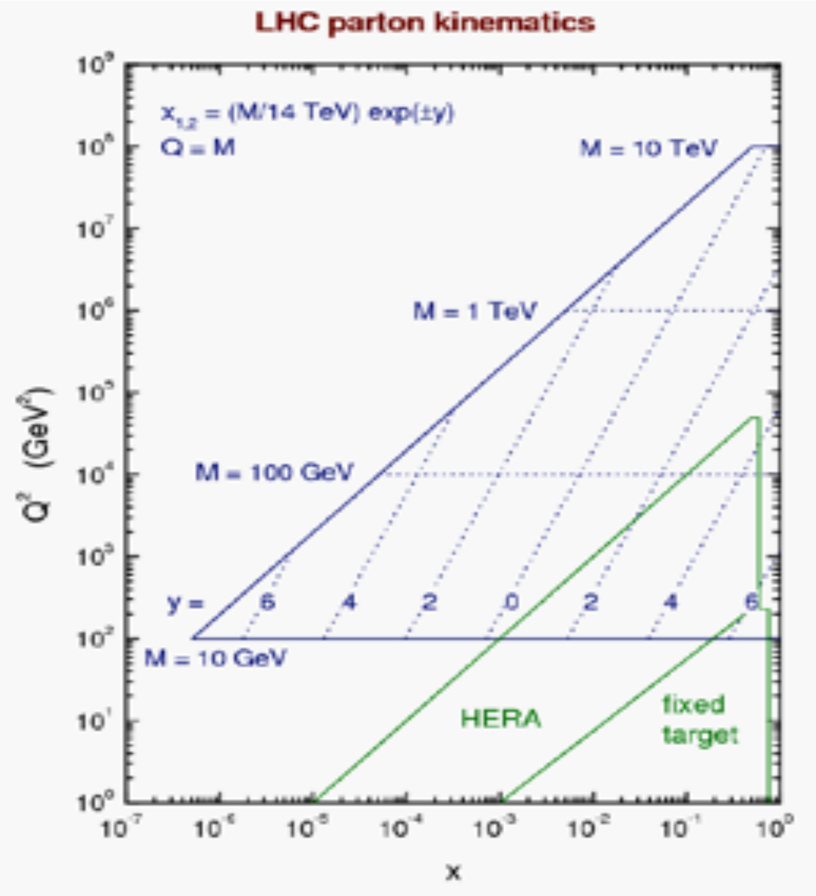

PDFs: vastly extended x,Q^2 landscape

- HERA —> LHC: evolution across up to 3 decades in Q^2
- M < 100 GeV physics: small x relevant
- TeV scale physics: large x relevant
- large angles/rapidities: extreme x

pQCD at the energy frontier

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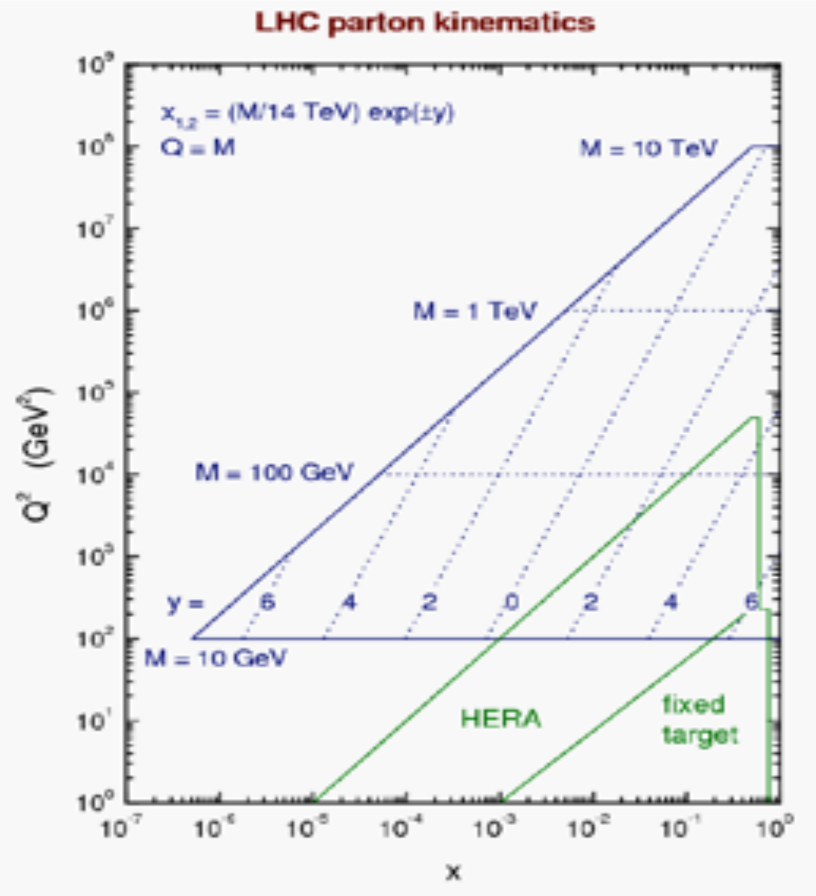

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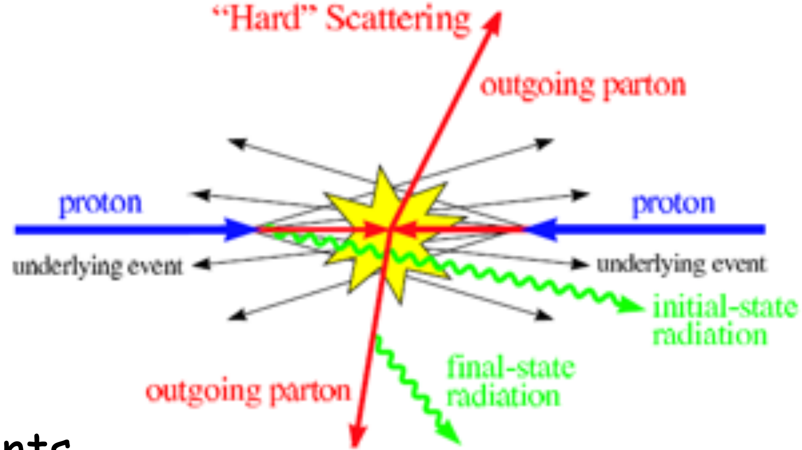

real events at the LHC are very messy:

- possible interactions of spectator partons leading to multiple interactions/underlying events

—> relies on event generators (Sherpa, Herwig, ...);
state-of-the-art: merge with NLO calculations (MC@NLO, POWHEG, ...)

Start your
business right
with Precision
Calculations
advise!

4-1

the Whys and Hows of
NLO Calculations & Beyond

why go beyond LO (and even NLO)?

recall factorization theorem for hadronic processes:

$$d\sigma = \sum_{ij} \int dx_i dx_j \, f_i(x_i, \mu^2) \, f_j(x_j, \mu^2) \, d\hat{\sigma}_{ij}(\alpha_s(\mu_r), Q^2, \mu^2, x_i, x_j)$$

non-perturbative but universal PDFs $\xleftrightarrow[\text{by } \mu]{\text{linked}}$ **hard scattering of two partons → pQCD**

☑ independence of physical $d\sigma$ on μ (and μ_r) has led us to powerful RGEs

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caveat: we work with a perturbative series truncated at LO, NLO, NNLO, …

→ at any fixed order N there will be a **residual scale dependence** in our theoretical prediction

→ since μ is completely arbitrary this limits the precision of our results

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simplest example:

$e^+e^- \to$ hadrons

$$\frac{d}{d\ln\mu_r}\sum_{n=1}^{N} c_n(\mu_r)\,\alpha_s^n(\mu_r) \sim \mathcal{O}\left(\alpha_s^{N+1}(\mu_r)\right)$$

uncertainty is formally of higher order

-> gets smaller if higher orders are known

applies in general also for μ_f

explicit example: scale dependence of e^+e^- —> jets

recall: at NLO we have

$$\sigma^{\mathrm{NLO}}(\mu_R) = \sigma_{q\bar{q}}\left(1 + c_1\, \alpha_s(\mu_R)\right)$$

LO result

NLO coefficient
independent of scale

all **scale uncertainty**
from strong coupling

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recall: $$\alpha_s(\mu_r^2) = \frac{\alpha_s(Q^2)}{1 + 2b_0\alpha_s(Q^2)\ln(\mu_r/Q)}$$ coupling small expand $$\approx \alpha_s(Q^2) - 2b_0\,\alpha_s^2(Q^2)\,\ln(\mu_r/Q)$$

plug back into $\sigma^{\rm NLO}$ $$= \sigma_{q\bar{q}}\left(1\ +\ c_1\,\alpha_s(Q) - 2c_1 b_0 \ln\frac{\mu_R}{Q}\,\alpha_s^2(Q) + \mathcal{O}\left(\alpha_s^3\right)\right)$$

explicit example: scale dependence of e^+e^- —> jets

recall: at NLO we have $$\sigma^{\mathrm{NLO}}(\mu_R) = \sigma_{q\bar{q}}\left(1 + c_1\, \alpha_s(\mu_R)\right)$$

LO result

NLO coefficient
independent of scale

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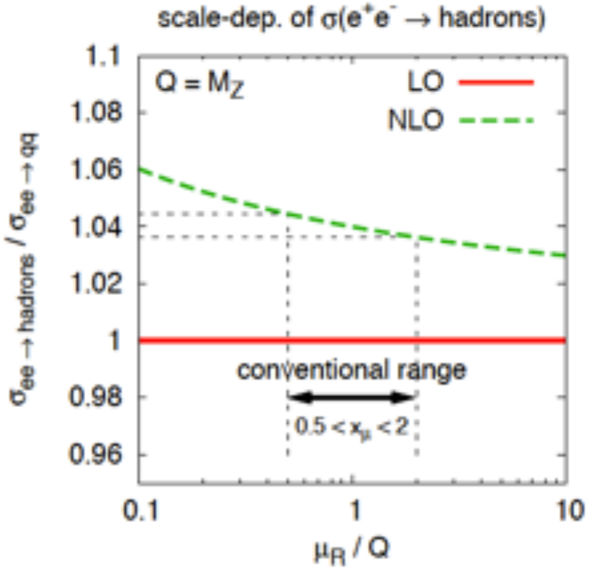

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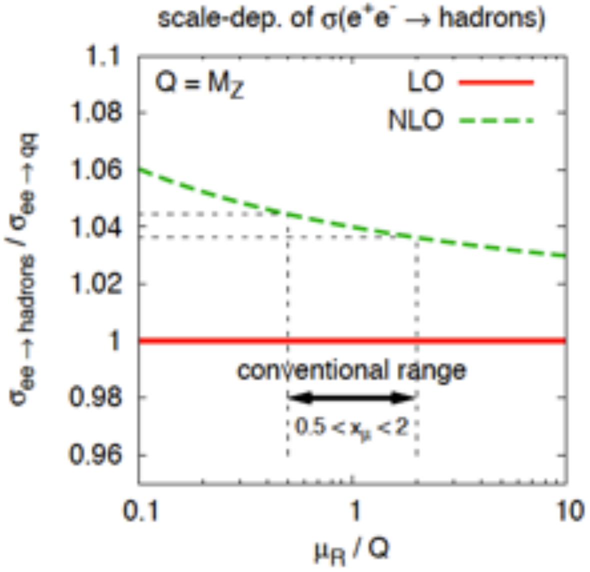

LO is a pure el-mag process, no α_s , no scales

note: the scale ambiguity gets amplified if we ask for more than two jets at LO

$\alpha_s^n(\mu_r) \approx \alpha_s^n(Q^2)(1 - 2n\, b_0\, \alpha_s(Q^2)\,\ln(\mu_r/Q) + \ldots)$

explicit example - cont'd

next calculate full NNLO result:

$$\sigma^{\mathrm{NNLO}}(\mu_R) = \sigma_{q\bar{q}}\ \left[1\ +\ c_1\,\alpha_s(\mu_R) + c_2(\mu_R)\alpha_s^2(\mu_R)\right]$$

NNLO term starts to depend on the scale

explicit example - cont'd

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in fact c_2 must (and will !) cancel the scale ambiguity found at NLO:

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explicit example - cont'd

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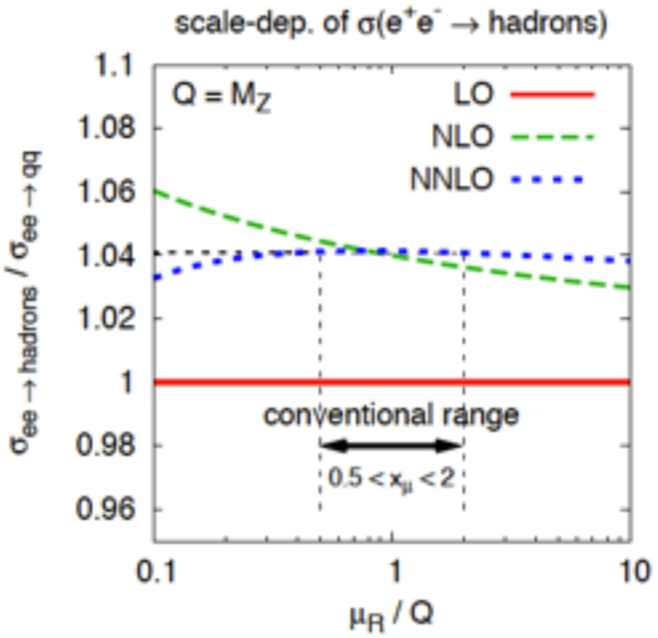

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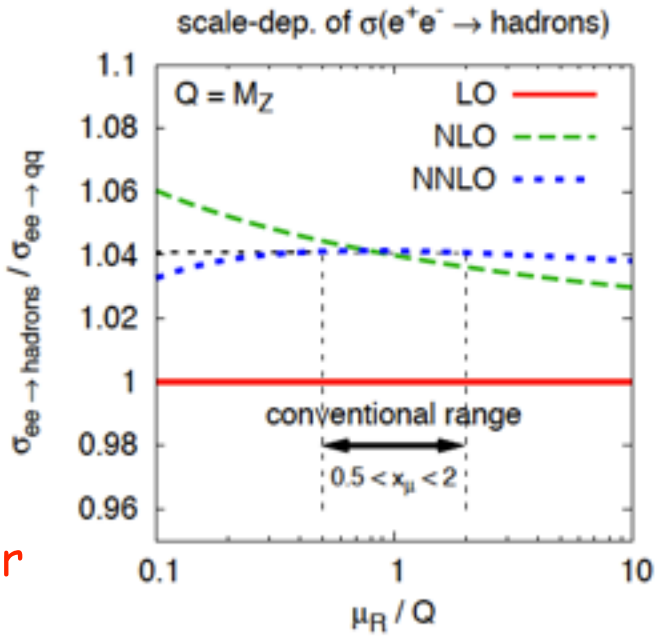

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explicit example - cont'd

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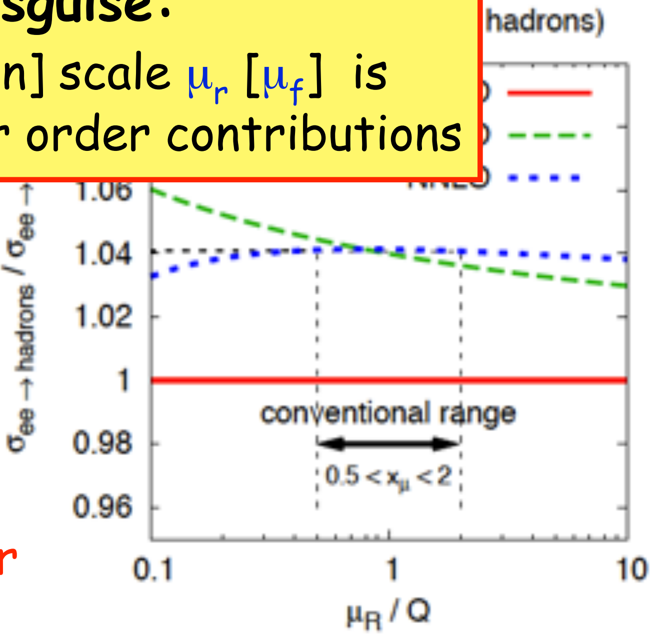
in fact c_2 must (and will !) cancel the scale ambiguity found at NLO:

scale "ambiguity" is a blessing in disguise:
varying the renormalization [factorization] scale μ_r [μ_f] is a way of guessing yet uncalculated higher order contributions



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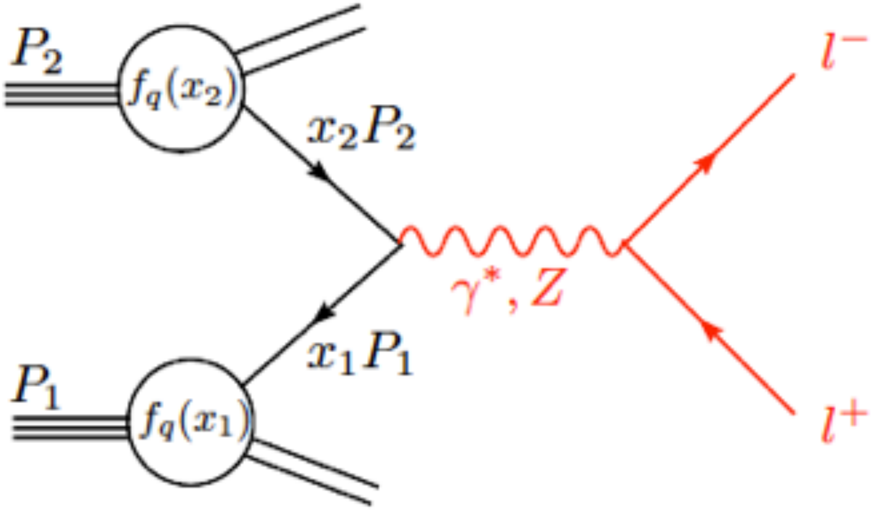
example from hadronic collisions

take the "classic" Drell Yan process



- dominated by quarks in the initial-state
- at LO no colored particles in the final-state
- clean experimental signature
- at LO an electromagnetic process (low rate)
- one of the best studied processes (known to NNLO)

as "clean" as it can get at a hadron collider

uncertainties for the Drell Yan process – cont'd

at NLO:

LO piece

$$\sigma^{\mathrm{NLO}}_{pp\to Z} = \sum_{i,j} \int dx_1 dx_2\, f_i(x_1,\mu_F^2)\, f_j(x_2,\mu_F^2)\, [\hat{\sigma}_{0,ij\to Z}(x_1,x_2) + \\ + \alpha_s(\mu_R)\hat{\sigma}_{1,ij\to Z}(x_1,x_2,\mu_F)]$$

- no α_s at LO but μ_F appears in PDFs
- α_s enters at NLO and hence μ_R
- NLO terms reduce dep. on μ_F

uncertainties for the Drell Yan process – cont’d

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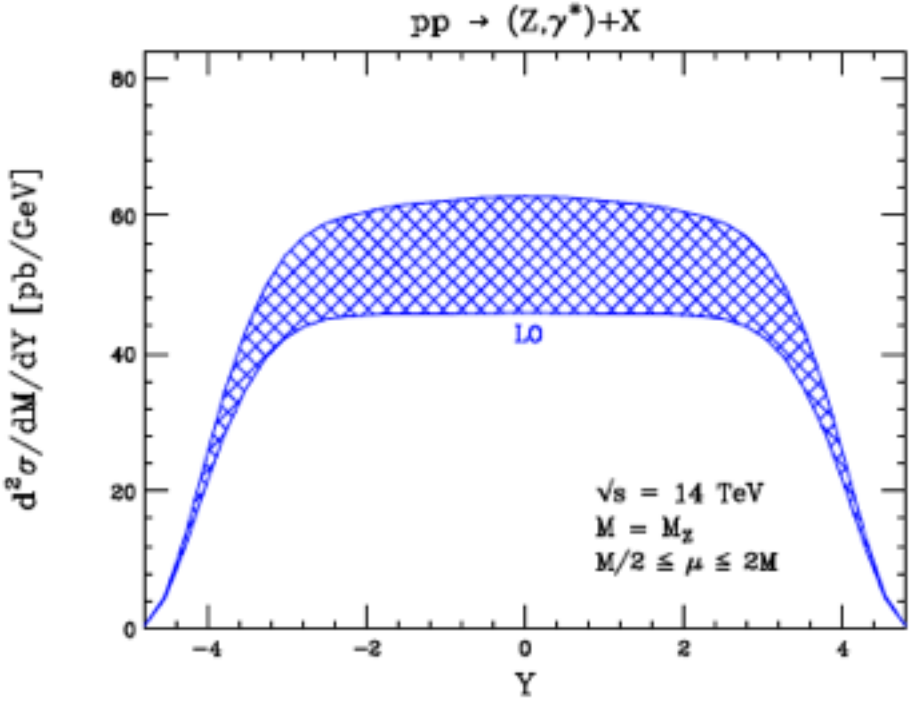

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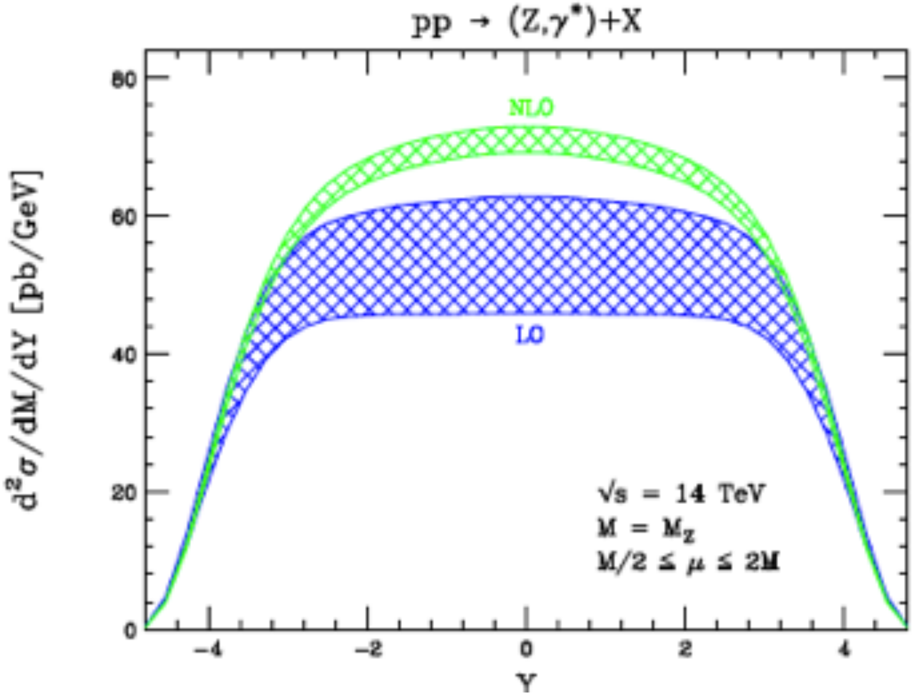

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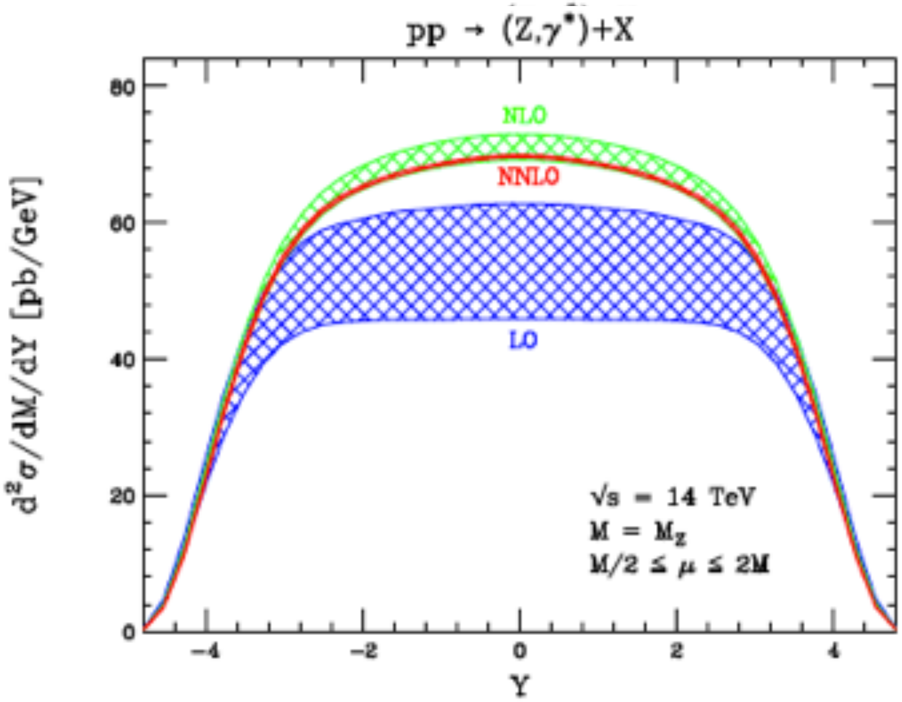

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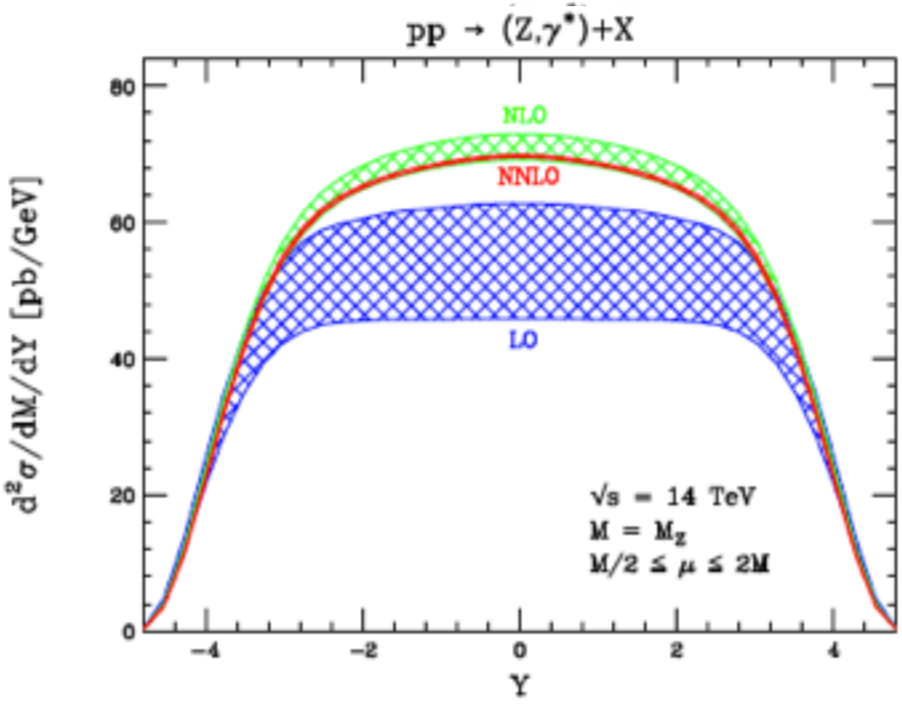
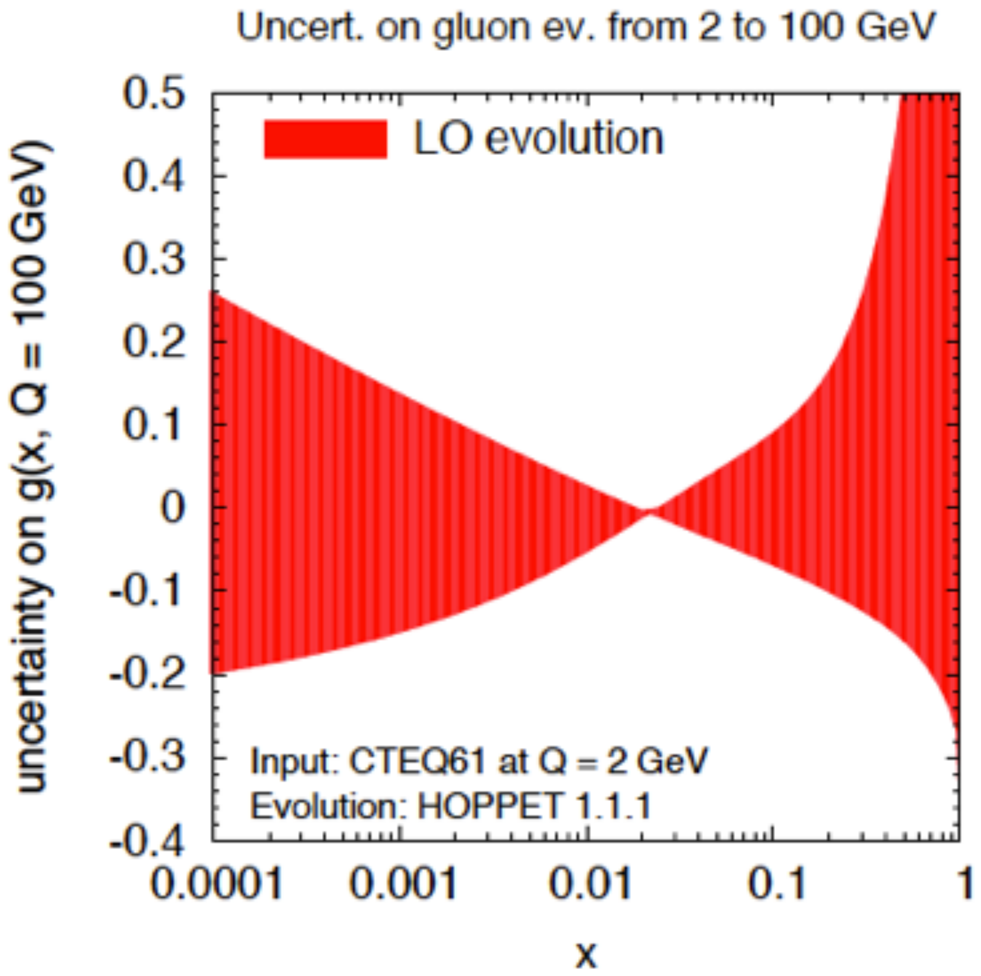


perturbative accuracy of O(percent) achieved

changing scales in DGLAP evolution

estimate by G. Salam: vary the scale of α_s in the DGLAP kernel



- about 30% in LO

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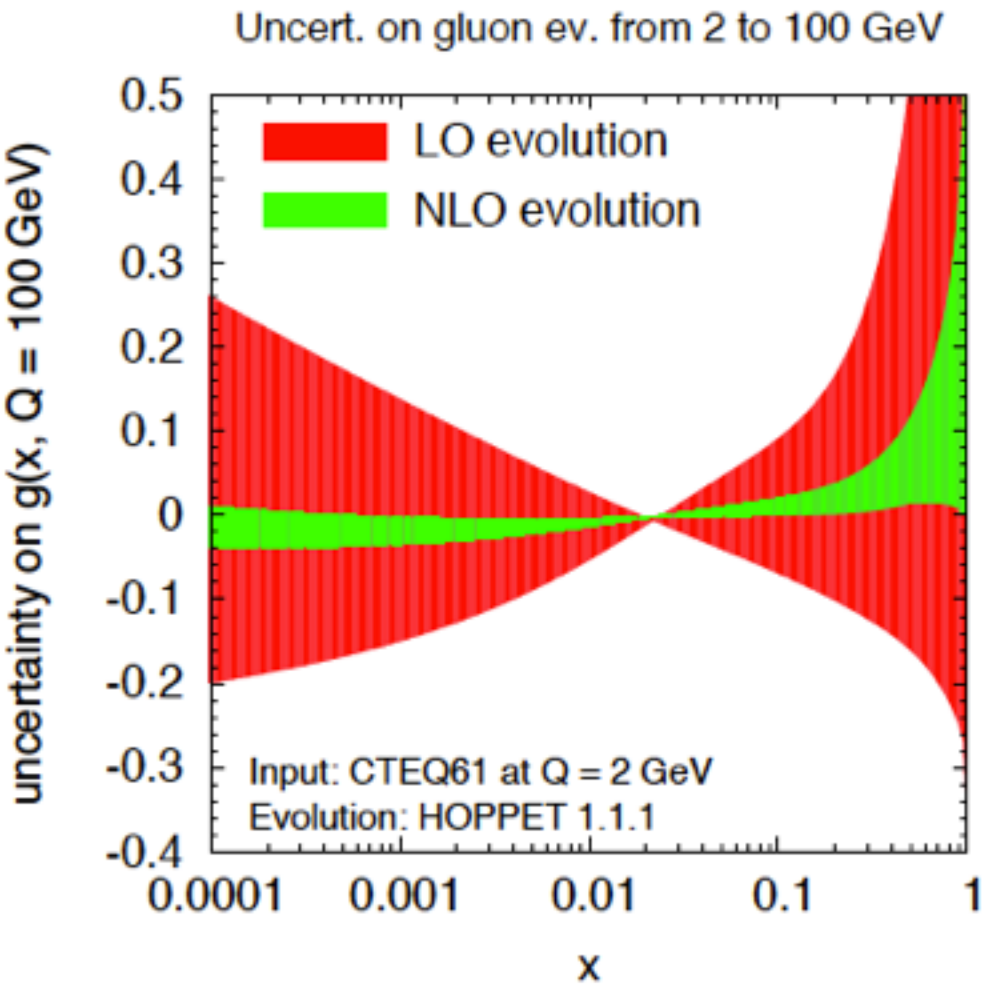

- about 30% in LO
- down to about 5% in NLO

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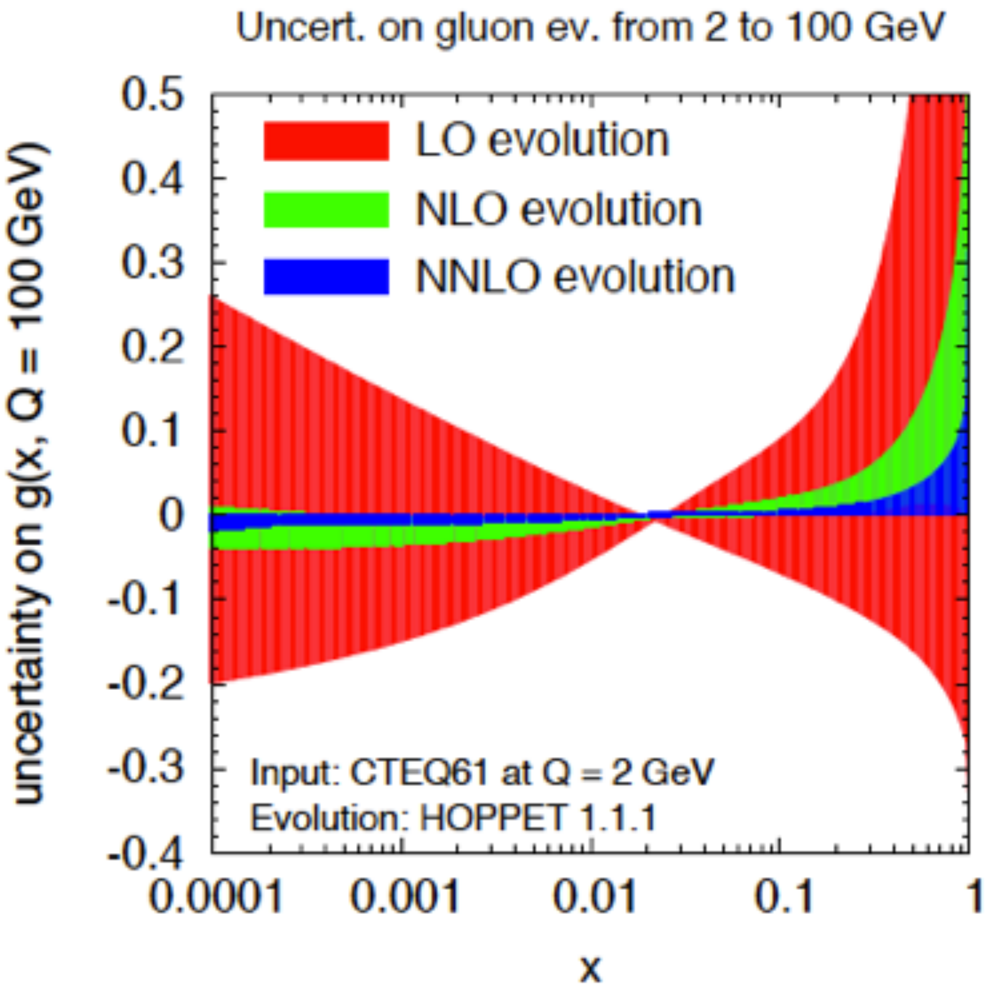

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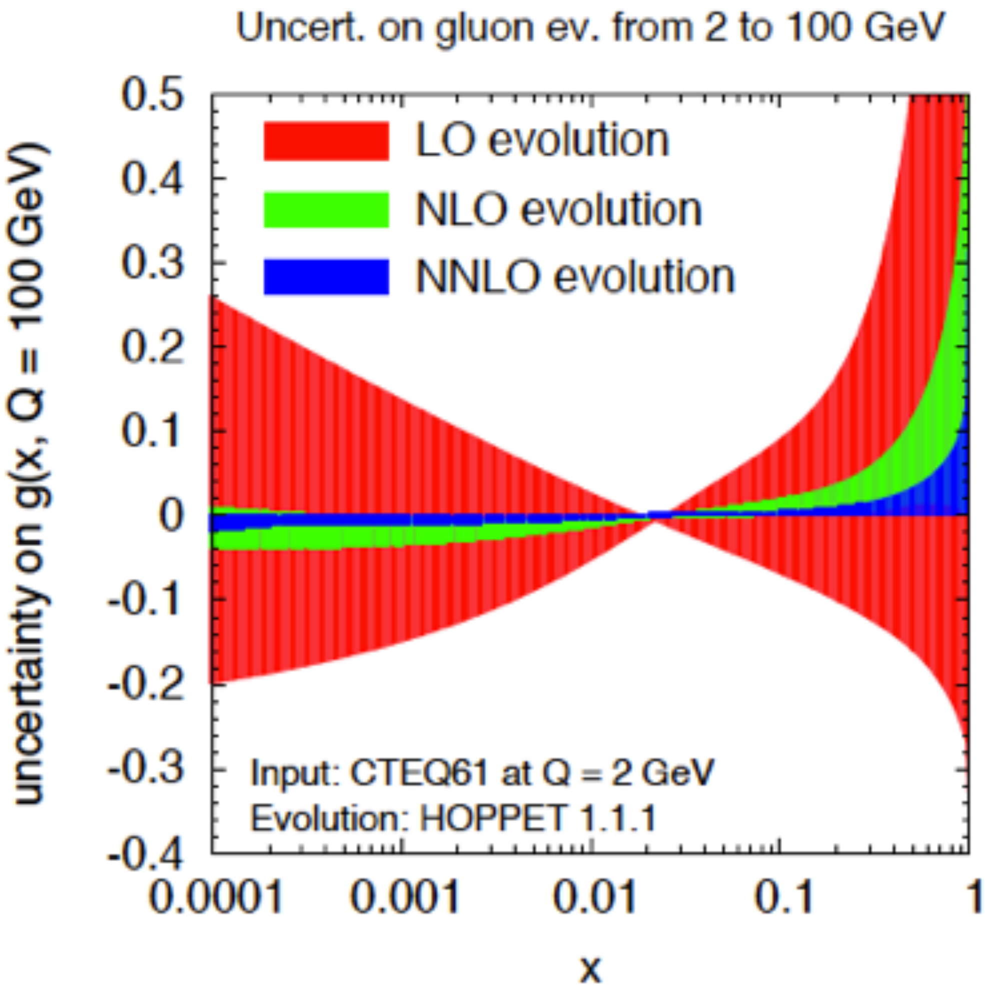

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  which is about the precision of the HERA DIS data

other motivations for NLO and beyond

- much more realistic final states, e.g., more partons can form a jet

LO NLO NNLO

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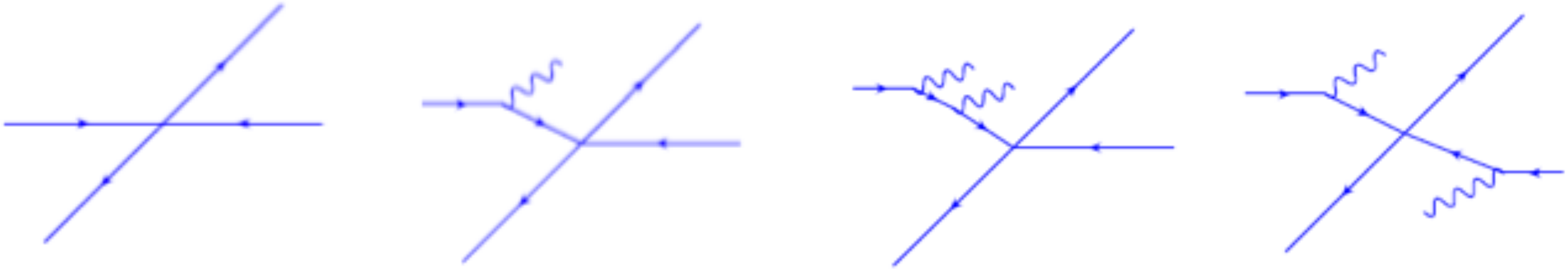
- higher orders generate non-trivial k_T effects/dependence



state of the art – the current precision frontier

	2->1	2->2	2->3	2->4	2->5	2->6
1	LO					
α	NLO	LO				
α	NNLO	NLO	LO			
α		NNLO	NLO	LO		
α			NNLO	NLO	LO	
α					NLO	LO

green: done

red-green: partially done

red: barely touched yet

table presumably
already outdated

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NLO some 2 → 3 results: pp → jjj, Hjj, VVV, …

some 2 → 4 results: pp → VVjj, Hjjj, ttbb, ttjj,Vjjj, VVbb; also Wjjjj

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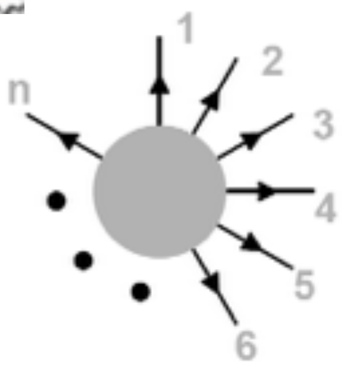
NNLO Drell-Yan type 2 → 1 processes (total and differential cross sections);
splitting functions; e^+e^- -> jjj; progress towards general 2 → 2 processes

NNLO including heavy flavor production (σ_{tot} at NNLO done)

new computational techniques & tools emerging

traditional Feynman diagram technique still going strong
but becomes very clumsy for high-multiplicity processes:



n	# diags
4	4
5	25
6	220
7	2485
8	34300
9	559405
10	10525900

rapid growth in complexity,
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→ new ways to compute amplitudes?

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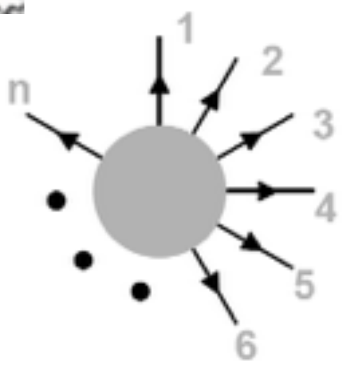

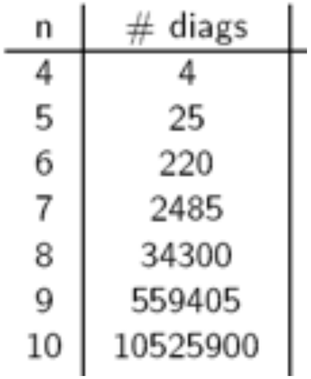

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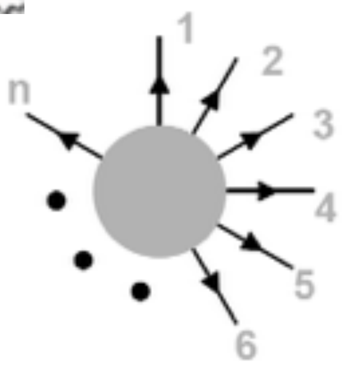

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amazing progress in a short time (few years) guided by two principles:

The best way to have a good idea is to have a lot of ideas --- Linus Pauling
Those are my principles, and if you don't like them … well, I have others --- Groucho Marx

currently aiming at full automatization at 1-loop level

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for some ideas, see:

Berends, Giele 1988 - recursion relations (off-shell)

Britto, Cachazo, Feng 2004 – recursion relations (on-shell) / unitarity

Cachazo, Svrcek, Witten 2004 – MHV amplitudes

Ossola, Pittau, Papadopolous 2006 – NLO loop integrals w/o doing integrals

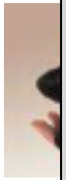
recent report on unitarity method:

Ellis, Kunszt, Melnikov, Zanderighi, arXiv:1105.4319

ple

les?

id



ols

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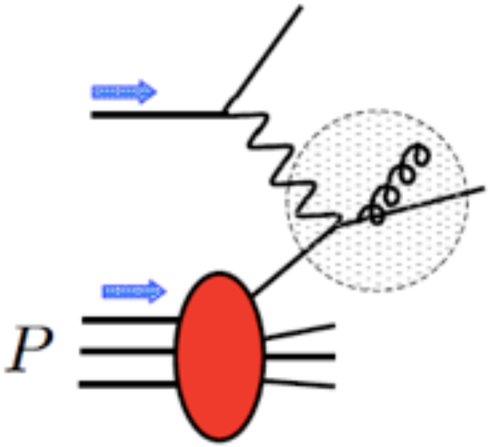
4-2

Anatomy of a Global QCD PDF Analysis

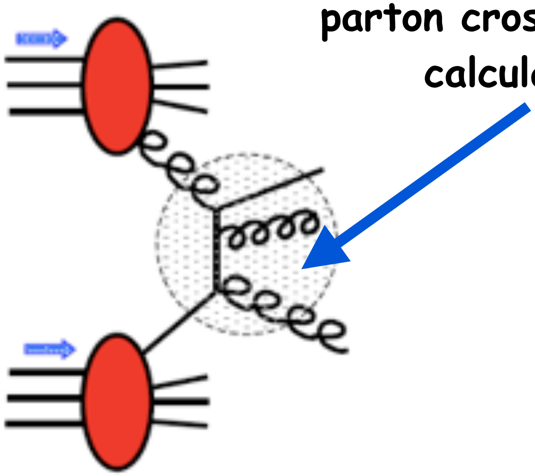
how to determine PDFs from data?

probes:



DIS

hard scale Q



parton cross section calculable

hadron-hadron

hard scale p_T

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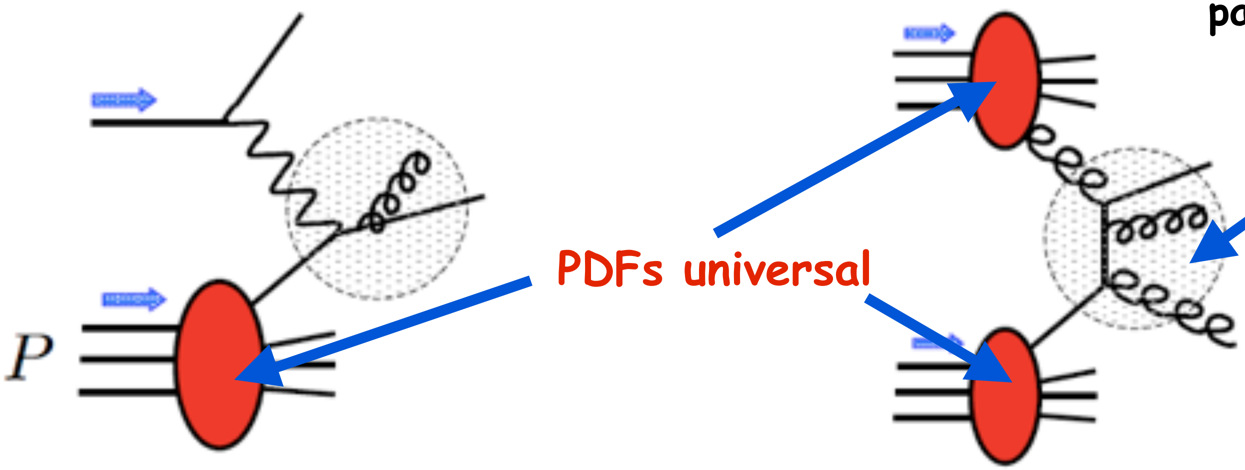

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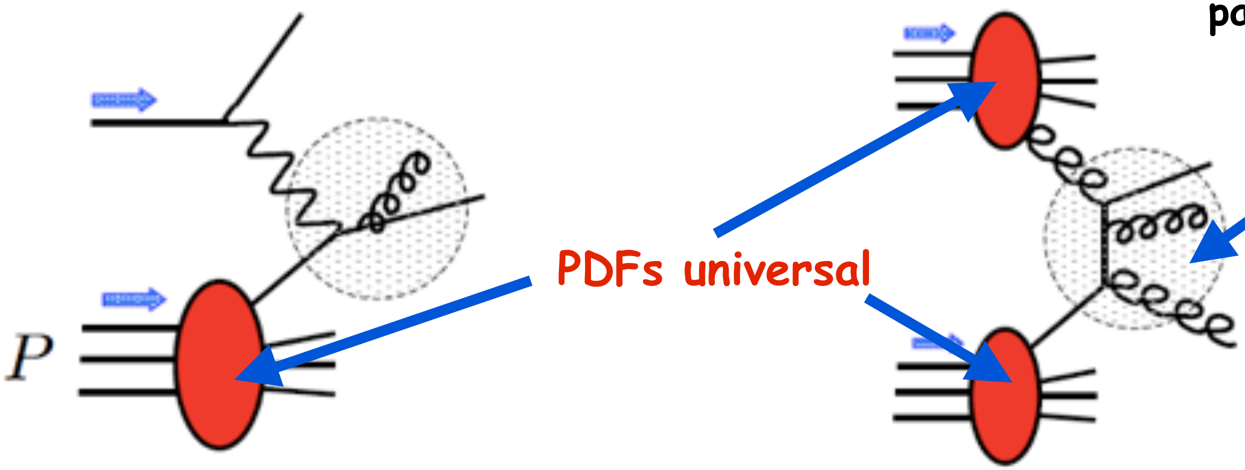


PDFs universal

parton cross section calculable

DIS

hard scale Q

hadron-hadron

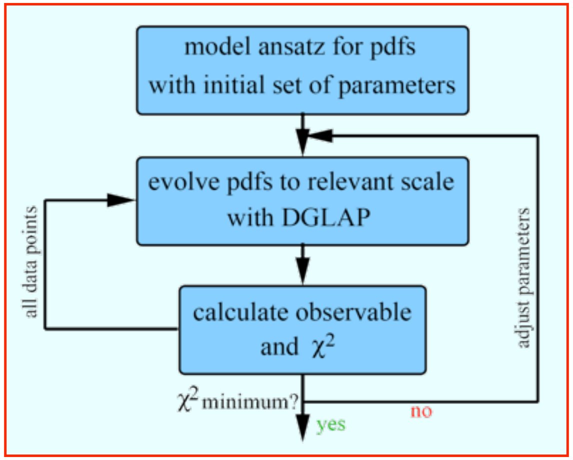
hard scale p_T

task: extract PDFs and their uncertainties (assume factorization)

- all processes tied together: universality of pdfs & Q^2 - evolution
- each reaction provides insights into different aspects and kinematics
- need at least NLO accuracy for quantitative analyses
- information on PDFs “hidden” inside complicated (multi-)convolutions

anatomy of global PDF analyses

obtain PDFs
through global χ^2 optimization



set of **optimum parameters**
for *assumed* functional form

computational challenge:

- up to O(20-30) parameters
- many sources of uncertainties
- very time-consuming NLO expressions

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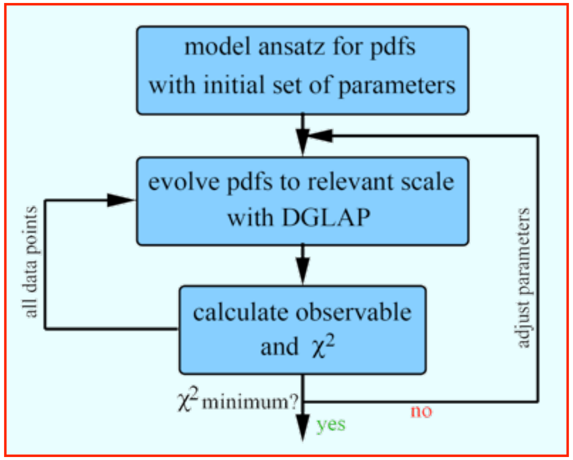

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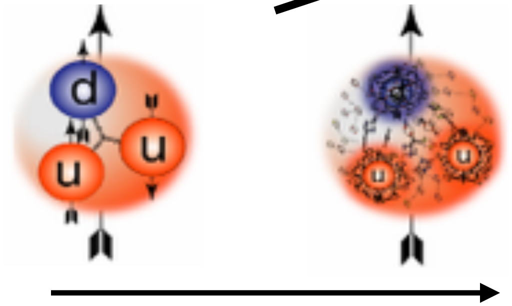
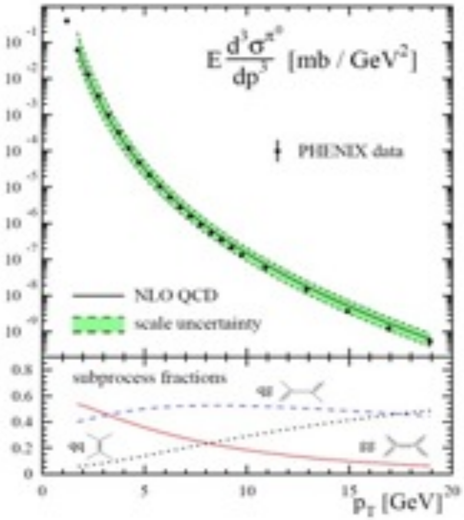
for *assumed* functional form

evolution kernels



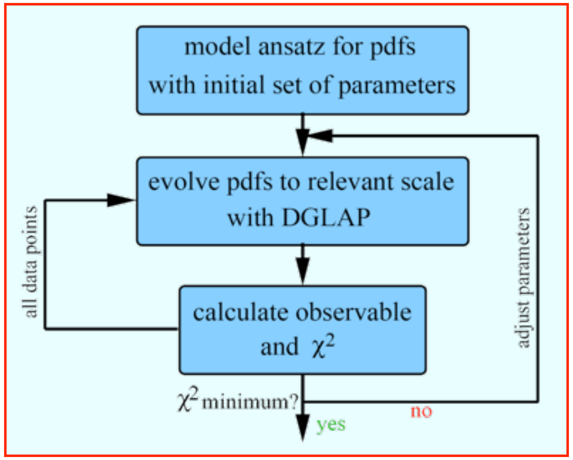
"resolution scale" μ

anatomy of global QCD analyses



cross sections at NLO

obtain PDFs
through global χ^2 optimization



set of **optimum parameters**
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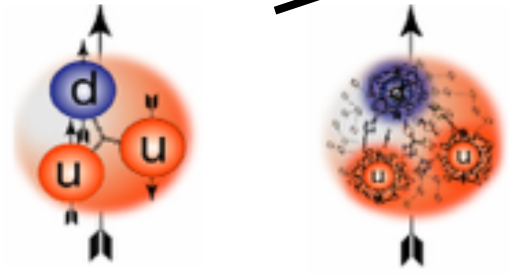

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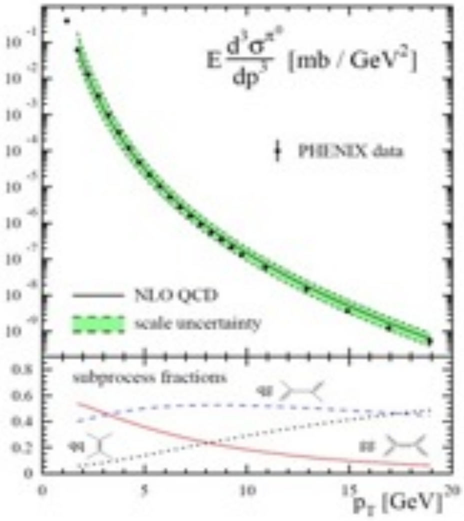

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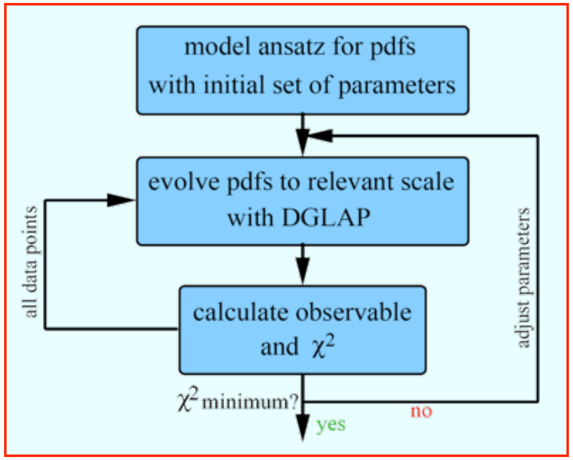

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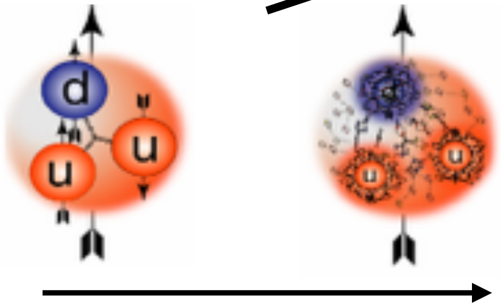
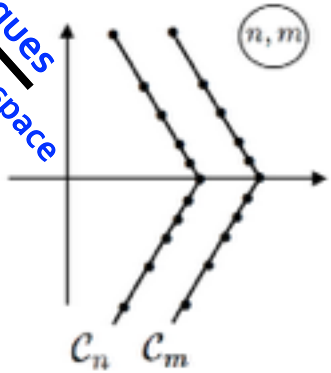
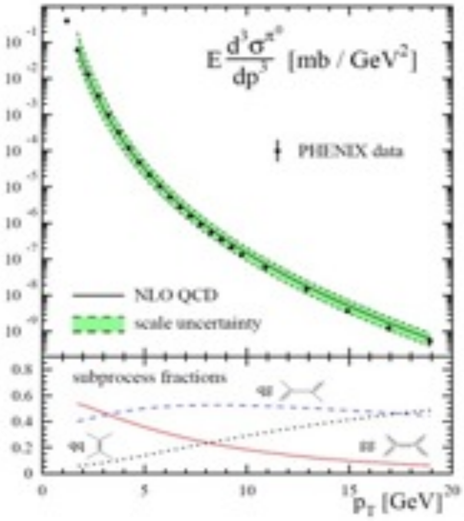


"resolution scale" μ

novel techniques
e.g. in complex Mellin space



anatomy of global QCD analyses



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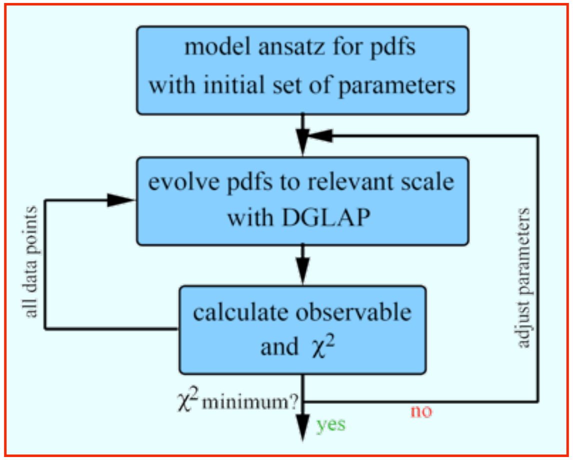
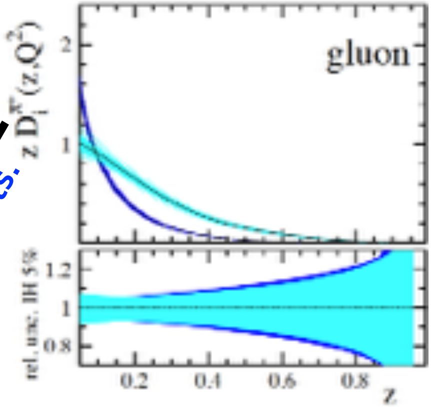


non-pert. inputs
e.g. frag. fcts.



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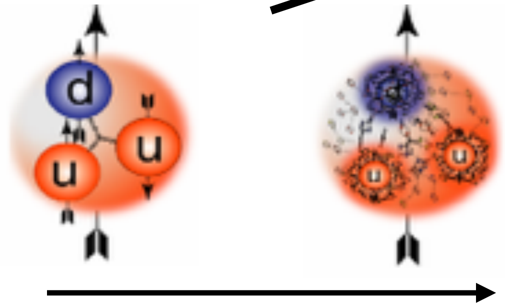

"resolution scale" μ

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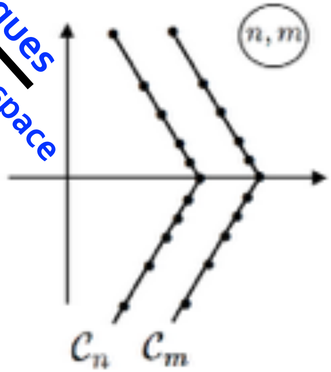
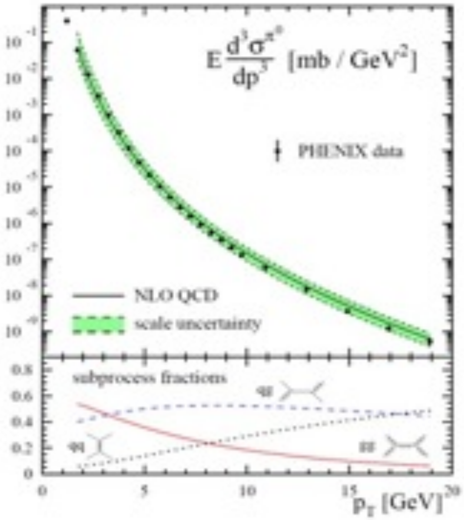


anatomy of global QCD analyses



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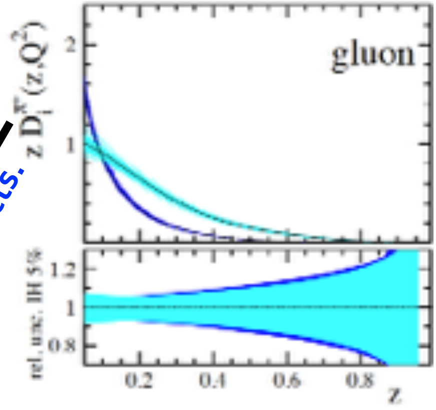

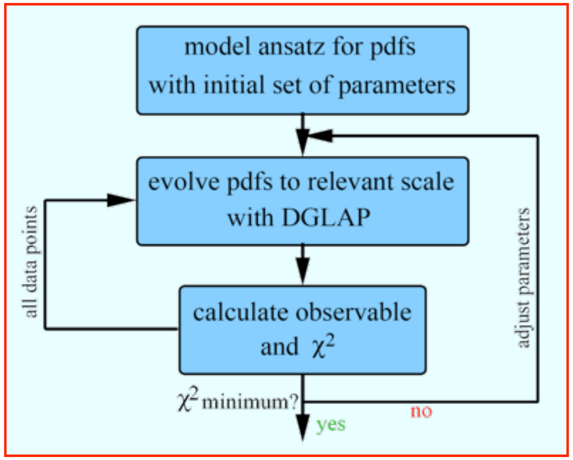

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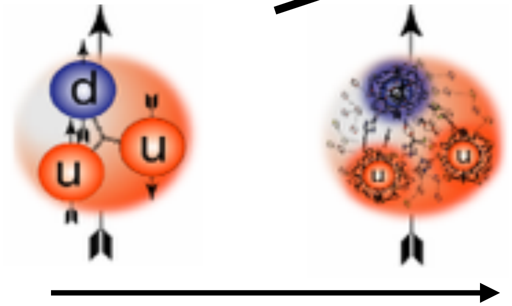
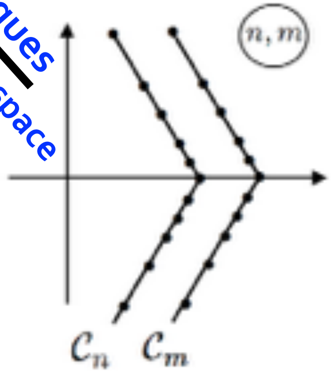


“resolution scale” μ

plus a prescription to
estimate & propagate
uncertainties



global analysis: computational challenge

- one has to deal with **O(2800) data points** from many processes and experiments
- need to determine **O(20-30) parameters** describing PDFs at μ_0
- NLO expressions often very complicated → computing time becomes excessive
  → develop **sophisticated algorithms & techniques**, e.g., based on Mellin moments
  Kosower; Vogt; Vogelsang, MS

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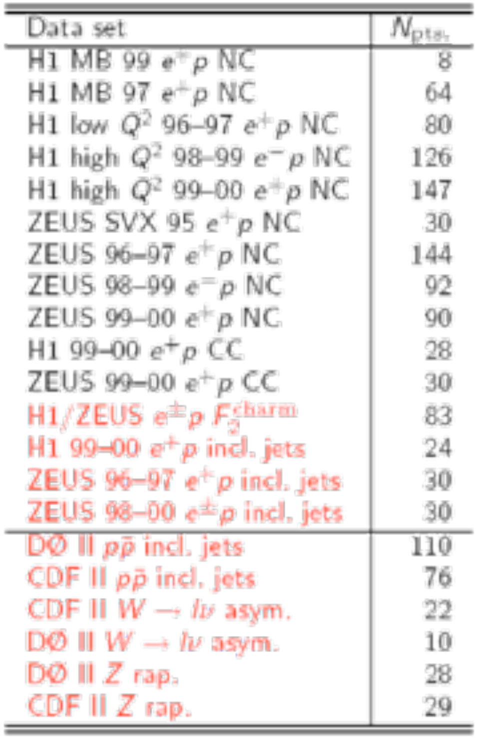
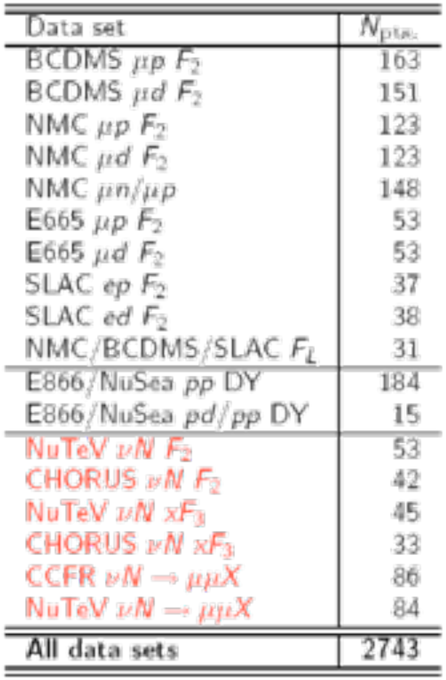
Kosower; Vogt; Vogelsang, MS

data sets & (x,Q^2) coverage used in MSTW fit

Martin, Stirling, Thorne, Watt, arXiv:0901.0002

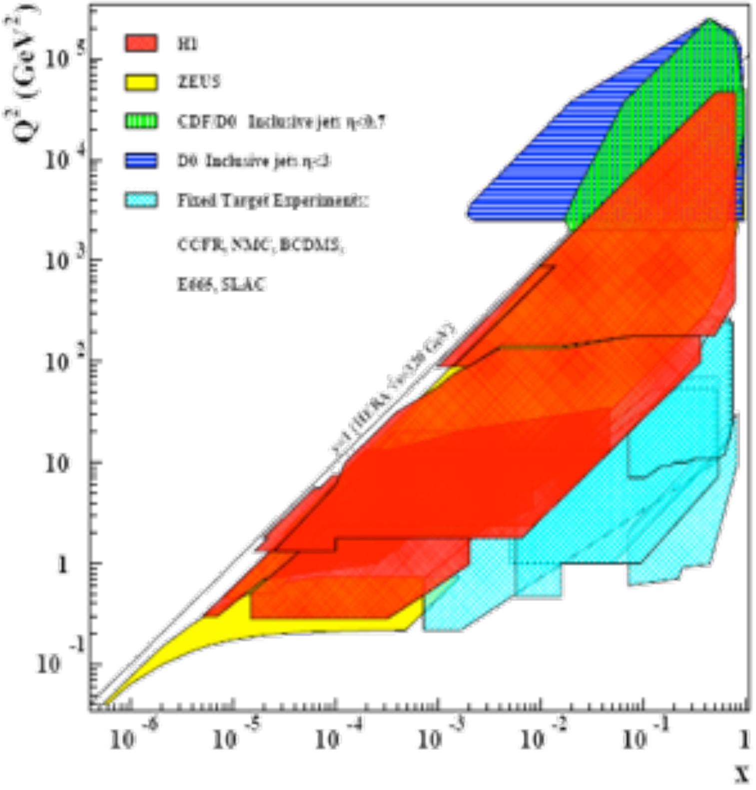


Data set	$N_{\rm pts.}$
H1 MB 99 e^+p NC	8
H1 MB 97 e^+p NC	64
H1 low Q^2 96–97 e^+p NC	80
H1 high Q^2 98–99 e^-p NC	126
H1 high Q^2 99–00 e^+p NC	147
ZEUS SVX 95 e^+p NC	30
ZEUS 96–97 e^+p NC	144
ZEUS 98–99 e^-p NC	92
ZEUS 99–00 e^+p NC	90
H1 99–00 e^+p CC	28
ZEUS 99–00 e^+p CC	30
H1/ZEUS $e^\pm p$ $F_2^{\rm charm}$	83
H1 99–00 e^+p incl. jets	24
ZEUS 96–97 e^+p incl. jets	30
ZEUS 98–00 $e^\pm p$ incl. jets	30
DØ II $p\bar{p}$ incl. jets	110
CDF II $p\bar{p}$ incl. jets	76
CDF II $W \to l\nu$ asym.	22
DØ II $W \to l\nu$ asym.	10
DØ II Z rap.	28
CDF II Z rap.	29

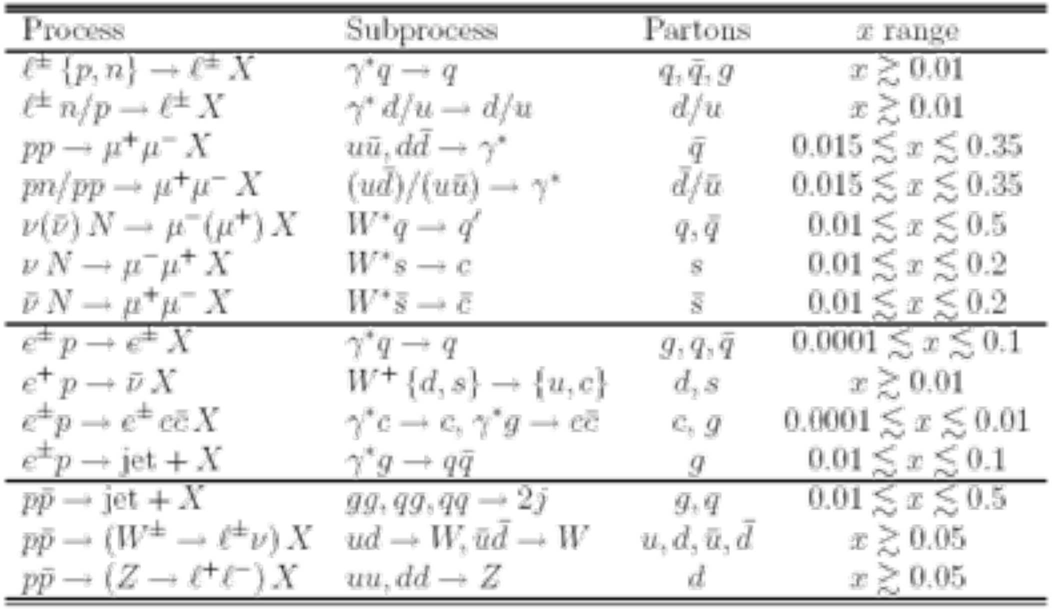


Data set	$N_{\rm pts.}$
BCDMS μp F_2	163
BCDMS μd F_2	151
NMC μp F_2	123
NMC μd F_2	123
NMC $\mu n/\mu p$	148
E665 μp F_2	53
E665 μd F_2	53
SLAC ep F_2	37
SLAC ed F_2	38
NMC/BCDMS/SLAC F_L	31
E866/NuSea pp DY	184
E866/NuSea pd/pp DY	15
NuTeV νN F_2	53
CHORUS νN F_2	42
NuTeV νN xF_3	45
CHORUS νN xF_3	33
CCFR $\nu N \to \mu\mu X$	86
NuTeV $\nu N \to \mu\mu X$	84
All data sets	**2743**

- Red = New w.r.t. MRST 2006 fit.



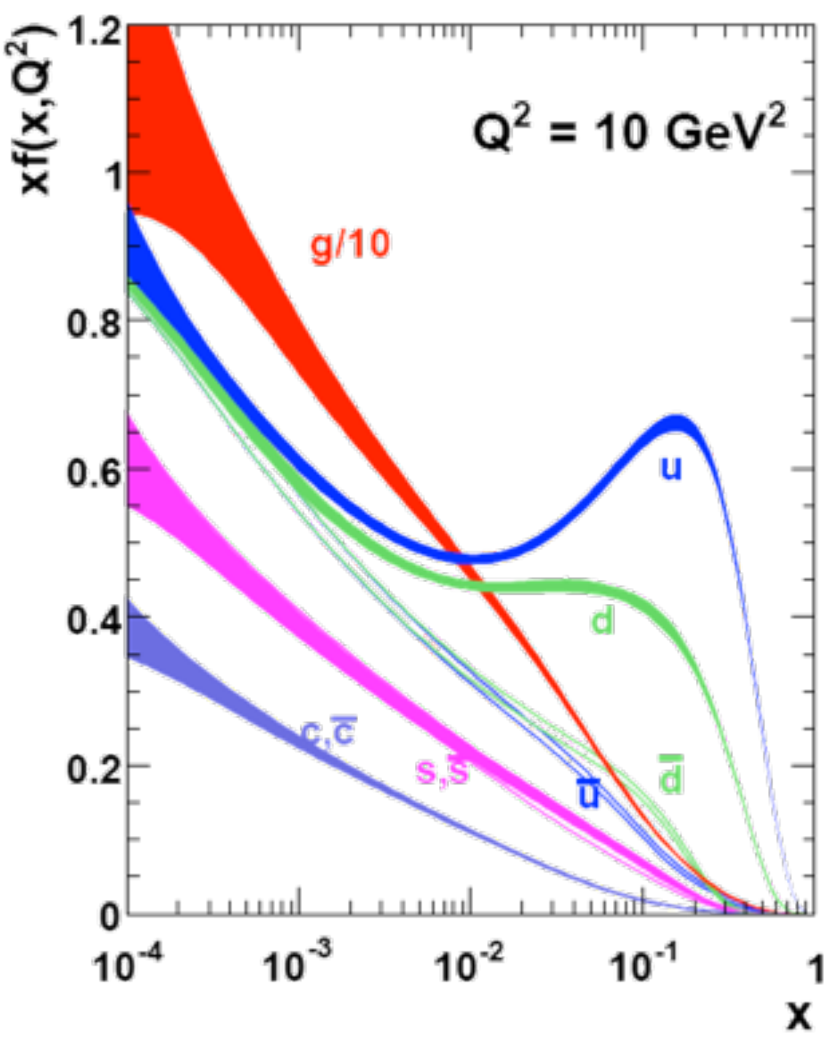
which data sets determine which partons



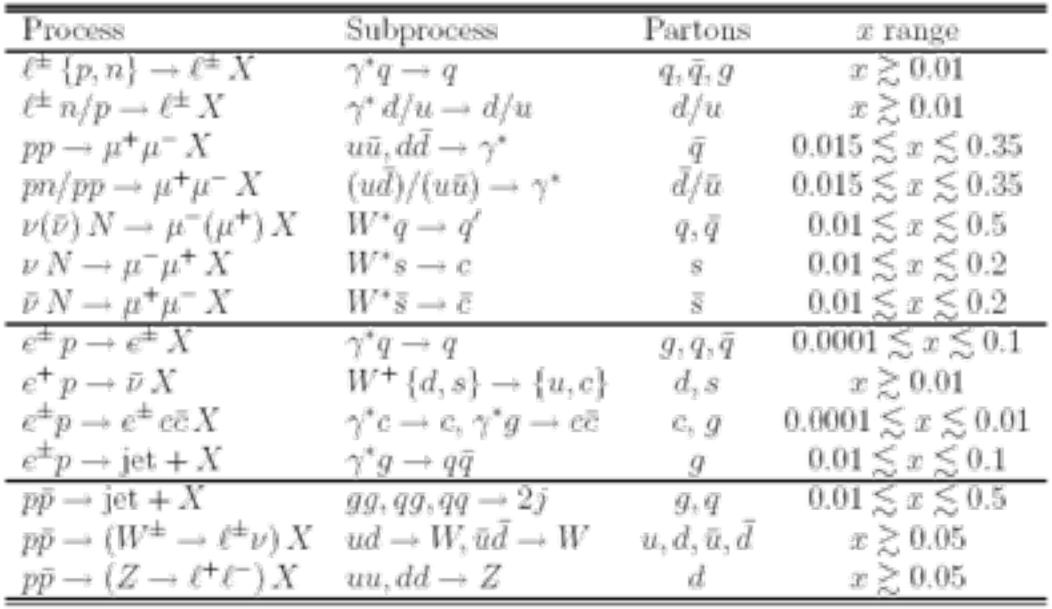
Process	Subprocess	Partons	x range
$\ell^\pm \{p,n\} \to \ell^\pm X$	$\gamma^* q \to q$	$q, \bar{q}, g$	$x \gtrsim 0.01$
$\ell^\pm n/p \to \ell^\pm X$	$\gamma^* d/u \to d/u$	d/u	$x \gtrsim 0.01$
$pp \to \mu^+\mu^- X$	$u\bar{u}, d\bar{d} \to \gamma^*$	$\bar{q}$	$0.015 \lesssim x \lesssim 0.35$
$pn/pp \to \mu^+\mu^- X$	$(u\bar{d})/(u\bar{u}) \to \gamma^*$	$\bar{d}/\bar{u}$	$0.015 \lesssim x \lesssim 0.35$
$\nu(\bar{\nu})\, N \to \mu^-(\mu^+)\, X$	$W^* q \to q'$	$q, \bar{q}$	$0.01 \lesssim x \lesssim 0.5$
$\nu\, N \to \mu^-\mu^+ X$	$W^* s \to c$	s	$0.01 \lesssim x \lesssim 0.2$
$\bar{\nu}\, N \to \mu^+\mu^- X$	$W^* \bar{s} \to \bar{c}$	$\bar{s}$	$0.01 \lesssim x \lesssim 0.2$
$e^\pm p \to e^\pm X$	$\gamma^* q \to q$	$g, q, \bar{q}$	$0.0001 \lesssim x \lesssim 0.1$
$e^+ p \to \bar{\nu} X$	$W^+ \{d,s\} \to \{u,c\}$	d, s	$x \gtrsim 0.01$
$e^\pm p \to e^\pm c\bar{c} X$	$\gamma^* c \to c,\ \gamma^* g \to c\bar{c}$	c, g	$0.0001 \lesssim x \lesssim 0.01$
$e^\pm p \to \text{jet} + X$	$\gamma^* g \to q\bar{q}$	g	$0.01 \lesssim x \lesssim 0.1$
$p\bar{p} \to \text{jet} + X$	$gg, qg, qq \to 2j$	g, q	$0.01 \lesssim x \lesssim 0.5$
$p\bar{p} \to (W^\pm \to \ell^\pm \nu)\, X$	$ud \to W, \bar{u}\bar{d} \to W$	$u, d, \bar{u}, \bar{d}$	$x \gtrsim 0.05$
$p\bar{p} \to (Z \to \ell^+\ell^-)\, X$	$uu, dd \to Z$	d	$x \gtrsim 0.05$

Martin, Stirling, Thorne, Watt, arXiv:0901.0002

NLO fit, 68% C.L.



which data sets determine which partons



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$e^{\pm} p \to e^{\pm} c\bar{c} X$	$\gamma^* c \to c, \gamma^* g \to c\bar{c}$	c, g	$0.0001 \lesssim x \lesssim 0.01$
$e^{\pm} p \to \text{jet} + X$	$\gamma^* g \to q\bar{q}$	g	$0.01 \lesssim x \lesssim 0.1$
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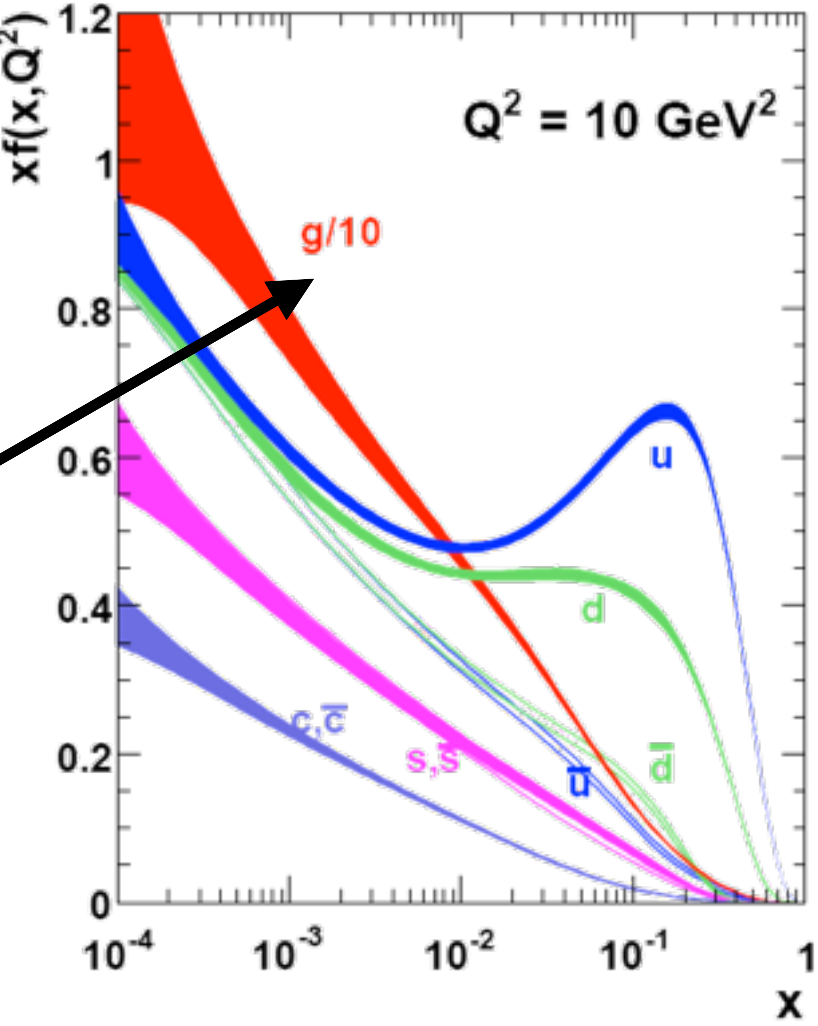


- notice the huge gluon distribution
- quality of the fit:

χ^2/ #data pts.

- 2543/2699 **NLO**
- 3066/2598 **LO**

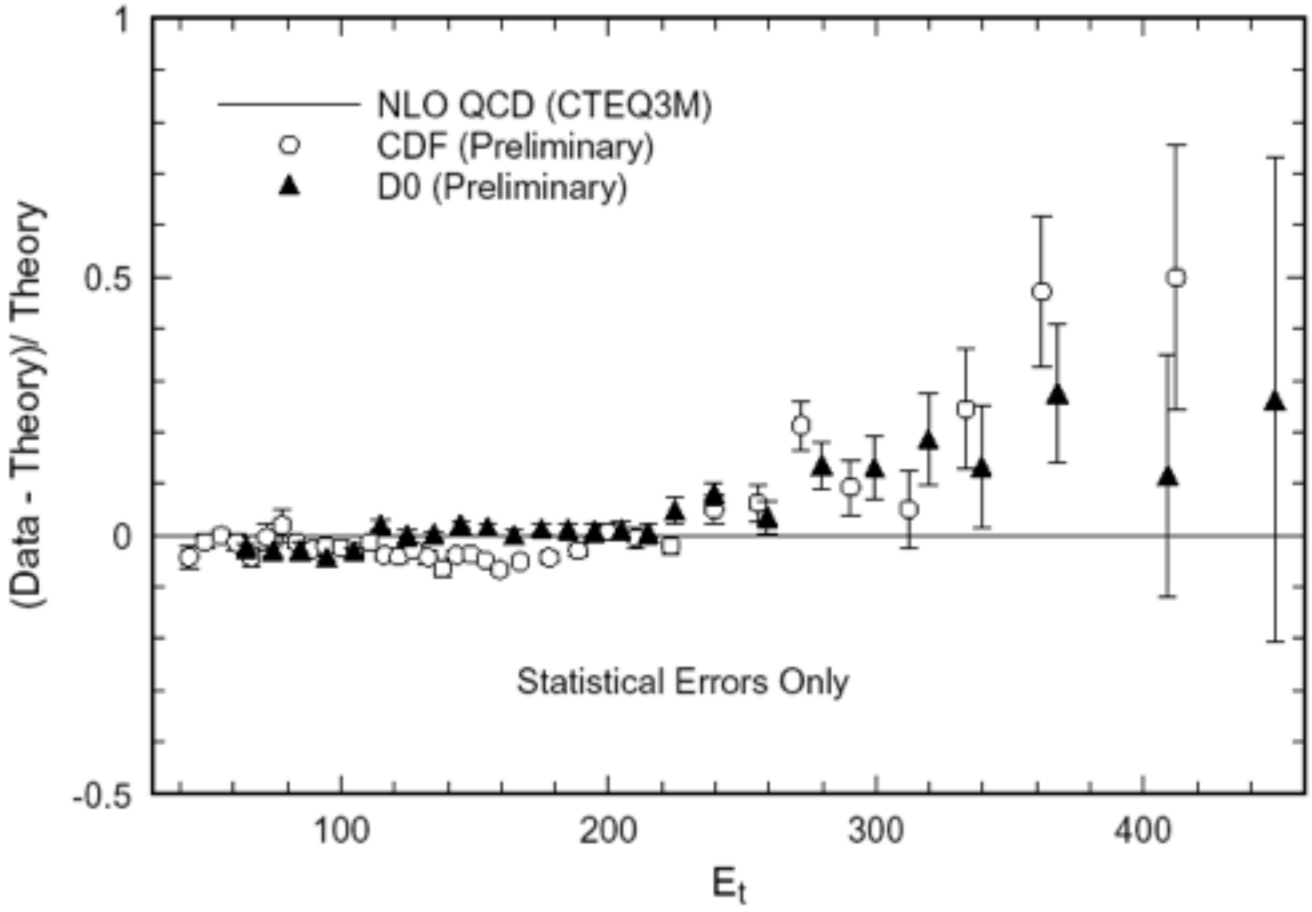
interplay of many data sets crucial


PARTON
Drive carefully
Burial place of
James Clerk Maxwell

from R.D. Ball

new physics or PDF uncertainties?

important lesson from the past: (in)famous TeVatron "excess" in jet yield



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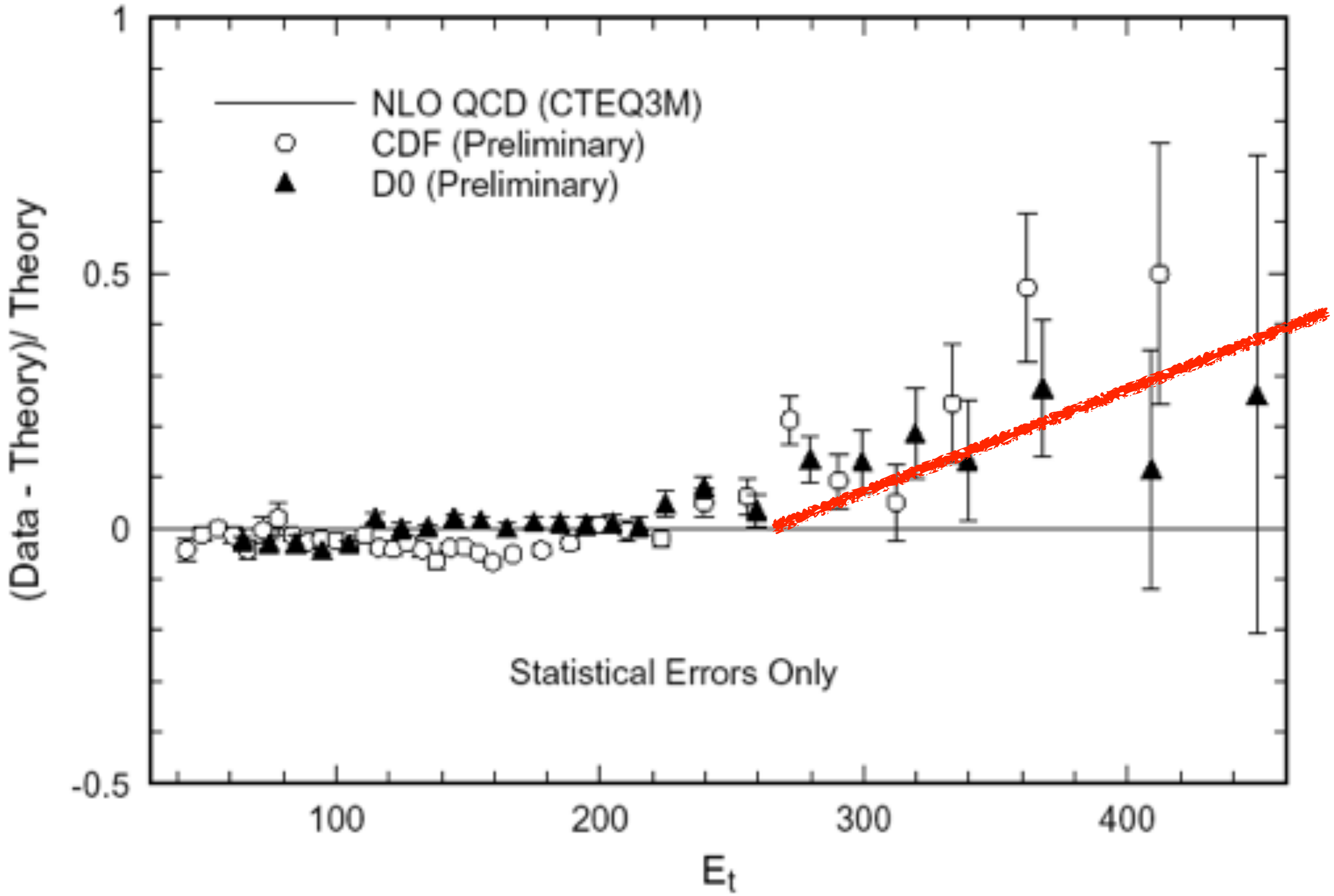

huge excess at large p_T

—> new particle with mass of a few 100 GeV?

arXiv got cluttered with New Physics papers

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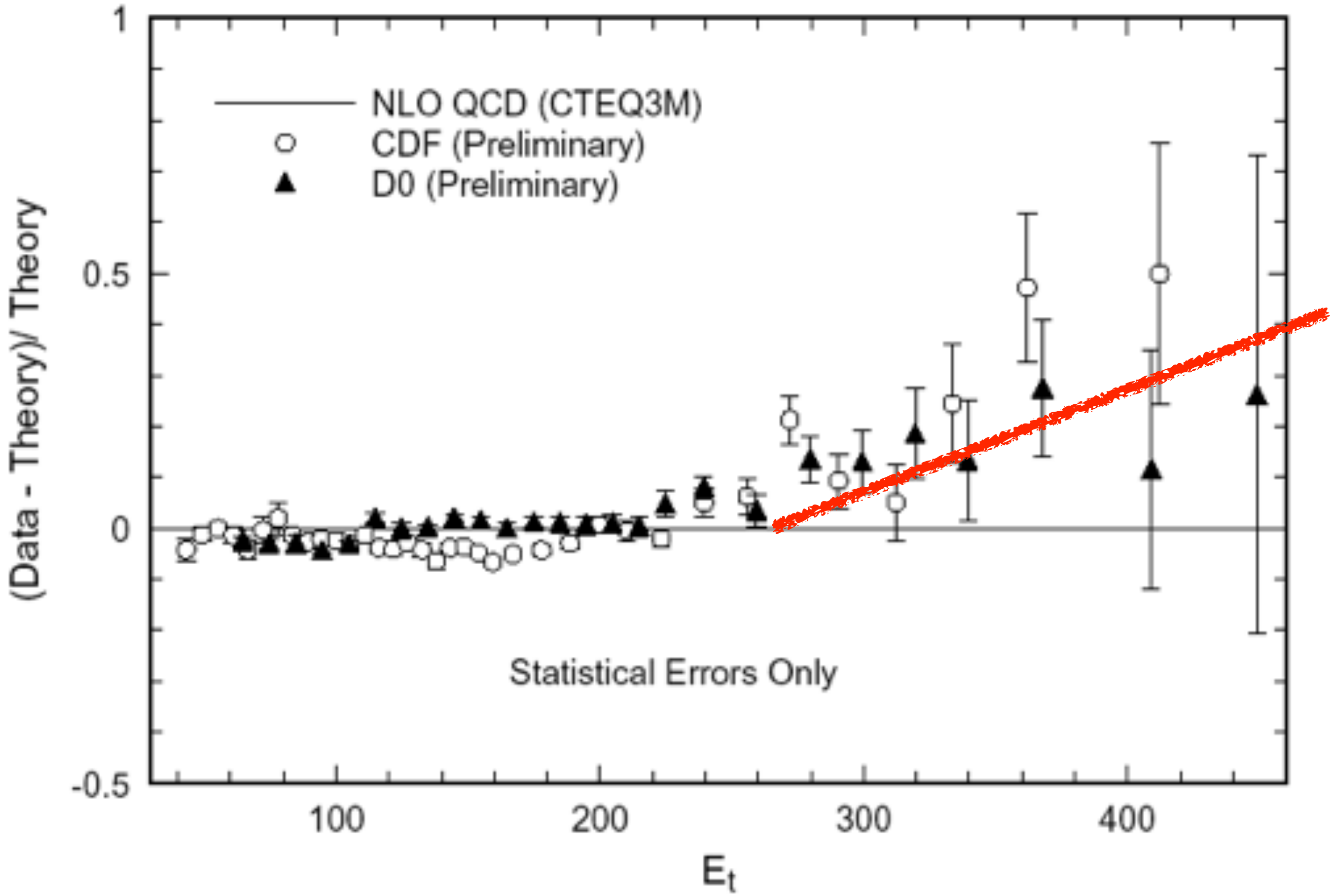

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… but you better think twice - a MUCH less mundane explanation is usually at work

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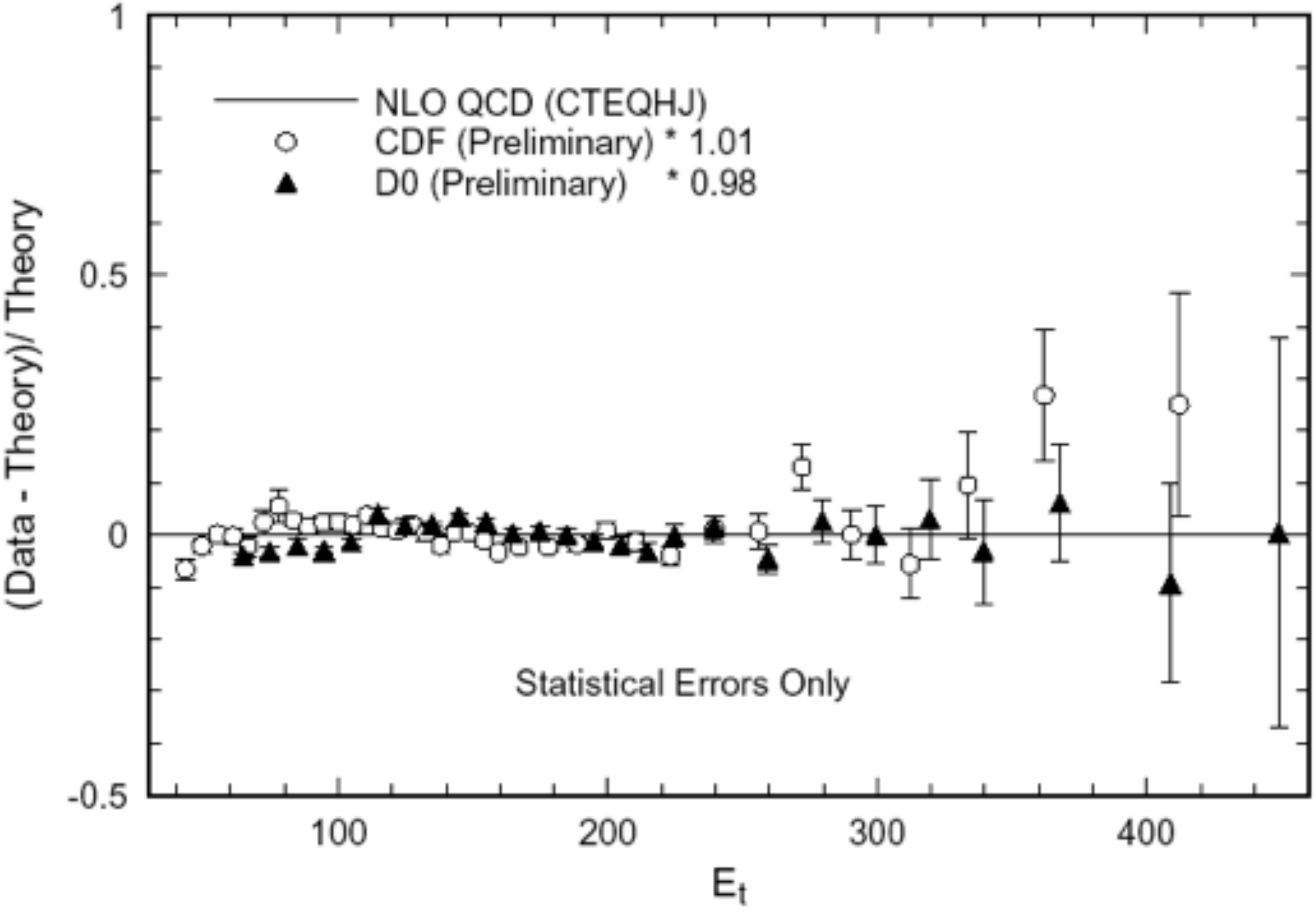

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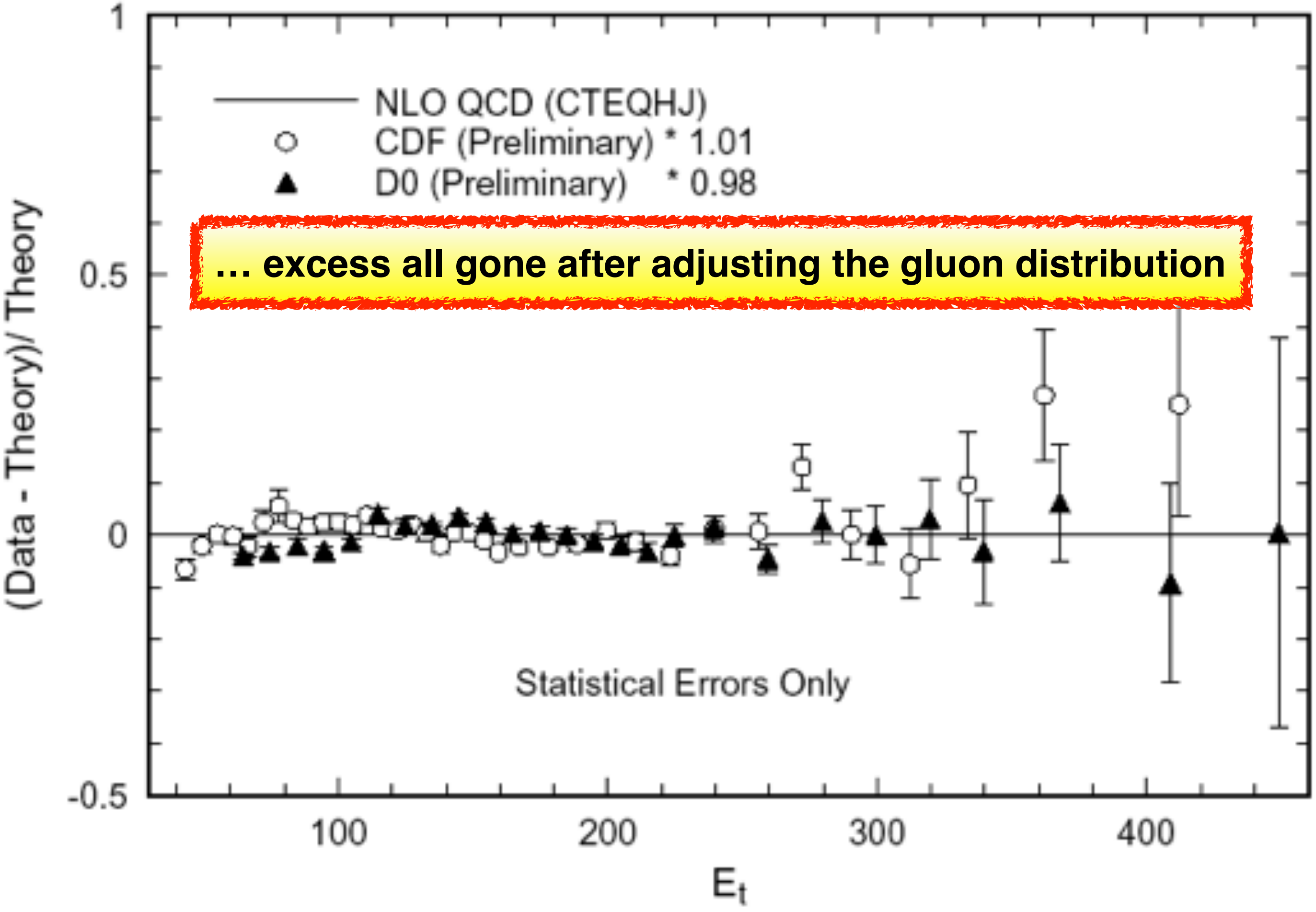


… excess all gone after adjusting the gluon distribution

what’s on the market?

set	H.O.	data	$\alpha_s(M_Z)@NNLO$	uncertainty	HQ	Comments
MSTW 2008	NNLO	DIS+DY+Jets	0.1171	Hessian (dynamical tolerance)	GM-VFN (ACOT+TR’)	old HERA DIS
CT10	NNLO	DIS+DY+Jets	0.118	Hessian (dynamical tolerance)	GM-VFN (SACOT-X)	New HERA DIS
NNPDF	NNLO	DIS+DY+Jets +LHC	0.1174	Monte Carlo	GM-VFN (FONLL)	New HERA DIS
ABKM	NNLO	DIS+DY(f.t.) +DY-tT(LHC)	0.1132	Hessian	FFN BMSN	New HERA DIS
(G)JR	NNLO	DIS+DY(f.t.)+ some jet	0.1124	Hessian	FFN (VFN massless)	valence like input pdfs
HERA PDF	NNLO	only DIS HERA	0.1176	Hessian	GM-VFN (ACOT+TR’)	Latest HERA DIS

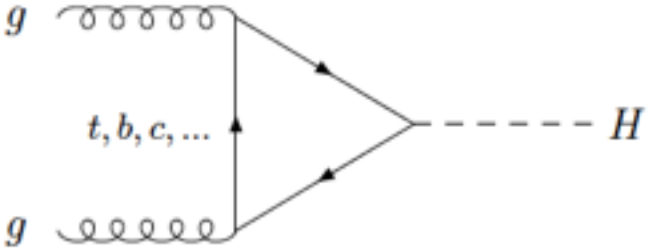
compilation by D. de Florian (DIS 2014)

PDF's and the LHC

important example: Higgs production through gluon-gluon fusion

PDF uncertainty: look a **parton-parton luminosities**

$$\mathcal{L}_{ij}(\tau \equiv M_X^2/S) = \frac{1}{S}\int_\tau^1 \frac{dx}{x} f_i(x, M_X^2) f_j(\tau/x, M_X^2)$$



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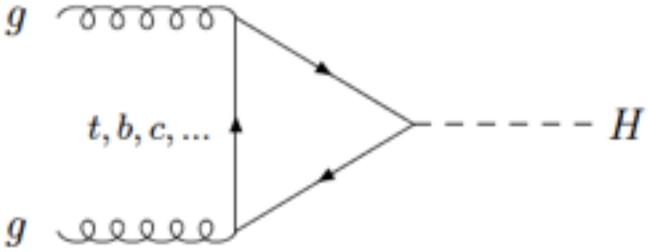

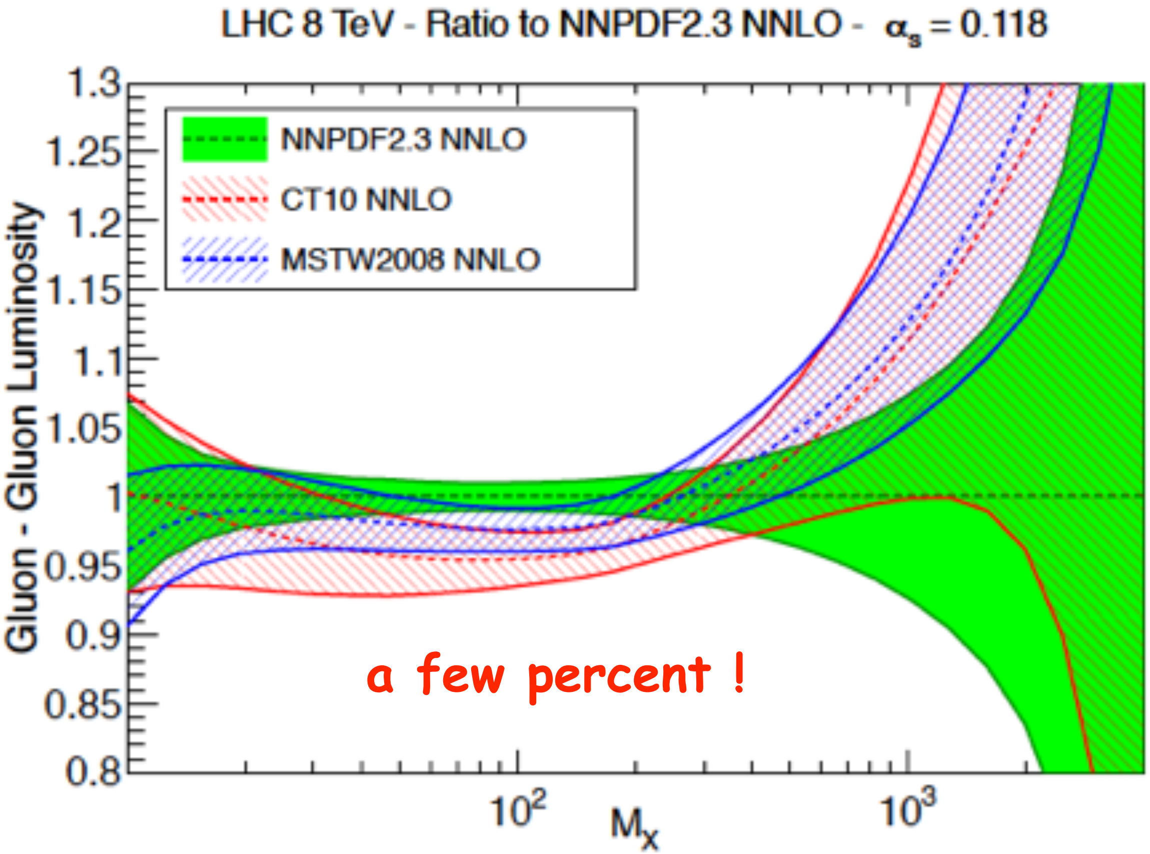

PDF4LHC group
1211.5142

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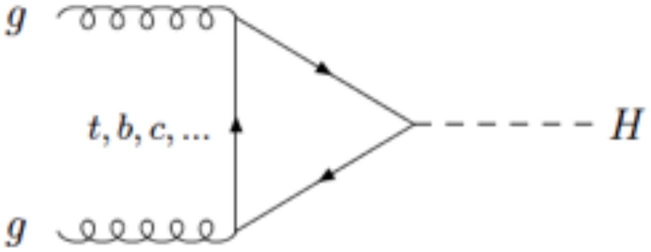

another culprit is the strong coupling:

- optimum α_s in global fits varies by about 5%

  error much larger than for "PDG average"

PDF's and the LHC

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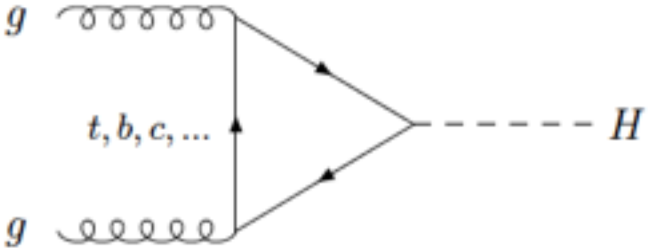

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current (theory) status for total Higgs cross section:

scale variation PDF & α_s variation

$$\sigma(\mathrm{M_H} = \mathbf{125}\,\mathrm{GeV}) = \mathbf{19.27}\,{}^{+\mathbf{7.2}\%}_{-\mathbf{7.8}\%}\,{}^{+\mathbf{7.5}\%}_{-\mathbf{6.9}\%}\,\mathrm{pb}$$

de Florian, Grazzini

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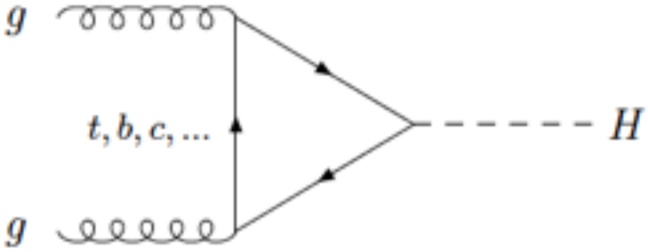

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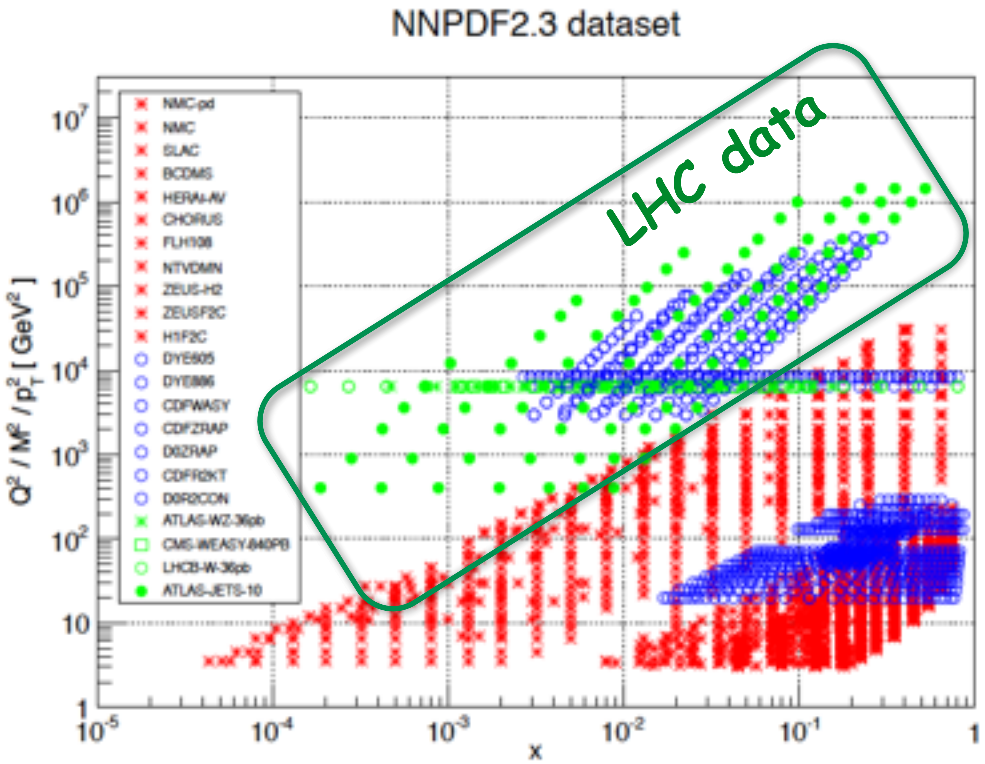
de Florian, Grazzini

precise LHC data important for validating and improving PDF and α_s determinations

improving PDF's at the LHC

efforts have already started

example: NNPDF 2.3 fit
1207.1303

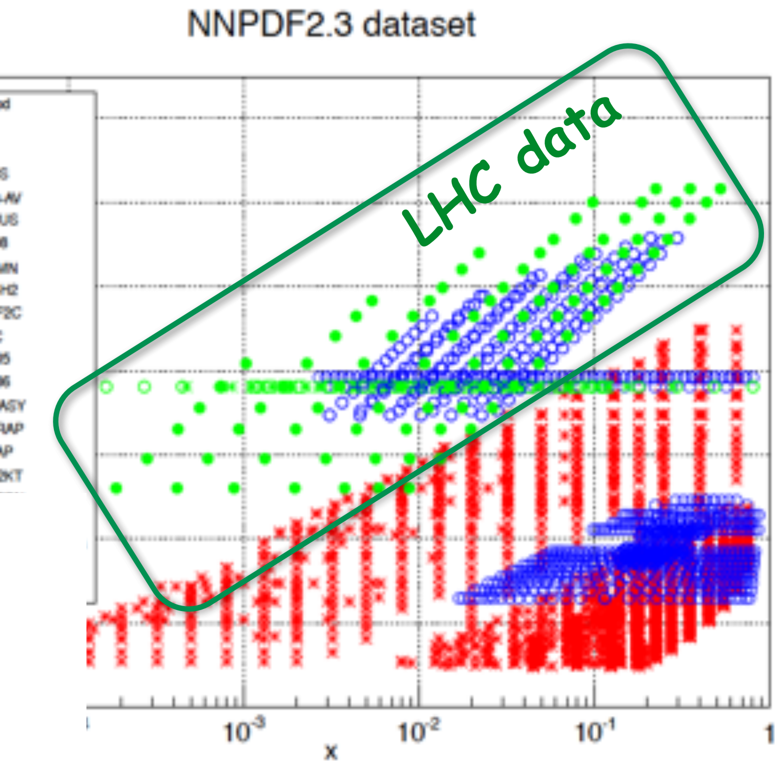


improving PDF's at the LHC

efforts have already started

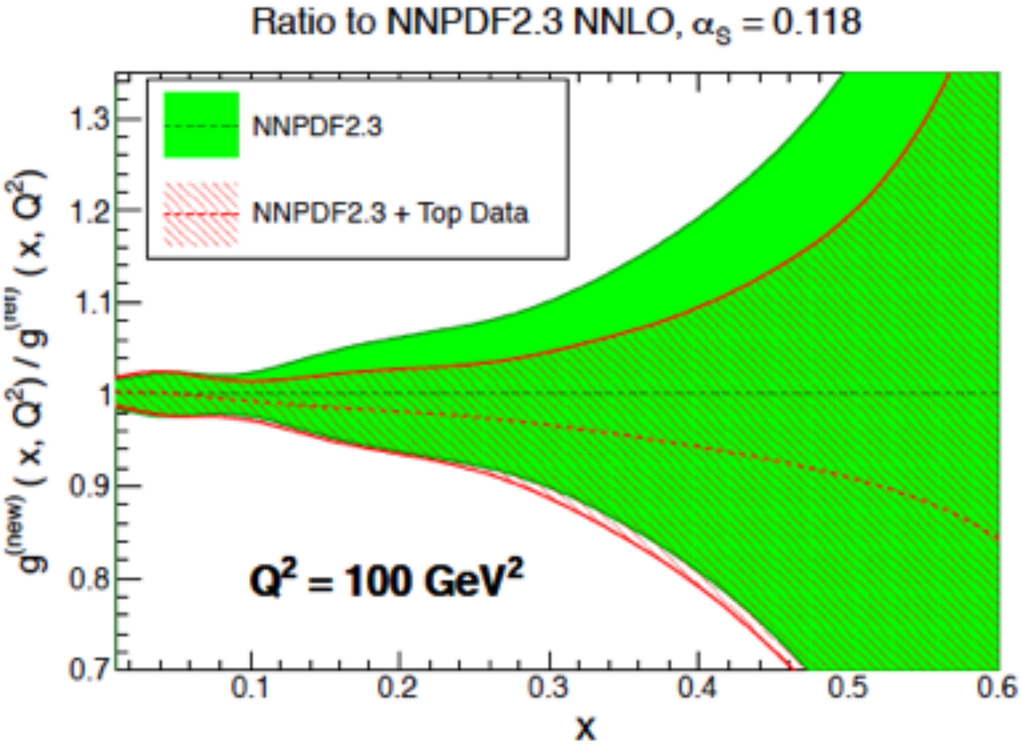
example: NNPDF 2.3 fit
1207.1303

NNPDF2.3 dataset



LHC data

Ratio to NNPDF2.3 NNLO, $\alpha_S = 0.118$

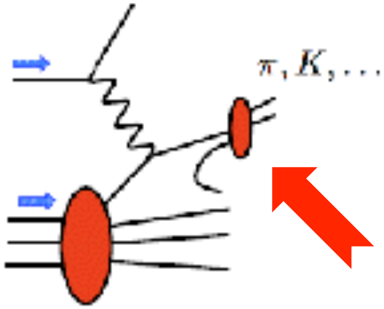


most recently: make use of recent NNLO results for top-pair production

Czakon, Mangano, Mitov, Rojo 1303.7215

find: about 20% error reduction for gluon at x values between 0.15 and 0.5

status of fragmentation functions



recall:

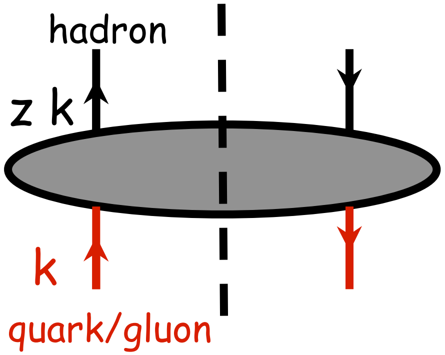
crucial for pQCD interpretation (factorization)
of all data with detected (identified) hadrons, e.g.,
SIDIS (HERMES, COMPASS), $pp \to \pi X$ (PHENIX, STAR, ALICE, ...)

global QCD analysis of fragmentation functions

very similar to PDFs:

- non-perturbative but universal
- pQCD predicts scale evolution
- describe the collinear transition of a parton "i" into a massless hadron "h" carrying fractional momentum z

$$\mathbf{D_i^h(z,\mu^2)}$$



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  Collins, Soper ’81, ’83

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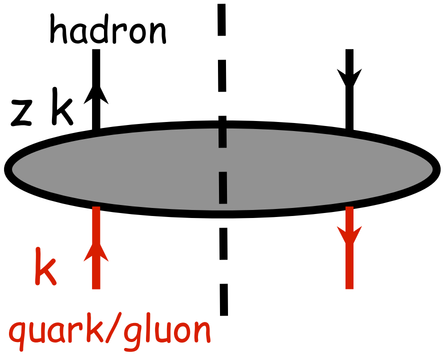

no inclusive final-state

no local OPE —> no lattice formulation

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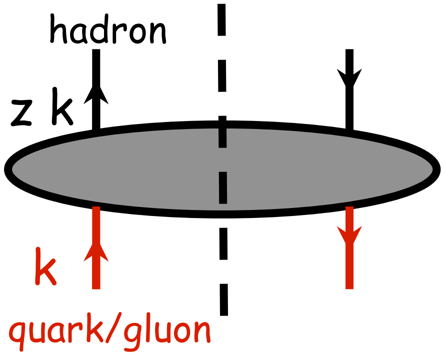

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- key process is e^+e^- **annihilation** to hadrons (plays similar role than DIS for PDFs)

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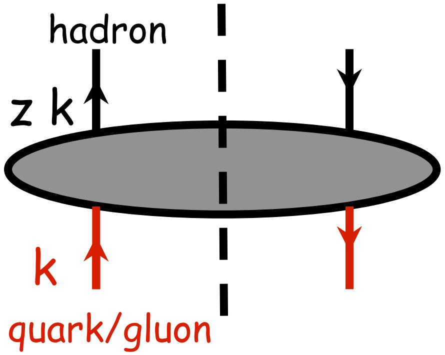

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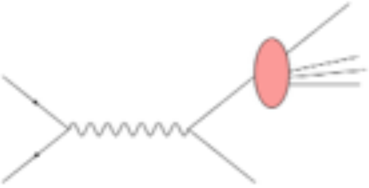
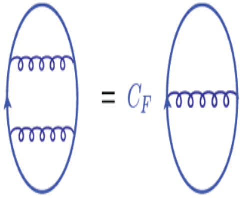
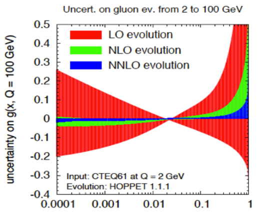


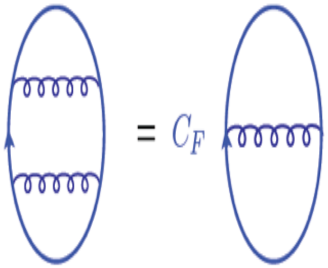
- key process is e^+e^- **annihilation** to hadrons (plays similar role than DIS for PDFs)
- **semi-inclusive DIS** provides flavor separation
- **pp data** (RHIC, LHC) important for gluon FF

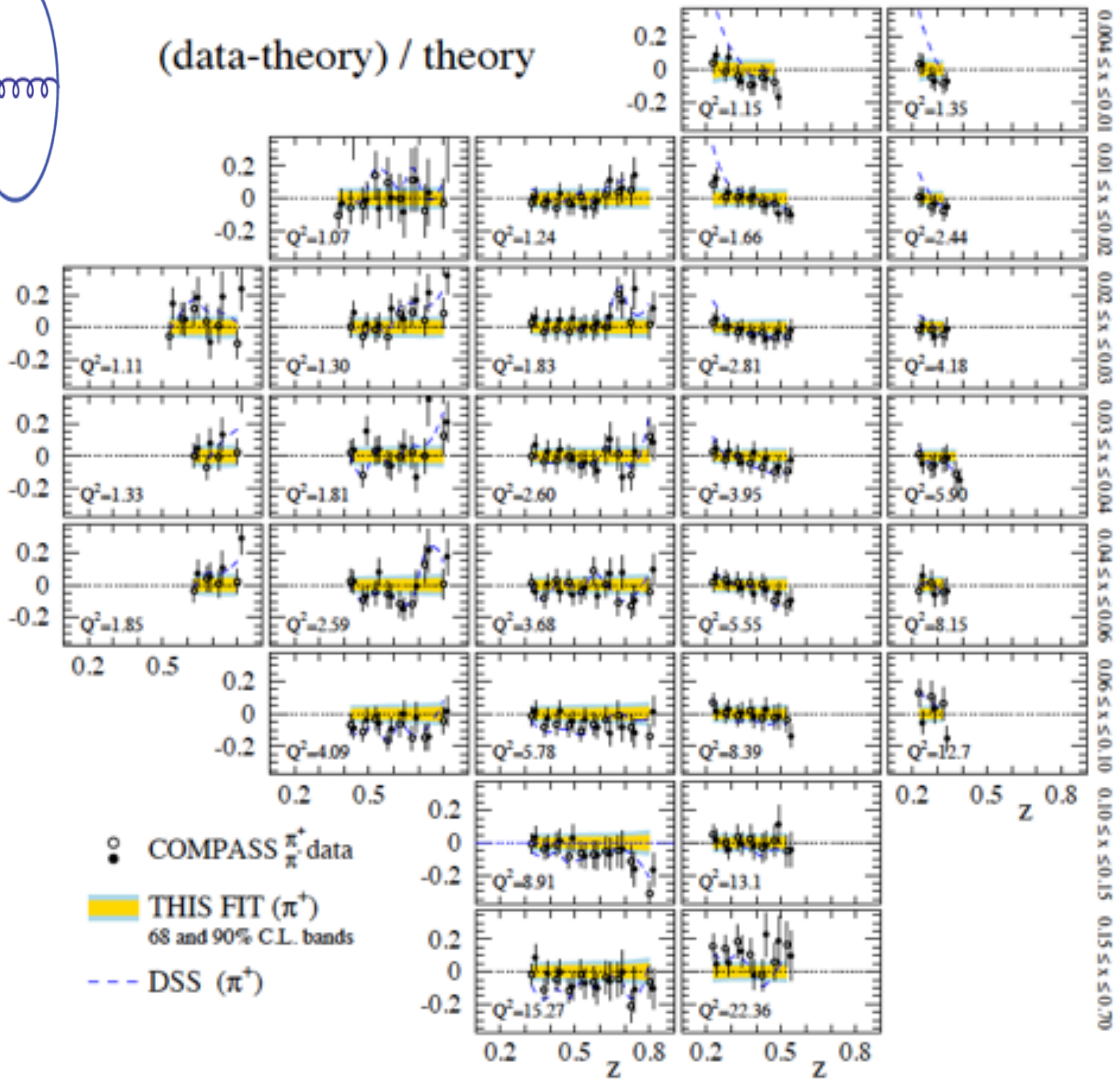




sneak preview of new global QCD analysis

de Florian, Epele, Hernandez-Pinto, Sassot, MS





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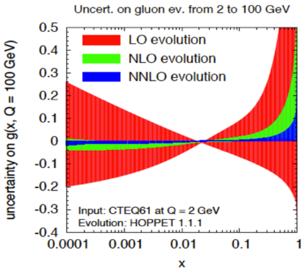

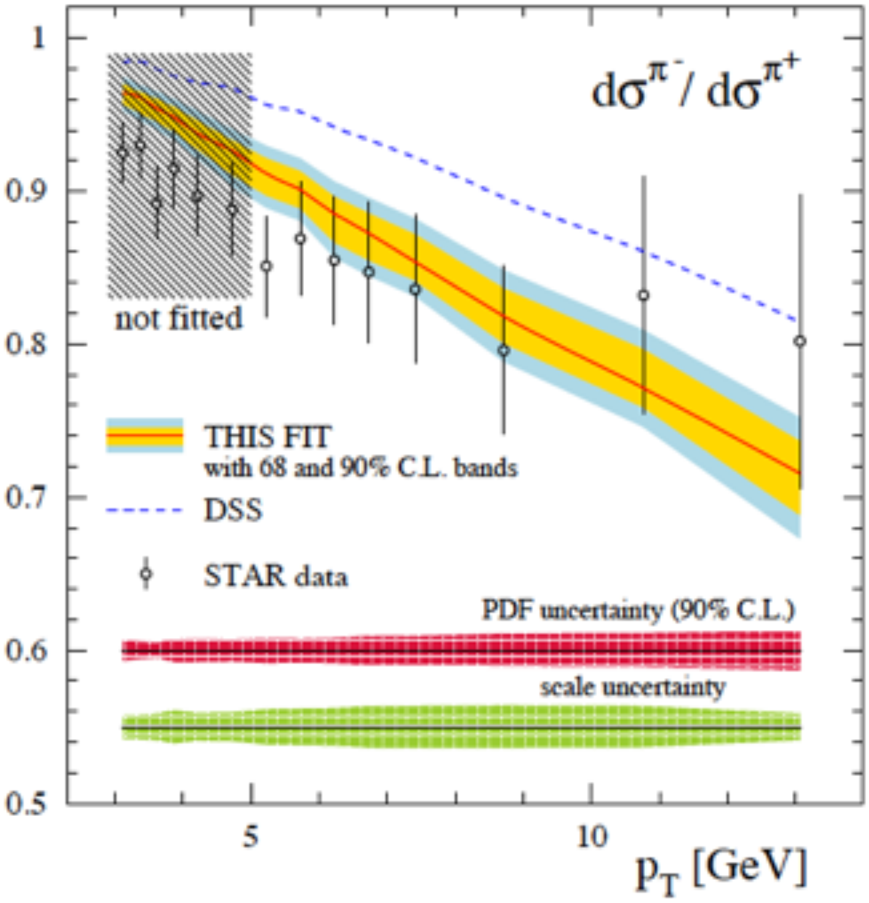

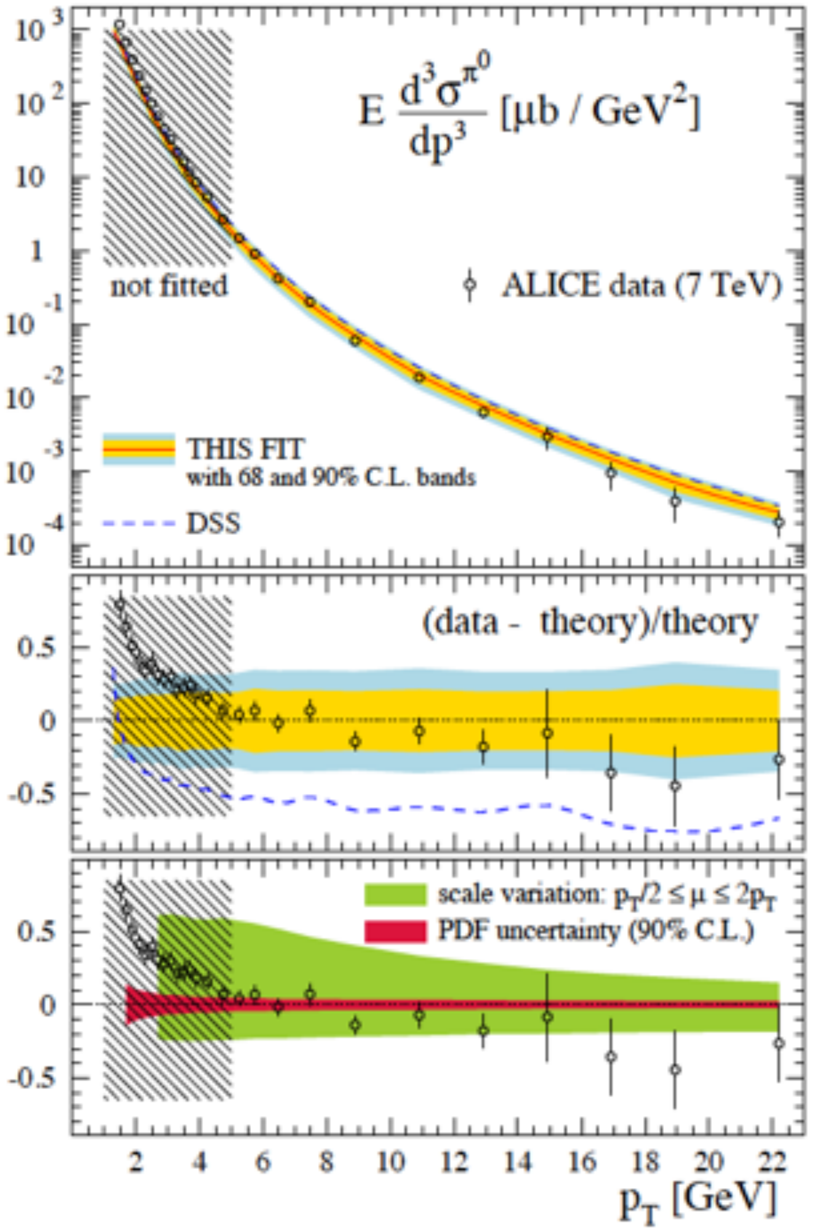

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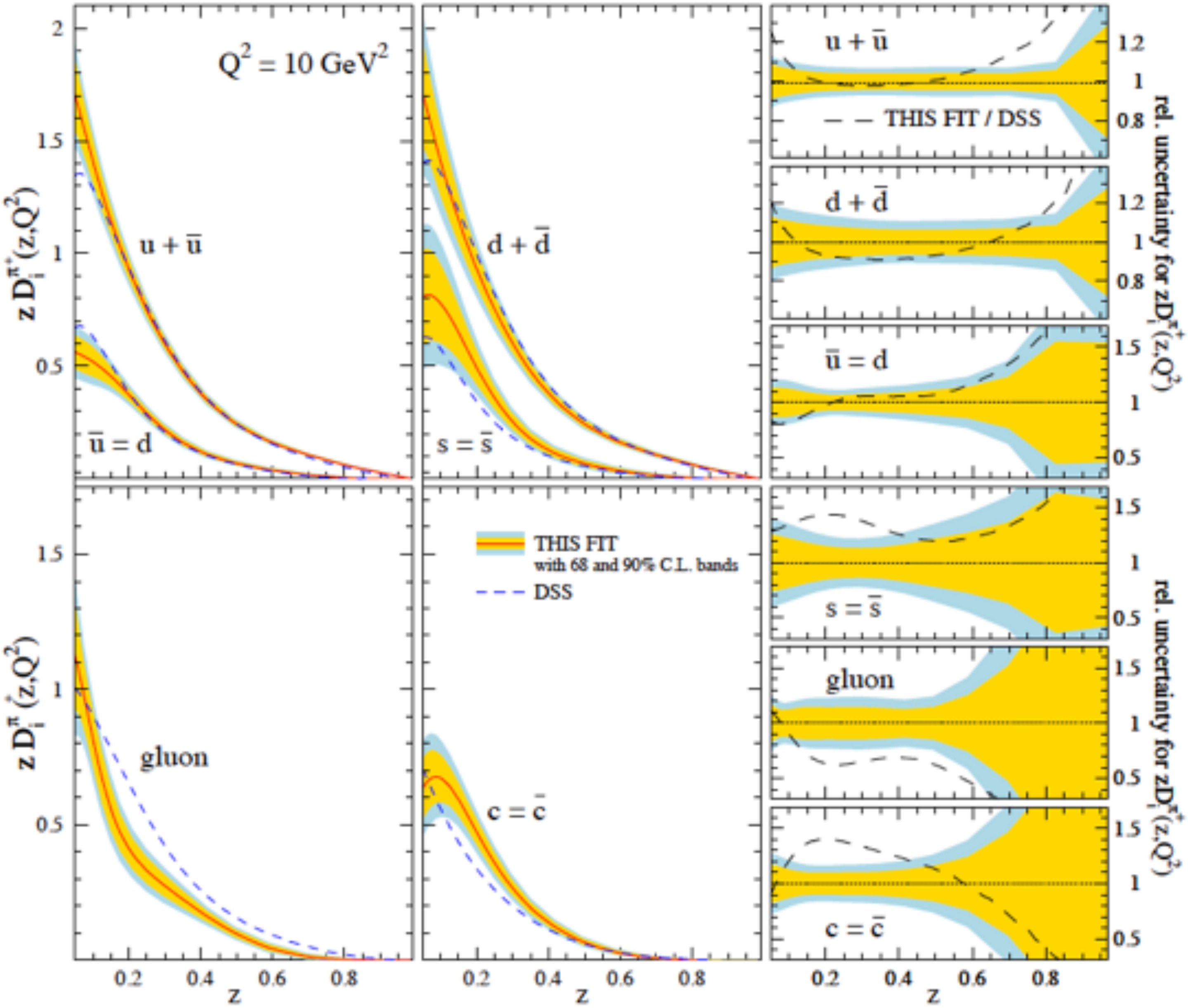

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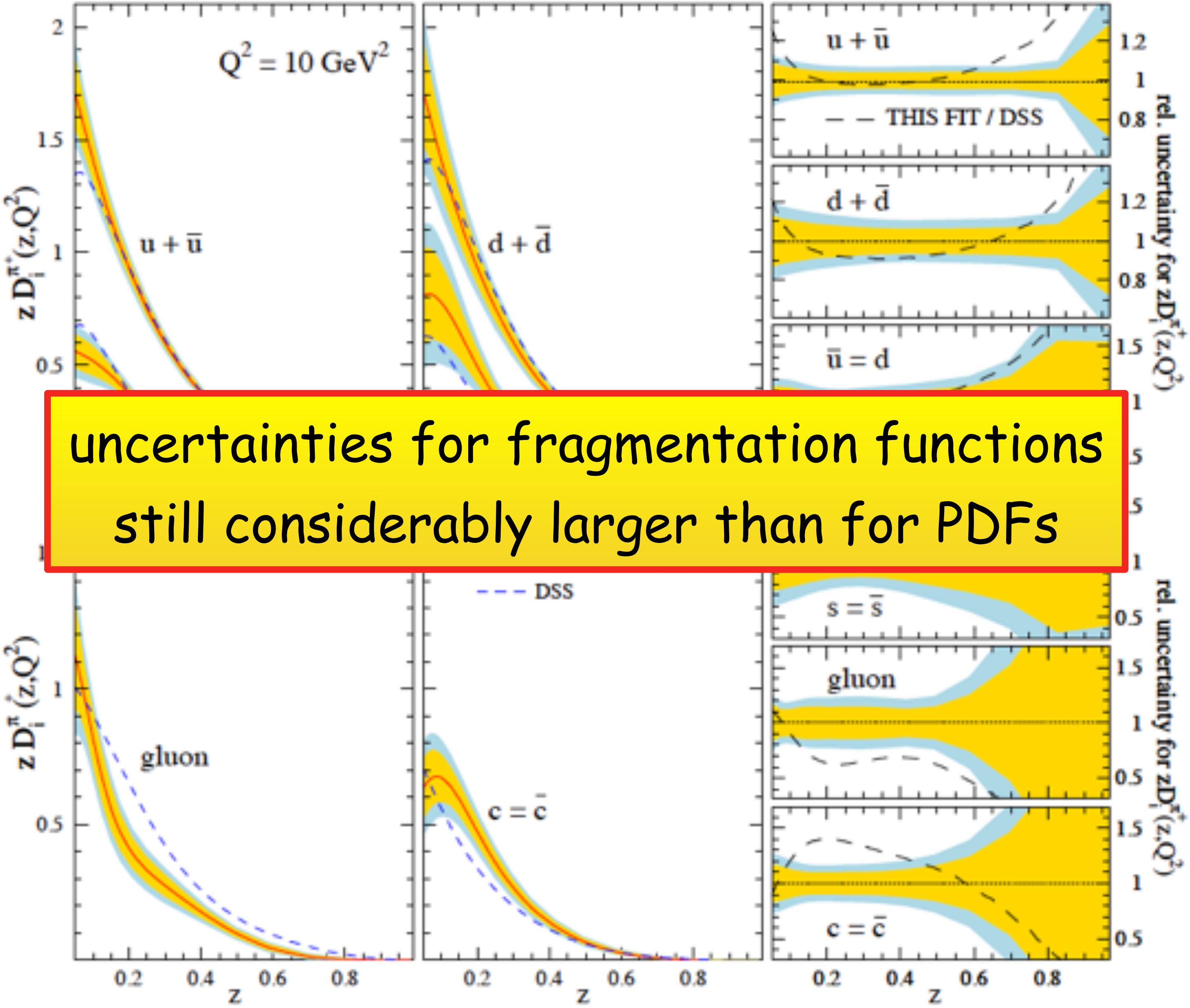
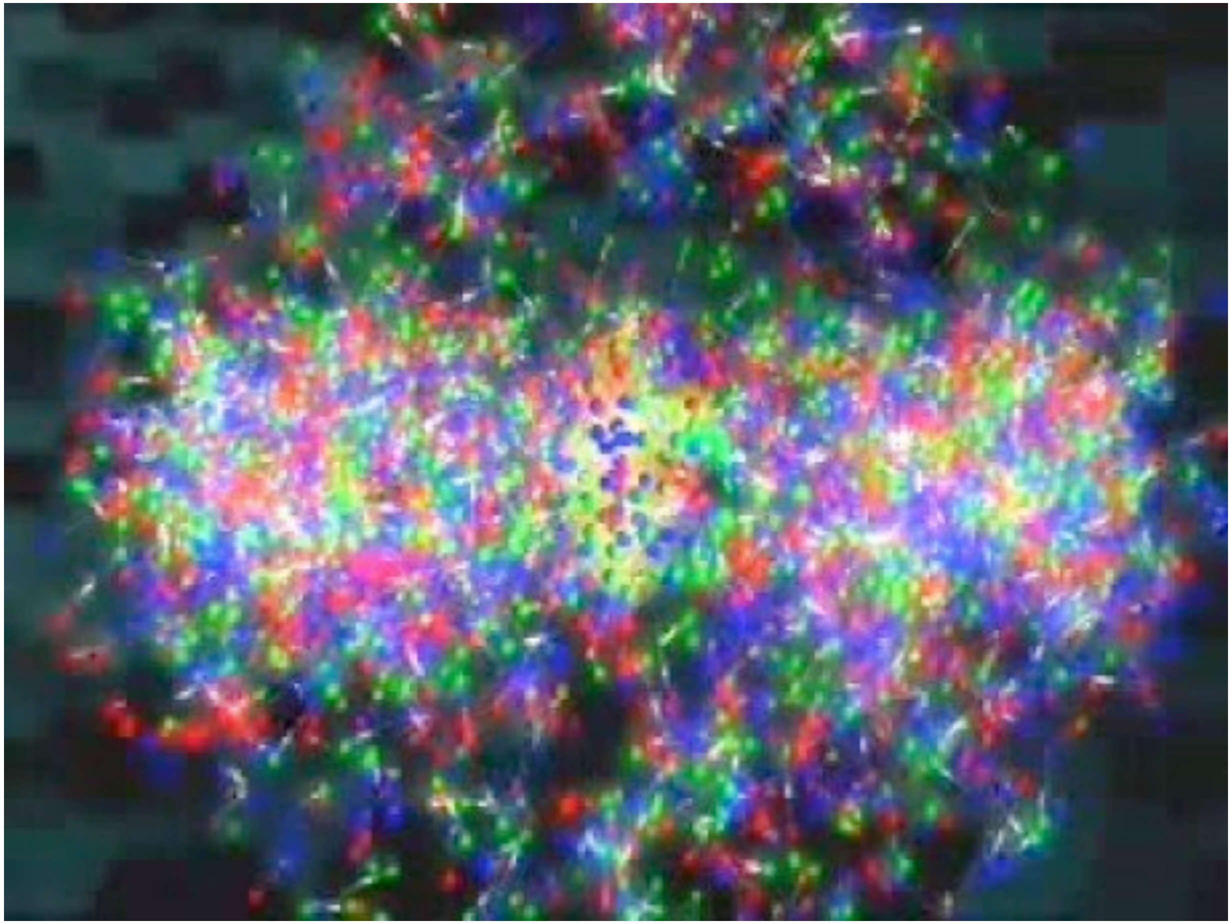


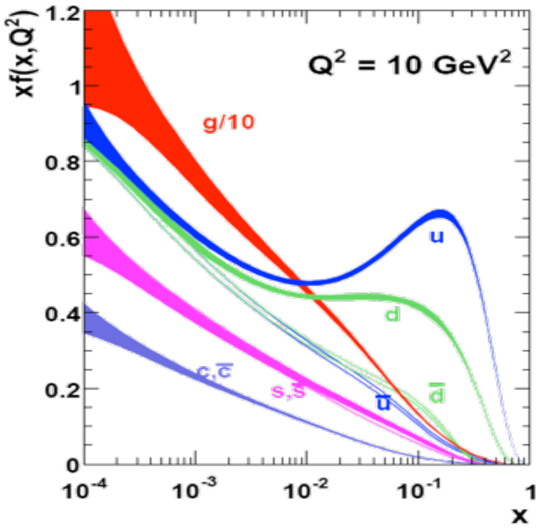
uncertainties for fragmentation functions still considerably larger than for PDFs



4-3

when there is not enough room: gluons at small x

what drives the growth of the gluon density



observe that only 2 splitting fcts are singular at small x

$$P_{gq}(x)\big|_{x\to 0} \approx \frac{2C_F}{x} \qquad P_{gg}(x)\big|_{x\to 0} \approx \frac{2C_A}{x}$$

-> small x region dominated by gluons

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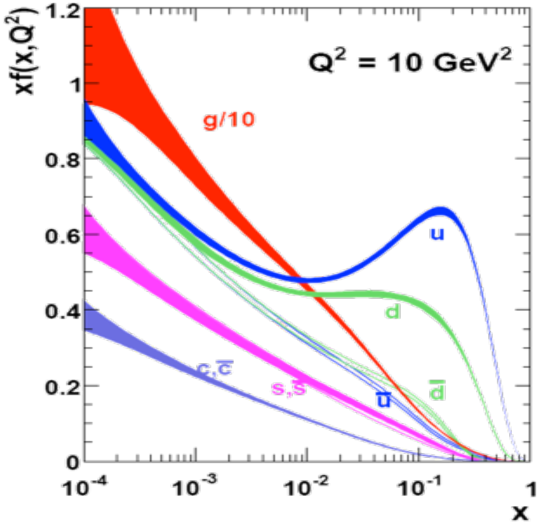

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- write down **"gluon-only" DGLAP equation** **only valid for small x and large Q^2**

$$\frac{dg(x,\mu^2)}{d\log\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\frac{2C_A}{z} g(x/z,\mu^2)$$

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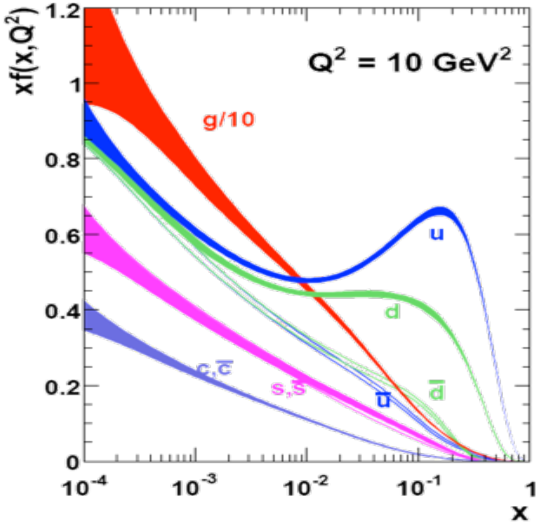

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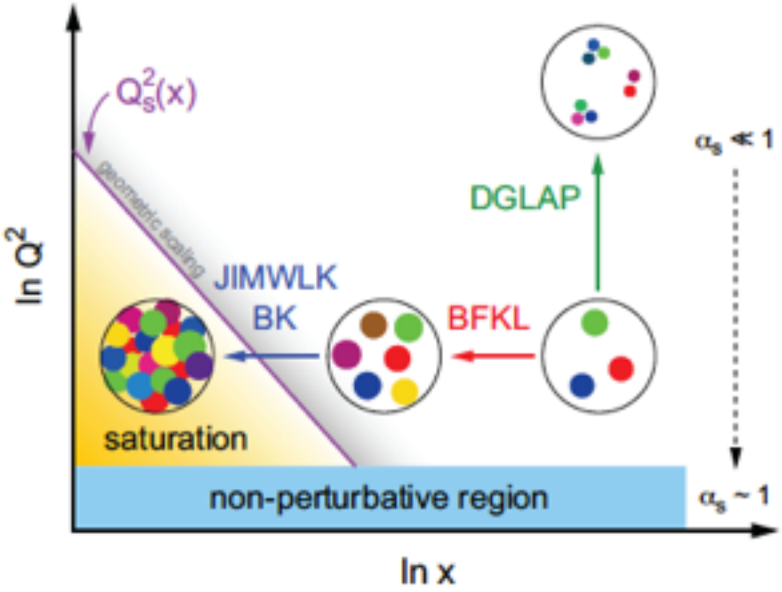
- for fixed coupling this leads to **"double logarithmic approximation"**

$$xg(x,Q^2) \sim \exp\left(2\sqrt{\frac{\alpha_S C_A}{\pi}\log(1/x)\log(Q^2/Q_0^2)}\right)$$

predicts rise that is faster than $\log^a(1/x)$ but slower than $(1/x)^a$

gluon occupancy



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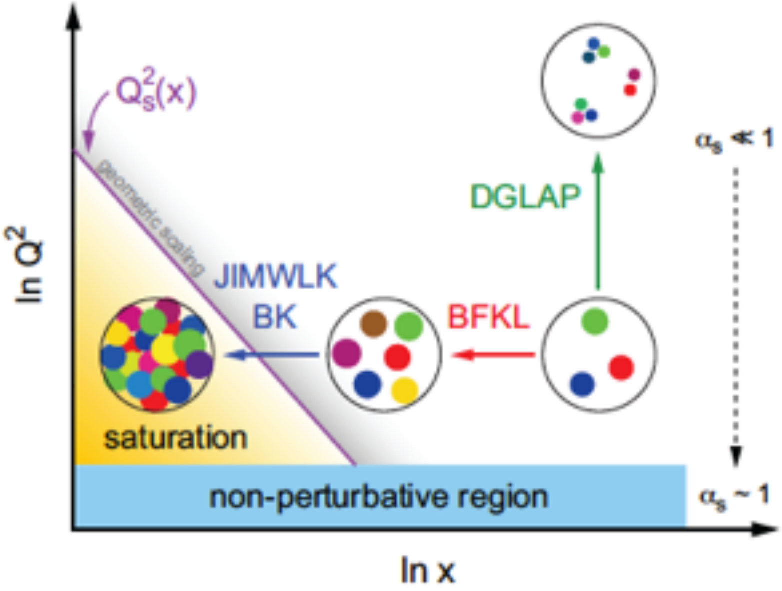
transverse size of partons ≈ 1/Q

gluon occupancy



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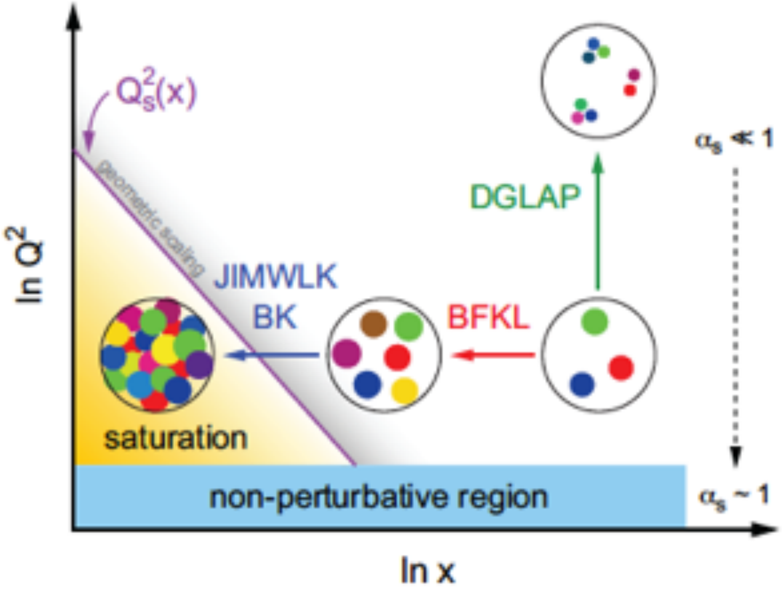

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"high-energy (Regge) limit of QCD"

- aim to resum terms ≈ $\alpha_s \log(1/x)$
- Balitsky-Fadin-Kuraev-Lipatov (**BFKL**) equation: evolves in x not Q^2
- BFKL predicts a power-like growth $xg(x, Q^2) \sim (1/x)^{\alpha_P - 1}$

much faster than in DGLAP

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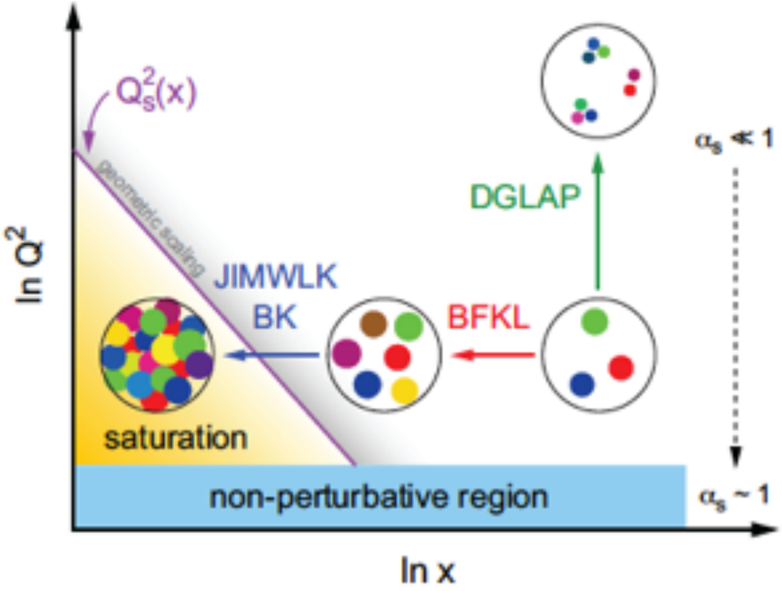

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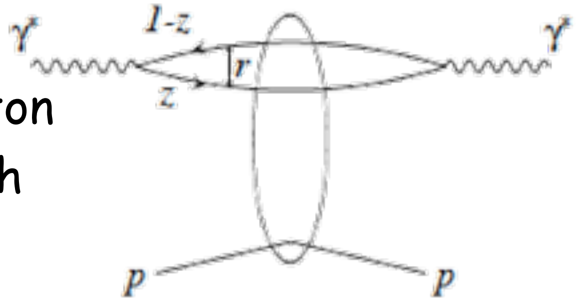
BIG problem

- proton quickly fills up with gluons (transverse size now fixed !)
- hadronic cross sections violate ln^{2}s bound (Froissart-Martin) and grow like a power

color dipole model

make progress by viewing, e.g., DIS from a "different angle"



DIS in the **proton rest frame** can be viewed as the photon splitting into a quark-antiquark pair ("**color dipole**") which scatters off the proton (= "slow" gluon field)

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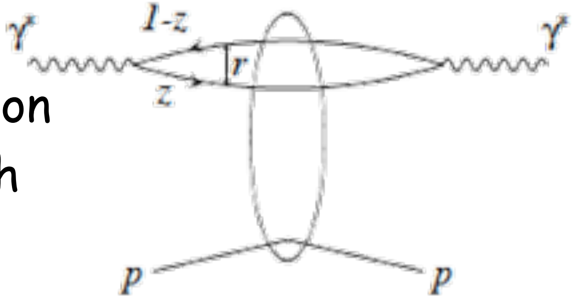

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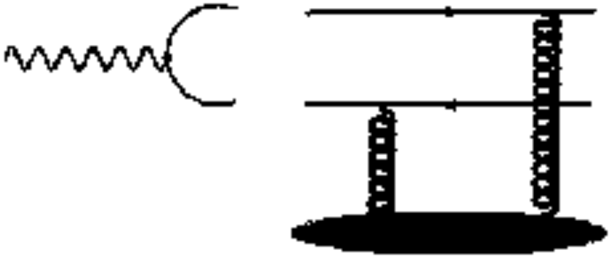

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QED QCD

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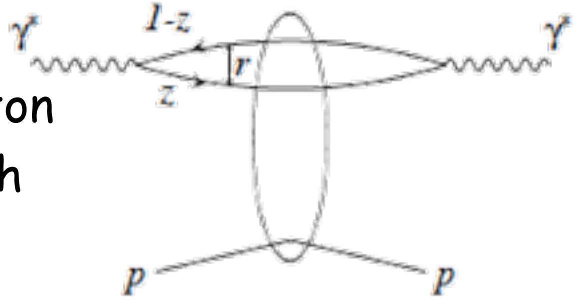

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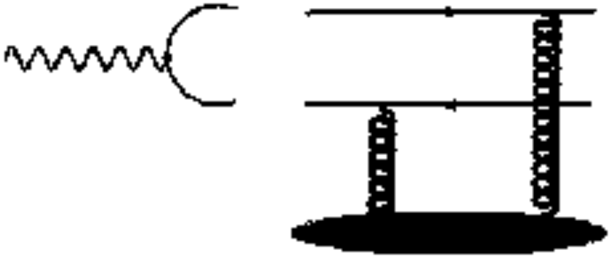

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- introduces **dipole-nucleon scattering amplitude N** as fund. building block
- energy dependence of N described by Balitsky-Kovchegov equation

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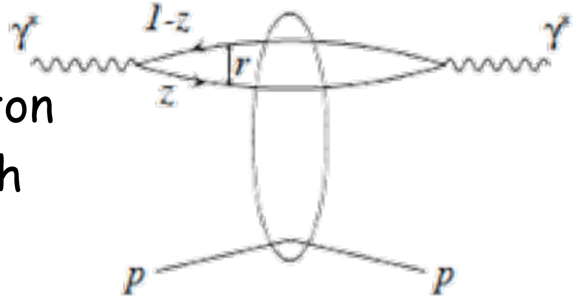

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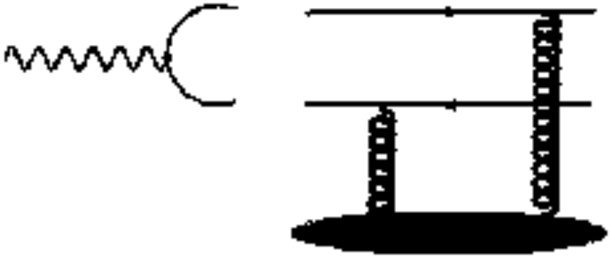

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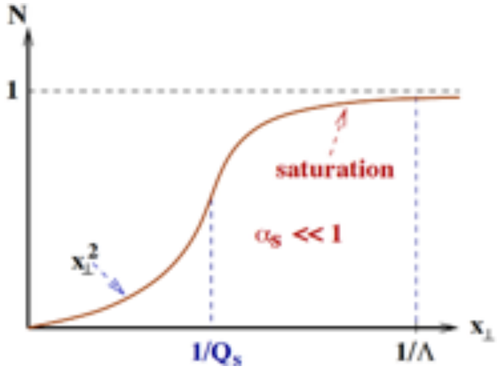
QED **QCD**

- introduces **dipole-nucleon scattering amplitude N** as fund. building block
- energy dependence of N described by Balitsky-Kovchegov equation
  - **non-linear** -> includes multiple scatterings for unitarization
  - generates saturation scale Q_s
  - suited to treat collective phenomena (shadowing, diffration)
  - impact parameter dependence



4-4

when N^{x}LO is not enough: all order resummations

when a N^{x}LO calculation is not good enough

observation: fixed N^{x}LO order QCD calculations are not necessarily reliable
this often happens at low energy fixed-target experiments
and can be an issue also at colliders, even the LHC

reason: structure of the perturbative series and IR cancellation

at partonic threshold / near exclusive boundary:

- just enough energy to produce, e.g., high-p_T parton
- "inhibited" radiation (general phenomenon for gauge theories)

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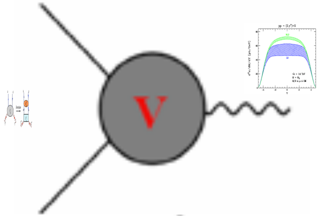
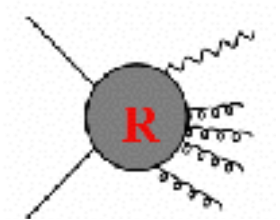
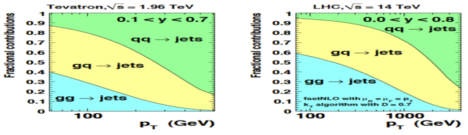
simple example:
Drell-Yan process



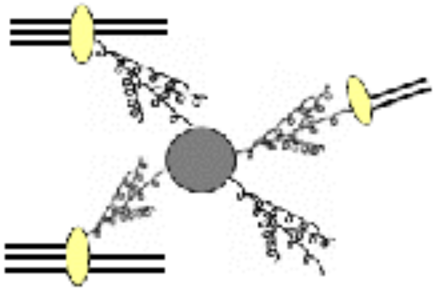
$$z \equiv \frac{Q^2}{\hat{s}} = 1$$





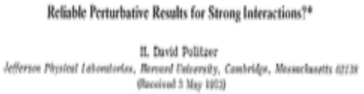
"imbalance" of real and virtual contributions: **IR cancellation leaves large log's**

all order structure of partonic cross sections



let's consider pp scattering:

logarithms related to
partonic threshold



Reliable Perturbative Results for Strong Interactions?*

H. David Politzer

Jefferson Physical Laboratories, Harvard University, Cambridge, Massachusetts 02138

(Received 3 May 1973)

general structure of partonic cross sections at the kth order:

$$p_T^3 \frac{d\hat{\sigma}_{ab}}{dp_T} = p_T^3 \frac{d\hat{\sigma}_{ab}^{\mathrm{Born}}}{dp_T} \Bigg[1 + \underbrace{\mathcal{A}_1 \, \alpha_s \ln^2\left(1-\hat{x}_T^2\right) + \mathcal{B}_1 \, \alpha_s \ln\left(1-\hat{x}_T^2\right)}_{\text{NLO}}$$

$$+ \ldots + \mathcal{A}_k \, \alpha_s^k \ln^{2k}\left(1-\hat{x}_T^2\right) + \ldots \Bigg] + \ldots$$

"threshold logarithms"

all order structure of partonic cross sections

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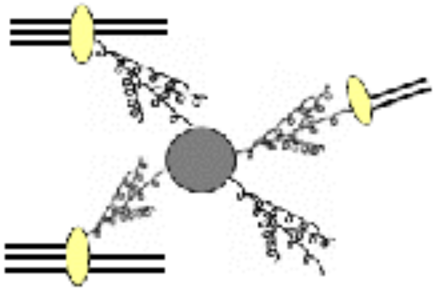

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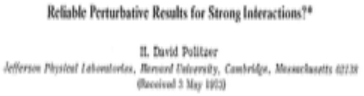

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$$\left. + \ldots + \; \mathcal{A}_k \, \alpha_s^k \, \ln^{2k} \left(1 - \hat{x}_T^2\right) \; + \; \ldots \right] \; + \; \ldots$$

"threshold logarithms"

where relevant? … convolution with steeply falling parton luminosity L_{ab}:

$$d\sigma \propto \sum_{a,b} \int_\tau^1 \frac{dz}{z} \mathcal{L}_{ab}\left(\frac{\tau}{z}\right) d\hat{\sigma}_{ab}(z)$$

z = 1 emphasized,
in particular as τ → 1

large at small τ/z

→ important for fixed target phenomenology: threshold region more relevant (large τ)

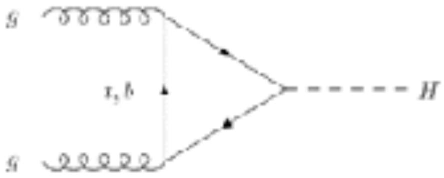
resummations – how are they done

$\alpha_s^k \ln^{2k}(1-\hat{x}_T^2)$ **may spoil perturbative series - unless taken into account to all orders**

resummation of such terms has reached a high level of sophistication

Sterman; Catani, Trentadue; Laenen, Oderda, Sterman;
Catani et al.; Sterman, Vogelsang; Kidonakis, Owens; ...

- worked out for most processes of interest at least to NLL
- well defined class of higher-order corrections
- often of much phenomenological relevance even for high mass particle production at the LHC



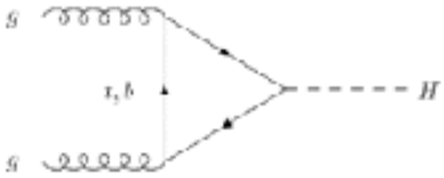
resummations – how are they done

$\alpha_s^k \ln^{2k}(1-\hat{x}_T^2)$ **may spoil perturbative series - unless taken into account to all orders**

resummation of such terms has reached a high level of sophistication

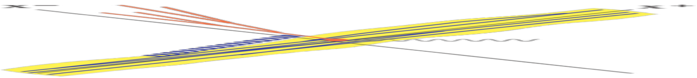
Sterman; Catani, Trentadue; Laenen, Oderda, Sterman; Catani et al.; Sterman, Vogelsang; Kidonakis, Owens; ...

- worked out for most processes of interest at least to NLL
- well defined class of higher-order corrections
- often of much phenomenological relevance even for high mass particle production at the LHC



resummation (= exponentiation) occurs when "right" moments are taken:

Mellin moments for threshold logs



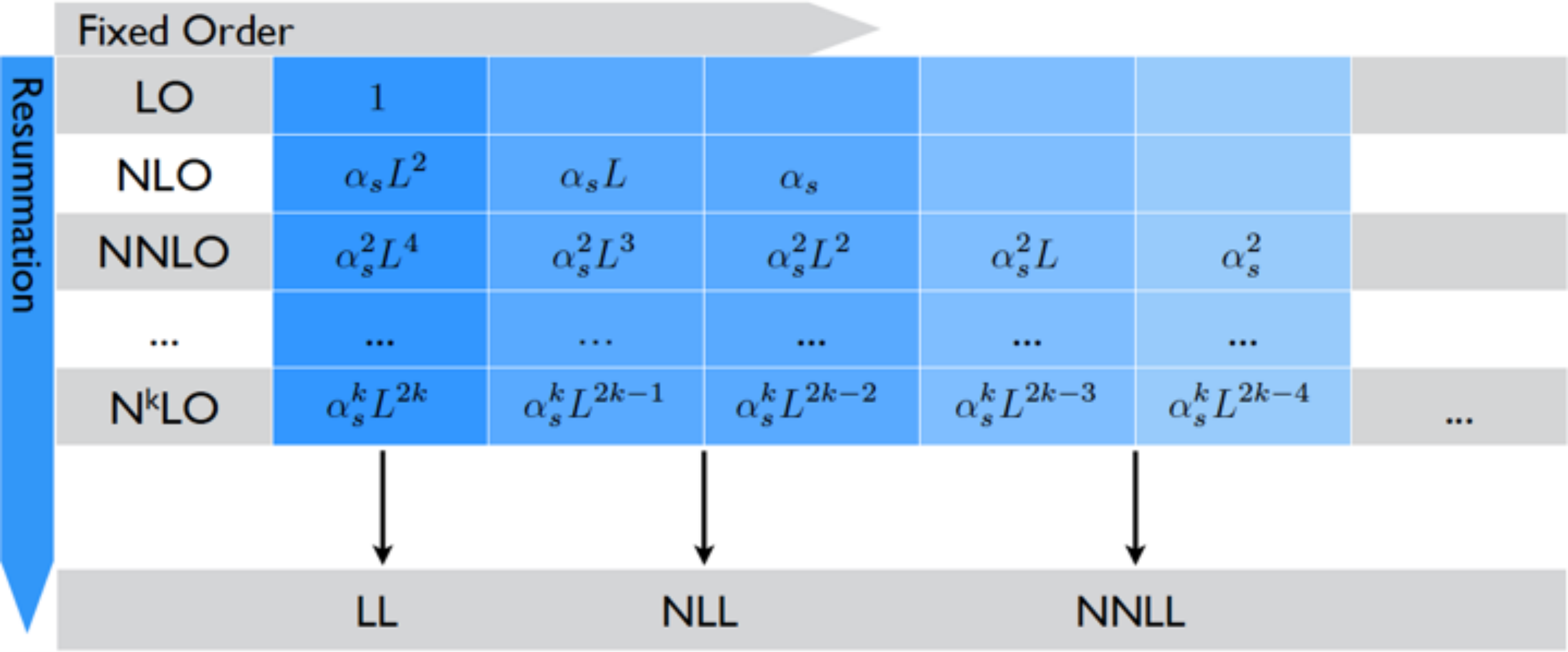
- fixed order calculations needed to determine "coefficients"
- the more orders are known, the more subleading logs can be resummed

resummations – terminology

Fixed Order

LO	1					
NLO	$\alpha_s L^2$	$\alpha_s L$	α_s			
NNLO	$\alpha_s^2 L^4$	$\alpha_s^2 L^3$	$\alpha_s^2 L^2$	$\alpha_s^2 L$	α_s^2	
...	...	$\cdots$	...	...	...	
N^{k}LO	$\alpha_s^k L^{2k}$	$\alpha_s^k L^{2k-1}$	$\alpha_s^k L^{2k-2}$	$\alpha_s^k L^{2k-3}$	$\alpha_s^k L^{2k-4}$	...

resummations – terminology



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LL NLL NNLL

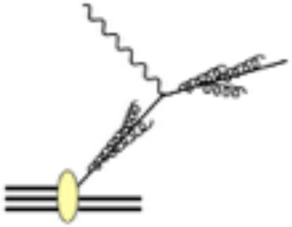
some leading log exponents

(assuming fixed α_s for simplicity)

color factors for soft gluon radiation matter:

unobserved parton Sudakov "suppression"

DIS



$$\exp\left[\frac{C_F\,\alpha_s}{\pi}\ \ln^2(N)\ -\ \frac{C_F\,\alpha_s}{\pi}\ \frac{1}{2}\ \ln^2(N)\right]$$

moderate enhancement, unless x_{Bj} large

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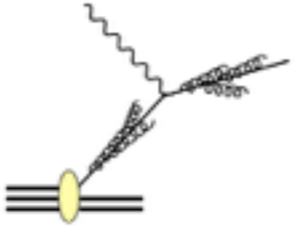

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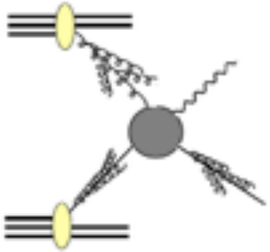
prompt photons



$q\bar{q}\to\gamma g$ $$\exp\left[\left(C_F + C_F - \frac{1}{2}C_A\right)\frac{\alpha_s}{\pi}\ln^2(N)\right]$$

$qg\to\gamma q$ $$\exp\left[\left(C_F + C_A - \frac{1}{2}C_F\right)\frac{\alpha_s}{\pi}\ln^2(N)\right]$$

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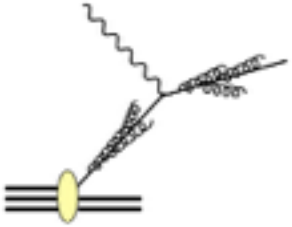

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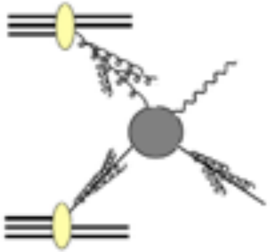

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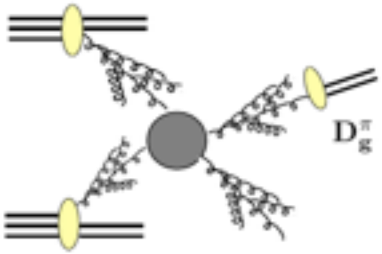
exponents positive ⟶ enhancement

inclusive hadrons



e.g.

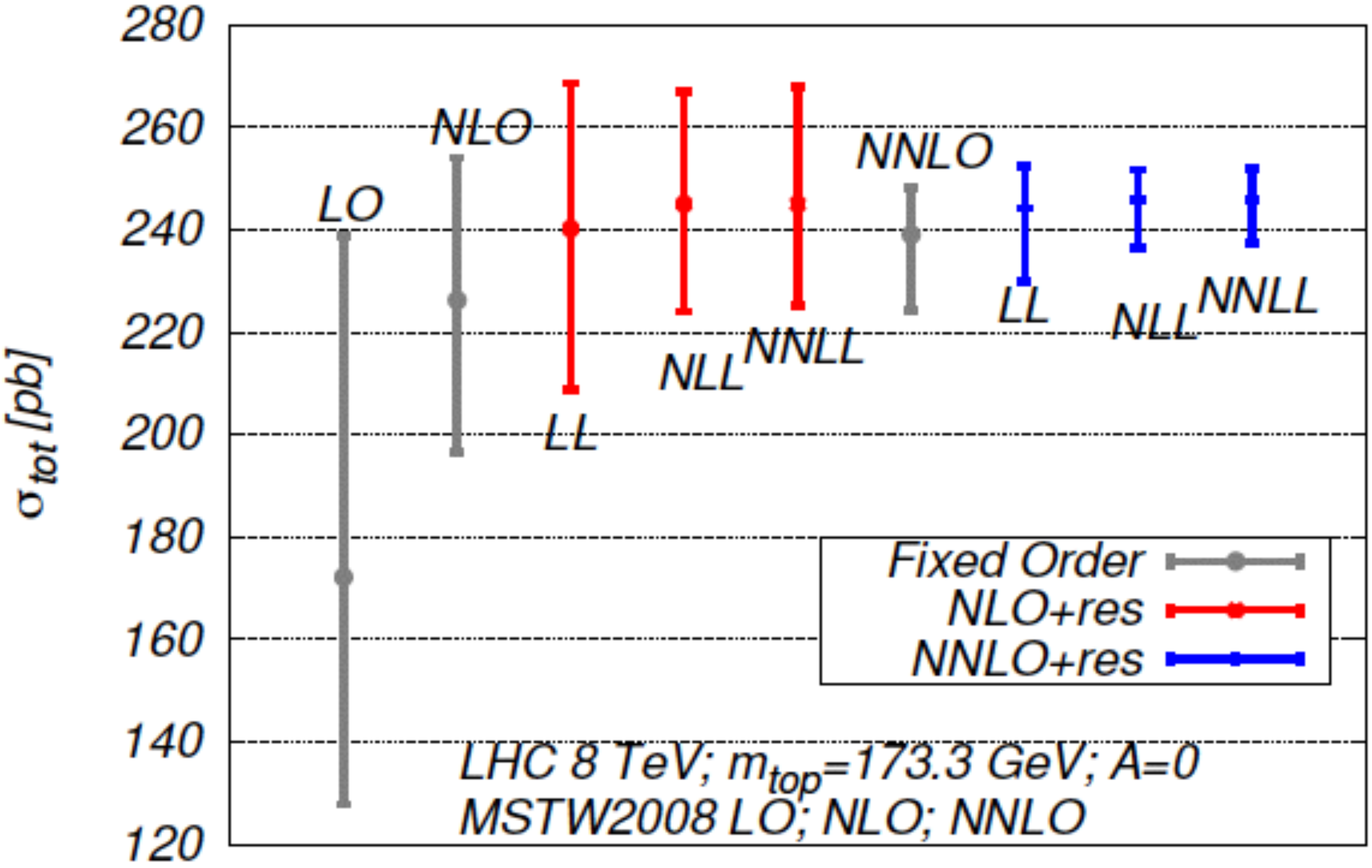
observed partons

unobserved

$gg\to gg$ $$\exp\left[\left(C_A+C_A+C_A-\frac{1}{2}C_A\right)\frac{\alpha_s}{\pi}\ln^2(N)\right]$$

expect much larger enhancement

one recent example: top-pair production



Czakon, Fiedler, Mitov, Rojo 1305.3892

resummations: window to non-perturbative regime

important technical issue:

resummations are sensitive to strong coupling regime

→ need some "minimal prescription" to avoid Landau pole (where $\alpha_s \to \infty$)

Catani, Mangano, Nason, Trentadue:

define resummed result such that series is asymptotic
w/o factorial growth associated with power corrections
[achieved by particular choice of Mellin contour]

→ power corrections may be added afterwards if pheno. needed
studying power corrections prior to resummations makes no sense

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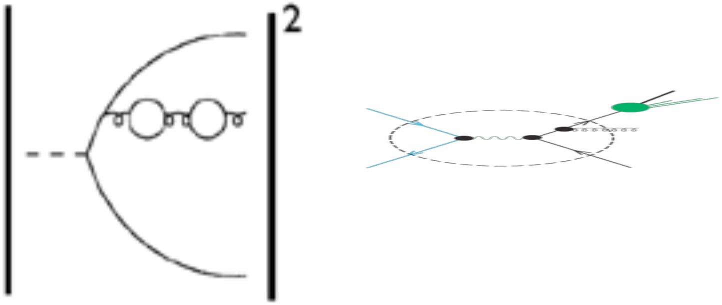
window to the non-perturbative regime so far little explored

“convergence” of an asymptotic series

see, “Renormalons” review by M. Beneke, hep-ph/9807443

suppose we keep calculating
higher and higher orders



factorial
growth

→ **big trouble**: the perturbative series is not convergent but only asymptotic

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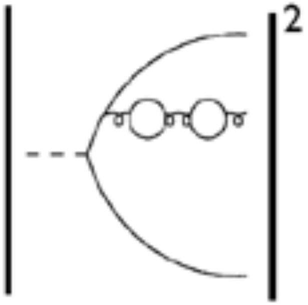
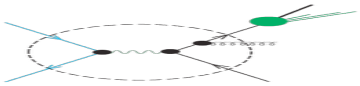




factorial growth

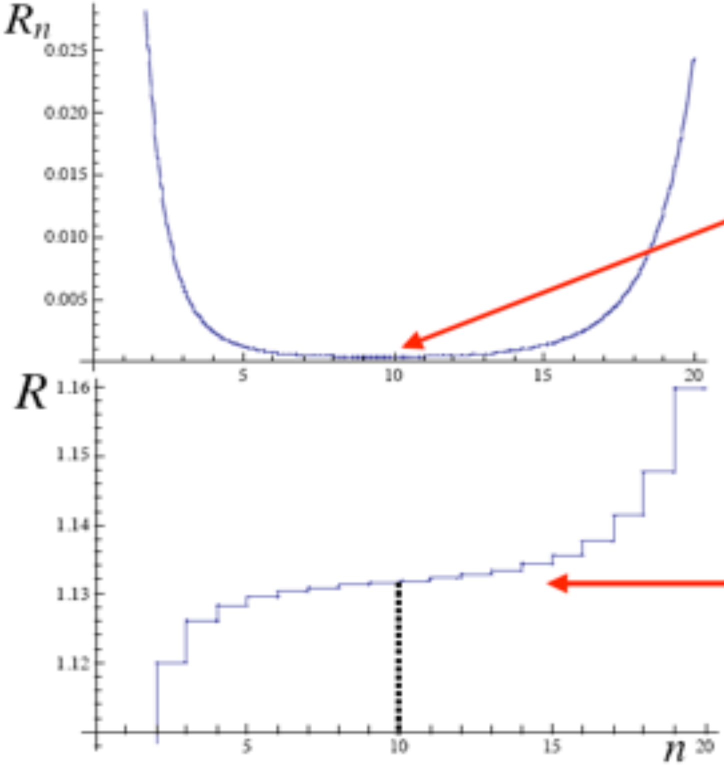
→ **big trouble**: the perturbative series is not convergent but only asymptotic

illustration:

try resumming

$$R = \sum_{n=0}^{\infty} \alpha_s^n n!$$

[with α_s= 0.1]



minimal term

$R_{min} = 1/\alpha_s$

asymptotic value of the sum:

$$R_{asymp} = \sum_{n=0}^{n_{min}} \alpha_s^n n!$$

taken from M. Cacciari

pQCD – non-perturbative bridge

- "renormalon ambiguity" ↔ incompleteness of pQCD series

  → we can only define what the sum of the perturbative series is like truncating it at the minimal term

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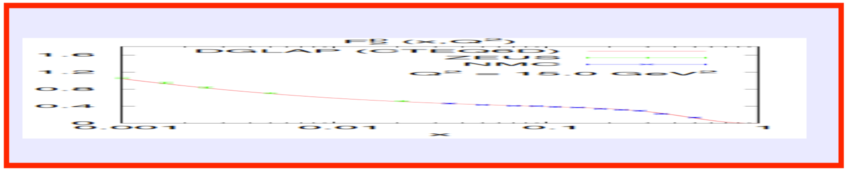

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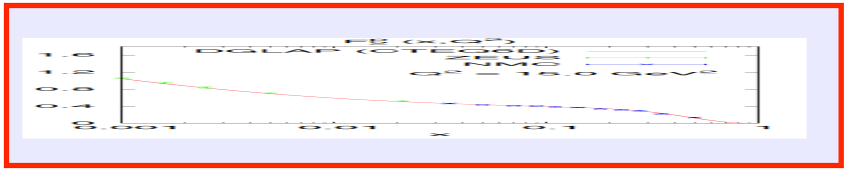
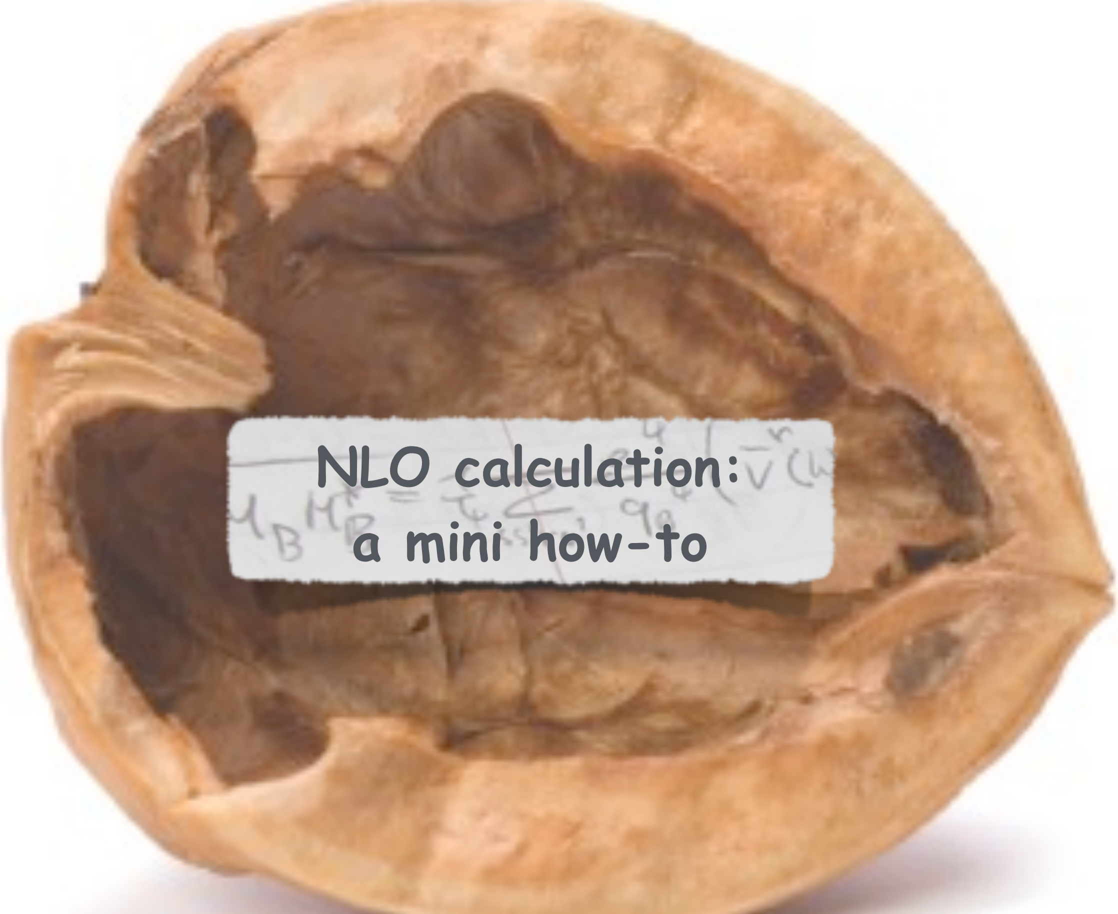


- QCD: NP corrections are power suppressed:

the value of p depends on the process and can sometimes be predicted

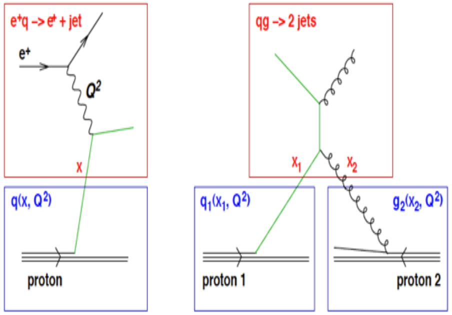


NLO calculation:
a mini how-to

4-5

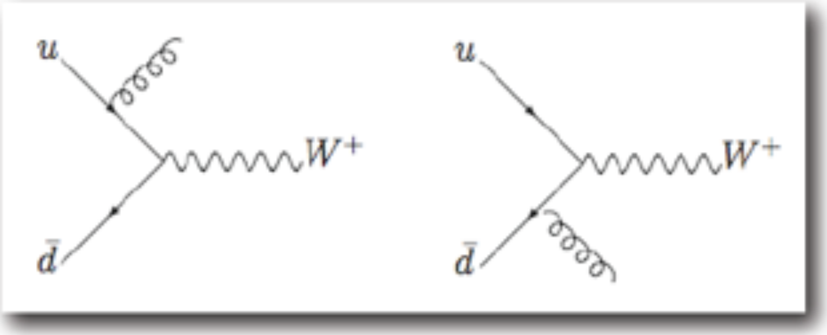
NLO calculation in a nutshell: Drell-Yan

at NLO we need to compute two contributions:



$\sigma_{ep} = \sigma_{eq} \otimes q$

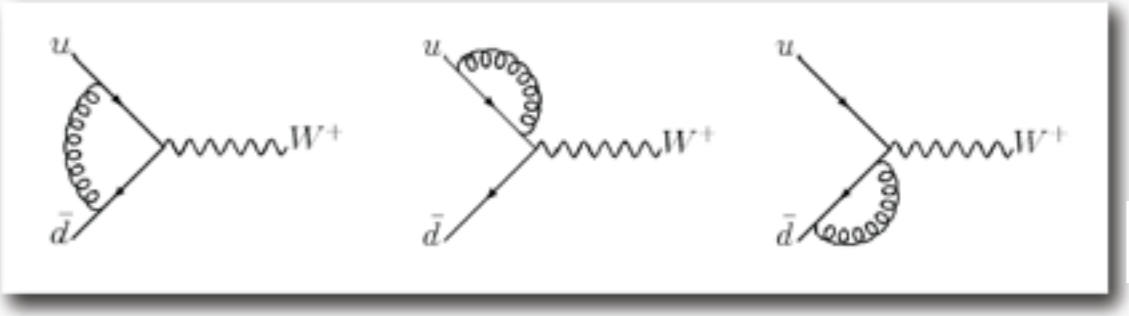
$\sigma_{pp\to 2\,jets} = \sigma_{qg\to 2\,jets} \otimes q_1 \otimes g_2 + \cdots$



real radiation corrections

one extra parton in final-state

$$|\mathcal{M}_{W+g}|^2 \sim (g_s)^2$$



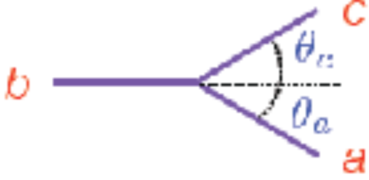
one-loop virtual corrections

only interference with Born contributes at NLO

$$(\mathcal{M}_{W,1-\mathrm{loop}} \times \mathcal{M}_{W,\mathrm{tree}}) \sim g_s^2 \times 1$$

NLO in a nutshell: real radiation

recall: collinear/soft kinematics



$$p_a - z p_b \ , p_c - (1 - z) p_b$$

$$\longrightarrow \ E_a - z E_b \ ; E_c - (1 - z) E_b$$

$$z\theta_a - (1 - z)\theta_c - 0 \quad \longrightarrow \ \theta_a - (1 - z)(\theta_a \quad \theta_c)$$

rewrite in terms of $\frac{d}{d\ln\mu}\begin{pmatrix} q(x,\mu) \\ g(x,\mu) \end{pmatrix} = \int_x^1 \frac{dz}{z} \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix}_{(z,\alpha_s)} \cdot \begin{pmatrix} q(x/z,\mu) \\ g(x/z),\mu) \end{pmatrix}$ $\begin{aligned} R_{u,d,s,c,b} &= R_{u,d,s,c} + 3 \times \left(-\frac{1}{3}\right)^2 \\ &= \frac{11}{3} \end{aligned}$

then $d\sigma_{(\ldots)ac} \sim \int |\mathcal{M}_{(\ldots)ac}|^2 E_a^2 dE_a \, \theta_a d\theta_a \sim d\sigma_{(\ldots)b} \left(\frac{\alpha_s}{2\pi}\right) \frac{dt}{t} P_{ab}(z) \, dz$

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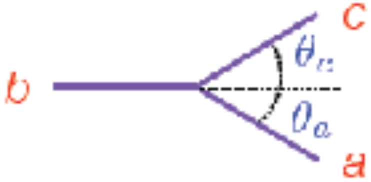

$$p_a = z p_b \,, p_c = (1-z) p_b$$

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since we cannot calculate with infinities we need to regularize them:

this time we choose **dimensional regularization** (i.e. work in d=4-2ε dimensions)

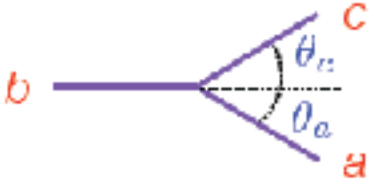
$$E_a^2 dE_a \, \theta_a d\theta_a \to E_a^{2-2\epsilon} dE_a \, \theta_a^{1-2\epsilon} d\theta_a = E_a^2 dE_a \, \theta_a d\theta_a \, z^{-\epsilon}(1-z)^{-\epsilon}\, t^{-\epsilon}$$

and obtain

$$d\sigma^{4-2\epsilon}_{(\ldots)ac} = d\sigma_{(\ldots)b} \left(\frac{\alpha_s}{2\pi}\right) \frac{dt}{t^{1+\epsilon}} P_{ab}(z) z^{-\epsilon}(1-z)^{-\epsilon}\, dz$$

NLO in a nutshell: real radiation

recall: collinear/soft kinematics



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and obtain

familiar splitting function accompanying IR term

$$d\sigma_{(\ldots)ac}^{4-2\epsilon} = d\sigma_{(\ldots)b} \left(\frac{\alpha_s}{2\pi}\right) \frac{dt}{t^{1+\epsilon}} \boxed{P_{ab}(z)} z^{-\epsilon} (1-z)^{-\epsilon} \, dz$$

integrals now finite

NLO in a nutshell: poles

we can now see how the singularities are regularized in d dimensions

$$d\sigma^{4-2\epsilon}_{(\ldots)ac} = d\sigma_{(\ldots)b}\left(\frac{\alpha_s}{2\pi}\right)\frac{dt}{t^{1+\epsilon}}P_{ab}(z)z^{-\epsilon}(1-z)^{-\epsilon}\,dz$$

- **collinear pole** $\int \frac{dt}{t^{1+\epsilon}} \to \frac{1}{\epsilon}$

recall: such a factor is present in P_{qq} (and P_{gg})

- **soft pole** $\int dz(1-z)^{-\epsilon}\left(\frac{1}{1-z}\right) \to \frac{1}{\epsilon}$

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putting all together one obtains the following (general) structure

$$d\sigma_{W+g} = \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \frac{2}{\epsilon} P_{qq} + \mathcal{O}(\epsilon^0)\right) d\sigma_{W,\text{tree}}$$

+ finite terms

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+ finite terms

IR poles will cancel in sum with virtual corrections

needs to be absorbed into bare PDFs by factorization

NLO in a nutshell: virtual corrections

only one loop diagram to consider at NLO
(selfenergy on massless external lines zero in d dimensions)

obtain for amplitude:

$$\int \frac{d^{4-2\epsilon}\ell \quad \mathcal{N}}{\ell^2(\ell+p_{\bar{d}})^2(\ell+p_{\bar{d}}+p_u)^2}$$

with some complicated Dirac structure in numerator

$$\mathcal{N} = [\bar{u}(p_{\bar{d}})\gamma^\alpha \not{\ell}\gamma^\mu(\not{\ell}+\not{p}_{\bar{d}}+\not{p}_u)\gamma_\alpha u(p_u)]\, V_\mu(p_W)$$

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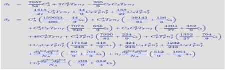
inspect denominator:

$$\ell^2(\ell+p_{\bar{d}})^2(\ell+p_{\bar{d}}+p_u)^2 \xrightarrow{\text{can shift momenta}} \ell^2(\ell-p_{\bar{d}})^2(\ell+p_u)^2$$



- soft singularity for l -> 0
- singularities for l collinear to quark lines

regularize again in d dimensions

NLO in a nutshell: loop integration

can decompose Dirac structure into given set of simpler scalar integrals
(**Passarino Veltman decomposition**)

then: $$\frac{1}{\ell^2(\ell+p_{\bar{d}})^2(\ell+p_{\bar{d}}+p_u)^2}$$

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then:

$$\frac{1}{\ell^2(\ell+p_{\bar{d}})^2(\ell+p_{\bar{d}}+p_u)^2}$$

$$= 2\int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \, \frac{\delta(x_1+x_2+x_3-1)}{\left[x_1\ell^2+x_2(\ell+p_{\bar{d}})^2+x_3(\ell+p_{\bar{d}}+p_u)^2\right]^3}$$

need to combine different terms in denominator
with help of **Feynman parameter integrals**

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then:

$$\frac{1}{\ell^2(\ell+p_{\bar{d}})^2(\ell+p_{\bar{d}}+p_u)^2}$$

need to combine different terms in denominator with help of **Feynman parameter integrals**

$$= 2\boxed{\int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3} \frac{\delta(x_1+x_2+x_3-1)}{\left[x_1\ell^2 + x_2(\ell+p_{\bar{d}})^2 + x_3(\ell+p_{\bar{d}}+p_u)^2\right]^3}$$

$$F_2^n(x) = x\left(\frac{1}{9}d_n(x) + \frac{4}{9}u_n(x)\right) = x\left(\frac{4}{9}d_p(x) + \frac{1}{9}u_p(x)\right)$$

where

$$\begin{aligned} \left.\frac{d^2\hat{\sigma}}{dx dQ^2}\right|_{F_2} &\equiv \hat{F}_2^q \\ &= e_q^2 x \left[\delta(1-x) + \frac{\alpha_s(\mu_r)}{4\pi}\left[P_{qq}(x)\ln\frac{Q^2}{m_q^2} + C_2^q(x)\right]\right] \end{aligned}$$

this can be evaluated using:

$$\int \frac{d^d L}{(2\pi)^d} \frac{1}{(L^2-\Delta)^n} = i\frac{(-1)^n}{(4\pi)^{d/2}} \frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)} \Delta^{d/2-n}$$

NLO in a nutshell: loop integration

can decompose Dirac structure into given set of simpler scalar integrals
(**Passarino Veltman decomposition**)

then:
$$\frac{1}{\ell^2(\ell+p_{\bar{d}})^2(\ell+p_{\bar{d}}+p_u)^2}$$

need to combine different terms in denominator
with help of **Feynman parameter integrals**

$$= 2\int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \frac{\delta(x_1+x_2+x_3-1)}{\left[x_1\ell^2+x_2(\ell+p_{\bar{d}})^2+x_3(\ell+p_{\bar{d}}+p_u)^2\right]^3}$$

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and one obtains:
$$\int_0^1 dx_1\int_0^{1-x_1} dx_3\,(-2x_1x_3\,p_u\cdot p_{\bar{d}})^{-1-\epsilon} = (-2p_u\cdot p_{\bar{d}})^{-1-\epsilon}\int_0^1 dx_1\,x_1^{-1-\epsilon}\left(-\frac{1}{\epsilon}\right)x_1^{-\epsilon}$$

$$= (-2p_u\cdot p_{\bar{d}})^{-1-\epsilon}\left(-\frac{1}{\epsilon}\right)\frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} = (-2p_u\cdot p_{\bar{d}})^{-1-\epsilon}\left(\frac{1}{\epsilon^2}\right)\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

IR singularity

NLO in a nutshell: final result

once all scalar integrals are computed and put together, find:

$$d\sigma_{W,1-\text{loop}} = \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + \text{finite} \right) d\sigma_{W,\text{tree}}$$

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recall:
factorized into PDFs

$$d\sigma_{W+g} = \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \frac{2}{\epsilon} P_{qq} + \mathcal{O}(\epsilon^0) \right) d\sigma_{W,\text{tree}}$$

and one ends up with the finite NLO result (where d->4) ✓

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and one ends up with the finite NLO result (where d->4) ✓

this is one of the simplest loop calculations !

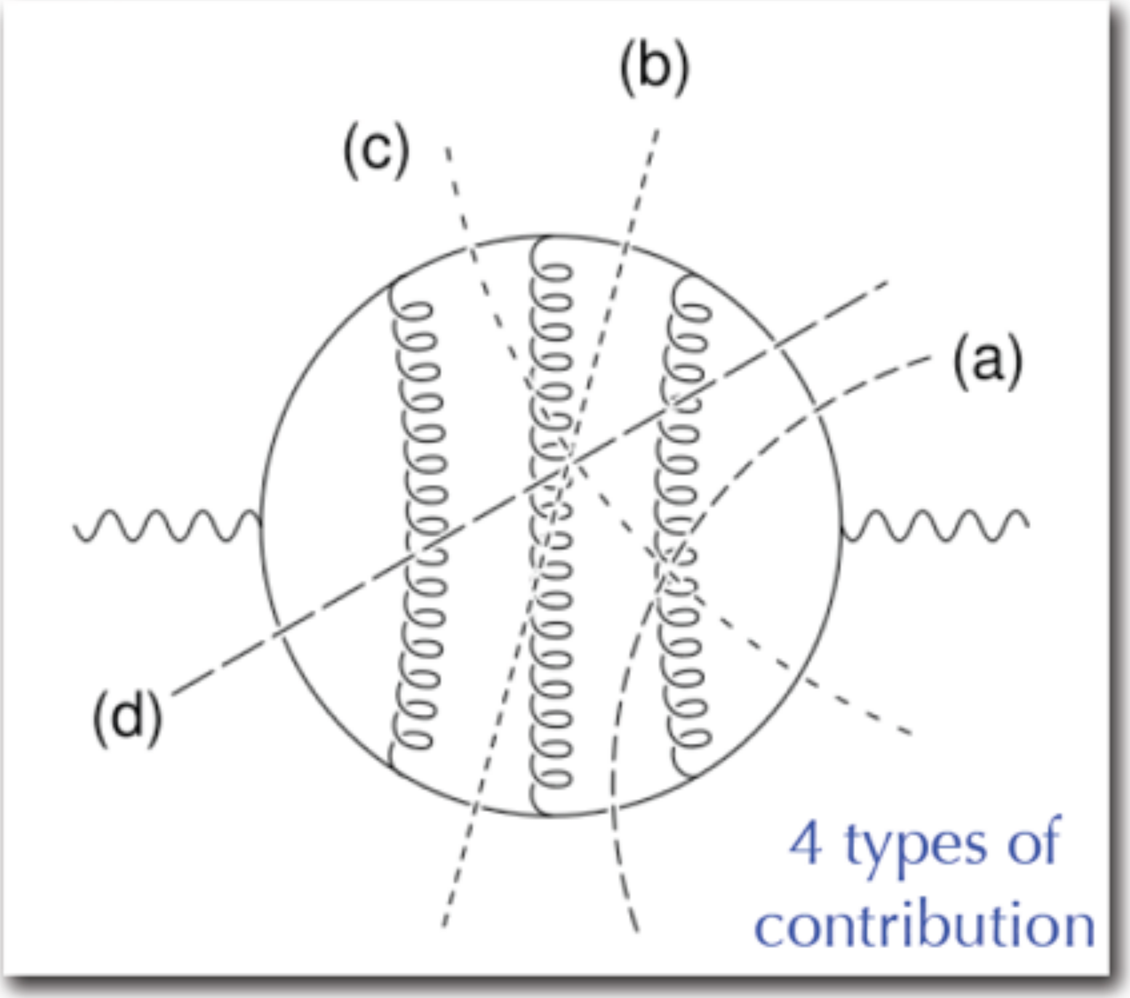
in general it is much more complicated

but the general ideas are the same

for high multiplicity final states one needs novel methods "beyond" Feynman diagrams

NNLO complexity

one can envision the contributions to a NNLO calculation by considering all possible cuts to a 3-loop diagram:



example:

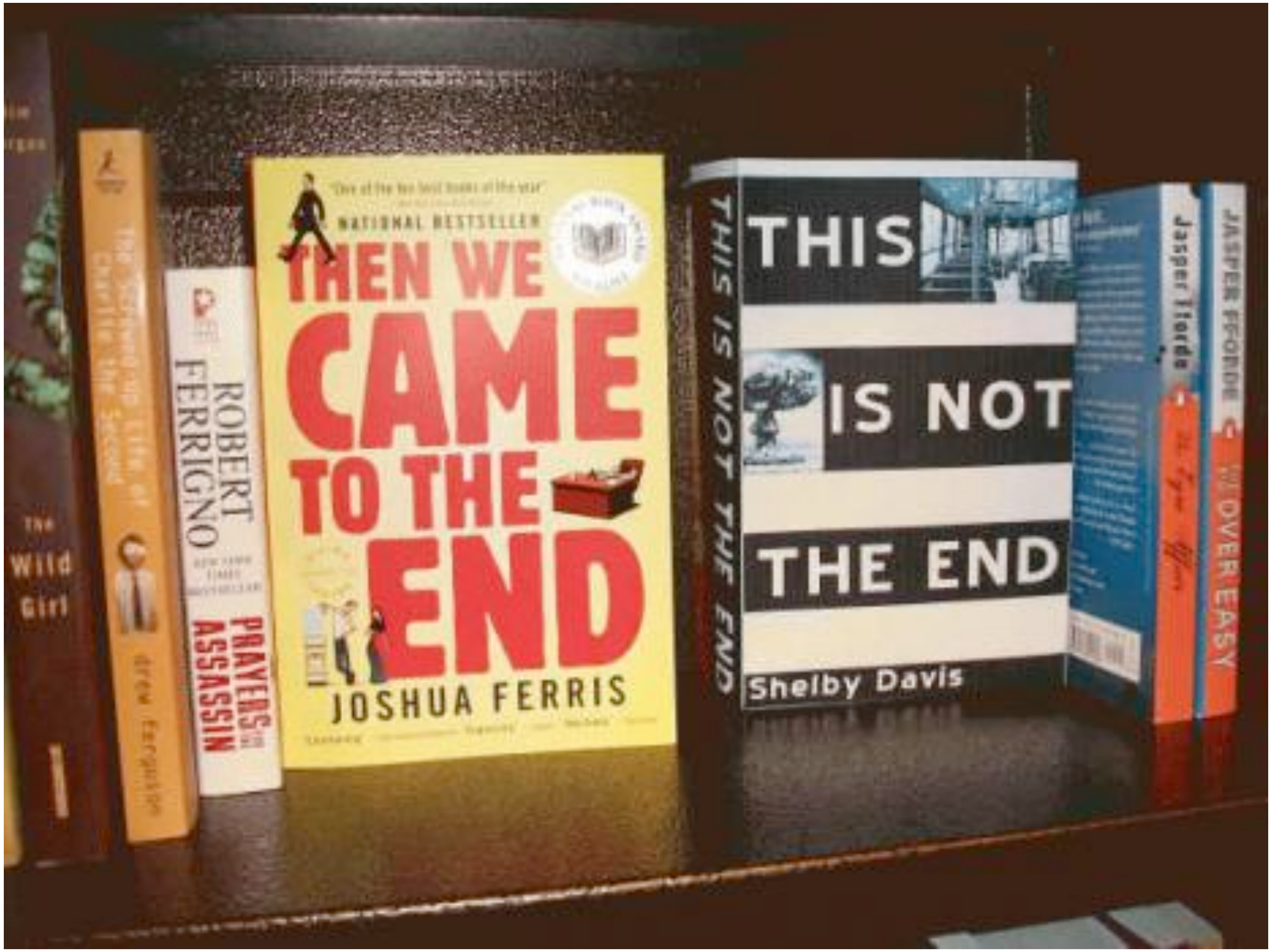
3 jet production in e^+e^-

(a) **two-loop virtual correction**

(b) **one-loop x one-loop**

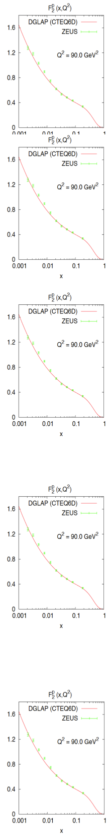
(c) **one-loop x real**
both with an extra parton

(d) **real**
with two extra partons



SUMMARY & OUTLOOK

QCD: the most perfect gauge theory (so far)



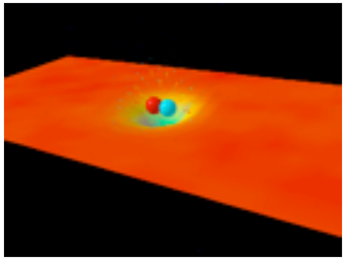
simple $\mathcal{L}$ but rich & complex phenomenology; few parameters

in principle complete up to the Planck scale
(issue: CP, axions?)

highly non-trivial ground state responsible
for all the structure in the visible universe

emergent phenomena: confinement,
chiral symmetry breaking, hadrons

confinement



D. Leinweber

non-perturbative
structure of hadrons

e.g. through lattice QCD

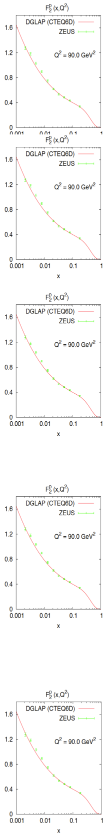
interplay between
High Energy and
Hadron Physics

asymptotic freedom

hard scattering
cross sections
and
renormalization group

perturbative methods

QCD: the most perfect gauge theory (so far)



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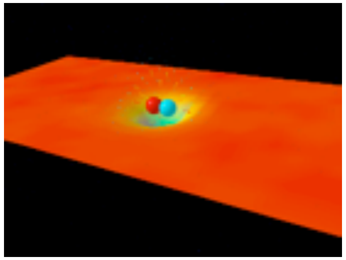

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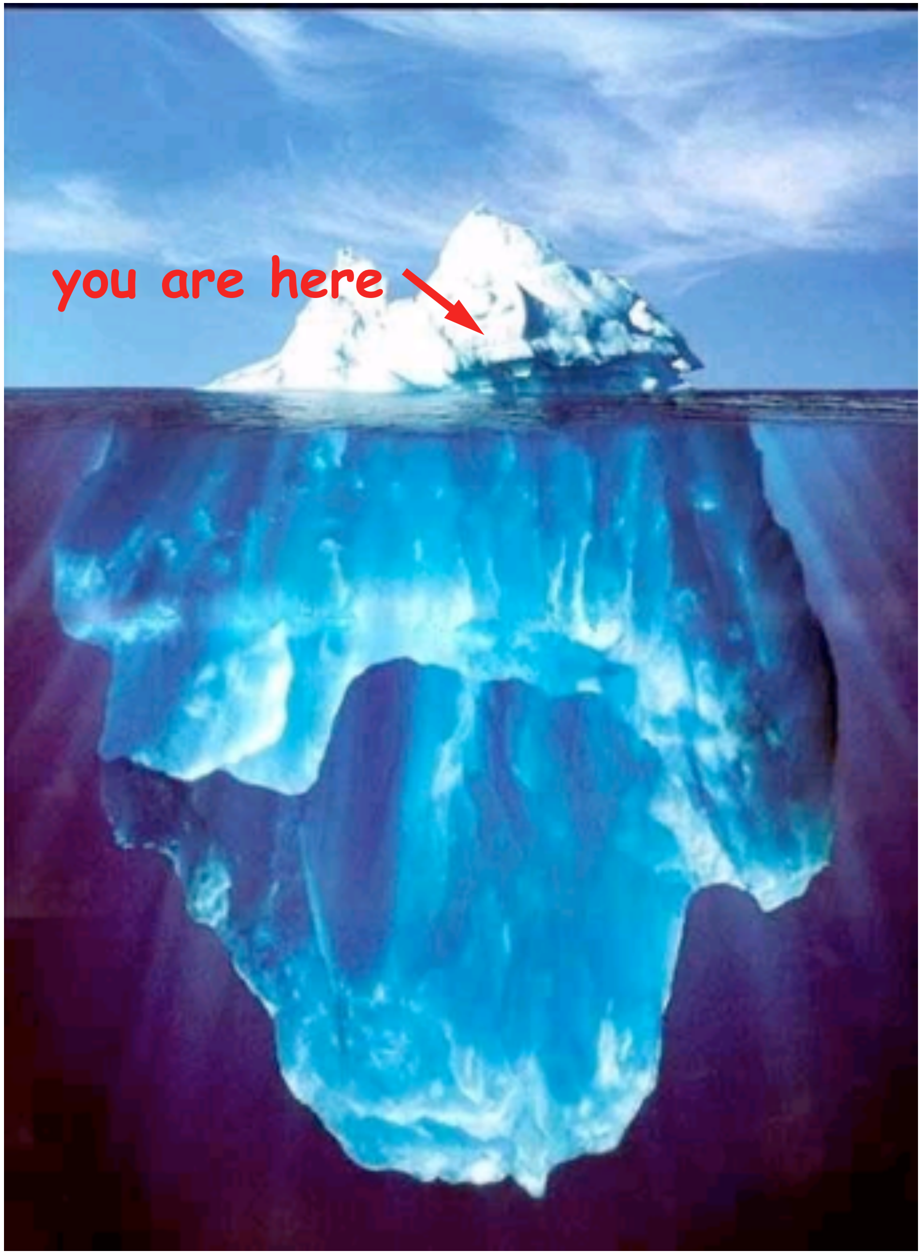
we have just explored the tip of the iceberg



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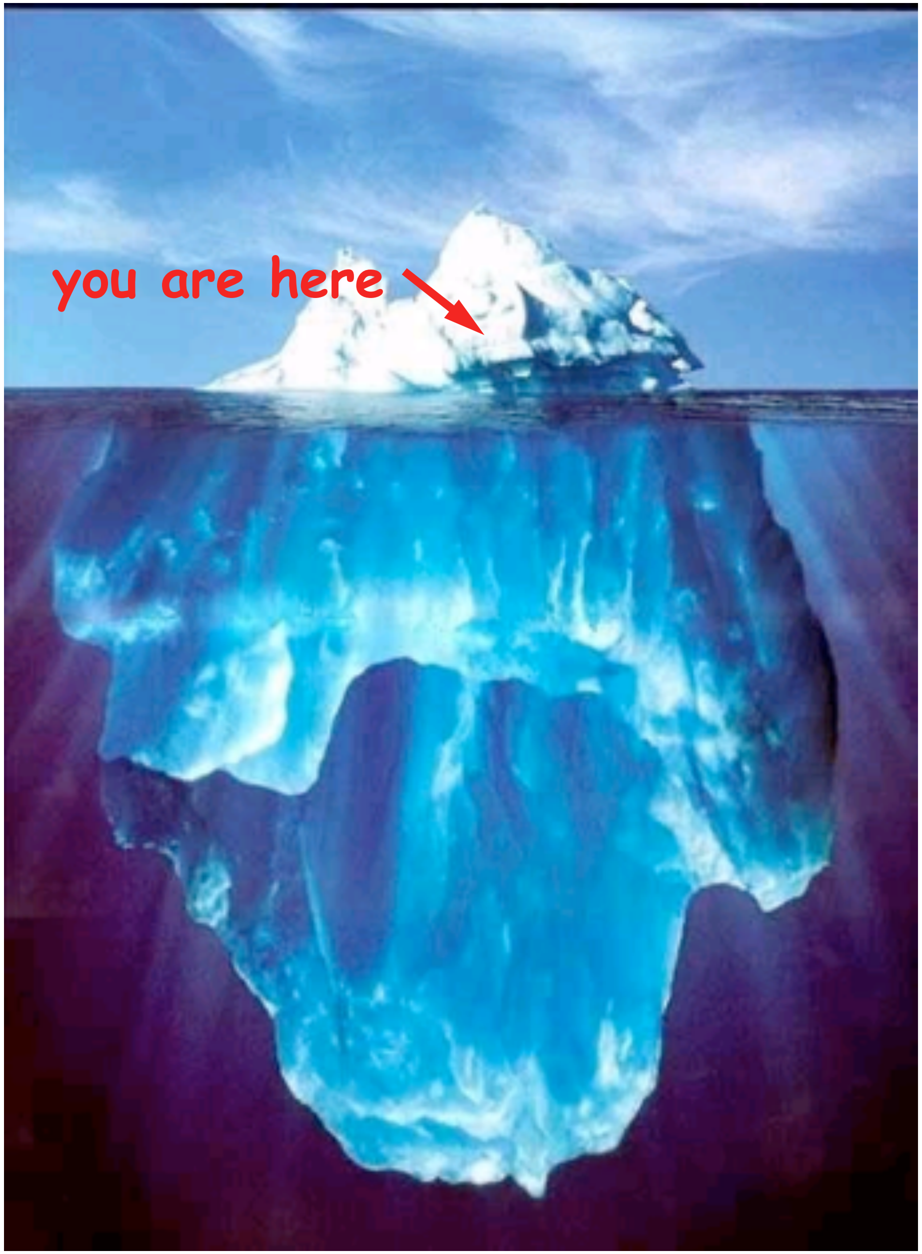

enjoy the other lectures !