Higgs mass in noncommutative geometry

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Introduction

$$\begin{array}{ccc} \text{commutative spectral triple} & \rightarrow & \text{noncommutative spectral triple} \\ \updownarrow & & \downarrow \\ \text{Riemannian geometry} & & \text{non-commutative geometry [NCG]} \end{array}$$

Suitable framework to describe the standard model of elementary particles [SM] together with (Euclidean) general relativity in a common geometrical framework.

Space-(time) is the product of a commutative geometry (gravitational degrees of freedom) by a noncommutative geometry (quantum degrees of freedom).

The Lagrangian of the SM and Einstein-Hilbert action follow from a single action formula: the SM is a gravity theory.

Bonus: the Higgs field comes out as the noncommutative part of the connection.

- **170 GeV:** prediction of the Higgs mass from NCG. Ruled out by Tevratron in August 2008.
- **126 GeV:** mass of the Higgs-Brout-Englert boson, official since July 2012.

“God not only play dices, but also Russian roulette.”

$$V(H) = -\frac{\mu}{2}H^2 + \frac{\lambda}{4}H^4$$

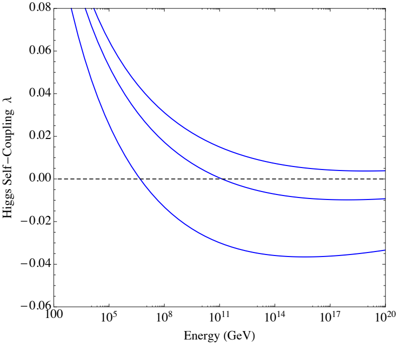


FIG. 2: Higgs self-coupling λ as a function of energy, for different values of the Higgs mass from 2-loop RG evolution. Lower curve is for $m_H = 116\,\mathrm{GeV}$, middle curve is for $m_H = 126\,\mathrm{GeV}$, and upper curve is for $m_H = 130\,\mathrm{GeV}$. All other Standard Model couplings have been fixed in this plot, including the top mass at $m_t = 173.1\,\mathrm{GeV}$.

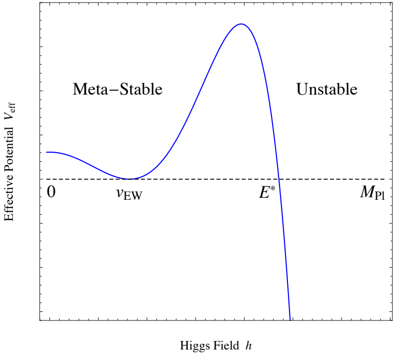
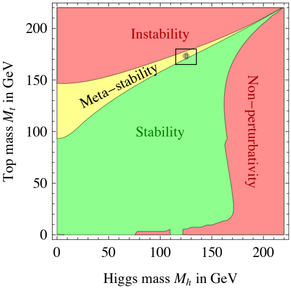
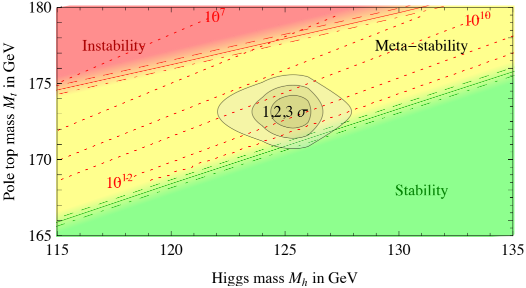


FIG. 3: Schematic of the effective potential V_{eff} as a function of the Higgs field h. This is *not* drawn to scale; for a Higgs mass in the range indicated by LHC data, the heirarchy is $v_{EW} \ll E^* \ll M_{\mathrm{Pl}}$, where each of these 3 energy scales is separated by several orders of magnitude.

M. P. Hertzberg, *A correlation between the Higgs mass and dark matter*, arXiv:1210.3624





Degrassi, Di Vita, Elias-Miro, Espinosa, Guidice, Isidori and A. Sturmia, *Higgs mass and Vacuum Stability in the SM at NNLO*, arXiv:1205.6497

The instability of the electroweak vacuum can be cured by introducing a new scalar field σ:[†]

$$V(H,\sigma) = \frac{1}{4}(\lambda H^4 + \lambda_\sigma \sigma^4 + 2\lambda_{H\sigma} H^2\sigma^2).$$

Incidentally, in the description of the SM in NCG, the field σ allows to pull m_H back to 126 GeV.

Resilience of the spectral SM, Chamseddine, Connes 2012

Is σ natural in NCG, or is it just an artifact for saving the model ?

[†] Elias-Miro, Espinosa, Guidice, Lee and Sturmia, *Stabilization of the Electroweak Vacuum by a Scalar Threshold effect*, JHEP **1206** (2012) 031; Degrassi, Di Vita, Elias-Miro, Espinosa, Guidice, Isidori and A. Sturmia, *Higgs mass and Vacuum Stability in the SM at NNLO, arXiv:1205.6497*; Chian-Shu Chen and Yong Tang, *Vacuum Stability, Neutrinos and Dark matter*, JHEP **1204** (2012) 019; Oleg Lebedev, *On Stability of the Higgs Potential and the Higgs Portal*, JHEP, arXiv:1203.0156.

I. Higgs mass in NCG: state of the art

- the spectral triple of the standard model
- spectral action and the 170 Gev prediction
- a new hope: field σ versus the first order condition ?

II. The grand algebra

- mixing of spinor and internal degrees of freedom

III. Reduction to the standard model

- the field σ
- emergence of geometry

I. State of the art (till 2012)

Spectral triple

A $*$-algebra $\mathcal{A}$, faithful representation on $\mathcal{H}$, operator D on $\mathcal{H}$ such that $[D, a]$ is bounded and $a[D - \lambda\mathbb{I}]^{-1}$ is compact for all $a \in \mathcal{A}$ and $\lambda \notin$ Spec D.

With extra-conditions (dimension, regularity, finitude, first order, orientability, reality, Poincaré duality):

Theorem Connes 1996-2008-2013

$\mathcal{M}$ compact Riemann spin manifold, then $(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \partial\!\!\!/)$ is a spectral triple.

$(\mathcal{A}, \mathcal{H}, D)$ a spectral triple with $\mathcal{A}$ unital commutative, then there exists a compact Riemannian spin manifold $\mathcal{M}$ such that $\mathcal{A} = C^\infty(\mathcal{M})$.

The spectral triple of the standard model: product of a manifold by

$$\mathcal{A}_{sm} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \quad \mathcal{H}_F = \mathbb{C}^{96=N\times 2\times 2\times 8} = \mathcal{H}_R \oplus \mathcal{H}_L \oplus \mathcal{H}_R^c \oplus \mathcal{H}_L^C,$$

$$D_F = \begin{pmatrix} 0_{8N} & M & M_R & 0_{8N} \\ M^\dagger & 0_{8N} & 0_{8N} & 0_{8N} \\ M_R^\dagger & 0_{8N} & 0_{8N} & \bar{M} \\ 0_{8N} & 0_{8N} & M^T & 0_{8N} \end{pmatrix} \quad N = 3 = \sharp \text{ generations.}$$

- M contains the quarks, leptons, neutrinos (Dirac) Yukawa couplings, as well as the quarks and neutrinos mixing parameters;
- M_R contains the Majorana neutrinos mass.

The total spectral triple of the SM is

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_{sm}, \quad \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F, \quad D = \partial\!\!\!/ \otimes \mathbb{I}_{96} + \gamma^5 \otimes D_F$$

$$\Gamma = \gamma^5 \otimes \gamma_F, \quad J = \mathcal{J} \otimes J_F$$

where $\mathcal{J}$ is the charge conjugation and

$$\gamma_F = \begin{pmatrix} \mathbb{I}_{8N} & & & \\ & -\mathbb{I}_{8N} & & \\ & & -\mathbb{I}_{8N} & \\ & & & \mathbb{I}_{8N} \end{pmatrix}, \quad J_F = \begin{pmatrix} 0_{16N} & \mathbb{I}_{16N} \\ \mathbb{I}_{16N} & 0_{16N} \end{pmatrix}.$$

Spectral action

$S = \mathrm{Tr}\, f(\frac{D_A}{\Lambda})$ yields the bosonic SM Lagrangian coupled with Einstein-Hilbert action (f is a smooth approximation of $\Xi_{[0,\Lambda]}$, with Λ a parameter fixing the mass scale).

Requiring a unique unification scale,

$$\frac{g_3^2 f_0}{2\pi^2} = \frac{1}{4}, \quad g_3^2 = g_2^2 = \frac{5}{3} g_1^2$$

($f_\beta = \int_0^\infty f(v) v^{\beta-1} dv$), the asymptotic expansion of S yields

$$\int_{\mathcal{M}} \sqrt{g} d^4x \; \Big(\frac{1}{\kappa_0^2} R + \alpha_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \gamma_0 + \tau_0 R^* R* \\ + \frac{1}{4} G^i_{\mu\nu} \bar{G}_i^{\mu\nu} + \frac{1}{4} F^\alpha_{\mu\nu} \bar{F}_\alpha^{\mu\nu} + \frac{1}{4} B_{\mu\nu} \bar{B}^{\mu\nu} \\ + \frac{1}{2} |D_\mu H|^2 - \mu_0^2 |H|^2 - \frac{1}{12} R|H|^2 + \lambda_0 |H|^4 \Big)$$

where

$$\frac{1}{\kappa_0^2} \sim (f_2 \Lambda^2, f_0), \quad \mu_0^2 \sim (\frac{f_2 \Lambda^2}{f_0}, 1), \quad \alpha_0 \sim \tau_0 \sim f_0, \quad \gamma_0 \sim (f_4 \Lambda^4, f_2 \Lambda^2, f_0), \quad \lambda_0 \sim \frac{1}{f_0}$$

- From the top mass, one gets the boundary condition $\lambda_0 = 0.356$ at $\Lambda = 10^{17}$GeV. Under the big desert hypothesis, this yields $\lambda(M_Z) \simeq 0.241$ hence $m_H = \sqrt{2\lambda}\, v_{EW} = 246 \sqrt{2 \times 0.241} = 170.8$ GeV.

A new hope

By turning the entry of the neutrino Majorana mass in D_F into a field,

$$k_R \to k_R \sigma$$

(one generation (k_t, k_ν) only) one gets the potential

$$V = \frac{1}{4}\left(\lambda \bar{h}^4 + 2\lambda_{h\sigma}\bar{h}^2 + \lambda_\sigma \bar{\sigma}^4\right) - \frac{2g^2}{\pi^2} f_2 \Lambda^2 \left(\bar{h}^2 + \bar{\sigma}^2\right)$$

where $H = \begin{pmatrix} 0 \\ h \end{pmatrix}$, $\bar{h} = |k_t|$, $\bar{\sigma} = |k_R|\sigma$ and, defining $k_\nu = \sqrt{n} k_t$,

$$\lambda = \frac{n^2+3}{(n+3)^2}(4g^2), \quad \lambda_{h\sigma} = \frac{2n}{n+3}(4g^2), \quad \lambda_\sigma = 2(4g^2).$$

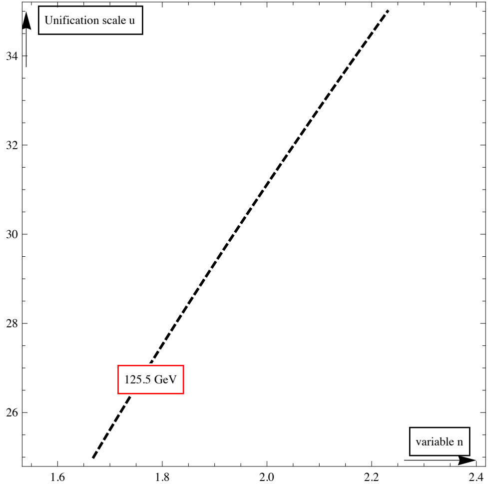
One finds, for $u = \ln \frac{\Lambda}{M_Z}$,

Resilience of the spectral SM, Chamseddine, Connes JHEP (2012)

$$m_H(u=0) = 246\sqrt{2\lambda(0)\left(1 - \frac{\lambda_{h\sigma}(0)}{\lambda_h(0)\lambda_\sigma(0)}\right)}.$$

Higgs mass as a function of n and the unification scale $u \in [25, 35]$, i.e.

$$m_Z e^{25} = 6.55245 \times 10^{12}\text{GeV} \quad \text{to} \quad m_Z e^{35} = 1.44327 \times 10^{17}\text{Gev}.$$



- For any unification scale u, there exists n such that $125 \leq m_H \leq 126\text{GeV}$.
- No instability: $\lambda_{h\sigma}^2 < \lambda_h \ \lambda_\sigma$.

Gauge fields and the first order condition

The gauge fields of the SM (including the Higgs) are obtained by fluctuation of the metric

$$[D, a] = [\partial\!\!\!/ \otimes \mathbb{I} + \gamma^5 \otimes D_F, f^i \otimes m_i],$$

allowing to turn the constant components of D_F into fields on the manifold $\mathcal{M}$.

Unfortunately, the first order conditon

$$[[D, a], JbJ^{-1}] = 0 \quad \forall a, b \in \mathcal{A}$$

prevents to do so for the field σ. Indeed, for D_R the Dirac with only the neutrino mass ($D_F = D_0 + D_R$),

$$[[D_R, a], JbJ^{-1}] = 0 \quad \forall a, b \in \mathcal{A}_{sm} \Longrightarrow [D_R, a] = 0.$$

- No way to obtain σ as the other bosonic fields within the spectral triple of the standard model, following the NCG rules.

II. The grand algebra

Connes, Chamseddine, Marcolli: the finite dimensional algebra is of the form

$$M_a(\mathbb{H}) \oplus M_{2a}(\mathbb{C}) \quad a \in \mathbb{N}$$

and acts on an Hilbert space of dimension $d = 2 \times (2 \times a)^2$.

- $a = 1$: too small to get the gauge group as unitaries of $M(\mathbb{H}) \oplus M_2(\mathbb{C})$.
- $a = 2$ yields $d = 32 = \#$particles per generation.

  Grading condition $[\Gamma, a] = 0$ (coming from the orientability axiom) imposes

  $$\mathcal{A}_F = M_2(\mathbb{H}) \oplus M_4(\mathbb{C}) \longrightarrow \mathcal{A}_{LR} = \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_4(\mathbb{C}).$$

  1st-order condition without neutrino mass further imposes

  $$\mathcal{A}_{LR} \longrightarrow \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C}) \oplus \mathbb{C}.$$

  1st-order condition with neutrino mass finally gives

  $$\mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C}) \oplus \mathbb{C}. \longrightarrow \mathbb{H}_L \oplus \mathbb{C}' \oplus M_3(\mathbb{C}) \oplus \mathbb{C}$$

  with $\mathbb{C} = \mathbb{C}'$. Hence the the reduction

  $$\mathcal{A}_F \to \mathcal{A}_{sm} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}).$$

- a=3: $d = 72$. No obvious relation with 32 particles/generation.

- a=4: $d=128=$ dimension 4×32 of the total Hilbert space for 1 generation:

$$\mathcal{H}=L^2(\mathcal{M},S)\otimes\mathcal{H}_F=L^2(\mathcal{M})\otimes\mathsf{H}_F$$

where

$$\mathsf{H}_F=\mathbb{C}^4\otimes\mathcal{H}_F=\mathbb{C}^4\otimes\mathbb{C}^{32}=\mathbb{C}^{128}.$$

By mixing the spin

$$s=l,r,\quad \dot{s}=\dot{0},\dot{1}$$

and the internal

$$C=p,a\quad \alpha=u_R,d_R,u_L,d_L\ (I=1,2,3),\ e_R,\nu_R,e_L,\nu_L\ (I=0)$$

degrees of freedom, the Hilbert space $\mathcal{H}$ of the standard model allows to represent the grand algebra

$$C^\infty(\mathcal{M})\otimes\mathcal{A}_G\quad\text{where}\quad\mathcal{A}_G=M_4(\mathbb{H})\oplus M_8(\mathbb{C})$$

without touching the particle contents of the SM.

Representation: a spinor in $\mathcal{H}$ is $\Psi^{CI}_{s\dot{s}\alpha}$

- Both $Q \in M_2(\mathbb{H})$ and $M \in M_4(\mathbb{C})$ are viewed as 4×4 complex matrices:

$$Q^{\beta}_{\alpha} \quad M^{I}_{J}$$

  $A = (Q, M) \in C^{\infty}(\mathcal{M}) \otimes (M_2(\mathbb{H}) \oplus M_4(\mathbb{C}))$ acts trivially on the spin indices:

$$A^{t\dot{t}CI\beta}_{s\dot{s}DJ\alpha} = \delta^{t}_{s}\delta^{\dot{t}}_{\dot{s}}\left(\delta^{C}_{0}\delta^{I}_{J}Q^{\beta}_{\alpha} + \delta^{C}_{1}M^{I}_{J}\delta^{\beta}_{\alpha}\right)$$

- Both $Q \in M_4(\mathbb{H})$ and $M \in M_8(\mathbb{C})$ are viewed as 2×2 block matrices, with block 4×4 complex matrices:

$$Q^{\dot{t}\beta}_{\dot{s}\alpha} = \begin{pmatrix} Q^{\dot{0}\beta}_{\dot{0}\alpha} & Q^{\dot{1}\beta}_{\dot{0}\alpha} \\ Q^{\dot{0}\beta}_{\dot{1}\alpha} & Q^{\dot{1}\beta}_{\dot{1}\alpha} \end{pmatrix}, \quad M^{tI}_{sJ} = \begin{pmatrix} M^{rI}_{rJ} & M^{lI}_{rJ} \\ M^{rI}_{lJ} & M^{lI}_{lJ} \end{pmatrix}.$$

  $A = (Q, M) \in C^{\infty}(\mathcal{M}) \otimes (M_2(\mathbb{H}) \oplus M_4(\mathbb{C}))$ has a non-diagonal action on the spin indices $s, \dot{s}$:

$$A^{t\dot{t}CI\beta}_{s\dot{s}DJ\alpha} = \left(\delta^{C}_{0}\delta^{t}_{s}\delta^{I}_{J}Q^{\dot{t}\beta}_{\dot{s}\alpha} + \delta^{C}_{1}M^{tI}_{sJ}\delta^{\dot{t}}_{\dot{s}}\delta^{\beta}_{\alpha}\right)$$

The Dirac matrices are

$$\gamma^{\mu} = \begin{pmatrix} 0_2 & \sigma^{\mu}{}^{\dot{t}}_{\dot{s}} \\ \overline{\sigma}^{\mu}{}^{\dot{t}}_{\dot{s}} & 0_2 \end{pmatrix}_{st} \qquad \gamma^5 = \begin{pmatrix} \mathbb{I}_2 & 0_2 \\ 0_2 & -\mathbb{I}_2 \end{pmatrix}_{st}$$

where for $\mu = 0, 1, 2, 3$ one defines $\sigma^{\mu} = \{\mathbb{I}_2, -i\sigma_i\}, \bar{\sigma}^{\mu} = \{\mathbb{I}_2, i\sigma_i\}$.

The chirality is

$$\Gamma = \gamma^5 \otimes \gamma_F = \eta^C_D\, \eta^t_s\, \delta^I_J\, \delta^{\dot{t}}_{\dot{s}}\, \eta^{\beta}_{\alpha}$$

where $\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The Dirac operator on $L^2(\mathcal{M}, S) \otimes \mathcal{H}_F = L^2(\mathcal{M}) \otimes H_F$ is unchanged

$$\begin{aligned} D &= \partial\!\!\!/ \otimes \mathbb{I}_{\mathcal{H}_F} + \gamma^5\, \mathbb{I}_{L^2(\mathcal{M},S)} \otimes D_F \\ &= \partial_{\mu} \otimes \gamma^{\mu}{}^{\dot{t}t}_{\dot{s}s}\, \delta^{CI\beta}_{DJ\alpha} + \mathbb{I}_{L^2(\mathcal{M})} \otimes \gamma^5{}^{\dot{t}t}_{\dot{s}s}\, D_F{}^{CI\beta}_{DJ\alpha}. \end{aligned}$$

III. Reduction to the standard model

The grading condition $[\Gamma, a] = 0$ imposes the reduction

$$\mathcal{A}_G = M_4(\mathbb{H}) \oplus M_8(\mathbb{C}) \longrightarrow \mathcal{A}'_G = (M_2(\mathbb{H})_L \oplus M_2(\mathbb{H})_R) \oplus (M_4(\mathbb{C})_l \oplus M_4(\mathbb{C})_r).$$

Solution of the 1^{st}-order condition of the Majorana Dirac operator $\mathbb{I}_{L^2(\mathcal{M})} \otimes \gamma^5 D_R$:

$$\mathcal{A}'_G \longrightarrow \mathcal{A}''_G = (\mathbb{H}_L \oplus \mathbb{H}'_L \oplus \mathbb{C}_R \oplus \mathbb{C}'_R) \oplus (\mathbb{C}_l \oplus M_3(\mathbb{C})_l \oplus \mathbb{C}_r \oplus M_3(\mathbb{C})_r)$$

with $\mathbb{C}_R = \mathbb{C}_r = \mathbb{C}_l$.

Proposition — Devastato, Lizzi, Martinetti 2013

For $a \in \mathcal{A}''_G, \quad [\mathbb{I}_{L^2(\mathcal{M})} \otimes \gamma^5 D_R, a]$ is not necessarily zero.

The further reduction $\mathbb{C}'_R = \mathbb{C}_R$, $\mathbb{H}'_L = \mathbb{H}_L$, $M_3(\mathbb{C})_l = M_3(\mathbb{C})_r$, that is

$$\mathcal{A}''_G \to \mathcal{A}_{sm} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}),$$

then satisfies the 1st order condition for the full Dirac operator

$$D = \partial\!\!\!/ \otimes \mathbb{I}_{\mathcal{H}_F} + \gamma^5\, \mathbb{I}_{L^2(\mathcal{M},S)} \otimes D_F \qquad D_F = D_0 + D_R.$$

- Starting with the grand algebra

$$\mathcal{A}_G = M_4(\mathbb{H}) \oplus M_8(\mathbb{C})$$

  (reduced to $\mathcal{A}'_G = (M_2(\mathbb{H})_L \oplus M_2(\mathbb{H})_R) \oplus (M_4(\mathbb{C})_l \oplus M_4(\mathbb{C})_r)$ by the grading condition) one generates the field σ by a fluctuation of the Majorana Dirac operator D_R, respecting the 1st-order condition imposed by D_R. The latter yields the reduction to

$$\mathcal{A}''_G = (\mathbb{H}_L \oplus \mathbb{H}'_L \oplus \mathbb{C}_R \oplus \mathbb{C}'_R) \oplus (\mathbb{C}_l \oplus M_3(\mathbb{C})_l \oplus \mathbb{C}_r \oplus M_3(\mathbb{C})_r).$$

- σ does not satisfy the 1st-order condition imposed by the free Dirac $\partial\!\!\!/$. The latter yields the reduction of $\mathcal{A}''_G$ to the algebra of the standard model

$$\mathcal{A}_{sm} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}).$$

- Hopefully the reduction $\mathcal{A}''_G \to \mathcal{A}_{sm}$ might be understood dynamically, as a minimum of the spectral action for the free Dirac operator. σ would then be the "Higgs field" corresponding to this breaking.

- Almost simultaneously, Chamseddine, Connes and van Suijlekom proposed a definition of inner fluctuation without first order condition.

  Starting with $M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$, they generate the field σ, and retrieve the 1st-order condition dynamically, by minimizing the spectral action.

Conclusion

NCG proposes a description of the SM as a pure gravity theory, on a slightly noncommutative version of space(-time). Compatible with $m_H = 126\text{GeV}$ as soon as one makes the first-order condition flexible. So far, two proposals:

- CCvS: no problem with the continuous part but with the finite part.
- Grand algebra: no problem with the finite part, but with the continuous part.

The fluctuation of the Dirac Majorana only might suggest a cosmological model where the right neutrino is a "primordial particle" that generates σ, then the smooth structure ($\partial\!\!\!/$) emerges. But any cosmological interpretation of the spectral action is plagued by the Euclidean signature:

- put causal structure into the game,
- time comes from the noncommutativity: thermal-time of Connes-Rovelli.

The Higgs field acquires a metric interpretation: it gives the distance between two copies of the manifold, indexed by the pure state of $\mathbb{C}$ and the pure state of $\mathbb{H}$.

Grand symmetry, spectral action and the Higgs mass,
Devastato, Lizzi, P. M., JHEP **01** (2014) 042 arXiv:1304.0415 [hep-th]
Inner fluctuations in NCG without first order condition,
Chamseddine, Connes, Suijlekom, J. Geo. Phys. **73** (2013) arXiv: 1304.7583 [math-ph]
Beyond the spectral standard model: emergence of Pati-Salam unification,
Chamseddine, Connes, Suijlekom, JHEP **132** (2013) 11 arXiv: 1304.8050 [hep-th]

Covariant Dirac operator

$$D_A = D + A$$

with

$$A = \sum_i a^i \, [D, b_i] = A^* \quad a^i, b_i \in \mathcal{A}.$$

- D_A is called the covariant Dirac operator.

$$\left.\begin{array}{lcl} \mathcal{A} & = & C_0^\infty(\mathcal{M}) \otimes \mathcal{A}_F \\ \mathcal{H} & = & L_2(\mathcal{M}, S) \otimes \mathcal{H}_F \\ D & = & \partial\!\!\!/ \otimes \mathbb{I}_I + \gamma^5 \otimes D_F \end{array}\right\} \Longrightarrow A = H - i\gamma^\mu A_\mu.$$

- H: scalar field on $\mathcal{M}$ with value in $\mathcal{A}_I$ $\quad\rightarrow$ Higgs.
- A_μ: 1-form field with value in $Lie(U(\mathcal{A}_F))$ $\quad\rightarrow$ gauge field.

1. Dimension: D^{-1} is an infinitesimal of order $\frac{1}{m}$.

2. Regularity: for any $a \in \mathcal{A}$, a and $[D, a]$ belong to the intersection of the domains of all the powers δ^k of the derivation $\delta(b) \doteq [|D|, b]$, where b belongs to the algebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$.

3. Finitude: $\mathcal{A}$ is a pre-C^*-algebra and the set $\mathcal{H}^\infty \doteq \bigcap_{k\in\mathbb{N}}$ Dom D^k of smooth vectors of $\mathcal{H}$ is a finite projective module.

4. First order: the representation of $\mathcal{A}^\circ$ commutes with $[D, \mathcal{A}]$

$$[[D, a], Jb^* J^{-1}] = 0 \text{ for all } a, b \in \mathcal{A}.$$

5. Orientability: there exists a Hochschild cycle $c \in Z_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ)$ such that $\pi(c) = \Gamma$.

6. Reality $(\mathcal{A} \otimes \mathcal{A}^\circ, \mathcal{H}, D, \Gamma, J)$ is a KR^n-cycle with $[a, Jb^* J^{-1}] = 0$. J is called the *real structure*. That is

- J is a anti-unitary bijection on $\mathcal{H}$ that implements the involution, i.e. $JaJ^{-1} = a^*$ for all $a \in \mathcal{A}$;
- if n is even, there is a graduation Γ of $\mathcal{H}$ that commutes with $\mathcal{A}$ and anticommutes with D;
- the following table holds

n mod 8	0	1	2	3	4	5	6	7
$J^2 = \pm\mathbb{I}$	+	+	-	-	-	-	+	+
$JD = \pm DJ$	+	-	+	+	+	-	+	+
$J\Gamma = \pm\Gamma J$	+		-		+		-	

For odd n, one sets $\Gamma = \mathbb{I}$.

7. Poincaré duality: the additive coupling on $K_*(\mathcal{A})$ coming from the index of the Dirac operator is non-degenerated.