

Discrete parafermions and quantum-group symmetries

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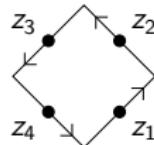
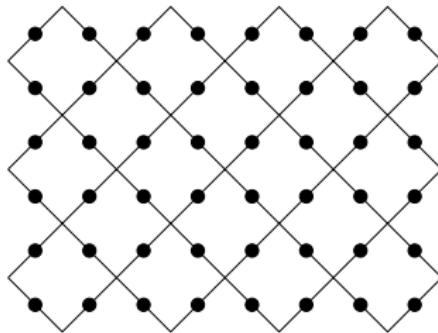
Outline

1. Introduction
2. The Bernard-Felder construction
3. Mapping to loop models

1. Introduction

Discretely holomorphic functions

- Discrete function: $F(z)$ on midpoints of square lattice \mathcal{L}



- Discrete “Cauchy-Riemann” equation:

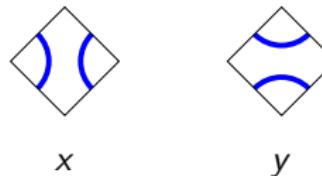
$$e^{\frac{i\pi}{4}} F(z_1) + e^{\frac{3i\pi}{4}} F(z_2) + e^{\frac{5i\pi}{4}} F(z_3) + e^{\frac{7i\pi}{4}} F(z_4) = 0$$

- Short-hand notation: $\sum_{\diamond} F(z) \delta z = 0$

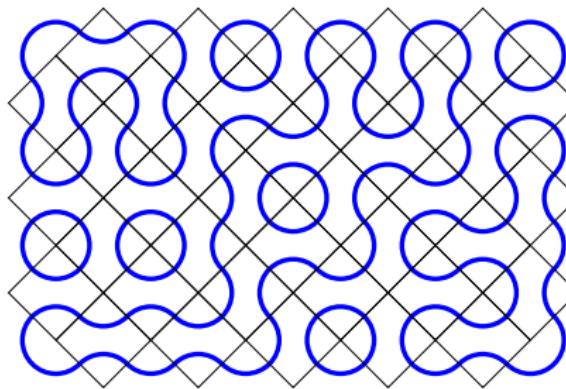
Loop models in Statistical Mechanics

The Temperley-Lieb loop model

- ▶ Plaquette configurations:



- ▶ Lattice configurations:



- ▶ Boltzmann weights: $W(C) = x^{N_x(C)} y^{N_y(C)} n^{N_\ell(C)}$

- ▶ Partition function: $Z = \sum_{\text{config. } C} W(C)$

Loop models in Statistical Mechanics

Correlation functions

- ▶ Averaging on Boltzmann weights:

$$\langle f(C) \rangle := \frac{1}{Z} \sum_C W(C) f(C).$$

- ▶ Two-leg correlation function:

$$G(z_1, z_2) := \frac{1}{Z} \sum_{C \mid z_1, z_2 \in \text{same loop}} W(C)$$

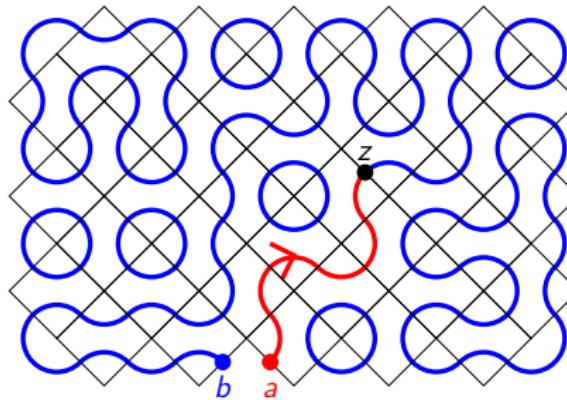
- ▶ Phases in scaling limit:

- ▶ Non-critical phase: $G(z_1, z_2) \sim \exp(-|z_1 - z_2|/\xi)$

- ▶ Critical phase: $G(z_1, z_2) \sim |z_1 - z_2|^{-2X_2}$

- ▶ “Coulomb-gas” studies \Rightarrow TL model is critical for $0 < n \leq 2$.

Discretely holomorphic observables in loop models



- ▶ Pick a pair of boundary points (a, b) \longrightarrow define BC.
- ▶ Define correlation function:

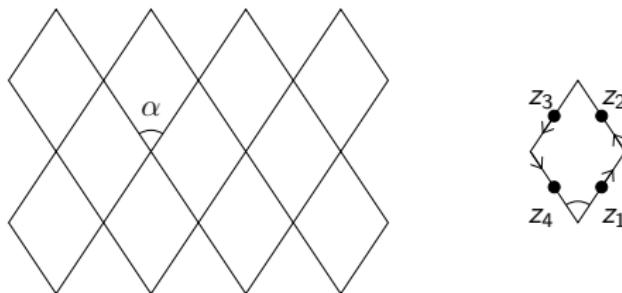
$$F_s(z) := \frac{1}{Z} \sum_{C \mid z \in \text{open path}} W(C) e^{is\theta_{a \rightarrow z}(C)}$$

$[\theta_{a \rightarrow z} := \text{winding angle of red arc from } a \text{ to } z]$

- ▶ **Theorem:** $n = 2 \sin \frac{\pi s}{2} \Rightarrow \forall \diamond \in \Omega, \sum_{\diamond} F_s(z) \delta z = 0.$

Algebraic structure behind discrete holomorphicity?

- Discretely holomorphic observables like F_s exist in various models: TL, $O(n)$, \mathbb{Z}_N clock models ...
- Rhombic lattice \Rightarrow additional parameter α



Modified Cauchy-Riemann equation:

$$e^{-\frac{i\alpha}{2}} F(z_1) + e^{\frac{i\alpha}{2}} F(z_2) - e^{-\frac{i\alpha}{2}} F(z_3) - e^{\frac{i\alpha}{2}} F(z_4) = 0 \quad (\text{CR}_\alpha)$$

- Observations :
 1. F_s satisfies CR_α when $W \equiv \text{integrable}$ Boltzmann weights
 2. $\alpha \equiv$ spectral parameter
- Q: general relation discrete holomorphicity \leftrightarrow integrability?

Discrete holomorphicity in Physics and Mathematics

- ▶ [Dotsenko,Polyakov 88] : Linear relations for fermions in Ising
- ▶ [Smirnov 01–06] : Conf. inv. for interfaces in perco+Ising
- ▶ [Cardy,Riva,Rajabpour,YI 06–09] : Discr. holo. in various lattice models, obs. relation to integrability
- ▶ [Smirnov,Chelkak,Hongler,Izyurov,Kytölä 09–12] : Scaling limit of interfaces+corr. func. in Ising
- ▶ [Duminil-Copin,Smirnov 10] : Proof of connectivity constant for SAW on honeycomb
- ▶ [Beaton,de Gier,Guttmann,Jensen 11–12] : Critical boundary parameter for SAW on honeycomb
- ▶ [Fendley 12] : Discr. holo. from topological QFT
- ▶ [Alam,Batchelor 12] : CR eq \leftrightarrow star-triangle in \mathbb{Z}_N models
- ▶ [Hongler,Kytölä,Zahabi 12] : Discr. holo. for non-local currents in Ising, transfer-matrix formalism

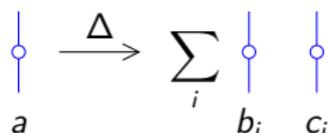
2. The Bernard-Felder construction

Hopf algebras

Bi-algebra structure

► Product $m : \begin{cases} \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A} \\ (a, b) \mapsto a.b \end{cases}$

► Coproduct $\Delta : \begin{cases} \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A} \\ a \mapsto \sum_i b_i \otimes c_i \end{cases}$



► $\Delta(a.b) = \Delta(a).\Delta(b), \quad \Delta(a + \lambda b) = \Delta(a) + \lambda\Delta(b)$

► $\Delta(\Delta(a)) = \sum_i \Delta(b_i) \otimes c_i = \sum_i b_i \otimes \Delta(c_i)$

► Example: enveloping algebra of a Lie algebra g

► g Lie algebra, with bracket $[X_a, X_b] = i f_{abc} X_c$

► $\mathcal{A} := U(g) = \text{span}[\text{words on alphabet } \{X_a\}]$

► bracket \equiv commutator ($[a, b] = ab - ba$)

► Trivial coproduct $\Delta(X_a) = X_a \otimes \mathbf{1} + \mathbf{1} \otimes X_a$

Hopf algebras

Tensor-product representations

- ▶ V finite-dimensional vector space

Map $\pi : \mathcal{A} \rightarrow \text{End}(V)$ is a representation of \mathcal{A} iff:

- ▶ π is linear and surjective,
- ▶ π is a morphism: $\pi(ab) = \pi(a)\pi(b)$.

- ▶ The coproduct defines higher-dim. representations:

$$\Delta(a) = \sum_i b_i \otimes c_i \quad \longrightarrow \quad \pi_{12}(a) := \sum_i \pi_1(b_i) \otimes \pi_2(c_i)$$

- ▶ Iterate:
-

- ▶ Example: $\mathcal{A} = U(g)$, for a Lie algebra g

$$\pi^{(L)}(\chi_a) = \sum_{m=1}^L \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \underset{\substack{\uparrow \\ m-\text{th}}}{\pi(\chi_a)} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$$

Hopf algebras

The R -matrix

- ▶ The two representations $V_1 \otimes V_2$ and $V_2 \otimes V_1$ are isomorphic.
- ▶ Intertwiner $R_{12} : V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$ such that:

$$\forall a \in \mathcal{A}, \quad R_{12} \pi_{12}(a) = \pi_{21}(a) R_{12}$$

- ▶ Expand coproduct [$\pi_{12}(a) = \sum_i \pi_1(b_i) \otimes \pi_2(c_i)$]:

$$\sum_i \begin{array}{c} V_2 & V_1 \\ \text{\scriptsize blue line} & \text{\scriptsize blue line} \\ b_i & c_i \\ \text{\scriptsize blue line} & \text{\scriptsize blue line} \\ V_1 & V_2 \end{array} = \sum_i \begin{array}{c} V_2 & V_1 \\ b_i & c_i \\ \text{\scriptsize blue line} & \text{\scriptsize blue line} \\ V_1 & V_2 \end{array}$$

R_{12}

- ▶ Consistency condition = Yang-Baxter equation:

$$(R_{23} \otimes \mathbf{1}) \cdot (\mathbf{1} \otimes R_{13}) \cdot (R_{12} \otimes \mathbf{1}) = (\mathbf{1} \otimes R_{12}) \cdot (R_{13} \otimes \mathbf{1}) \cdot (\mathbf{1} \otimes R_{23})$$

Non-local conserved currents

[Bernard-Felder, 91]

- Generators of \mathcal{A} : $\{J_1, J_2 \dots\}$ and $\{\mu_1, \mu_2 \dots\}$.

Assume the coproduct of \mathcal{A} has the following form:

$$\Delta(J_k) = J_k \otimes \mathbf{1} + \mu_k \otimes J_k \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \xrightarrow{\Delta} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} + \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}$$

$$\Delta(\mu_k) = \mu_k \otimes \mu_k \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \xrightarrow{\Delta} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}$$

- Iteration of coproduct \Rightarrow “conserved charges”:

$$Q_k := \Delta^{L-1}(J_k) = \sum_{m=1}^L \mu_k \otimes \cdots \otimes \mu_k \otimes \underset{\substack{\uparrow \\ m}}{J_k} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$$

- Non-local currents:

$$\begin{aligned} \psi_k(m) &:= \mu_k \otimes \cdots \otimes \mu_k \otimes \underset{\substack{\uparrow \\ m}}{J_k} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \\ &= \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \cdots \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \\ V_1 &\qquad \qquad \qquad V_m &\qquad \qquad \qquad V_L \end{aligned}$$

Commutation relations

- ▶ From intertwining relations $[R_{12} \pi_{12}(a) = \pi_{21}(a) R_{12}]$:

- ▶ For $a = J_k$:

$$\begin{array}{c} \text{Diagram: } \\ \text{Two vertical lines with a crossing between them. A blue square is at the top of the left line.} \end{array} + \begin{array}{c} \text{Diagram: } \\ \text{Two vertical lines with a crossing between them. A blue square is at the bottom of the right line.} \end{array} = \begin{array}{c} \text{Diagram: } \\ \text{Two vertical lines with a crossing between them. A blue square is at the top of the right line.} \end{array} + \begin{array}{c} \text{Diagram: } \\ \text{Two vertical lines with a crossing between them. A blue square is at the bottom of the left line.} \end{array}$$

- ▶ For $a = \mu_k$:

$$\begin{array}{c} \text{Diagram: } \\ \text{Two vertical lines with a crossing between them. The left line has a dashed segment above the crossing.} \end{array} = \begin{array}{c} \text{Diagram: } \\ \text{Two vertical lines with a crossing between them. The right line has a dashed segment above the crossing.} \end{array}$$

- ▶ Transfer matrix:

$$T = \begin{array}{ccccccc} & \text{X} & & \text{X} & & \text{X} & & \text{X} \\ & \diagdown & & \diagup & & \diagdown & & \diagup \\ \text{V} & V' & V & V' & \dots & V & V' \end{array}$$

- ▶ Conservation laws:

$$\forall a \in \mathcal{A}, \quad T \cdot \pi^{(L)}(a) = \pi^{(L)}(a) \cdot T$$

The affine quantum group $\mathcal{A} = U_q(\widehat{sl_2})$

- Generators: $E_0, E_1, F_0, F_1, T_0, T_1$

$\{E_0, E_1, F_0, F_1\}$ =raising/lowering ops, $\{T_0, T_1\}$ =diag. ops.

- Product rules:

$$[T_0, T_1] = 0$$

$$[E_i, F_j] = \delta_{ij} \frac{T_i - T_i^{-1}}{q - q^{-1}}$$

$$T_i E_j T_i^{-1} = q^{2(-1)^{\delta_{ij}}} E_j$$

$$T_i F_j T_i^{-1} = q^{2(-1)^{\delta_{ij}+1}} F_j$$

(+higher order rules ...)

- Coproduct rules:

$$\Delta(E_i) = E_i \otimes \mathbf{1} + T_i \otimes E_i$$

$$\Delta(F_i) = F_i \otimes T_i^{-1} + \mathbf{1} \otimes F_i$$

$$\Delta(T_i) = T_i \otimes T_i$$

- Introduce $\bar{E}_i := q T_i F_i \Rightarrow \Delta(\bar{E}_i) = \bar{E}_i \otimes \mathbf{1} + T_i \otimes \bar{E}_i$

- BF structure: $\{J_k\} = \{E_0, E_1, \bar{E}_0, \bar{E}_1\}$ $\{\mu_k\} = \{T_0, T_1\}$.

Evaluation representations of $\mathcal{A} = U_q(\widehat{\mathfrak{sl}_2})$

- ▶ Representations are labelled by a complex number u
 Explicit form:

$$\pi_u : \begin{cases} E_0 \mapsto \begin{bmatrix} 0 & 0 \\ u & 0 \end{bmatrix} & \bar{E}_0 \mapsto \begin{bmatrix} 0 & u^{-1} \\ 0 & 0 \end{bmatrix} & T_0 \mapsto \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix} \\ E_1 \mapsto \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} & \bar{E}_1 \mapsto \begin{bmatrix} 0 & 0 \\ u^{-1} & 0 \end{bmatrix} & T_1 \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix} \end{cases}$$

- ▶ Intertwiner: $R(u/v)\pi_{u,v} = \pi_{v,u}R(u/v)$

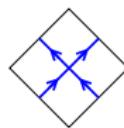
$$R(w) = \begin{bmatrix} qw - (qw)^{-1} & 0 & 0 & 0 \\ 0 & w - w^{-1} & q - q^{-1} & 0 \\ 0 & q - q^{-1} & w - w^{-1} & 0 \\ 0 & 0 & 0 & qw - (qw)^{-1} \end{bmatrix}$$

$$(w = u/v)$$

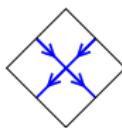
Application to the six-vertex model

- ▶ Use basis for V_u : $\{\uparrow, \downarrow\}$.

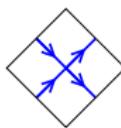
Plaquette configurations:



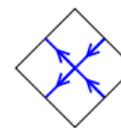
ω_1



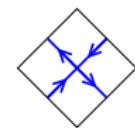
ω_2



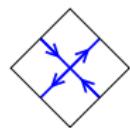
ω_3



ω_4



ω_5



ω_6

- ▶ Boltzmann weights:

$$R_{6V} = \begin{bmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_5 & \omega_4 & 0 \\ 0 & \omega_3 & \omega_6 & 0 \\ 0 & 0 & 0 & \omega_2 \end{bmatrix}$$

- ▶ When $R_{6V} \equiv R_{U_q(\widehat{sl}_2)}$, the 6V model is integrable.

3. Mapping to loop models

From the TL model to the 6V model

[Baxter, Kelland, Wu 73]

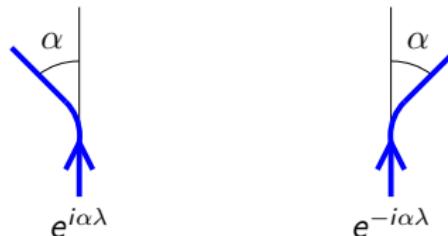
- ▶ Orient each loop independently:

$$\text{Diagram: A loop } C \text{ is divided into three regions by a vertical line. The left region has boundary } n = 2 \cos 2\pi\lambda, \text{ the middle region has boundary } e^{2i\pi\lambda}, \text{ and the right region has boundary } e^{-2i\pi\lambda}.$$

- ▶ Partition function:

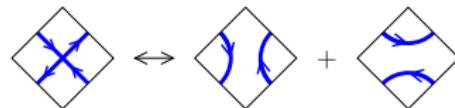
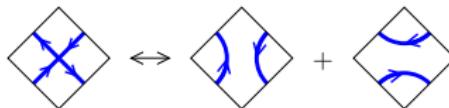
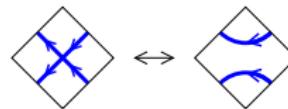
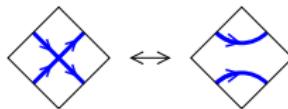
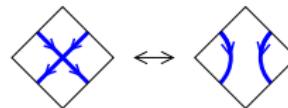
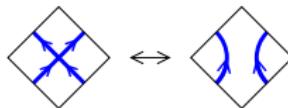
$$Z = \sum_C x^{N_x(C)} y^{N_y(C)} e^{2i\pi\lambda[N_\ell^+(C) - N_\ell^-(C)]}$$

- ▶ Distribute phase factors locally:



From the TL model to the 6V model (2)

- Vertex configurations:



- Six-vertex weights arising from loop model:

$$\omega_1 = \omega_2 = x, \quad \omega_3 = \omega_4 = y, \quad \begin{cases} \omega_5 = e^{+2i\lambda\alpha}x + e^{-2i\lambda(\pi-\alpha)}y \\ \omega_6 = e^{-2i\lambda\alpha}x + e^{+2i\lambda(\pi-\alpha)}y \end{cases}$$

- Set $q = -e^{2i\lambda\pi}$, $w = e^{-2i\lambda\alpha}$:

$$\omega_1 = \omega_2 = qw - \frac{1}{qw}, \quad \omega_3 = \omega_4 = w - \frac{1}{w} \Rightarrow \omega_5 = \omega_6 = q - \frac{1}{q}.$$

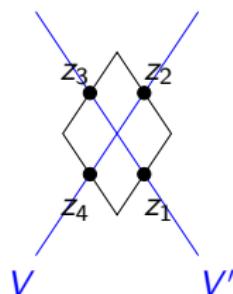
Conserved currents in the 6V model

- ▶ $\begin{cases} \Delta(E_0) = E_0 \otimes \mathbf{1} + T_0 \otimes E_0 \\ \Delta(T_0) = T_0 \otimes T_0 \end{cases} \Rightarrow \text{BF current } \psi_0$

$$\psi_0(m) = T_0 \otimes T_0 \otimes \cdots \otimes T_0 \otimes E_0 \underset{\substack{\uparrow \\ m-\text{th}}}{\otimes} \mathbf{1} \otimes \cdots \otimes \mathbf{1}$$

- ▶ Commutation with R -matrix \Rightarrow linear relation:

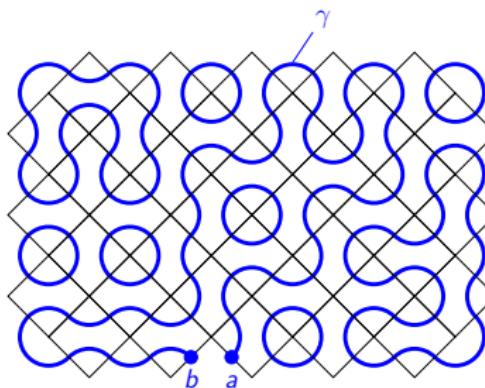
$$\psi_0(z_1) - \psi_0(z_2) - \psi_0(z_3) + \psi_0(z_4) = 0.$$



- ▶ Similar construction for $E_1, \bar{E}_0, \bar{E}_1 \rightarrow \psi_1, \bar{\psi}_0, \bar{\psi}_1$.

Mapping of conserved currents

What is the meaning of $\langle \psi_0(z) \rangle$ in terms of loops?



$\psi_0(z)$ cannot sit on a closed loop

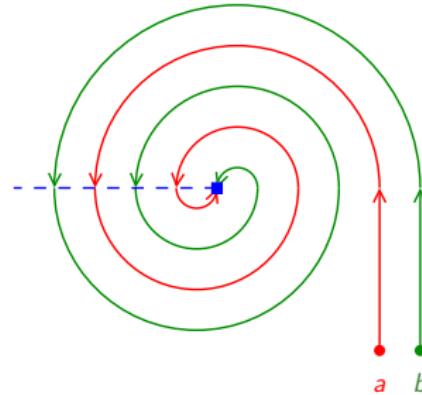


$$\Rightarrow \langle \psi_0(z) \rangle = \frac{u}{Z} \sum_{C| z \in \gamma} W(C) \times (\text{phase factor})$$

Mapping of conserved currents (2)

Identification of phase factors

► $\theta_{b \rightarrow z} = \theta_{a \rightarrow z} + \pi , \quad q = e^{i\pi(2\lambda-1)}$



► phase factor:

$$e^{i\lambda(\theta_{a \rightarrow z} + \theta_{b \rightarrow z})} \times q^{\frac{\theta_{a \rightarrow z} + \theta_{b \rightarrow z} - \pi}{2\pi}} = A e^{i(4\lambda-1)\theta_{a \rightarrow z}}$$

↑ ↑
turns $T_0 \otimes \dots \otimes T_0$

► $\Rightarrow \langle \psi_0(z) \rangle = \frac{uA}{Z} \sum_{C| z \in \gamma} W(C) e^{i(4\lambda-1)\theta_{a \rightarrow z}} = uA \times F_0(z)$

Mapping of conserved currents (3)

Cauchy-Riemann relation

- ▶ Set $u = 1/u' = w^{1/2} \Rightarrow u/u' = w = e^{-2i\lambda\alpha}$
- ▶ Conservation relation:

$$\psi_0(z_1) - \psi_0(z_2) - \psi_0(z_3) + \psi_0(z_4) = 0$$

$$\Rightarrow \sum_{\diamond} F_0(z) \delta z = 0$$

- ▶ Conservation of BF current $\Rightarrow \text{CR}_\alpha$ relation

Extension of the results

- ▶ What we have also obtained:
 - ▶ Holo. obs. in TL corresponding to $E_1, \bar{E}_0, \bar{E}_1$
 - ▶ Holo. obs. in *dilute O(n)* model $\rightarrow \mathcal{A} = U_q \left(A_2^{(2)} \right)$
 - ▶ Boundary CR equation \leftrightarrow integrable *K*-matrix
- ▶ For future work:
 - ▶ Construct *new* holo. obs. in other models
 - ▶ Study non-critical cases $\rightarrow (\partial - m)F = 0?$
 - ▶ Find “other half” of CR equations?

Thank you for your attention!