Refined Cauchy and Littlewood identities, plane partitions and partition functions in the six-vertex model

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Outline

1. Review: Cauchy and Littlewood identities, connection with plane partitions
2. More advanced examples: t-generalizations, and symplectic characters
3. Refined Cauchy and Littlewood identities, and refined plane partitions
4. Partition functions in the six-vertex model, connection with ASM symmetry classes

Schur polynomials and SSYT

- The Schur polynomials $s_\lambda(x_1, \ldots, x_n)$ are the characters of irreducible representations of $GL(n)$. They are given by the Weyl formula:

$$s_\lambda(x_1, \ldots, x_n) = \frac{\det\left(x_i^{\lambda_j - j + n}\right)_{1 \leqslant i,j \leqslant n}}{\prod_{1 \leqslant i < j \leqslant n}(x_i - x_j)}$$

- A *semi-standard Young tableau* of shape λ is an assignment of one symbol $\{1, \ldots, n\}$ to each box of the Young diagram λ, such that

  1. The symbols have the ordering $1 < \cdots < n$.
  2. The entries in λ increase weakly along each row and strictly down each column.

- The Schur polynomial $s_\lambda(x_1, \ldots, x_n)$ is also given by a weighted sum over semi-standard Young tableaux T of shape λ:

$$s_\lambda(x_1, \ldots, x_n) = \sum_T \prod_{k=1}^{n} x_k^{\#(k)}$$

SSYT and sequences of interlacing partitions

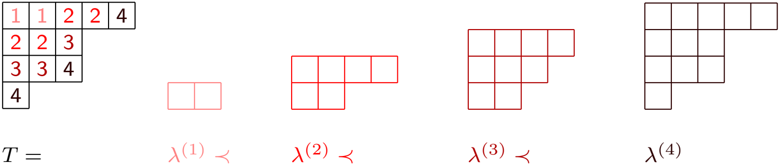
- Two partitions λ and μ *interlace*, written $\lambda \succ \mu$, if

$$\lambda_i \geqslant \mu_i \geqslant \lambda_{i+1}$$

  across all parts of the partitions. It is the same as saying λ/μ is a *horizontal strip*.
- One can interpret a SSYT as a sequence of interlacing partitions:

$$T = \{\emptyset \equiv \lambda^{(0)} \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(n)} \equiv \lambda\}$$

- The correspondence works by "peeling away" partition $\lambda^{(k)}$ from T, for all k:

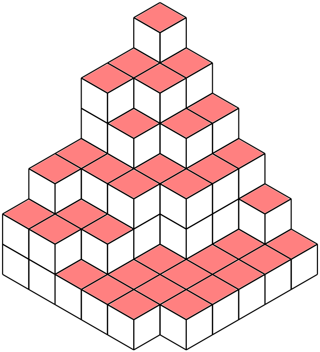


Plane partitions

- Plane partitions can be viewed as an *increasing then decreasing* sequence of interlacing partitions. They are equivalent to *conjoined* SSYT.

- We define the set

$$\boldsymbol{\pi}_{m,n} = \{\emptyset \equiv \lambda^{(0)} \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(m)} \equiv \mu^{(n)} \succ \cdots \succ \mu^{(1)} \succ \mu^{(0)} \equiv \emptyset\}$$



Cauchy identity and plane partitions

- The Cauchy identity for Schur polynomials,

$$\sum_{\lambda} s_{\lambda}(x_1,\dots,x_m) s_{\lambda}(y_1,\dots,y_n) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{1-x_i y_j}$$

can thus be viewed as a generating series of plane partitions:

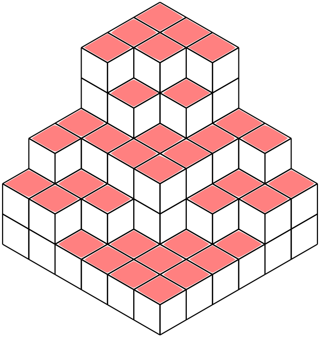
$$\sum_{\pi \in \boldsymbol{\pi}_{m,n}} \prod_{i=1}^{m} x_i^{|\lambda^{(i)}|-|\lambda^{(i-1)}|} \prod_{j=1}^{n} y_j^{|\mu^{(j)}|-|\mu^{(j-1)}|} = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{1-x_i y_j}$$

- Taking the q-specialization $x_i = q^{m-i+1/2}$ and $y_j = q^{n-j+1/2}$, we recover volume-weighted plane partitions:

$$\sum_{\pi \in \boldsymbol{\pi}_{m,n}} q^{|\pi|} = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{1-q^{m+n-i-j+1}} = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{1-q^{i+j-1}}$$

Symmetric plane partitions

- A *symmetric* plane partition is determined by an increasing sequence of interlacing partitions. (The decreasing part is obtained from the symmetry.)
- They are in one-to-one correspondence with SSYT.



Littlewood identities and symmetric plane partitions

- The three (simplest) Littlewood identities for Schur polynomials

$$\sum_{\lambda} s_\lambda(x_1,\dots,x_n) = \prod_{1\leqslant i<j\leqslant n} \frac{1}{1-x_i x_j} \prod_{i=1}^{n} \frac{1}{1-x_i}$$

$$\sum_{\lambda \text{ even}} s_\lambda(x_1,\dots,x_n) = \prod_{1\leqslant i<j\leqslant n} \frac{1}{1-x_i x_j} \prod_{i=1}^{n} \frac{1}{1-x_i^2}$$

$$\sum_{\lambda' \text{ even}} s_\lambda(x_1,\dots,x_n) = \prod_{1\leqslant i<j\leqslant n} \frac{1}{1-x_i x_j}$$

can each be viewed as generating series for symmetric plane partitions, with a (possible) constraint on the partition forming the main diagonal.

Hall–Littlewood polynomials

- Hall–Littlewood polynomials are t-generalizations of Schur polynomials. They can be defined as a sum over the symmetric group:

$$P_\lambda(x_1,\ldots,x_n;t) = \frac{1}{v_\lambda(t)} \sum_{\sigma\in S_n} \sigma\left(\prod_{i=1}^{n} x_i^{\lambda_i} \prod_{1\leqslant i<j\leqslant n} \frac{x_i - t x_j}{x_i - x_j} \right)$$

- Alternatively, the Hall–Littlewood polynomial $P_\lambda(x_1,\ldots,x_n;t)$ is given by a weighted sum over semi-standard Young tableaux T of shape λ:

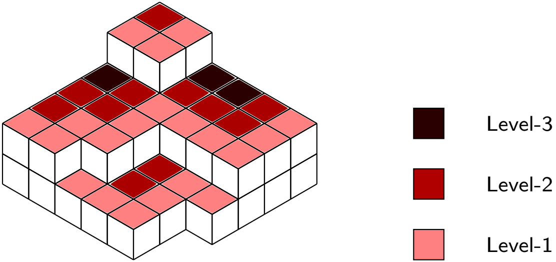
$$P_\lambda(x_1,\ldots,x_n;t) = \sum_{T} \prod_{k=1}^{n} \left(x_k^{\#(k)} \psi_{\lambda^{(k)}/\lambda^{(k-1)}}(t) \right)$$

where the function $\psi_{\lambda/\mu}(t)$ is given by

$$\psi_{\lambda/\mu}(t) = \prod_{\substack{i\geqslant 1 \\ m_i(\mu)=m_i(\lambda)+1}} \left(1 - t^{m_i(\mu)}\right)$$

Path-weighted plane partitions

- As Vuletić discovered, the effect of the t-weighting in tableaux has a nice combinatorial interpretation on plane partitions.
- The refinement is that all *paths* at level k receive a weight of $1 - t^k$.
- Example of a plane partition with weight $(1-t)^3(1-t^2)^4(1-t^3)^2$ shown below:



Hall–Littlewood Cauchy identity and path-weighted plane partitions

- The Cauchy identity for Hall–Littlewood polynomials,

$$\sum_{\lambda} \prod_{i=1}^{\infty} \prod_{j=1}^{m_i(\lambda)} (1-t^j) P_\lambda(x_1,\dots,x_m;t) P_\lambda(y_1,\dots,y_n;t) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1-tx_iy_j}{1-x_iy_j}$$

is thus a generating series of (path-weighted) plane partitions:

$$\sum_{\pi \in \boldsymbol{\pi}_{m,n}} \prod_{i\geqslant 1} (1-t^i)^{p_i(\pi)} \prod_{i=1}^{m} x_i^{|\lambda^{(i)}|-|\lambda^{(i-1)}|} \prod_{j=1}^{n} y_j^{|\mu^{(j)}|-|\mu^{(j-1)}|} = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1-tx_iy_j}{1-x_iy_j}$$

- Taking the same q-specialization as earlier, we obtain

$$\sum_{\pi \in \boldsymbol{\pi}_{m,n}} \prod_{i\geqslant 1} (1-t^i)^{p_i(\pi)} q^{|\pi|} = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1-tq^{i+j-1}}{1-q^{i+j-1}}$$

Littlewood identities for Hall–Littlewood polynomials

- The t-analogues of the previously stated Littlewood identities are

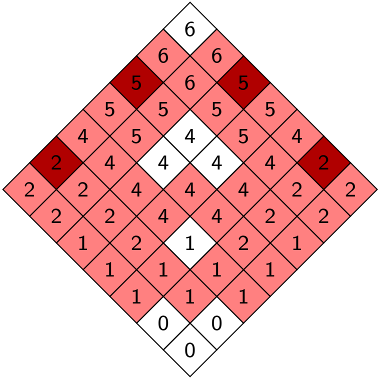
$$\sum_{\lambda} P_\lambda(x_1,\ldots,x_n;t) = \prod_{1\leqslant i<j\leqslant n} \frac{1-tx_ix_j}{1-x_ix_j} \prod_{i=1}^{n} \frac{1}{1-x_i}$$

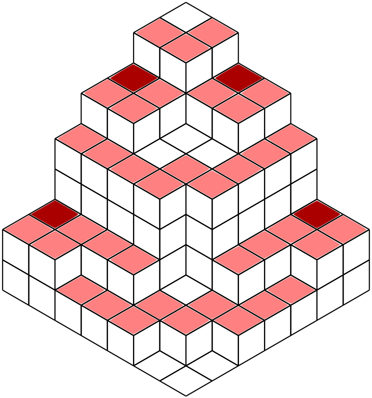
$$\sum_{\lambda \text{ even}} P_\lambda(x_1,\ldots,x_n;t) = \prod_{1\leqslant i<j\leqslant n} \frac{1-tx_ix_j}{1-x_ix_j} \prod_{i=1}^{n} \frac{1}{1-x_i^2}$$

$$\sum_{\lambda' \text{ even}} \prod_{i=1}^{\infty} \prod_{j \text{ even}}^{m_i(\lambda)} (1-t^{j-1}) P_\lambda(x_1,\ldots,x_n;t) = \prod_{1\leqslant i<j\leqslant n} \frac{1-tx_ix_j}{1-x_ix_j}$$

- These can be regarded as generating series for path-weighted symmetric plane partitions.
- Warning! Paths which intersect the main diagonal might not have a t-weight!

t-weighting of symmetric plane partitions





$$\sum_{\lambda' \text{ even}} \prod_{i=1}^{\infty} \prod_{j \text{ even}}^{m_i(\lambda)} (1 - t^{j-1}) P_\lambda(x_1, \ldots, x_n; t)$$

Symplectic Schur polynomials

- The symplectic Schur polynomials $sp_\lambda(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n)$ are the characters of irreducible representations of $Sp(2n)$. In this case, the Weyl formula gives

$$sp_\lambda(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n) = \frac{\det\left(x_i^{\lambda_j - j + n + 1} - \bar{x}_i^{\lambda_j - j + n + 1}\right)_{1 \leqslant i,j \leqslant n}}{\prod_{i=1}^{n}(x_i - \bar{x}_i) \prod_{1 \leqslant i < j \leqslant n}(x_i - x_j)(1 - \bar{x}_i \bar{x}_j)}$$

- A *symplectic tableau* of shape λ is an assignment of one symbol $\{1, \bar{1}, \ldots, n, \bar{n}\}$ to each box of the Young diagram λ, such that

  1. The symbols have the ordering $1 < \bar{1} < \cdots < n < \bar{n}$.
  2. The entries in λ increase weakly along each row and strictly down each column.
  3. All entries in row k of λ are at least k.

- The symplectic Schur polynomial $sp_\lambda(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n)$ is given by a weighted sum over symplectic tableaux $\overline{T}$ of shape λ:

$$sp_\lambda(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n) = \sum_{\overline{T}} \prod_{k=1}^{n} x_k^{\#(k) - \#(\bar{k})}$$

Symplectic tableaux as restricted interlacing sequences

- We can interpret a symplectic tableau $\overline{T}$ as a sequence of interlacing partitions, subject to an extra constraint:

$$\overline{T} = \{\emptyset \equiv \bar{\lambda}^{(0)} \prec \lambda^{(1)} \prec \bar{\lambda}^{(1)} \prec \cdots \prec \lambda^{(n)} \prec \bar{\lambda}^{(n)} \equiv \lambda \mid \ell(\bar{\lambda}^{(i)}) \leqslant i\}.$$

1	1	$\bar{1}$	$\bar{2}$	3
2	$\bar{2}$	3		
$\bar{3}$	$\bar{3}$	4		
4				

- The symplectic Schur polynomial can now be expressed as

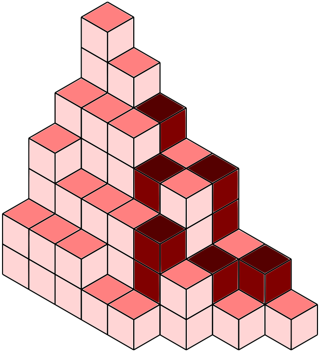
$$sp_{\lambda}(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n) = \sum_{\overline{T}} \prod_{i=1}^{n} x_i^{|\lambda^{(i)}| - |\bar{\lambda}^{(i-1)}|} \prod_{j=1}^{n} x_j^{|\lambda^{(j)}| - |\bar{\lambda}^{(j)}|}$$

$$= \sum_{\overline{T}} \prod_{i=1}^{n} x_i^{2|\lambda^{(i)}| - |\bar{\lambda}^{(i)}| - |\bar{\lambda}^{(i-1)}|}$$

A restricted class of plane partitions

- We consider the class of plane partitions formed by a conjoined SSYT and symplectic tableau.
- We define the set

$$\overline{\boldsymbol{\pi}}_{m,2n} = \{\emptyset \equiv \lambda^{(0)} \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(m)} \equiv \bar{\mu}^{(n)} \succ \mu^{(n)} \succ \cdots \succ \bar{\mu}^{(1)} \succ \mu^{(1)} \succ \bar{\mu}^{(0)} \equiv \emptyset\}$$



Symplectic Cauchy identity and associated plane partitions

- The Cauchy identity for symplectic Schur polynomials,

$$\sum_{\lambda} s_{\lambda}(x_1,\ldots,x_m) sp_{\lambda}(y_1,\bar{y}_1,\ldots,y_n,\bar{y}_n) = \frac{\prod_{1\leqslant i<j\leqslant m}(1-x_i x_j)}{\prod_{i=1}^{m}\prod_{j=1}^{n}(1-x_i y_j)(1-x_i\bar{y}_j)}$$

  can now be regarded as a generating series for the plane partitions defined:

$$\sum_{\pi\in\overline{\boldsymbol{\pi}}_{m,2n}} \prod_{i=1}^{m} x_i^{|\lambda^{(i)}|-|\lambda^{(i-1)}|} \prod_{j=1}^{n} y_j^{2|\mu^{(j)}|-|\bar{\mu}^{(j)}|-|\bar{\mu}^{(j-1)}|} =$$

$$\frac{\prod_{1\leqslant i<j\leqslant m}(1-x_i x_j)}{\prod_{i=1}^{m}\prod_{j=1}^{n}(1-x_i y_j)(1-x_i\bar{y}_j)}$$

- One can experiment with the parameters to find a "good" q-specialization. We choose $x_i = q^{m-i+3/2}$ and $y_j = q^{1/2}$, giving

$$\sum_{\pi\in\overline{\boldsymbol{\pi}}_{m,2n}} q^{|\pi_{\leqslant}|} q^{|\pi^{o}_{>}|-|\pi^{e}_{>}|} = \frac{\prod_{1\leqslant i<j\leqslant m}(1-q^{i+j+1})}{\prod_{i=1}^{m}(1-q^{i})^{n}(1-q^{i+1})^{n}}$$

Example 1(a): Refined Cauchy identity for Schur polynomials

Theorem

$$\sum_{\lambda} \prod_{i=1}^{n} (1 - t^{\lambda_i - i + n + 1}) s_\lambda(x_1, \ldots, x_n) s_\lambda(y_1, \ldots, y_n)$$

$$= \frac{1}{\Delta(x)_n \Delta(y)_n} \det \left\{ \frac{(1-t)}{(1 - t x_i y_j)(1 - x_i y_j)} \right\}_{1 \leqslant i,j \leqslant n}$$

Proof.

Expand the entries of the determinant as formal power series, and use Cauchy–Binet:

$$\det \left\{ \frac{(1-t)}{(1 - t x_i y_j)(1 - x_i y_j)} \right\}_{1 \leqslant i,j \leqslant n} = \det \left\{ \sum_{k=0}^{\infty} (1 - t^{k+1}) x_i^k y_j^k \right\}_{1 \leqslant i,j \leqslant n}$$

$$= \sum_{k_1 > \cdots > k_n \geqslant 0} \prod_{i=1}^{n} (1 - t^{k_i + 1}) \ \det \left\{ x_i^{k_j} \right\}_{1 \leqslant i,j \leqslant n} \det \left\{ y_j^{k_i} \right\}_{1 \leqslant i,j \leqslant n}$$

The proof follows after the change of indices $k_i = \lambda_i - i + n$. □

Example 1(b): Refined Cauchy identity for Hall–Littlewood polynomials

Theorem (Kirillov–Noumi, Warnaar)

$$\sum_{\lambda} \prod_{i=0}^{\infty} \prod_{j=1}^{m_i(\lambda)} (1-t^j) P_\lambda(x_1,\dots,x_n;t) P_\lambda(y_1,\dots,y_n;t)$$

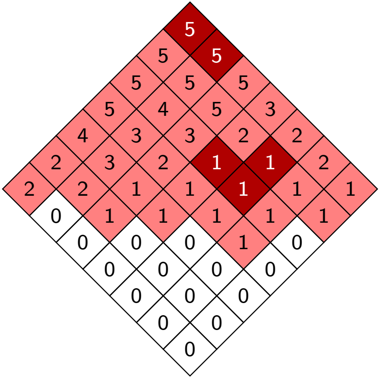
$$= \frac{\prod_{i,j=1}^{n}(1-tx_iy_j)}{\Delta(x)_n \Delta(y)_n} \det\left\{ \frac{(1-t)}{(1-tx_iy_j)(1-x_iy_j)} \right\}_{1\leqslant i,j\leqslant n}$$

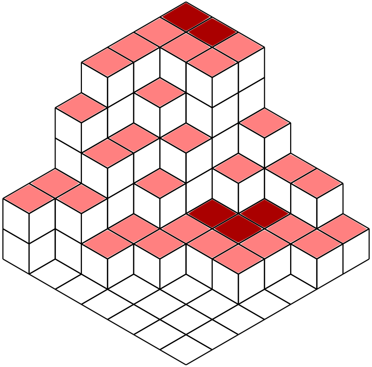
Proof.

The key idea is to act on the Hall–Littlewood Cauchy identity with a generating series of Macdonald's difference operators. The left hand side follows immediately. The right hand side follows after acting on the Cauchy kernel, and performing some manipulation. □

Example 1(b): Refined Cauchy identity for Hall–Littlewood polynomials

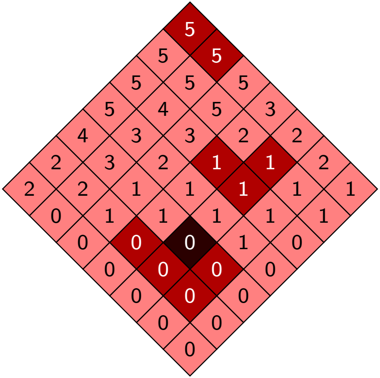
- Question: What does the refinement do at the level of plane partitions?

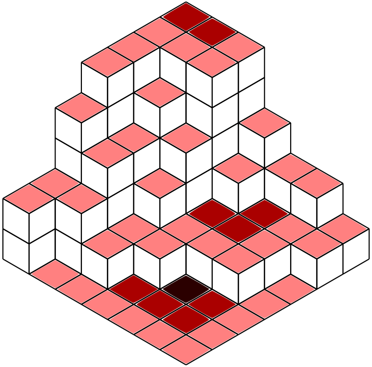




Example 1(b): Refined Cauchy identity for Hall–Littlewood polynomials

- Answer: The zero-height entries are treated like the rest!





Example 2(a): Refined Littlewood identity for Schur polynomials

Theorem (DB,MW)

$$\sum_{\lambda' \text{ even}} \prod_{i \text{ even}} (1 - t^{\lambda_i - i + 2n + 1}) s_\lambda(x_1, \ldots, x_{2n})$$
$$= \prod_{1 \leqslant i < j \leqslant 2n} \frac{1}{(x_i - x_j)} \mathrm{Pf}\left\{ \frac{(1-t)(x_i - x_j)}{(1 - t x_i x_j)(1 - x_i x_j)} \right\}_{1 \leqslant i < j \leqslant 2n}$$

Proof.

Expand the entries of the Pfaffian and use a Pfaffian analogue of Cauchy–Binet:

$$\mathrm{Pf}\left\{\cdots\right\}_{1 \leqslant i < j \leqslant 2n} = \mathrm{Pf}\left\{ \sum_{0 \leqslant k < l} \delta_{l,k+1} (1 - t^{k+1})(x_i^l x_j^k - x_i^k x_j^l) \right\}_{1 \leqslant i < j \leqslant 2n}$$
$$= \sum_{1 \leqslant s_1 < \cdots < s_{2n}} (-)^n \, \mathrm{Pf}\left\{ \delta_{s_i+1, s_j} (1 - t^{s_i}) \right\}_{1 \leqslant i < j \leqslant 2n} \det\left\{ x_i^{s_j - 1} \right\}_{1 \leqslant i,j \leqslant 2n}$$

The Pfaffian in the sum factorizes, to produce the correct (blue) factor and the restriction on the summation. □

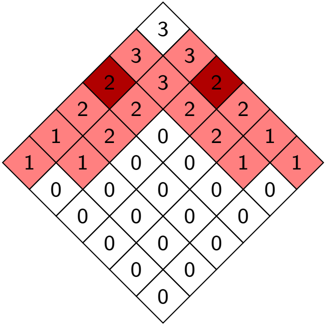
Example 2(b): Refined Littlewood identity for Hall–Littlewood polynomials

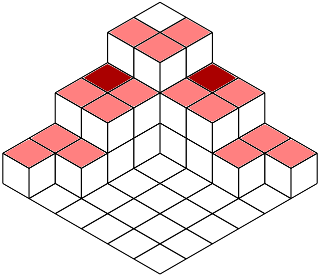
Conjecture (DB,MW)

$$\sum_{\lambda' \text{ even}} \prod_{i=0}^{\infty} \prod_{j \text{ even}}^{m_i(\lambda)} (1-t^{j-1}) P_\lambda(x_1,\dots,x_{2n};t) = \prod_{1 \leqslant i<j \leqslant 2n} \frac{(1-tx_i x_j)}{(x_i-x_j)} \mathrm{Pf}\left\{ \frac{(1-t)(x_i-x_j)}{(1-tx_ix_j)(1-x_ix_j)} \right\}_{1 \leqslant i<j \leqslant 2n}$$

Example 2(b): Refined Littlewood identity for Hall–Littlewood polynomials

- At the level of plane partitions, this is (again) a very simple refinement!





Example 2(b): Refined Littlewood identity for Hall–Littlewood polynomials

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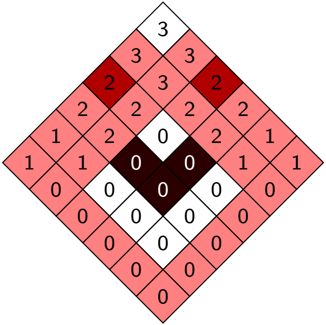
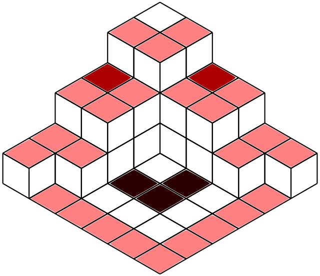




Example 3(a): Refined Cauchy identity for symplectic Schur polynomials

Theorem (DB,MW)

$$\sum_{\lambda} \prod_{i=1}^{n} (1 - t^{\lambda_i - i + n + 1}) s_{\lambda}(x_1, \ldots, x_n) sp_{\lambda}(y_1, \bar{y}_1, \ldots, y_n, \bar{y}_n) =$$

$$\frac{\prod_{i=1}^{n} (1 - t x_i^2)}{\Delta(x)_n \Delta(y)_n \prod_{i<j} (1 - \bar{y}_i \bar{y}_j)} \det \left\{ \frac{(1-t)}{(1 - t x_i y_j)(1 - t x_i \bar{y}_j)(1 - x_i y_j)(1 - x_i \bar{y}_j)} \right\}_{1 \leqslant i,j \leqslant n}$$

Proof.

Similar to the ordinary Schur case:

$$\prod_{i=1}^{n} (1 - t x_i^2)(y_i - \bar{y}_i) \det \{\cdots\}_{1 \leqslant i,j \leqslant n} = \det \left\{ \sum_{k=0}^{\infty} (1 - t^{k+1}) x_i^k (y_j^{k+1} - \bar{y}_j^{k+1}) \right\}_{1 \leqslant i,j \leqslant n}$$

$$= \sum_{k_1 > \cdots > k_n \geqslant 0} \prod_{i=1}^{n} (1 - t^{k_i + 1}) \det \left\{ x_i^{k_j} \right\}_{1 \leqslant i,j \leqslant n} \det \left\{ y_j^{k_i + 1} - \bar{y}_j^{k_i + 1} \right\}_{1 \leqslant i,j \leqslant n}$$

The proof follows after the change of indices $k_i = \lambda_i - i + n$. □

Example 3(b): Identity involving BC_n Hall–Littlewood functions

- Macdonald extended his theory of symmetric functions to other root systems. We will use the type BC_n Hall–Littlewood functions:

$$K_\lambda(y_1, \bar{y}_1, \ldots, y_n, \bar{y}_n; t) = \frac{1}{v_\lambda(t)} \sum_\omega \omega \left(\prod_{i=1}^n \frac{y_i^{\lambda_i}}{(1 - \bar{y}_i^2)} \prod_{1 \leqslant i < j \leqslant n} \frac{(y_i - t y_j)(1 - t \bar{y}_i \bar{y}_j)}{(y_i - y_j)(1 - \bar{y}_i \bar{y}_j)} \right)$$

where the sum is taken over the hyperoctahedral group, $\omega \in S_n \rtimes \mathbb{Z}_2^n$.

Conjecture (DB,MW)

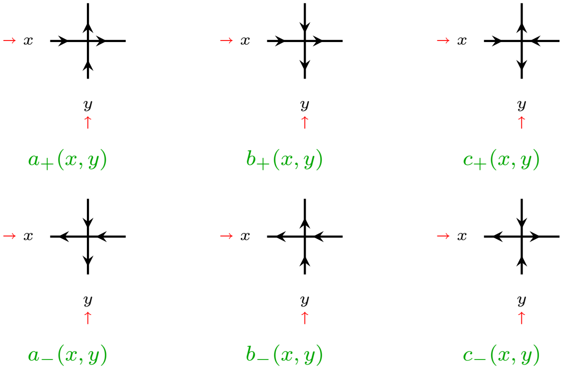
$$\sum_\lambda \prod_{i=0}^\infty \prod_{j=1}^{m_i(\lambda)} (1 - t^j) P_\lambda(x_1, \ldots, x_n; t) K_\lambda(y_1, \bar{y}_1, \ldots, y_n, \bar{y}_n; t) =$$

$$\frac{\prod_{i,j=1}^n (1 - t x_i y_j)(1 - t x_i \bar{y}_j)}{\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)(y_i - y_j)(1 - t x_i x_j)(1 - \bar{y}_i \bar{y}_j)}$$

$$\times \det \left\{ \frac{(1 - t)}{(1 - t x_i y_j)(1 - t x_i \bar{y}_j)(1 - x_i y_j)(1 - x_i \bar{y}_j)} \right\}_{1 \leqslant i,j \leqslant n}$$

The six-vertex model

- The vertices of the six-vertex model are

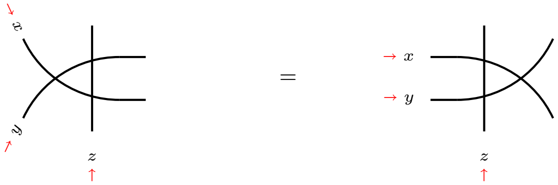


The six-vertex model

- The Boltzmann weights are given by

$$a_+(x,y) = \frac{1-tx/y}{1-x/y} \qquad a_-(x,y) = \frac{1-tx/y}{1-x/y}$$
$$b_+(x,y) = 1 \qquad b_-(x,y) = t$$
$$c_+(x,y) = \frac{(1-t)}{1-x/y} \qquad c_-(x,y) = \frac{(1-t)x/y}{1-x/y}$$

- The parameter t from Hall–Littlewood is now the crossing parameter of the model.
- The Boltzmann weights obey the *Yang–Baxter* equations (the $\mathcal{U}_q(\widehat{sl_2})$ solution):

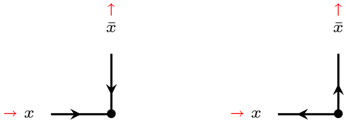


Boundary vertices

- In addition to the bulk vertices, we introduce U-turn vertices

$1/(1-x^2)$ $\qquad$ $1/(1-x^2)$

which depend on a single spectral parameter and are spin-conserving, and corner vertices

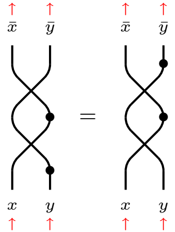


1 $\qquad$ 1

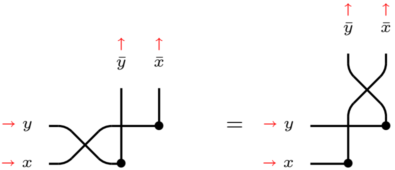
which *do not* depend on a spectral parameter and behave like sources/sinks.

Boundary vertices

- The U-turn vertices satisfy Sklyanin's reflection equation:

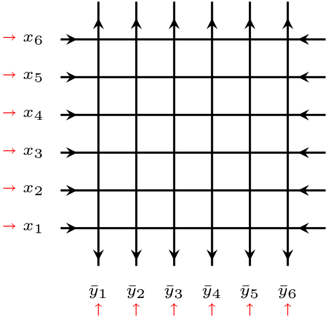


- The corner vertices satisfy their own variant of the reflection equation:



Domain wall boundary conditions

- The six-vertex model on a lattice with *domain wall* boundary conditions:



- This partition function is of fundamental importance in periodic quantum spin chains based on $\mathcal{Y}(sl_2)$ and $\mathcal{U}_q(\widehat{sl_2})$.
- Configurations on this lattice are in one-to-one correspondence with alternating sign matrices.

Domain wall boundary conditions

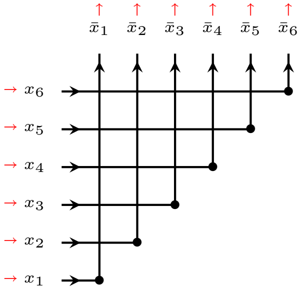
- The domain wall partition function was evaluated in determinant form by Izergin:

$$Z_{\mathrm{ASM}}(x_1,\dots,x_n;y_1,\dots,y_n;t) = \frac{\prod_{i,j=1}^{n}(1-tx_iy_j)}{\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(y_i-y_j)} \det\left[\frac{(1-t)}{(1-tx_iy_j)(1-x_iy_j)}\right]_{1\leqslant i,j\leqslant n}$$

- This is precisely what appears on the right hand side of Example 1(b).

Off-diagonally symmetric boundary conditions

- *Off-diagonally symmetric* boundary conditions were introduced by Kuperberg. They involve the corner vertices:



- Configurations on this lattice are in one-to-one correspondence with off-diagonally symmetric ASMs (OSASMs).

Off-diagonally symmetric boundary conditions

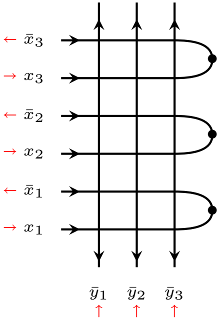
- Kuperberg evaluated this partition function as a Pfaffian:

$$Z_{\mathrm{OSASM}}(x_1,\ldots,x_{2n};t) = \prod_{1\leqslant i<j\leqslant 2n} \frac{(1-tx_ix_j)}{(x_i-x_j)} \mathrm{Pf}\left[\frac{(1-t)(x_i-x_j)}{(1-tx_ix_j)(1-x_ix_j)}\right]_{1\leqslant i<j\leqslant 2n}$$

- This is precisely what appears on the right hand side of Example 2(b).

Reflecting domain wall boundary conditions

- The six-vertex model has also been studied on a lattice with *reflecting* boundary conditions:



- This quantity is important in quantum spin-chains with open boundary conditions.
- Configurations on this lattice are in one-to-one correspondence with U-turn ASMs (UASMs).

Reflecting domain wall boundary conditions

- The partition function was evaluated in determinant form by Tsuchiya:

$$
Z_{\mathrm{UASM}}(x_1,\ldots,x_n;y_1,\bar{y}_1,\ldots,y_n,\bar{y}_n;t) = \frac{\prod_{i,j=1}^{n}(1-tx_iy_j)(1-tx_i\bar{y}_j)}{\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(y_i-y_j)(1-tx_ix_j)(1-\bar{y}_i\bar{y}_j)} \times \det\left[\frac{(1-t)}{(1-tx_iy_j)(1-tx_i\bar{y}_j)(1-x_iy_j)(1-x_i\bar{y}_j)}\right]_{1\leqslant i,j\leqslant n}
$$

- This matches what appears on the right hand side of Example 3(b).