Neutrino Data Analysis

Invisibles 14 School

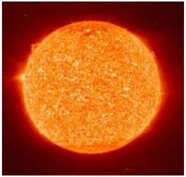
Thomas Schwetz-Mangold

8-13 July 2014, Chateau de Button, France

Global data on neutrino oscillations Debbie Harris' lecture

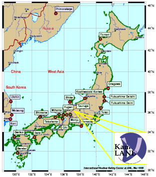
various neutrino sources and vastly different energy and distance scales:

sun



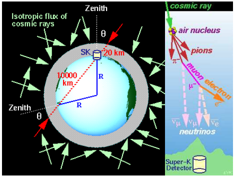
Homestake,SAGE,GALLEX
SuperK, SNO, Borexino

reactors



KamLAND, CHOOZ

atmosphere



SuperKamiokande

accelerators

K2K, MINOS, T2K

- global data fits nicely with the 3 neutrinos from the SM
- for this lecture I will ignore "anomalies" (at 2-3 σ) which do not fit the 3-flavour picture: LSND, MiniBooNE, reactor anomaly, no LMA MSW up-turn of solar neutrino spectrum

3-flavour oscillation parameters

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

$$\mathrm{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & e^{-i\delta} s_{13} \\ 0 & 1 & 0 \\ -e^{i\delta} s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Δm^2_{31} Δm^2_{21}

$$U = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}}_{\text{atm+LBL(dis)}} \underbrace{\begin{pmatrix} c_{13} & 0 & e^{-i\delta} s_{13} \\ 0 & 1 & 0 \\ -e^{i\delta} s_{13} & 0 & c_{13} \end{pmatrix}}_{\text{react+LBL(app)}} \underbrace{\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{solar+KamLAND}}$$

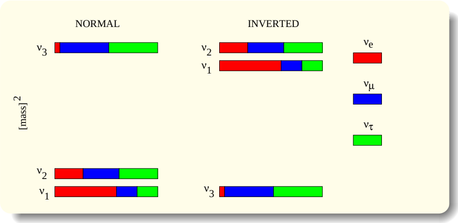
3-flavour effects are suppressed: $\Delta m^2_{21} \ll \Delta m^2_{31}$ and $\theta_{13} \ll 1$ ($U_{e3} = s_{13} e^{-i\delta}$)

$\Rightarrow$ CP-violation is suppressed by θ_{13}

$\Rightarrow$ dominant oscillations are well described by effective two-flavour oscillations

$\Rightarrow$ present data requires already to go beyond two-flavour description

Neutrino mass states and mixing



3-flavour oscillation parameters, ranges at 1σ (3σ) NuFit 1.3 [$\theta_{ij}, \delta_{\rm CP}$ in °]

$$\Delta m^2_{21} = 7.5 \pm 0.18 \ \left({}^{+0.56}_{-0.47}\right) \times 10^{-5}\ \mathrm{eV}^2$$

$$\Delta m^2_{31}(\mathrm{N}) = 2.46^{+0.05}_{-0.05} \ \left({}^{+0.14}_{-0.14}\right) \times 10^{-3}\ \mathrm{eV}^2$$

$$|\Delta m^2_{32}|(\mathrm{I}) = 2.49^{+0.05}_{-0.05} \ \left({}^{+0.14}_{-0.14}\right) \times 10^{-3}\ \mathrm{eV}^2$$

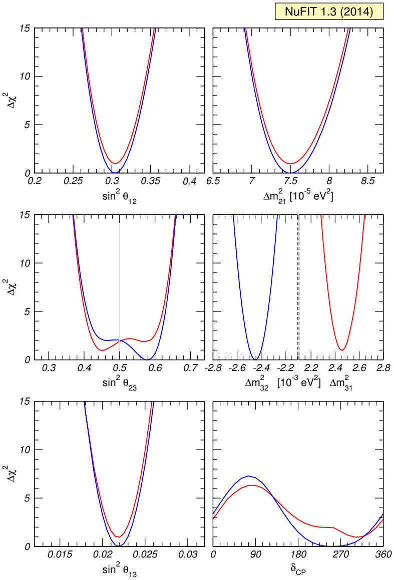
$$\theta_{12} = 33.5^{+0.77}_{-0.74} \ \left({}^{+2.4}_{-2.2}\right)$$

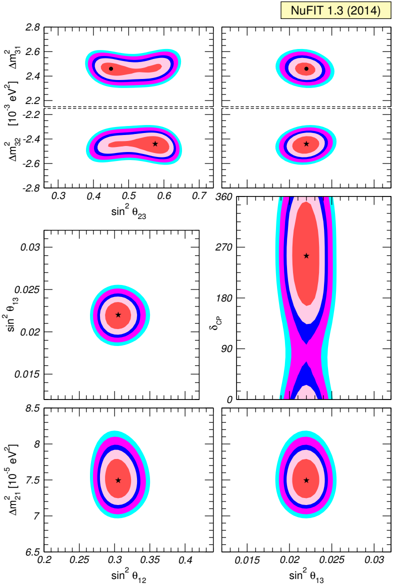
$$\theta_{23} = \begin{cases} (\mathrm{N})\ 42.1^{+3.2}_{-1.5} \ \left({}^{+11.1}_{-3.7}\right) \\ (\mathrm{I})\ 49.4^{+1.6}_{-2.0} \ \left({}^{+3.9}_{-11.0}\right) \end{cases}$$

$$\theta_{13} = 8.5^{+0.19}_{-0.17} \ \left({}^{+0.6}_{-0.5}\right)$$

$$\delta_{\rm CP} = \begin{cases} (\mathrm{N})\ 300^{+45}_{-45} \ \left({}^{+60}_{-300}\right) \\ (\mathrm{I})\ 251^{+67}_{-59} \ \left({}^{+109}_{-251}\right) \end{cases}$$

Global 3-flavour fit





These lectures

- mention some features of global fits of present and (a bit of) upcoming data
- discuss technical issues of how to do such type of analyses
- statistics techniques (complementary to Glen Cowan's lecture)
  - oriented towards practice in context of neutrino data fitting
  - recommend to look up relevant parts in Glen's lecture and make contact
- build on lectures by Debbie Harris and Renata Zukanovich-Funchal

Outline

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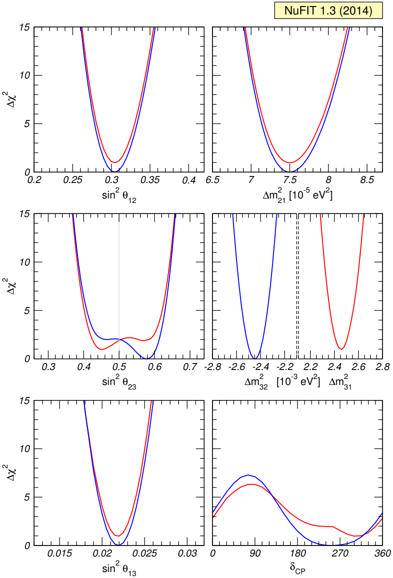
Global 3-flavour fit



- very robust determination of
  - $\Delta m^2_{21}, \theta_{12}$: solar, KamLAND
  - θ_{13}: Daya Bay, RENO, DoubleC
- ambiguity in sign of Δm^2_{31} ($\Delta\chi^2 \approx 1$) $\rightarrow$ mass ordering ("hierarchy")
- θ_{23}: rather broad allowed range
  non-significant indications about non-maximality/octant
  results of other groups differ slightly
  Capozzi et al., 1312.2878
  Forero et al. 1405.7540
- slight preference for $\delta_{CP} \sim -\pi/2$
  T2K $\nu_\mu \rightarrow \nu_e$ + Daya Bay
  not significant yet!

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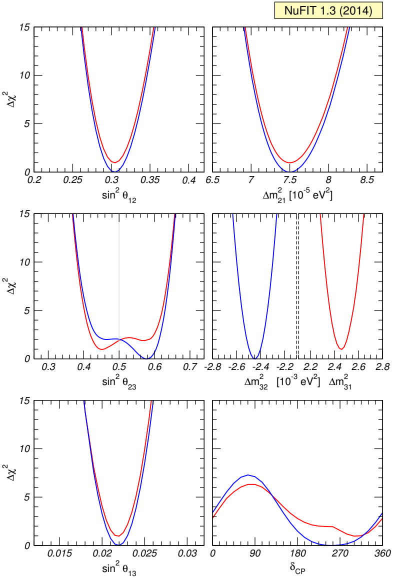

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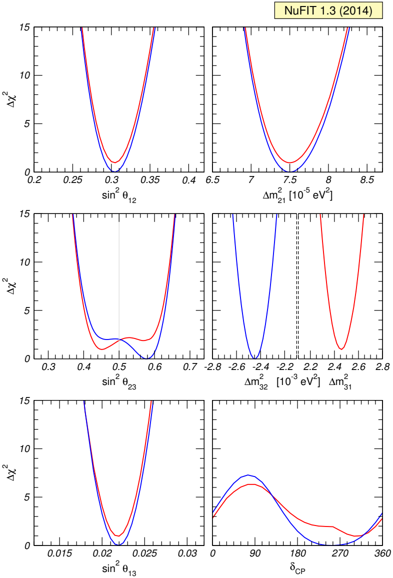

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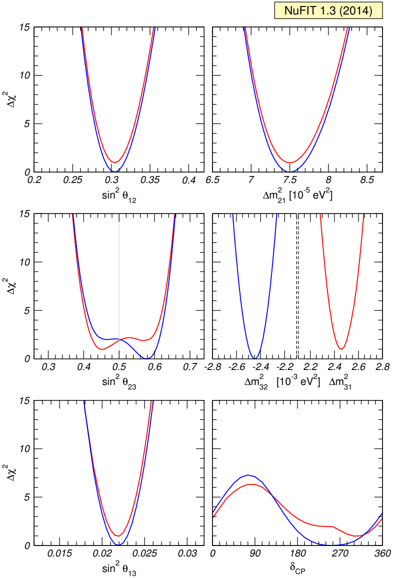

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The LBL appearance oscillation probability:

$$
\begin{aligned}
P_{\mu e} \quad \simeq \quad & \sin^2 2\theta_{13} \sin^2\theta_{23} \frac{\sin^2(1-A)\Delta}{(1-A)^2} \\
+ \quad & \sin 2\theta_{13}\, \hat{\alpha}\, \sin 2\theta_{23} \frac{\sin(1-A)\Delta}{1-A} \frac{\sin A\Delta}{A} \cos(\Delta + \delta_{\rm CP}) \\
+ \quad & \hat{\alpha}^2 \cos^2\theta_{23} \frac{\sin^2 A\Delta}{A^2}
\end{aligned}
$$

with $\quad \Delta \equiv \frac{\Delta m^2_{31} L}{4E_\nu}\,, \quad \hat{\alpha} \equiv \frac{\Delta m^2_{21}}{\Delta m^2_{31}} \sin 2\theta_{12}\,, \quad A \equiv \frac{2E_\nu V}{\Delta m^2_{31}}$

anti-ν: $\delta_{\rm CP} \to -\delta_{\rm CP}$, $A \to -A$, $\quad P_{e\mu}$: $\delta_{\rm CP} \to -\delta_{\rm CP}$

other mass ordering: $\Delta \to -\Delta$, $A \to -A$, $\hat{\alpha} \to -\hat{\alpha}$

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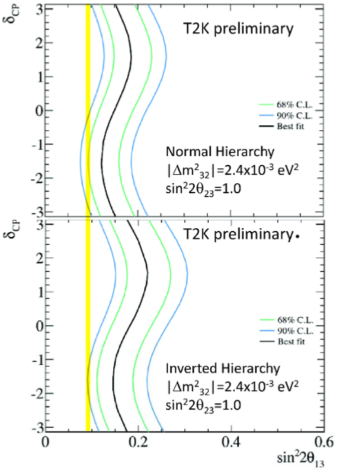
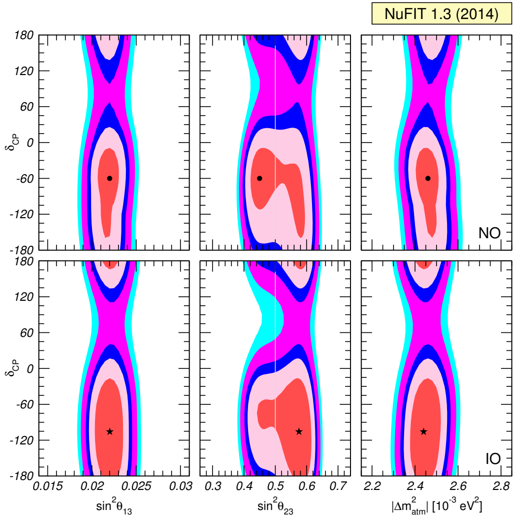
ν_e disappearance at $L \sim 1$ km:

$$\boxed{P_{ee} = 1 - \sin^2 2\theta_{13} \sin^2\Delta + \mathcal{O}(\alpha^2)}$$

Combining T2K/MINOS appearance with θ_{13} reactors





Degeneracies

Suppose that the true osc. params. in nature are

$$\hat{\boldsymbol{\theta}} = (\Delta\hat{m}^2_{21}, \Delta\hat{m}^2_{31}, \hat{\theta}_{12}, \hat{\theta}_{23}, \hat{\theta}_{13}, \hat{\delta}_{\rm CP})$$

A $\nu_\mu \to \nu_e$ appearance experiment will observe a number of events $\hat{N}$ corresponding to the probability $\hat{P}_{\mu e} = P_{\mu e}(\hat{\boldsymbol{\theta}})$

For fixed $\hat{N}$ there are other values of $\theta_{13} \neq \hat{\theta}_{13}$ and $\delta_{\rm CP} \neq \hat{\delta}_{\rm CP}$, which lead to the same osc. probability:

$$\hat{P}_{\mu e} = P_{\mu e}(\theta_{13}, \delta_{\rm CP})$$

and similar for anti-neutrinos:

$$\hat{\bar{P}}_{\mu e} = \bar{P}_{\mu e}(\theta_{13}, \delta_{\rm CP})$$

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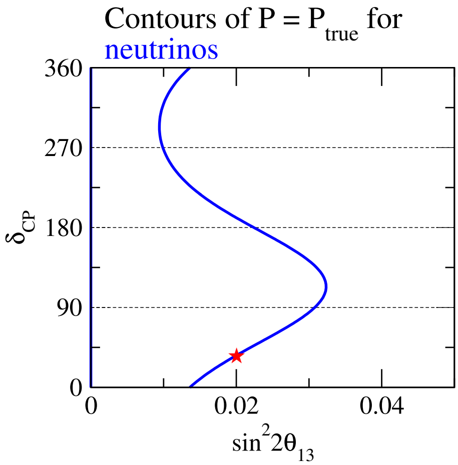
"Intrinsic" degeneracy

numerical example ("historical" plots, note θ_{13} value):



$\Delta\hat{m}^2_{21} = 7.9 \times 10^{-5}\text{eV}^2$
$\Delta\hat{m}^2_{31} = 2.5 \times 10^{-3}\text{eV}^2$
$\sin^2\hat{\theta}_{12} = 0.3$
$\sin^2\hat{\theta}_{23} = 0.4$
$\sin^2 2\hat{\theta}_{13} = 0.02$
$\hat{\delta}_{\mathrm{CP}} = 36^\circ$

$E_\nu = 2.2$ GeV
$L = 812$ km
(**NOνA**)

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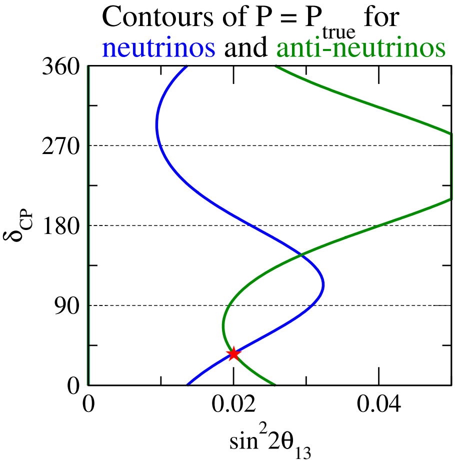

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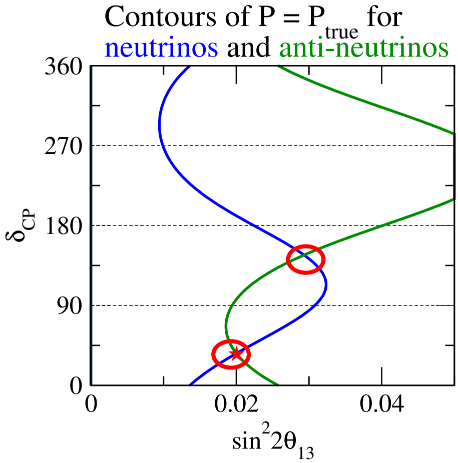

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Sign Δm^2_{31} degeneracy Minakata, Nunokawa, JHEP 10 (2001) 001

Exercise: show that the oscillation probability $P_{\mu e}$ in vaccum is invariant under the transformation

$$\Delta m^2_{31} \to -\Delta m^2_{31}\,, \qquad \delta_{\rm CP} \to \pi - \delta_{\rm CP}$$

for small matter effect ($A \ll 1$) the linear order in A cannot break the degeneracy $\to$ need to enter the regime of "strong" matter effect $A \sim 1$, i.e., observe the resonance

(see Schwetz, hep-ph/0703279)

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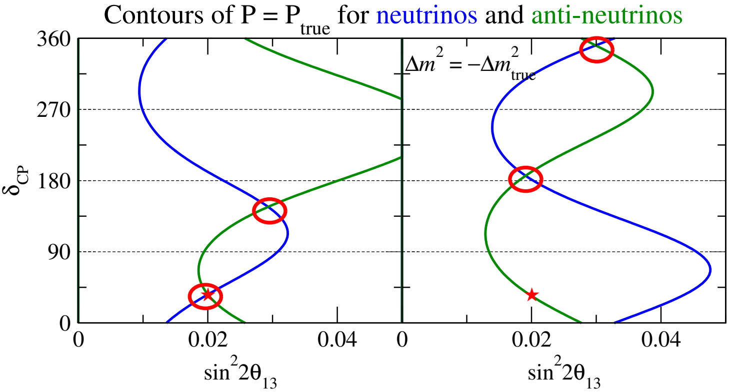

Octant degeneracy Fogli, Lisi, PRD54 (1996) 3667

θ_{23} is determined dominantly from ν_μ disappearance experiments (SK-atmospheric, T2K, MINOS)

0th approximation: $$P_{\mu\mu} \approx 1 - \sin^2 2\theta_{23} \sin^2 \frac{\Delta m^2_{\rm atm} L}{4E}$$

$\longrightarrow$ degeneracy between $\sin^2\theta_{23}$ and $(1 - \sin^2\theta_{23})$

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better approximation:

$$P_{\mu\mu} \approx 1 - 4|U_{\mu 3}|^2(1 - |U_{\mu 3}|^2) \sin^2 \frac{\Delta m^2_{\rm atm} L}{4E} \qquad |U_{\mu 3}|^2 = \sin^2\theta_{23}\cos^2\theta_{13}$$

$\longrightarrow$ degeneracy between $\sin^2\theta_{23}$ and $(\frac{1}{\cos^2\theta_{13}} - \sin^2\theta_{23})$

$\frac{1}{\cos^2\theta_{13}} \approx 1.02$

Octant degeneracy Fogli, Lisi, PRD54 (1996) 3667

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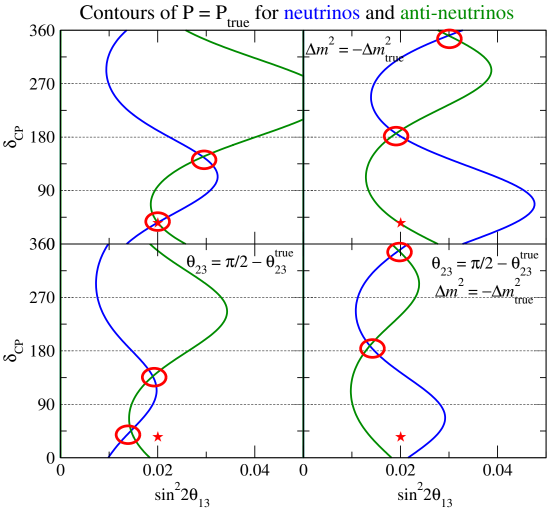
$\longrightarrow$ degeneracy between $\sin^2\theta_{23}$ and $(1-\sin^2\theta_{23})$

BUT: appearance probability depends on θ_{23} in a non-symmetric way:

$$\begin{aligned} P_{\mu e} \;\; &\simeq \;\; \sin^2 2\theta_{13} \sin^2\theta_{23} \frac{\sin^2(1-A)\Delta}{(1-A)^2} \\ &+ \;\; \sin 2\theta_{13}\,\hat\alpha\, \sin 2\theta_{23} \frac{\sin(1-A)\Delta}{1-A} \frac{\sin A\Delta}{A} \cos(\Delta + \delta_{\rm CP}) \end{aligned}$$

The eight-fold degeneracy Barger, Marfatia, Whisnant, PRD 02



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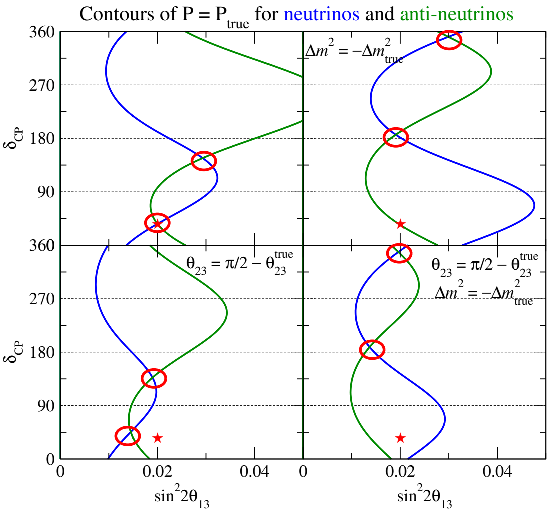

- ambiguities in determination of θ_{13} and $\delta_{\rm CP}$
- can involve an ambiguity between CP conserving and CP violating values of $\delta_{\rm CP}$
- sign(Δm^2_{31}) is not determined (neutrino mass ordering)
- the octant of θ_{23} is not determined

Resolving the degeneracies

several possibilities to resolve the degeneracies are known:

- combining information from detectors at different baselines
- using additional oscillation chanels ($\nu_e \to \nu_\tau$)
- spectral information (wide band beam)
- adding information on θ_{13} from a reactor experiment
- adding information from (Mt scale) atmospheric neutrino experiments
- ...

... many of them work quite well for large θ_{13}!

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Octant degeneracy - beams versus reactor

Fogli, Lisi, 96; Minakata, Sugiyama, Yasuda, Inoue, Suekane, 02; ...

fix θ_{13} by a reactor experiment and use an appearance experiment to determine the octant of θ_{23}

$$
\begin{aligned}
P_{\mu e} \quad &\simeq \quad \sin^2 2\theta_{13} \sin^2 \theta_{23} \, \frac{\sin^2(1-A)\Delta}{(1-A)^2} \\
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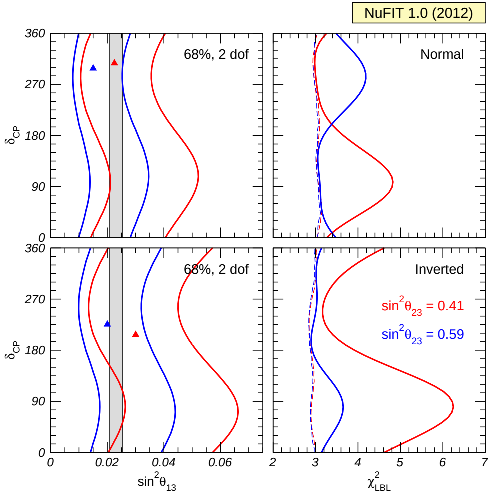
Combining T2K/MINOS appearance with θ_{13} reactors



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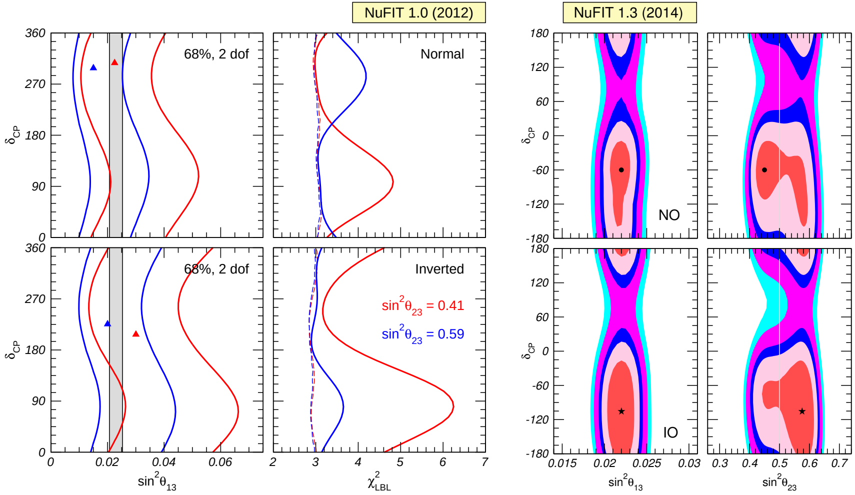
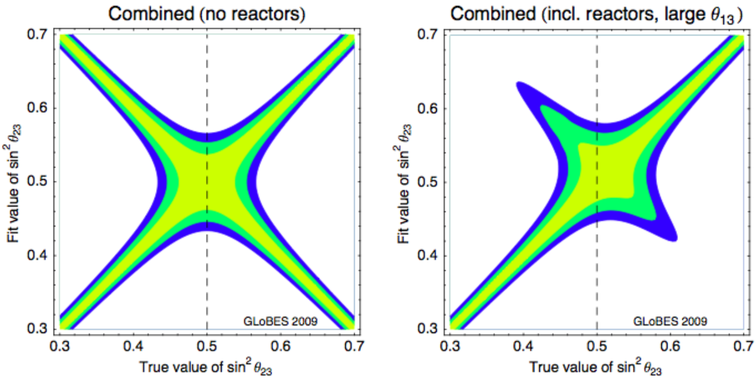


Octant degeneracy - simulated data

T2K + NOvA   T2K + NOvA + Daya Bay



Huber, Lindner, Schwetz, Winter, 0907.1896

Determination of the mass ordering

- matter effect in the 13-sector: resonance condition for $\nu_\mu \to \nu_e$ oscillations:

$$A \equiv \pm \frac{2EV}{\Delta m^2_{31}} = \cos 2\theta_{13} \approx 1$$

can be fulfilled for

neutrinos if $\Delta m^2_{31} > 0$ (normal ordering)
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(see my talk at the workshop next week)

Determination of the mass ordering

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Degeneracies 2014

after Daya Bay θ_{13} is no longer a "free" parameter

the relevant degrees of freedom are θ_{23} and $\delta_{\rm CP}$ times $\mathrm{sign}(\Delta m^2_{31})$
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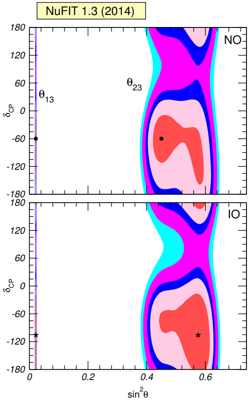

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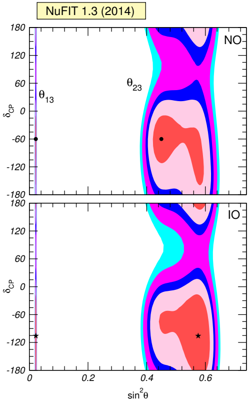

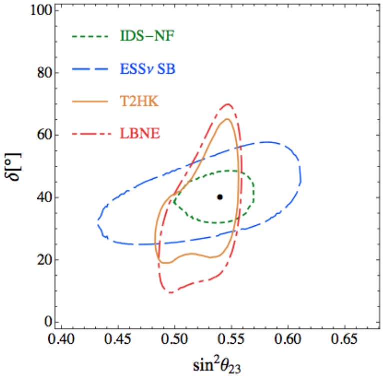

How to analyze data from neutrino oscillation experiments

Basic steps towards an analysis

- Suppose a given experiment divides the range of observation into N bins. The outcome is reported in number of observed events in each bin n_i.
  (Expect Poisson distribution for the number of events in each bin.)

- For given oscillation parameters

$$\boldsymbol{\theta} = (\theta_{12}, \theta_{13}, \theta_{23}, \delta_{\rm CP}, \Delta m^2_{21}, \Delta m^2_{31}) \qquad (P = 6)$$

  we can predict the expected number of events per bin $\mu_i(\boldsymbol{\theta})$.

- Build a χ^2, e.g. (more details later):

$$\chi^2(\boldsymbol{\theta}) = \sum_{i=1}^{N} \left[\frac{\mu_i(\boldsymbol{\theta}) - n_i}{\sigma_i}\right]^2$$

- Use $\chi^2(\boldsymbol{\theta})$ to perform a statistical analysis

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Analysis of present oscillation data and beyond
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Event rates in oscillation experiments
- Reactor experiments
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Event rates in oscillation experiments

number of events in a $\nu_\alpha \to \nu_\beta$ oscillation experiment:

$$N(\boldsymbol{\theta}) = T\mathcal{N} \int dE_\nu \, \phi_{\nu_\alpha}(E_\nu) \, P_{\alpha\beta}(E_\nu; \boldsymbol{\theta}) \, \sigma_{\nu_\beta}(E_\nu)$$

T	exposure time
$\mathcal{N}$	number of target particles
ϕ_{ν_α}	neutrino flux of flavour α at detector
$P_{\alpha\beta}$	$\nu_\alpha \to \nu_\beta$ oscillation probability
σ_{ν_β}	detection cross section of neutrino ν_β

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- in more realistic situations we need to take into account the characteristics of the particular experiment
- consider in more detail the actual observables
- typically it will involve more integrals
  Ex.: atmospheric neutrinos: integrate also over zenith angle, production height in atmosphere, ...

Example: Reactor experiments

- source of $\bar{\nu}_e$ with few MeV $\rightarrow$ $\bar{\nu}_e$ disappearance
- detection reaction: inverse beta-decay

$$\bar{\nu}_e + p \rightarrow n + e^+$$

  observe positron and neutron in coincedence

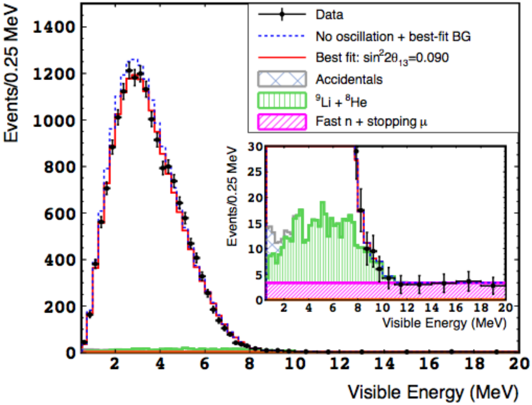
- visible energy:

$$E_{\rm vis} \approx E_{\rm kin}^{e^+} + 2m_e = E_\nu - (m_n - m_p) + m_e + \mathcal{O}(E_\nu^2/m_n)$$
$$E_{\rm vis} \approx E_\nu - 0.8\,\mathrm{MeV}$$

  $\rightarrow$ one-to-one relation between $E_{\rm vis}$ and E_ν

- accurate spectral information: number of inverse beta-decay events binned in visible energy

Ex.: DoubleChooz energy spectrum 1406.7763



Number of events per bin

ideal experiment:

$$N_i(\boldsymbol{\theta}) = T\mathcal{N} \int_{E_{\rm vis}^{low,i}}^{E_{\rm vis}^{up,i}} dE_\nu \; \phi(E_\nu)\, P_{ee}(E_\nu;\boldsymbol{\theta})\, \sigma(E_\nu) \qquad E_\nu \approx E_{\rm vis} + 0.8\,\mathrm{MeV}$$

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BUT: need to take into account energy resolution: a "true" $E_{\mathrm{vis}}^{\mathrm{true}}$ is reconstructed as E_{vis} with a certain probability distribution $R(E_{\mathrm{vis}}, E_{\mathrm{vis}}^{\mathrm{true}})$

$$N_i(\boldsymbol{\theta}) = T\mathcal{N} \int_{E_{\mathrm{vis}}^{low,i}}^{E_{\mathrm{vis}}^{up,i}} dE_{\mathrm{vis}} \int dE_\nu \, \phi(E_\nu)\, P_{ee}(E_\nu;\boldsymbol{\theta})\, \sigma(E_\nu) R(E_{\mathrm{vis}}, E_{\mathrm{vis}}^{\mathrm{true}})$$

$$E_\nu \approx E_{\mathrm{vis}}^{\mathrm{true}} + 0.8\,\mathrm{MeV}$$

can write this as

$$N_i(\boldsymbol{\theta}) = T\mathcal{N} \int dE_\nu \, \phi(E_\nu) \, P_{ee}(E_\nu; \boldsymbol{\theta}) \, \sigma(E_\nu) \, R_i(E_\nu)$$

$$R_i(E_\nu) \equiv \int_{E_{\rm vis}^{low,i}}^{E_{\rm vis}^{up,i}} dE_{\rm vis} R(E_{\rm vis}, E_{\rm vis}^{\rm true}) \qquad E_\nu \approx E_{\rm vis}^{\rm true} + 0.8\,\mathrm{MeV}$$

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often it is a good approximation to assume a Gaussian resolution function:

$$R(E_{\rm vis}, E_{\rm vis}^{\rm true}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(E_{\rm vis} - E_{\rm vis}^{\rm true})^2}{2\sigma^2}\right] \qquad \sigma = \sigma(E_{\rm vis}^{\rm true})$$

$$R_i(E_\nu) = \frac{1}{2}\left[\text{erf}\left(\frac{E_{\rm vis}^{up,i} - E_{\rm vis}^{\rm true}}{\sqrt{2}\sigma}\right) - \text{erf}\left(\frac{E_{\rm vis}^{low,i} - E_{\rm vis}^{\rm true}}{\sqrt{2}\sigma}\right)\right]$$

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to compare with observation add expected background in each bin:

$$\mu_i(\boldsymbol{\theta}) = N_i(\boldsymbol{\theta}) + B_i$$

$\rightarrow$ can be used to build χ^2, for example:

$$\chi^2(\boldsymbol{\theta}) = \sum_{i=1}^{N} \frac{[\mu_i(\boldsymbol{\theta}) - n_i]^2}{n_i}$$

includes only statistical errors $\rightarrow$ on systematics see later

Example: long-baseline experiment

- consider a $\nu_\mu \to \nu_e$ appearance experiment with $E_\nu \sim 1$ GeV (e.g., T2K, NOvA)
- detection reaction: $\nu_e + N \to e + X$
  significant energy is carried away by hadronic scattering products X

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assume only electron is observed and events are binned in electron energy

$$N_i(\boldsymbol{\theta}) = T\mathcal{N} \int dE_\nu \; \phi(E_\nu) \, P_{\mu e}(E_\nu; \boldsymbol{\theta}) \; \int_{E_e^{low,i}}^{E_e^{up,i}} dE_e \frac{d\sigma}{dE_e}(E_\nu)$$

$\longrightarrow$ double integral even before including resolution function

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some detectors can use info on X to reconstruct $E_\nu \to$ bins in E_ν^{rec}

may require complicated cuts introducing energy dependent efficiences,...

Detector response function - migration matrix

$$N_i(\boldsymbol{\theta}) = T\mathcal{N} \int dE_\nu \, \phi(E_\nu) \, P_{\mu e}(E_\nu; \boldsymbol{\theta}) \, \sigma(E_\nu) \, \mathcal{R}_i(E_\nu)$$

$\mathcal{R}_i(E_\nu)$: detector response function

- describes the probability that an event with neutrino energy E_ν is reconstructed in the bin i
- the bins may label any observable (e.g., lepton energy, reconstr. neutrino energy, ...)
- $\mathcal{R}_i(E_\nu)$ can include many effects related to the detector (energy resolution, energy dep. efficiencies, differential cross sections, ...)
- if the integral over true neutrino energy is discretized $\mathcal{R}_i(E_\nu)$ becomes a matrix $\mathcal{R}_{ij} \to$ "migration matrix"

Detector response function - migration matrix

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$\mathcal{R}_i(E_\nu)$: detector response function

can be conveniently done with the GLoBES software package
Huber, Lindner, Winter, hep-ph/0407333; Huber et al., hep-ph/0701187
http://www.mpi-hd.mpg.de/lin/globes/

Example: atmospheric neutrinos

consider an experiment observing muons induced by atmospheric neutrinos (e.g., INO):

$$N_{ij}(\boldsymbol{\theta}) = T\mathcal{N} \int dE_\nu \int d\Omega\, \sigma(E_\nu)\, \mathcal{R}_{ij}(E_\nu, \Omega) \times \\ [\phi_\mu(E_\nu, \Omega)\, P_{\mu\mu}(E_\nu, \Omega; \boldsymbol{\theta}) + \phi_e(E_\nu, \Omega)\, P_{e\mu}(E_\nu, \Omega; \boldsymbol{\theta})]$$

i	bin in muon energy
j	bin in muon zenith angle
$\phi_\alpha(E_\nu, \Omega)$	flux of ν_α with given E_ν and solid angle Ω

$\mathcal{R}_{ij}(E_\nu, \Omega)$: probability to reconstruct muon from a neutrino with energy E_ν coming from a solid angle Ω into the muon bin ij (includes double differential cross section)

(still simplified in several respects....)

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- Can define:

$$\chi^2 = \sum_{i=1}^{N} \frac{[\mu_i(\boldsymbol{\theta}) - n_i]^2}{\mu_i(\boldsymbol{\theta})} \quad \text{or} \quad \sum_{i=1}^{N} \frac{[\mu_i(\boldsymbol{\theta}) - n_i]^2}{n_i}$$

- If the number of events is small in some bins ("Poisson χ^2"):

$$\chi^2 = 2\sum_{i=1}^{N} \left[\mu_i(\boldsymbol{\theta}) - n_i + n_i \log \frac{n_i}{\mu_i(\boldsymbol{\theta})}\right]$$

- If statistical errors include the ones from a subtracted background:

$$\chi^2 = \sum_{i=1}^{N} \left[\frac{\mu_i(\boldsymbol{\theta}) - n_i}{\sigma_i}\right]^2$$

- If there is correlation between bins:

$$\chi^2 = \sum_{i,j=1}^{N} [\mu_i(\boldsymbol{\theta}) - n_i] V_{ij}^{-1} [\mu_j(\boldsymbol{\theta}) - n_j]$$

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Systematic uncertainties

Assume we have N experimental data points n_i with statistical error σ_i and theoretical predictions μ_i for each of the data points:

$$\chi^2 = \sum_{i=1}^{N} \frac{(\mu_i - n_i)^2}{\sigma_i^2}$$

$\mu_i(\theta)$ depends on the parameters of the model θ.

Consider the situation that μ_i depends also on additional parameters ξ, describing systematical uncertainties ("nuisance parameters"): $\mu_i(\theta, \xi)$

We may have some knowledge on ξ: mean values $\langle \xi_\alpha \rangle = \hat{\xi}_\alpha$ and uncertainty σ_α^ξ

Example

$$\mu_i(\theta) = \xi_1\,(\xi_2\,N_i(\theta) + \xi_3\,B_i) \qquad \xi_\alpha = 1 \pm x_\alpha\%$$

$$\approx (1+\delta_1+\delta_2)N_i(\theta) + (1+\delta_1+\delta_3)B_i \qquad \delta_\alpha = \xi_\alpha - 1$$

ξ_1 overall detector normalization

ξ_2 overall signal normalization (e.g., flux uncertainty)

ξ_3 background normalization

can be generalized to more complicated systematics, including energy dependent uncertainties (shape), energy scale,...

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Consider ξ at the same level as θ and add info to χ^2

$$\chi^2(\theta,\xi) = \sum_{i=1}^{N} \frac{[\mu_i(\theta,\xi) - n_i]^2}{\sigma_i^2} + \sum_{\alpha} \frac{(\xi_\alpha - \hat{\xi}_\alpha)^2}{(\sigma_\alpha^\xi)^2}$$

$$\chi^2(\theta) = \min_{\xi} \chi^2(\theta,\xi)$$

$\chi^2(\theta)$ is distributed as usual with $N = (N-P) + P$ dof

no conceptual issue also for $P \gtrsim N$

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Linearize the problem

$$\mu_i(\theta,\xi) \approx \mu_i(\theta,\hat{\xi}) + \sum_\alpha \frac{\partial \mu_i}{\partial \xi_\alpha}(\xi_\alpha - \hat{\xi}_\alpha)$$

define: $\mu_i(\theta,\hat{\xi}) \equiv \hat{\mu}_i(\theta)\,, \qquad \xi'_\alpha \equiv \dfrac{\xi_\alpha - \hat{\xi}_\alpha}{\sigma^\xi_\alpha}\,, \qquad R_{i\alpha} \equiv \sigma^\xi_\alpha \dfrac{\partial \mu_i}{\partial \xi_\alpha}$

$$\chi^2(\theta,\xi') = \sum_i \frac{[\hat{\mu}_i(\theta) + \sum_\alpha R_{i\alpha}\xi'_\alpha - n_i]^2}{\sigma_i^2} + \sum_\alpha {\xi'_\alpha}^2$$

$\chi^2(\theta,\xi')$ is quadratic in ξ' $\Rightarrow$ $\frac{\partial \chi^2}{\partial \xi_\alpha} = 0$ is a linear system of equations
$\Rightarrow$ solve the system to obtain ξ_{min} and obtain $\chi^2(\theta) = \chi^2(\theta,\xi_{min})$

- this proceedure works fine if $\xi'_\alpha \lesssim 1$ and $(R\xi')_i \ll \mu_i$
- if $(R\xi')_i \sim \mu_i$, the prediction can become negative

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Equivalence of pull and covariance approaches

- "pull" approach:

$$\chi^2_{\text{pull}}(\theta) = \min_{\xi} \chi^2(\theta, \xi)$$

- "covariance" approach:

$$V_{ij} = \sum_{\alpha} \frac{\partial \mu_i}{\partial \xi_\alpha} \frac{\partial \mu_j}{\partial \xi_\alpha} (\sigma_\alpha^\xi)^2 = \sum_\alpha R_{i\alpha} R_{j\alpha}$$

$$\chi^2_{\text{cov}}(\theta) = \sum_{ij} [\hat{\mu}_i(\theta) - n_i]^T S_{ij}^{-1} [\hat{\mu}_j(\theta) - n_j] \quad \text{with} \quad S_{ij} \equiv \sigma_i^2 \delta_{ij} + V_{ij}$$

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Exercise: proof that $\chi^2_{\text{pull}}(\theta) \equiv \chi^2_{\text{cov}}(\theta)$

Fogli, Lisi, Marrone, Montanino, Palazzo, PRD02 [hep-ph/0206162]

Simple example

Consider the case of a single systematic describing an over-all normalization uncertainty

$$\chi^2(\theta, \xi) = \sum_i \left[\frac{\mu_i(\theta)(1+\xi) - n_i}{\sigma_i}\right]^2 + \left(\frac{\xi}{\sigma_\xi}\right)^2$$

$$R_i = \mu_i(\theta)$$

covariance matrix for the covariance method: $S_{ij} = \delta_{ij}\sigma_i^2 + \mu_i\mu_j\sigma_\xi^2$

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Exercise:

- minimize the χ^2 and calculate ξ_{min} and $\chi^2(\theta, \xi_{min})$
- consider the same systematic using the Poisson χ^2 (check that your solution makes sense!)

Simple example

Consider the case of a single systematic describing an over-all normalization uncertainty

$$\chi^2(\theta,\xi) = \sum_i \left[\frac{\mu_i(\theta)(1+\xi) - n_i}{\sigma_i}\right]^2 + \left(\frac{\xi}{\sigma_\xi}\right)^2$$

$$R_i = \mu_i(\theta)$$

covariance matrix for the covariance method: $S_{ij} = \delta_{ij}\sigma_i^2 + \mu_i\mu_j\sigma_\xi^2$

for $\sigma_\xi \to \infty$ this corresponds to a shape-only analysis (free normalization)

exactly this method has been used recently by the Daya Bay collaboration for their analysis based on near-far comparison

Real-life example Daya Bay 1203.1669

The value of $\sin^2 2\theta_{13}$ was determined with a χ^2 constructed with pull terms accounting for the correlation of the systematic errors [28],

$$\chi^2 = \sum_{d=1}^{6} \frac{\left[M_d - T_d\left(1 + \varepsilon + \sum_r \omega_r^d \alpha_r + \varepsilon_d\right) + \eta_d\right]^2}{M_d + B_d} + \sum_r \frac{\alpha_r^2}{\sigma_r^2} + \sum_{d=1}^{6}\left(\frac{\varepsilon_d^2}{\sigma_d^2} + \frac{\eta_d^2}{\sigma_B^2}\right), \qquad (2)$$

where M_d are the measured IBD events of the d-th AD with backgrounds subtracted, B_d is the corresponding background, T_d is the prediction from neutrino flux, MC, and neutrino oscillations [29], ω_r^d is the fraction of IBD contribution of the r-th reactor to the d-th AD determined by baselines and reactor fluxes. The uncertainties are listed in Table III. The uncorrelated reactor uncertainty is σ_r (0.8%), σ_d (0.2%) is the uncorrelated detection uncertainty, and σ_B is the background uncertainty listed in Table II. The corresponding pull parameters are $(\alpha_r, \varepsilon_d, \eta_d)$. The detector- and reactor-related correlated

Real-life example Daya Bay 1203.1669

Exercise: study the χ^2 used in the Daya Bay paper

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Pull versus covariance approaches

- Pull approach requires to solve a linear system of equations of dimension P (number of pulls)
- Covariance approach requires to invert the $N \times N$ covariance matrix (N number of bins)
- Depending on whether N is larger or smaller than P one or the other method may be preferred (often $P \ll N$)
- Pull method allows for more diagnostics of the fit, e.g.:
  - look at $\xi_{\alpha min}$ to identify a systematic with large "pull",
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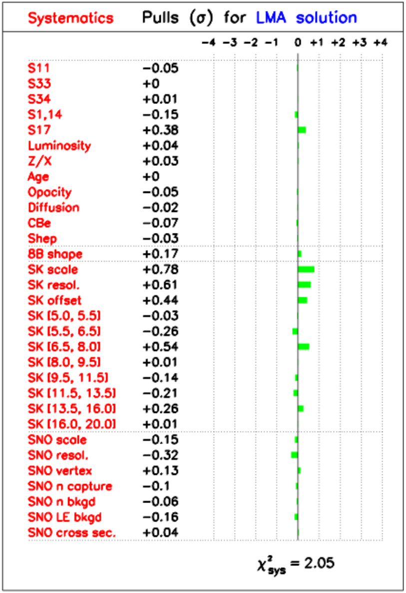
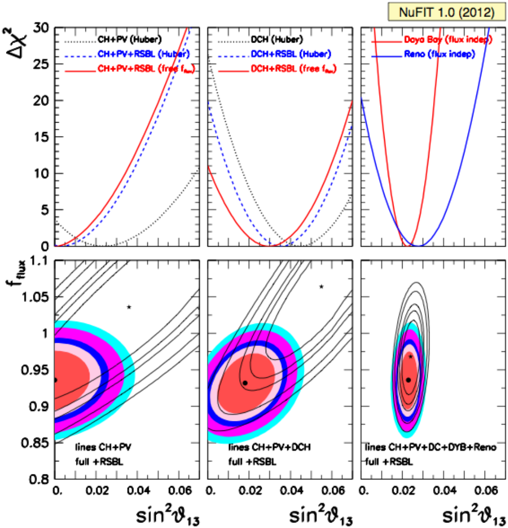
Example for "pull diagram" from solar neutrino fit



Fogli et al hep-ph/0206162

Correlations between reactor flux normalization and θ_{13}



Poisson χ^2

The pull method can be generalized to the Poissonian form of the χ^2 which should be used in case of small event numbers per bin:

$$\chi^2(\theta, \xi_\alpha) = 2 \sum_{i=1}^{N} \left[\mu_i(\theta, \xi_\alpha) - n_i + n_i \log \frac{n_i}{\mu_i(\theta, \xi_\alpha)} \right] + \sum_\alpha \xi_\alpha^2$$

- allows to introduce correlated errors in the Poisson χ^2
- $\mu(\theta, \xi)$ can still be linearized in ξ, but the χ^2 will no longer be a quadratic function in $\xi \Rightarrow$ have to use numerical or semi-analytic methods to do the minimization

Comments - 1

- straight forward to generalize to correlated data and/or pulls:

$$\chi^2(\theta,\xi) = \sum_{i,j=1}^{N} [\mu_i(\theta,\xi) - n_i] V_{ij}^{-1} [\mu_j(\theta,\xi) - n_j] + \sum_{\alpha,\beta} (\xi_\alpha - \hat{\xi}_\alpha) W_{\alpha\beta}^{-1} (\xi_\beta - \hat{\xi}_\beta)$$

- can also be applied in the framework of likelihood analysis

$$\mathcal{L}(\theta,\xi) = \mathcal{L}_{\mathrm{data}}(\theta,\xi) \times \mathcal{L}_{\mathrm{nuis}}(\xi)$$
$$\mathcal{L}(\theta) = \max_\xi \mathcal{L}(\theta,\xi)$$

$\mathcal{L}_{\mathrm{nuis}}(\xi)$ contains all information we have on the nuisance parameters

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Comments - 2

- The methods discussed here for the treatment of systematic erros assume that systematic uncertainties are of statistical nature. Their effects on the analysis are encoded by assuming some random distribution for them (often Gaussian).
- Sometimes these assumptions are justified e.g. when the origin of the uncertainty is some measurment (e.g., normalization uncertainty).
- Sometimes these assumptions are not justified, in case of true "theoretical uncertainties" (e.g. nuclear matrix elements for neutrino-less double-beta decay).
- Frequentist interpretation in the strict sense is not clear
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Referenzes on pull method in neutrino context

- in the context of solar neutrinos
  G. L. Fogli, E. Lisi, A. Marrone, D. Montanino and A. Palazzo, Phys. Rev. D **66** (2002) 053010 [hep-ph/0206162]
- in the context of short-baseline oscillation experiments
  T. Schwetz, PhD thesis, Univ. Vienna 2002, see appendix A, available at http://www.cern.ch/schwetz
- in the context of SuperKamiokande atmospheric neutrinos
  M. C. Gonzalez-Garcia and M. Maltoni, Phys. Rept. **460** (2008) 1 [arXiv:0704.1800], see appendix A
- in the context of future long-baseline oscillation experiment simulation
  P. Huber, M. Mezzetto and T. Schwetz, JHEP **0803** (2008) 021 [arXiv:0711.2950]

Outline

Using the χ^2

in the "χ^2 appoximation":

$\chi^2(\boldsymbol{\theta})$	$=$	$\chi^2_{\text{min}}(\hat{\boldsymbol{\theta}})$	$+$	$\Delta\chi^2(\boldsymbol{\theta})$
N		$N - P$		P
		parameter estimation, goodness of fit		confidence intervall

- The parameter values $\hat{\theta}_\alpha$ which minimize the χ^2 (usually called "best fit values") are estimators of the "true values".
- χ^2_{min} follows a χ^2-distribution with $N - P$ d.o.f. and can be used to evaluate the goodness of fit.
- The $\Delta\chi^2$ relative to the minimum follows a χ^2-distribution with P d.o.f. and can be used to determine confidence intervalls (or regions) for the parameters $\boldsymbol{\theta}$.

Confidence regions from $\Delta\chi^2$

A P-dimensional region in the space $\boldsymbol{\theta}$ at given CL is obtained by requiring $\Delta\chi^2(\boldsymbol{\theta}) < X(\mathrm{CL})$ (contours in $\Delta\chi^2$)

d.o.f. \ CL	68%(1σ)	90%	95%(2σ)	99%	99.73%(3σ)
1	1	2.71	4	6.64	9
2	2.28	4.61	5.99	9.21	11.8
3	3.51	6.25	7.82	11.4	14.2

Confidence regions from $\Delta\chi^2$

Suppose you want to show regions at a CL β for p parameters x, and you are not interested in $q = P - p$ parameters y:

- use p d.o.f. and minimize wrt to y:
  "the p-dimensional region for x, irrespective of the values of y"
- use p d.o.f. and fix y to some values:
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- use P d.o.f. and show a projection of the P-dimensional volume onto the p-dimensional x-space. (This is not a β CL region for x!)

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$$
\boxed{
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N & & N - P & & P \\
 & & & & \\
\Delta\chi^2(x, y) & = & \Delta\chi^2_{\text{min},y}(x) & + & \delta\chi^2(x, y) \\
P & & p = P - q & & q
\end{array}
}
$$

$$
\Delta\chi^2_{\text{min},y}(x) \equiv \min[\Delta\chi^2(x, y); y] \qquad (p \text{ d.o.f.})
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comment: this is exactly what is used for the pull-method to include systematics

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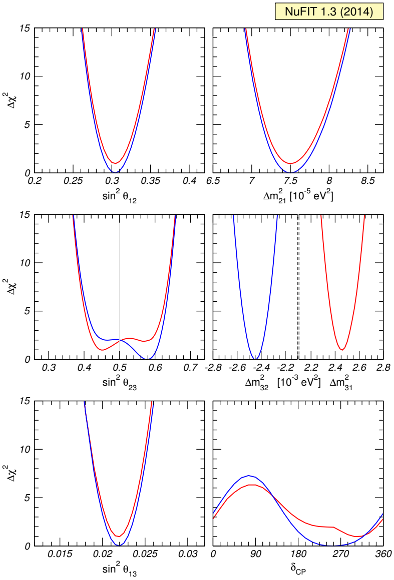
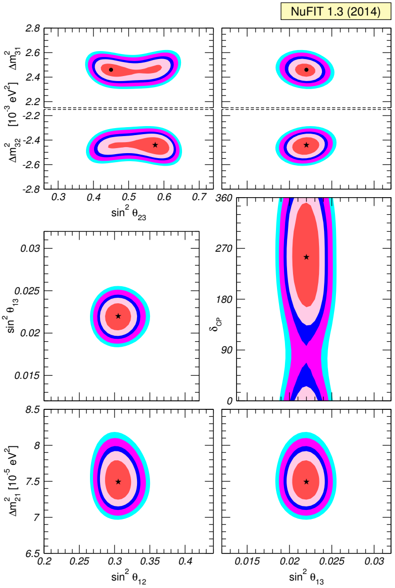
Example: 1-dim and 2-dim projections





Combining several experiments

- consider M experiments.
- experiment *ex* consists of N_{ex} data points.
- each experiment has its own χ^2 function: $\chi^2_{ex}(\boldsymbol{\theta})$
- the combined χ^2 is simply

$$\chi^2_{glob}(\boldsymbol{\theta}) = \sum_{ex=1}^{M} \chi^2_{ex}(\boldsymbol{\theta}) \qquad \#\,\text{d.o.f.} = \sum_{ex=1}^{M} N_{ex}$$

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Comments on Gaussian approximation

- In the Gaussian approximation ("χ^2 approximation")
  - 1-dimensional χ^2 projections will be parabolas
  - *p*-dimensional regions will be *p*-dimensional ellipsoids
  - inclination of the ellipse in a 2-dim plane gives the correlation between those two parameters
- In the θ_{12}, θ_{13}, Δm^2_{21} space we are close to Gaussian
- non-Gaussianities are relevant:
  - mass ordering degeneracy Δm^2_{31}
  - octant degeneracy $\chi^2(\theta_{23})$
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Outline

Open questions in neutrino oscillations

- Neutrino mass ordering (sign of Δm^2_{31})
- θ_{23}: maximality and octant
- CP violation, range of δ

How to estimate the sensitivity of a given proposed experiment?

Consider an experiment which has data n_i:

$$\chi^2(\boldsymbol{\theta}) = \sum_{i=1}^{N} \frac{[\mu_i(\boldsymbol{\theta}) - n_i]^2}{n_i}$$

$\mu_i(\boldsymbol{\theta})$: theoretical prediction of event rates, including experimental details, resolutions, backgrounds,...

future experiment: replace data n_i by predicted event rate assuming some "true" values of the oscillation parameters $\boldsymbol{\theta}^{tr}$

$$\chi^2(\boldsymbol{\theta}; \boldsymbol{\theta}^{tr}) = \sum_{i=1}^{N} \frac{[\mu_i(\boldsymbol{\theta}) - \mu_i(\boldsymbol{\theta}^{tr})]^2}{\mu_i(\boldsymbol{\theta}^{tr})}$$

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Including statistical fluctuations

Simulate statistical fluctuations by using your random number generator (Monte Carlo simulation):

$$n_i^{sim}(\boldsymbol{\theta}^{tr}) = P[\mu_i(\boldsymbol{\theta}^{tr})]\,, \qquad P[\mu] : \text{ Poisson distribution}$$

- For fixed θ^{tr} and a given statistical realisation a certain sensitivity will be obtained.
- Have to simulate many experiments to obtain a "distribution of sensitivities".

Example CP violation

- want to discover CPV at 99.9% CL
- simulate many experiments (at fixed $\boldsymbol{\theta}^{tr}$)
- a certain fraction β of those will discover CPV at 99.9% CL

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Type I and II errors see Glen Cowan's lecture

To quantify the sensitivity we need to quote two numbers: the CL at which we want to make a discovery and the probability with which a given proposed experiment will reach this.

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- Type I error: probability that CP is conserved although we claim discovery of CPV (0.1% in the previous example)
- Type II error: probability that CP is violated although our experiment does not find it ($1-\beta$ in the previous example)

for discussions in the context of neutrino oscillations see
Schwetz, Phys.Lett. B648 (2007) 54-59 [hep-ph/0612223]
Blennow, Coloma, Huber, Schwetz, JHEP 1403 (2014) 028 [1311.1822]

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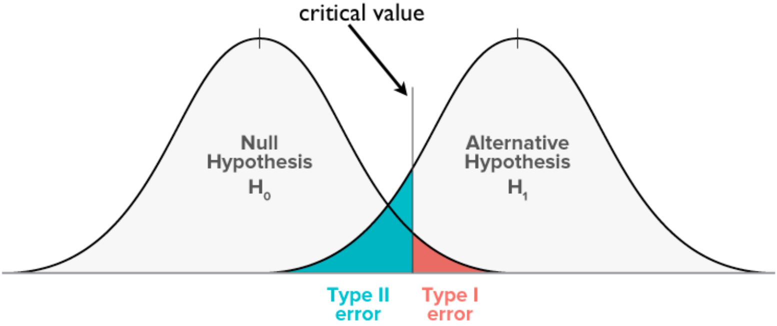


Including statistical fluctuations

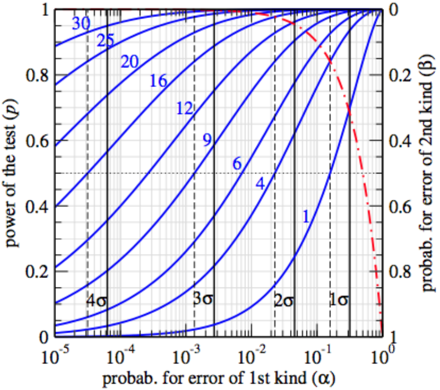
In practice Monte Carlo simulations can be quite "expensive":

- have to simulate *many* experiments,
- calculate sensitivity for each of them,
- do this for each choice of $\boldsymbol{\theta}^{tr}$

Analytical approximations

In some cases it may be possible to use analytic expressions ("Gaussian approximation") for the relevant probability distributions functions to calculate type I and type II errors

Example:
mass ordering sensitivity
Blennow et al., 1311.1822
simple expressions in terms of error function



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Instead of type I and II errors one can also quote the median sensitivity and its spread (again two numbers)

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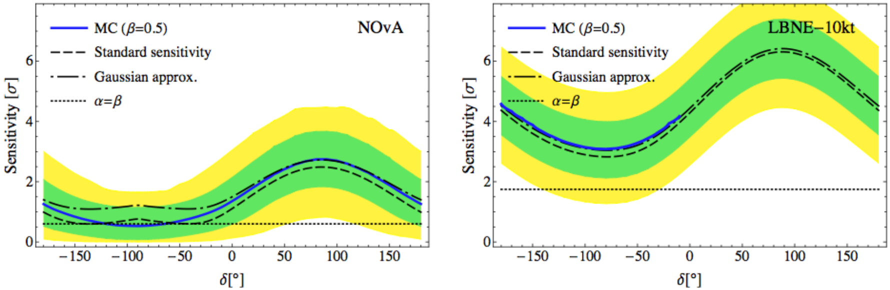


Median experiment

Coming back to the χ^2 using the predicted event rate as "data" (no statistical fluctuation):

$$\chi^2(\boldsymbol{\theta};\boldsymbol{\theta}^{tr}) = \sum_{i=1}^{N} \frac{[\mu_i(\boldsymbol{\theta}) - \mu_i(\boldsymbol{\theta}^{tr})]^2}{\mu_i(\boldsymbol{\theta}^{tr})}$$

$n_i = \mu_i(\boldsymbol{\theta}^{tr})$ can be considered as "most probable outcome" or the result of the "median experiment"

- interpret sensitivities based on the above χ^2 as median sensitivity, i.e., type II error rate of 50%.
  holds only approximately, in general needs to be checked by MC
  Schwetz, hep-ph/0612223, Blennow et al., 1311.1822

Median experiment

Coming back to the χ^2 using the predicted event rate as "data" (no statistical fluctuation):

$$\chi^2(\boldsymbol{\theta}; \boldsymbol{\theta}^{tr}) = \sum_{i=1}^{N} \frac{[\mu_i(\boldsymbol{\theta}) - \mu_i(\boldsymbol{\theta}^{tr})]^2}{\mu_i(\boldsymbol{\theta}^{tr})}$$

$n_i = \mu_i(\boldsymbol{\theta}^{tr})$ can be considered as "most probable outcome" or the result of the "median experiment"

- this is by far the most common method in the literature to calculate sensitivities of neutrino oscillation epxeriments

GLoBES software is designed primarily for this purpose
Huber, Lindner, Winter, hep-ph/0407333; Huber et al., hep-ph/0701187
http://www.mpi-hd.mpg.de/lin/globes/

Sensitivity calculations

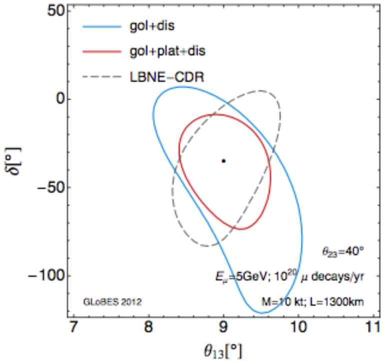
"doubling" of the parameter space: $\chi^2(\boldsymbol{\theta}; \boldsymbol{\theta}^{tr})$

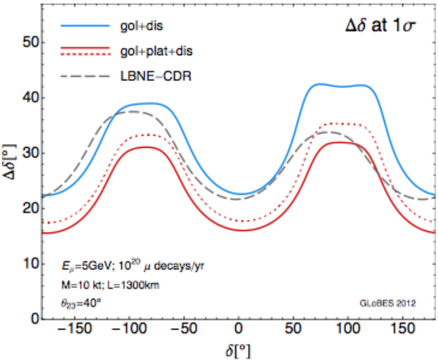
for each choice of $\boldsymbol{\theta}^{tr}$ one has to perform a fit to the "data", similar as one would do in case of real data

sensitivity depends on the assumed $\boldsymbol{\theta}^{tr} \to$ have to scan $\boldsymbol{\theta} \times \boldsymbol{\theta}^{tr}$ space

Example: CP phase δ

Christensen, Coloma, Huber, 1301.7727





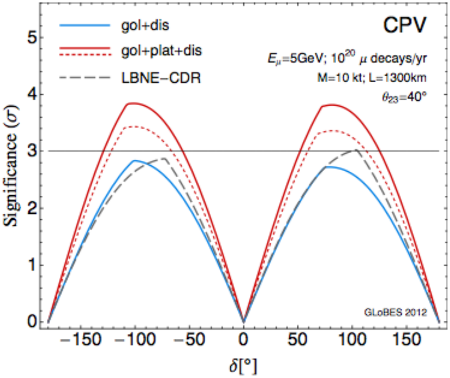
Example: CP violation

define

$$\chi^2_{\rm CP} = \min\left[\chi^2(\delta_{\rm CP} = 0; \boldsymbol{\theta}^{tr}), \chi^2(\delta_{\rm CP} = \pi; \boldsymbol{\theta}^{tr})\right]$$

(minimize wrt to all other parameters except $\delta_{\rm CP}$, incl. degeneracies)

Christensen, Coloma, Huber, 1301.7727



$\sqrt{\chi^2_{\rm CP}}$ corresponds to the number of σ with which CP-conserving values of δ can be excluded (1 d.o.f.)