Beautiful CCWZ formulation with "induced representation" of the chiral symmetry — $\xi \to \xi' = L\xi u(L,R,\xi)^\dagger = u(L,R,\xi)\xi R^\dagger$

Anomalies — two problems with the fancy chiral EFT for low energy hadron physics

$\pi^0 \to \gamma\gamma$ decay

η' and the chiral $U(1)$ symmetry

the η' problem is easier to state but at least historically it took much longer to understand, so I will state it briefly, then discuss $\pi^0 \to \gamma\gamma$, then go back to η' (again briefly)

η' and the chiral $U(1)$ symmetry

We have been assuming that the non-Abelian flavor symmetry of low energy hadrom physics is $SU(3) \times SU(3)$ spontaneously broken down to Gell-Mann's $SU(3)$. There is also the $U(1)$ baryon number which is unbroken (at least by low energy physics), but it doesn't act on the mesons so we usually don't even talk about it.

But classically, 3-flavor QCD with massless quarks doesn't looks quite different —

$$\mathcal{L} = i\bar{q}\, \not{D}\, q - \frac{1}{4} G_a^{\mu\nu} G_{a\mu\nu} = i\bar{q}_L\, \not{D}\, q_L + i\bar{q}_R\, \not{D}\, q_R - \frac{1}{4} G_a^{\mu\nu} G_{a\mu\nu}$$

It falls apart into completely separate L and R pieces and therefore has a chiral $U(3) \times U(3)$ symmetry, which contains separate "chiral baryon numbers" for the left handed and right-handed quarks.

Why didn't we include the chiral $U(1)$?

DESPERATION — THE THEORY DOESN'T WORK RIGHT IF WE DO!

If it existed, the chiral $U(1)$ would have to be spontaneously broken, because we don't see chiral pairs of baryons. But then there would be another Goldstone boson and the mass relation would not work. It is actually much worse than that recall that without the chiral $U(1)$ we got good GB masses

$$\mathcal{L} = f^2 \left(\frac{1}{4} \mathrm{tr}\left(D^\mu U^\dagger D_\mu U \right) + \mathrm{tr}\left(U \mu \mathcal{M} + \text{h.c.} \right) + \cdots \right)$$

$$\Pi = \pi_a T_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix} \qquad \begin{array}{c} K^- = K^{+\dagger} \\ \overline{K}^0 = K^{0\dagger} \end{array}$$

$U = \exp(2i\Pi/f)$ and $D^\mu U = \partial^\mu U + i\ell^\mu U - iUr^\mu$ (gives the currents) and

symmetry breaking by $\mathcal{M} = M$

$\bar{q}_L M q_R + \text{h.c.}$ gives “GB” masses

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

for μ and M real, the linear term cancels and the quadratic term is

$$-2\mathrm{tr}(\mu M \Pi^2),$$

which corresponds to a mass term (like baryons but only one term)

$$4\mathrm{tr}(\mu M \Pi^2)$$

for the Π — evaluate these masses in the limit of isospin invariance, ignoring weak and electromagnetic interactions and setting $m_u = m_d = m$

$$m_\pi^2 = 4\mathrm{tr}\left(\mu M \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = 2\mu m$$

$$m_K^2 = 4\mathrm{tr}\left(\mu M \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}\right) = \mu(m + m_s)$$

$$m_\eta^2 = 4\mathrm{tr}\left(\mu M \begin{pmatrix} \frac{1}{12} & 0 & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}\right) = \frac{2\mu}{3}(m + 2m_s)$$

The m^2 determined in this way satisfy the Gell-Mann Okubo relation

$$3m_\eta^2 + m_\pi^2 = 4m_K^2$$

but here we are using the momentum expansion rather than expanding in powers of the symmetry breaking — this depends on the $SU(3)$ symmetric part of M and makes sense even for $m_\pi \to 0$ — and explains why GMO for mesons works much better for m^2 than for m

notice that M gives a potential $-\frac{f^2}{2}\mathrm{tr}\left(U^\dagger \mu M\right) - \frac{f^2}{2}\mathrm{tr}\left(U \mu M\right)$ that is minimized for $U = I$ — until we turned on M all U were equally good vacua

This would look completely different with a chiral $U(1)$ - we could write the same terms

$$\mathcal{L} = f^2 \left(\frac{1}{4} \mathrm{tr}\left(D^\mu U^\dagger D_\mu U \right) + \mathrm{tr}\left(U \mu \mathcal{M} + \text{h.c.} \right) + \cdots \right)$$

but Π is not traceless - because the trace is GB of the $U(1)$

$$\Pi = \pi_a T_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi_{u\bar{u}}/\sqrt{2} & \pi^+ & K^+ \\ \pi^- & \pi_{d\bar{d}}/\sqrt{2} & K^0 \\ K^- & \overline{K}^0 & \pi_{s\bar{s}}/\sqrt{2} \end{pmatrix} \qquad \begin{array}{c} K^- = K^{+\dagger} \\ \overline{K}^0 = K^{0\dagger} \end{array}$$

symmetry breaking by $\mathcal{M} = M$

$\bar{q}_L M q_R + \text{h.c.}$ gives "GB" masses

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

now the mass term $4\mathrm{tr}(\mu M \Pi^2)$ gives $m_{q\bar{q}'} \propto m_q + m_{q'}$ — totally unlike what we see - even isospin doesn't work

Another term possible, $\mathrm{tr}\left(D^\mu U^\dagger \right) \mathrm{tr}\left(D_\mu U \right)$ — but no hope because the symmetry is wrong — as m_u and $m_d \to 0$, $U(2)$ remains unbroken so 4 GBs .

This is the chiral $U(1)$ problem!

$\pi^0 \to \gamma\gamma$ decay — another EM effect — could be produced by

$$D^\mu U = \partial^\mu U + ieA^\mu[Q, U].$$

but this confused people for many years — this simple substitution and related quantum effects are not very efficient in producing the decay

angular momentum and parity $\Rightarrow \gamma\gamma$ in $l = 1,\ j = 0$ state associated with the pseudoscalar operator

$$\epsilon_{\mu\nu\lambda\sigma}\, F^{\mu\nu}\, F^{\lambda\sigma}.$$

an effective interaction of the following form would produce the decay

$$\pi^0 \epsilon_{\mu\nu\lambda\sigma}\, F^{\mu\nu}\, F^{\lambda\sigma}$$

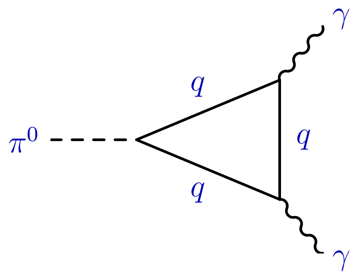
but the quark charge matrix of QCD commutes with T_3 because both are diagonal — Q doesn't break the chiral T_3 symmetry — so (the old slightly wrong argument called the Sutherland theorem goes) all interactions produced by Q must have the chiral symmetry and so in our beautiful GB formalism, they must be derivative interaction from Lagrangian terms invariant under the chiral symmetry under which the π^0 field transforms inhomogeneously we could write invariant interaction terms with derivatives, such as

$$i\epsilon_{\mu\nu\lambda\sigma}\, F^{\mu\nu}\, F^{\lambda\sigma}\, \mathrm{tr}\left(Q^2 U^\dagger \partial^\alpha \partial_\alpha U\right) + \mathrm{h.c.}$$

or terms involving the explicit chiral symmetry breaking, such as

$$i\epsilon_{\mu\nu\lambda\sigma}\, F^{\mu\nu}\, F^{\lambda\sigma}\, \mathrm{tr}\left(Q^2 \mu M U\right) + \mathrm{h.c.}$$

but the contribution of these terms to the π^0 decay amplitude is suppressed by a power of m_π^2 over the dimensional constant that governs higher order terms in the derivative expansion — which is expected to be of the order of 1 GeV — these terms are just too small to explain the observed decay this is particularly embarrassing because it seems to contradict a trivial calculation that gives the right answer — take our perturbative model of an octet of GBs — with the 3×3 Φ field, and couple it to three colors of quarks with an $SU(3) \times SU(3)$ invariant interaction $g(\bar{q}_L \Phi q_R + \text{h.c.})$ which after symmetry breaking gives $m_q = gf$ — in the effective theory below the mass of the quarks there a coupling between the GB π^0 and $\gamma\gamma$ generated by matching from the quark triangle diagram — this is the Steinberger calculation we started with!



we did this in the first lecture (with different charges) and the result here is

$$\frac{\alpha}{8\pi f}\pi^0\,\epsilon_{\mu\nu\alpha\beta}\,F^{\mu\nu}\,F^{\alpha\beta}$$

notice that the coupling to the quarks has gone away because the amplitude had a g in the numerator from the $ig\pi^0\lambda_3\bar{q}\gamma_5 q$ coupling and $m_q = gf$ in the denominator — this suggests that there is something general going on

What is going on here? is there something wrong with our fancy treatment of GBs with only derivative interactions?

In fact, the problem is not with the fancy treatment itself — the problem is that our fancy treatment not quite equivalent to the perturbative version where the fields transform linearly. We are missing something.

3×3 matrix Φ $\quad \Phi \to L\Phi R^\dagger \quad L = \exp\left(i l^a T_a^L\right)$

$SU(3) \times SU(3)$ symmetry $\quad q_L \to L q_L \quad q_R \to R q_R \quad R = \exp\left(i r^a T_a^R\right)$

$$\mathcal{L} = \mathrm{tr}\left(D_\mu \Phi^\dagger D^\mu \Phi\right) - \frac{\lambda_1}{2}\left(\mathrm{tr}(\Phi\Phi^\dagger) - \tfrac{3f^2}{4}\right)^2 - \frac{\lambda_2}{2}\mathrm{tr}\left(\left(\Phi\Phi^\dagger - \tfrac{f^2}{4}\right)^2\right)$$

$$-\frac{\lambda_3}{2}\mathrm{tr}\left(\mathrm{cof}\,\Phi - \tfrac{f}{2}\Phi^*\right)\left(\mathrm{cof}\,\Phi^\dagger - \tfrac{f}{2}\Phi^T\right) + i\bar{q}\,\not{D}\,q - g(\bar{q}_L \Phi q_R + \bar{q}_R \Phi^\dagger q_L)$$

$\langle\Phi\rangle = fI/2$ breaks $SU(3) \times SU(3)$ $\quad \xi \to L\xi u(L,R,\xi)^\dagger$

$\to SU(3) \to$ Goldstone bosons $\quad = u(L,R,\xi)\xi R^\dagger$

$\Phi = \xi\, e^{i\phi/f}\,(h + fI/2)\,\xi$ where $h = h^\dagger$ $\quad \mathcal{Q}_L = \xi^\dagger q_L \quad \mathcal{Q}_R = \xi q_R$

ξ is a chiral transformation of vacuum $\Rightarrow$ GBs in ξ have only $\partial_\mu \Pi$ interactions

$\left(\mathrm{tr}(\Phi\Phi^\dagger) - \frac{3f^2}{4}\right)^2$	$\mathrm{tr}\left(\left(\Phi\Phi^\dagger - \frac{f^2}{4}\right)^2\right)$	$\mathrm{tr}\left(\mathrm{cof}\,\Phi - \frac{f}{2}\Phi^*\right)\left(\mathrm{cof}\,\Phi^\dagger - \frac{f}{2}\Phi^T\right)$
mass to $\mathrm{tr}h$ $SO(18)$	mass to h chiral $U(1)$	mass to ϕ, h just $SU(3) \times SU(3)$

at low energies only GBs survive $\quad \mathcal{L} = \frac{f^2}{4}\mathrm{tr}\left(D_\mu U^\dagger D^\mu U\right)$ where $U = \xi^2$

None of this matters except the red part! Space-time dependent chiral transformation (ξ depends on x) generates derivative couplings.

So we have done a space-time-dependent chiral gauge transformation to get from the theory with γ_5 couplings of Π to derivative couplings — but in the process we have lost the $\pi F\tilde{F}$ coupling in the low energy theory.

$g\bar{q}\gamma_5 T_3 q\,\pi$ coupling to fermions with mass from SSB	triangle graph gives $\pi F\tilde{F}$ in low energy theory

space-time-dependent chiral transformation $\rightarrow$

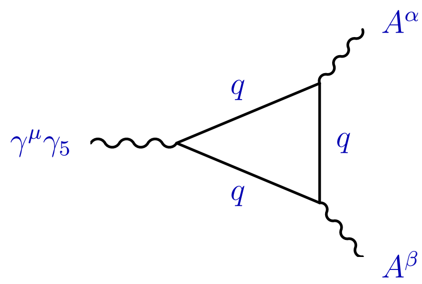
$g\bar{\mathcal{Q}}\gamma^\mu\gamma_5 T_3\mathcal{Q}\,\partial_\mu\pi/f$ coupling to massive $\mathcal{Q}$	triangle graph **does not give** $\pi F\tilde{F}$ in low energy theory

It must be that what is going on is that the chiral transformation itself in the presences of the fermion fields generates the $\pi F\tilde{F}$ terms that was previously generated by the triangle diagram in the theory with $\gamma_5\pi$ couplings

But that means that the space-time-dependent chiral tranformation produces an addtional change in the Lagrangian besides the derivative interactions that we expect from the classical Lagrangian. This in turn means that there is an extra quantum correction to the divergence of the corresponding chiral current.

This is the anomaly!

historically, this showed up in the ABJ triangle anomaly



calculate straightforwardly but use Lorentz invariance to pull out a piece proportional to several factors of momentum so the loop integration is finite

contract with p_μ to look at divergence $\partial_\mu j_5^\mu$ — you would expect to get the triangle graph for $2m\bar{q}\gamma_5 q$

but instead you find an extra term $\propto F\tilde{F}$ — independent of the mass — anomalous divergence — even for massless fermions

What really matters is the fermion transformation law.

chiral fermions
transforming linearly
$q_L \to L q_L, q_R \to R q_R$

$$i\bar{q}\ \not\partial\, q$$

space-time-dependent chiral transformation $\to$

fermions in SSB vacuum
transforming nonlinearly
$\mathcal{Q} \to u(L, R, \xi)\mathcal{Q}$

$$i\bar{\mathcal{Q}}\ \not\partial\, \mathcal{Q}$$
$$+\pi F \tilde{F}$$

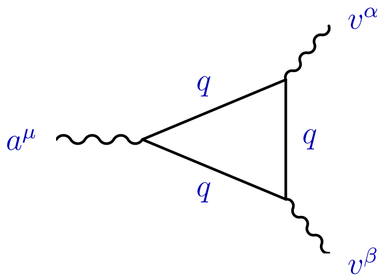
Fujikawa and the fermion measure

In the general situation in which we include all possible $SU(3) \times SU(3)$ classical gauge fields, to get from QCD in which chiral transformations act on the quark fields to the beautiful CCWZ low energy theory in which all the chiral transformation properties are carried by the GB fields and the matter fields transform only under the unbroken symmetry we have to add the extra term that depends on the gauge fields and the GB fields and is produced by the space-time dependent chiral transformation that gets us the CCWZ form for the matter fields.

various ways of seeing what this extra term is for general sources v^μ and a^μ,

$$v^\mu = (\ell^\mu + r^\mu)/2 \qquad a^\mu = (\ell^\mu - r^\mu)/2$$

as usual just using the classical sources as tools to calculate n-point functions — call the term $W(U, v^\mu, a^\mu)$ — the easiest way to find W use the triangle anomaly to find the change in W under an infinitesimal chiral transformation $\xi = e^{ic} = 1 + ic + \cdots$



Wess and Zumino determined its form from the $SU(3) \times SU(3)$ symmetry and Bardeen calculated it directly by looking at all the relevant loop graphs — both imposing conservation of the vector currents — the result for the change looks like this:

$$dW = F(c,\, v^\mu,\, a^\mu) = \int f(c(x),\, v^\mu(x),\, a^\mu(x))\, d^4x \quad \text{where}$$

$$\begin{aligned} f = -\frac{1}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}\Big(& c(3v^{\mu\nu}v^{\alpha\beta} + a^{\mu\nu}a^{\alpha\beta} \\ & -8i(a^\mu a^\nu v^{\alpha\beta} + a^\mu v^{\nu\alpha}a^\beta + v^{\mu\nu}a^\alpha a^\beta) \\ & -32a^\mu a^\nu a^\alpha a^\beta)\Big) \end{aligned}$$

with

$$v^\mu = (\ell^\mu + r^\mu)/2 \quad a^\mu = (\ell^\mu - r^\mu)/2$$

$$\begin{aligned} v^{\mu\nu} &= \partial^\mu v^\nu - \partial^\nu v^\mu + i[v^\mu,\, v^\nu] + i[a^\mu,\, a^\nu] \\ a^{\mu\nu} &= \partial^\mu a^\nu - \partial^\nu a^\mu + i[v^\mu,\, a^\nu] + i[a^\mu\, v^\nu] \end{aligned}$$

there are other forms like the so-called Fujikawa form that do not satisfy ordinary $SU(3)$ gauge invariance, but they can be obtained from this with the addition of polynomials in v^μ and a^μ to the action — these ambiguities are UV effects associated with the different schemes for defining the currents as composite operators like s^2 counterterm that one needs in a scalar theory for the source s of the composite operator ϕ^2 — the anomaly itself is an IR effect and is unambigous — you can move its effects around with different UV counterterms, but you can't get rid of it now we can put the GB fields — define

$$
\begin{gathered}
U(s) = e^{2is\Pi/f} \qquad \xi(s) = e^{is\Pi/f} \\
\ell^\mu(s) = \xi^\dagger(1-s)\ell^\mu\xi(1-s) - i\xi^\dagger(1-s)\partial^\mu\xi(1-s) \\
r^\mu(s) = \xi(1-s)r^\mu\xi^\dagger(1-s) - i\xi(1-s)\partial^\mu\xi^\dagger(1-s) \\
v^\mu(s) = \Big(\ell^\mu(s) + r^\mu(s)\Big)/2 \quad a^\mu(s) = \Big(\ell^\mu(s) - r^\mu(s)\Big)/2.
\end{gathered}
$$

different values of s are related by chiral gauge transformations, which depend on the Goldstone boson field Π — in particular

$$
\begin{aligned}
&\frac{d}{ds} W\Big(U(s),\ v^\mu(s),\ a^\mu(s)\Big)\ ds \\
&= \delta_{ds\Pi/f} W = ds\ F\Big(\Pi/f,\ v^\mu(s),\ a^\mu(s)\Big).
\end{aligned}
$$

we can integrate this!

$$
\begin{aligned}
W\Big(U,\ v^\mu,\ a^\mu\Big) &= W\Big(U(1),\ v^\mu(1),\ a^\mu(1)\Big) \\
&= \int_0^1 F\Big(\Pi/f,\ v^\mu(s),\ a^\mu(s)\Big)\, ds - W\Big(U(0),\ v^\mu(0), a^\mu(0)\Big)
\end{aligned}
$$

$$W\Big(U,\ v^\mu,\ a^\mu\Big) = W\Big(U(1),\ v^\mu(1),\ a^\mu(1)\Big)$$
$$= \int_0^1 F\Big(\Pi/f,\ v^\mu(s),\ a^\mu(s)\Big)\,ds - W\Big(U(0),\ v^\mu(0), a^\mu(0)\Big)$$

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$$r^\mu(s) = \xi(1-s)r^\mu\xi^\dagger(1-s) - i\xi(1-s)\partial^\mu\xi^\dagger(1-s)$$
$$v^\mu(s) = \Big(\ell^\mu(s) + r^\mu(s)\Big)/2 \quad a^\mu(s) = \Big(\ell^\mu(s) - r^\mu(s)\Big)/2.$$

$$U(0) = 1 \qquad \ell^\mu(0) = \xi^\dagger\ell^\mu\xi - i\xi^\dagger\partial^\mu\xi \qquad r^\mu(0) = \xi r^\mu\xi^\dagger - i\xi\partial^\mu\xi^\dagger$$

$v^\mu(0)$ and $a^m(0)$ are the V^μ and A^μ we discussed earlier which transform only under the u $SU(3)$

$$V^\mu = -\frac{i}{2}\left(\xi^\dagger(\partial^\mu + i\ell^\mu)\xi + \xi(\partial^\mu + ir^\mu)\xi^\dagger\right) = v^\mu(0)$$

$$V^\mu \to u(L,R,\xi)V^\mu u(L,R,\xi)^\dagger - i\,u(L,R,\xi)\partial^\mu u(L,R,\xi)^\dagger$$

$$A^\mu = -\frac{i}{2}\left(\xi^\dagger(\partial^\mu + i\ell^\mu)\xi - \xi(\partial^\mu + ir^\mu)\xi^\dagger\right) = a^\mu(0)$$

$$A^\mu \to u(L,R,\xi)A^\mu u(L,R,\xi)^\dagger$$

$W(1,\ v^\mu(0),\ a^\mu(0))$ is invariant under ordinary $SU(3)$ gauge transformations!

thus $W(1,\ \ell^\mu(0),\ v^\mu(0))$ can be absorbed into the ordinary terms in the action that are invariant under $SU(3)_L \times SU(3)_R$ and we can write the extra term that incorporates the effect of the anomaly as

$$W\Big(U,\ v^\mu,\ a^\mu\Big) = \int_0^1 F\Big(\Pi/f,\ v^\mu(s),\ a^\mu(s)\Big)\, ds$$

$$U(s) = e^{2is\Pi/f} \qquad \xi(s) = e^{is\Pi/f}$$
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$$v^\mu(s) = \Big(\ell^\mu(s) + r^\mu(s)\Big)/2 \quad a^\mu(s) = \Big(\ell^\mu(s) - r^\mu(s)\Big)/2$$

$$F(c,\ v^\mu,\ a^\mu) = \int f(c(x),\ v^\mu(x),\ a^\mu(x))\, d^4x$$

$$\text{where} \quad f = -\frac{1}{16\pi^2}\epsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}\Big(c\,(3v^{\mu\nu}v^{\alpha\beta} + a^{\mu\nu}a^{\alpha\beta}$$

$$-8i(a^\mu a^\nu v^{\alpha\beta} + a^\mu v^{\nu\alpha}a^\beta + v^{\mu\nu}a^\alpha a^\beta) - 32a^\mu a^\nu a^\alpha a^\beta)\Big)$$

$\Pi F \tilde{F} \Pi^3 \tilde{F} \epsilon \Pi^5$

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$$\Pi^3\tilde{F}$$

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$$\epsilon\Pi^5$$

Fun example of massless particles in 1+1 dimensions — much simpler than 3+1

$g^{00} = -g^{11} = 1,\ \epsilon^{01} = -\epsilon^{10} = -\epsilon_{01} = \epsilon_{10} = 1.$

From the defining properties $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $\gamma^5 = -\frac{1}{2}\epsilon_{\mu\nu}\gamma^\mu\gamma^\nu$, it follows that $\gamma^\mu\gamma^5 = -\epsilon^{\mu\nu}\gamma_\nu$ and $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \epsilon^{\mu\nu}\gamma^5$.

We will use the representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the components ψ_1 and ψ_2 describe a right-moving and left-moving fermion, respectively.
anomaly for free massless fermions in 1+1 dimensions with only one flavor ????

source s^μ $\quad \bar{\psi}(i\,\partial\!\!\!/ - s\!\!\!/)\psi \qquad j_5^\mu = \epsilon^{\mu\nu} j_\nu \qquad j_5^0 = -j^1 \qquad j_5^1 = -j^0$

clasically $\quad \partial_\mu j^\mu = \partial_\mu j_5^\mu = 0$

Exercise 7. Consider the "massless scalar 2D propagator"

$$D(t,x) = -\frac{i}{4\pi} \ln\left(-t^2 + x^2 + i\epsilon\right) + \text{constant}$$

The quotation marks are because massless scalars "don't exist" in 2D, and this is related to the divergent constant. But never mind all that. Find

$$k(t,x) = \left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}\right) D(t,x)$$

This should be a 2D δ-function, $\delta(t)\,\delta(x)$. But this should appear somewhat puzzling, because you found (I hope) that $k(t,x)$ depends only on $t^2 - x^2$! Explain what is happening here! It may help to graph $k(t,x)$ for non-zero ϵ. I think that this is a nice exercise because it will remind you of what complicated things the objects we are always manipulating in QFT really are.

Exercise 8. Consider a theory of free massless fermions in 1+1 dimensions. Unlike the massless scalars of the previous exercise, these things do exist. Calculate the 01 component of the 2-point function of a product of two currents using momentum space Feynman diagrams. The result should be finite and unambiguous. Now complete the calculation of the full tensor structure in two ways:

1. assuming that the vector current is conserved;
2. assuming that the calculation can be done independently for the LH and RH fermions.

These are the 2D analogs of the Bardeen form and the Fujikawa form of the anomaly in 4D.

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Anomalies of the Axial-Vector Current in Two Dimensions*

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In a world with one space and one time dimension, it is shown that the axial-vector current in a vector-gluon model exhibits anomalies in perturbation theory analogous to those found by Adler in fermion electrodynamics. The analysis is extended to include a four-fermion (Thirring) interaction; this model has been solved exactly by Sommerfield, permitting an explicit verification of the perturbation-theory calculations. In analogy to results in four dimensions, a model is presented in which the anomalous properties of the axial-vector current, both in its divergence and in its commutation relations, follow immediately from the canonical structure of the theory.

I. INTRODUCTION

It has been shown by Adler[1] that in massless spinor electrodynamics the axial-vector current is not conserved, in contradiction with the prediction of Noether's theorem. Perturbation-theory arguments[1–4] indicate that the correct divergence condition reads

$$\partial^\mu j^5{}_\mu=\partial^\mu(\bar{\psi}\gamma_\mu\gamma_5\psi)=\frac{\alpha}{2\pi}\bar{F}^{\mu\nu}F_{\mu\nu}\,, \tag{1.1}$$

where $F_{\mu\nu}$ and $\bar{F}^{\mu\nu}$ denote the electromagnetic field strength tensor and its dual, respectively; i.e.,

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,,\quad \bar{F}^{\mu\nu}=\tfrac{1}{2}\epsilon^{\mu\nu\sigma\rho}F_{\sigma\rho}\,.$$

this case, the offending diagram (the "bubble" diagram) is only logarithmically divergent and hence can be evaluated unambiguously. An important distinction between the two-dimensional and four-dimensional cases is that the former can be solved exactly (this fact motivated our restriction to massless fermions). In fact, the model with an additional current-current (Thirring[6]) interaction has been solved exactly by Sommerfield.[7] We extend our perturbation-theory analysis to include this case and explicitly verify our conclusions via Ref. 7.

In the four-dimensional case, one can construct a model of a pseudoscalar meson interacting nonminimally with the electromagnetic field in which the divergence condition (1.1) and the anomalous commuta-

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I would be very grateful if you could send me a copy of your paper:

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which appeared in

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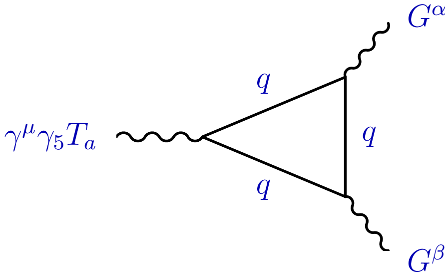
The anomaly and the $U(1)$ problem — after this long song and dance about $\pi^0 \to \gamma\gamma$, we can give a simple (though naive) answer to the question of why we left out the chiral $U(1)$ which is good enough for me. The gluons of QCD are gauge particles as the photons are, so there are anomalies that depend on the gluon fields.



$SU(3)$ axial currents with $\mathrm{tr}T_a$=0 have a gluon anomaly, but $T_0 \propto I$ does, and that means that a $U(1)$ chiral tranformation gives an extra term $\propto G\tilde{G}$ — Noether's theorem would then suggest that a global $U(1)$ chiral transformation, in addition to changing the phase of your changes the coefficient of a dimension 4 coupling in the Lagrangian, the θ parameter.

$$\theta \frac{\alpha_s}{8\pi} G^a \tilde{G}^a$$

$$\theta \frac{\alpha_s}{8\pi} G^a \tilde{G}^a$$

Very naively, one might expect θ to be irrelevant because $G\tilde{G}$ is a total divergence (albeit of a gauge variant current). This means that any affects do not show up in perturbation theory. But nonperturbatively, we believe that the theory is confining and so it is reasonable (if still naive) to say simply that this means that the gauge fields in the non-perturbative vacuum do not fall of at infinity and therefore the θ parameter is not irrelevant and the chiral $U(1)$ symmetry is not a symmetry and the corresponding Goldstone boson should not be included in the low energy theory. This is all, as I say, very naive, but it seems to work, and is consistent with various fancier (though not obviously more convincing) arguments — like instantons, lattice calculations, AdS-QCD, etc. As I say this is good enough for me. Of course, this solution leaves us with θ as an extra parameter in the low-energy hadron theory that is puzzlingly small. This is the strong CP puzzle. But like fine tuning, this is a puzzle - not a problem.

Two things that I probably won't have time to talk about.

HQET — sectors, superselection rules and symmetries

Like the heavy particle effective theory we discussed for nuclei, but the energy scale is different. Here we are interested in the properties of heavy quark currents in and effective theory at a scale large compared to the QCD scale but small compared to the masses of the heavy quarks. This is a beautiful example of a heavy fermion theory and of the emergence of symmetry in a low energy theory. What is thrown away is the antiquarks and the QCD interactions that could change the quark's direction.

It is not a surprise that at energies small compared to the mass of a heavy quark we get an $SU(2)$ spin symmetry. It is more surprising that if we have two heavy quarks with very different masses, like the c and the b, we get an $SU(4)$ symmetry that comprisess complex rotations among the 4 spin and flavor states of the two quarks and we get a separate $SU(4)$ symmetry of this kind for each possible quark velocity.

Simple and useful, but the connection with Lorentz invariance ("reparameterization invariance") is a bit confusing.

SCET — and effective field theory (maybe) of jets — I wanted to advertise a couple of recent papers byt Ilya Feige and Matt Schwartz that I think represent real progress in understanding this. I can't do much more than advertise them, because I am far from understanding the subject myself, but this a really interesting.