symmetries, currents and effective field theory —

My favorite examples of effective field theories beyond perturbation theory involve QCD. This is because I fell in love with the bizarre zoo of low energy particle physics 50 years ago and I have grown up with this physics. If I were learning it today, it would be much easier, because I know about QCD and effective field theory. So I am going to give some "practical" examples of applications to QCD that illustrate important features of EFT.

Isospin and low energy β-decay of nuclei.

low energy chiral dynamics of mesons and baryons

heavy quark effective theory ?

soft-colinear-effective-theory ???

current-current weak interactions from W exchange

$$\frac{G_F}{\sqrt{2}}\, j^\mu\, j^\dagger_\mu$$

j^μ is the charged, left-handed "weak iso-spin" current — more tensor products — charge and family #

$$\ell = \begin{pmatrix} \ell_{11} \\ \ell_{21} \\ \ell_{12} \\ \ell_{22} \\ \ell_{13} \\ \ell_{23} \end{pmatrix} = \begin{pmatrix} \nu_e \\ e^- \\ \nu_\mu \\ \mu^- \\ \nu_\tau \\ \tau^- \end{pmatrix} \qquad q = \begin{pmatrix} q_{11} \\ q_{21} \\ q_{12} \\ q_{22} \\ q_{13} \\ q_{23} \end{pmatrix} = \begin{pmatrix} u \\ d \\ c \\ s \\ t \\ b \end{pmatrix}$$

then the current is $j^\mu = j_1^\mu + i\, j_2^\mu$ — electroweak "isospin"

$$j_a^\mu = \underbrace{\bar{\ell}\, Q_a\, \gamma^\mu\, (1+\gamma_5)\ell}_{\text{leptonic } j^\mu_{\ell a}} + \underbrace{\bar{q}\, \mathcal{V}^\dagger\, Q_a\, \gamma^\mu\, (1+\gamma_5)\, \mathcal{V}\, q}_{\text{hadronic } j^\mu_{ha}}$$

$Q_a = \sigma_a/2$ weak isospin generators

$$\mathcal{V} = \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix} \qquad V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = (V^\dagger)^{-1}$$

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$$j_\ell^\mu = \bar{\nu}_e \gamma^\mu (1+\gamma_5) e^- + \bar{\nu}_\mu \gamma^\mu (1+\gamma_5) \mu^- + \bar{\nu}_\tau \gamma^\mu (1+\gamma_5) \tau^-$$

$$j_h^\mu = (\,\bar{u}, \bar{c}, \bar{t}\,)\, \gamma^\mu (1+\gamma_5) V \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

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$$j_h^\mu = (\,\bar u, \bar c, \bar t\,)\,\gamma^\mu(1+\gamma_5)V \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

$$\frac{G_F}{\sqrt{2}}\, j^\mu\, j_\mu^\dagger = \frac{G_F}{\sqrt{2}}\,(j_\ell^\mu\, j_{\ell\mu}^\dagger + j_\ell^\mu\, j_{h\mu}^\dagger + j_h^\mu\, j_{\ell\mu}^\dagger + j_h^\mu\, j_{h\mu}^\dagger)$$

$j^\mu\, j_\mu^\dagger$ — "leptonic weak interactions" — perturbation theory works great

$j_h^\mu\, j_{h\mu}^\dagger$ — "hadronic weak interactions" — perturbation theory doesn't work at all for light quarks and is complicated even for heavy quarks

$j_\ell^\mu\, j_{h\mu}^\dagger + j_h^\mu\, j_{\ell\mu}^\dagger$ — "semi-leptonic weak interactions" — perturbation theory is useful — to lowest order in electroweak ints, amplitudes factor into a leptonic part which is calculable times the matrix element of a hadronic current

$$\mathcal{M} = \frac{G_F}{\sqrt{2}}\ \langle \ell\ \mathrm{out}|\mathrm{j}_\ell^\mu|\ell\ \mathrm{in}\rangle\,\langle h\ \mathrm{out}|\mathrm{j}_{\mathrm{h}\mu}^\dagger|\mathrm{h\ in}\rangle \quad \text{or} \quad \langle \ell\ \mathrm{out}|\mathrm{j}_{\ell\mu}^\dagger|\ell\ \mathrm{in}\rangle\,\langle h\ \mathrm{out}|\mathrm{j}_{\mathrm{h}}^\mu|\mathrm{h\ in}\rangle$$

thus we are interested in matrix elements of hadronic currents — for the light quarks u, d and s, these are highly constrained by symmety

$$\mathcal{L} = (i\bar{q}\,\not{D}\;q - \bar{q}\,M_q\,q) - \frac{1}{4}G_a^{\mu\nu}\,G_{a\mu\nu} - \bar{q}\,\gamma^\mu\,(v_\mu + a_\mu\gamma_5)\;q$$

quark field q vector
in 3D color space
+ 6D flavor space

$D^\mu = \partial^\mu + igT_aG_a^\mu$
is "covariant
derivative"

$igT_aG_a^{\mu\nu} = [D^\mu,\ D^\nu]$
$G_a^{\mu\nu}$ is "gluon
field–strength"

T_a 3×3 traceless
Hermitian matrices
in color space

$$\mathrm{Tr}(T_aT_b) = \frac{1}{2}\delta_{ab}$$

M_q diagonal
in flavor space
u, d, c, s, t, b

color
gauge
symmetry

$T_aG_a^\mu \to UT_aG_a^\mu U^\dagger - \frac{i}{g}U\partial^\mu U^\dagger$

$q \to Uq \qquad G^{\mu\nu} \to UG^{\mu\nu}U^\dagger$

T_a's are "color" charges like EM charge in QED binds quarks and antiquarks into color–neutral combinations like photon exchange binds charged particles into electrically neutral atoms — color–neutral combinations are

$\bar{q}q$
mesons

and

$\epsilon_{jkl}q_jq_kq_l$
baryons

$v^\mu = v_\alpha^\mu t_\alpha$ and $a^\mu = a_\alpha^\mu t_\alpha$
sources for the "vector"
and "axial vector" currents

where t_α are 6×6
hermitian flavor
generators

simple first step — focus on the vector current — we will come back to the axial vector — and look at a world with only the u and d quarks — just isospin

$$\mathcal{L} = (i\bar{q}\,\not{\mathcal{D}}\;q - \bar{q}\,M_q\,q) - \frac{1}{4}G_a^{\mu\nu}\,G_{a\mu\nu} - \bar{q}\,\gamma_\mu\,t_\alpha v_\alpha^\mu\,q$$

quark field q	$\mathcal{D}^\mu = D^\mu + it_\alpha v_\alpha^\mu$	$igT_aG_a^{\mu\nu} + it_\alpha v_\alpha^{\mu\nu} = [\mathcal{D}^\mu,\,\mathcal{D}^\nu]$
3D color space	$= \partial^\mu + igT_aG_a^\mu$	$G_a^{\mu\nu}$ is "gluon field–strength"
2D flavor space	$+it_\alpha v_\alpha^\mu$	$v^{\mu\nu}$ is isospin field–strength

T_a 3×3 traceless	$\mathrm{Tr}(T_aT_b) = \delta_{ab}/2$	M_q diagonal
t_α 2×2 traceless		in flavor space
$= \sigma_\alpha/2$	$\mathrm{Tr}(t_\alpha t_\beta) = \delta_{\alpha\beta}/2$	u, d

v_α^μ is a classical gauge field! — classical flavor gauge symmetry in addition to the quantum color gauge symmetry

color gauge sym	$T_aG_a^\mu \to UT_aG_a^\mu U^\dagger - \frac{i}{g}U\partial^\mu U^\dagger$	$q \to Uq$	$G^{\mu\nu} \to UG^{\mu\nu}U^\dagger$
isospin	$t_\alpha v_\alpha^\mu \to \mathcal{U}t_\alpha v_\alpha^\mu\mathcal{U}^\dagger - \frac{i}{g}\mathcal{U}\partial^\mu\mathcal{U}^\dagger$	$q \to \mathcal{U}q$	$v^{\mu\nu} \to \mathcal{U}v^{\mu\nu}\mathcal{U}^\dagger$

broken by mass term if $m_u \neq m_d$

isospin at low energies — β-decay within a multiplet - say $({}^{34}\mathrm{Ar}\,,{}^{34}\mathrm{Cl}\,,{}^{34}\mathrm{S})$

the useful languages of group representations and heavy particle EFT

first consider a world in which $m_u = m_d$, and $\alpha = G_F = 0$ so isospin is unbroken and there are many stable particles

look at VERY low energies (say less that 10 MeV) at some isolated multiplet with isospin $\mathcal{I}$ of (say for simplicity) sinless nuclei — Φ — a $2\mathcal{I}+1$ component vector in isospin space — mass $M \gg 10\ MeV$

BUT - you say - why am I interested in Φ at all if I am stuck at low energies?

one answer is stability — somebody gave you a Φ and you can't get rid of it!

no relativity — preferred frame in which your Φ is at rest —

$$\mathcal{L} = \Phi^\dagger \left(i\mathcal{D}^0 + \frac{\mathcal{D}^j\mathcal{D}^j}{2M} + a\frac{t_\alpha[\mathcal{D}^j, v_\alpha^{0j}]}{\Lambda^2} + \frac{(\mathcal{D}^j\mathcal{D}^j)^2}{8M^3} + \cdots \right) \Phi$$

where

$\mathcal{D}^\mu = \partial^\mu + i\, t_\alpha^{\mathcal{I}} v_\alpha^\mu$ $\quad$ $t_\alpha^{\mathcal{I}}$ generate $2\mathcal{I}+1$ dim rep $\quad$ $[t_\alpha^{\mathcal{I}}, t_\beta^{\mathcal{I}}] = i\,\epsilon_{\alpha\beta\gamma} t_\gamma^{\mathcal{I}}$

convenient to label states and matrices by $[t_3^{\mathcal{I}}]$, raising and lowering operators, $\cdots$

$$\mathcal{L} = \Phi^\dagger \left(i\mathcal{D}^0 + \frac{\mathcal{D}^j\mathcal{D}^j}{2M} + a\frac{t_\alpha[\mathcal{D}^j, v_\alpha^{0j}]}{\Lambda^2} + \frac{(\mathcal{D}^j\mathcal{D}^j)^2}{8M^3} + \cdots \right) \Phi$$

motivation for this $\mathcal{L}$ — want to remove the time dependence , e^{-iMt} from the boring kinematical part of the Hamiltonian, the constant M, and focus on the interesting low-energy physics notice that Ms appear only as $1/M$ and there are no more $\mathcal{D}^0$s after the initial term — it is useful to see how this works in a free theory

$$\mathcal{L} = \phi^\dagger(-\partial^2 - M^2)\phi \quad \text{step one} \quad \phi(x) \to e^{-iMt}\tilde{\Phi}(x)/\sqrt{2M}$$

$$\to \tilde{\Phi}^\dagger \left(-\frac{(\partial^0 - iM)^2 + \vec{\partial}\cdot\vec{\partial} - M^2}{2M} \right) \tilde{\Phi} = \tilde{\Phi}^\dagger \left(i\partial^0 + \frac{\vec{\partial}\cdot\vec{\partial}}{2M} - \frac{\partial^0\partial^0}{2M} \right) \tilde{\Phi}$$

at this point, we have actually not changed the physics — the equation of motion for $\tilde{\Phi}$ is still quadratic in ∂^0 — and therefore has two solutions

$$\left(i\partial^0 + \frac{\vec{\partial}\cdot\vec{\partial}}{2M} - \frac{\partial^0\partial^0}{2M} \right) \tilde{\Phi} = 0 \quad \text{or} \quad E + \frac{E^2}{2M} - \frac{\vec{p}^{\,2}}{2M} = 0$$

$$\left(i\partial^0 + \frac{\vec{\partial}\cdot\vec{\partial}}{2M} - \frac{\partial^0\partial^0}{2M} \right)\tilde{\Phi} = 0 \quad \text{or} \quad E + \frac{E^2}{2M} - \frac{\vec{p}^{\,2}}{2M} = 0$$

$$E = \sqrt{\vec{p}^{\,2} + M^2} - M \quad \text{or} \quad E = -\sqrt{\vec{p}^{\,2} + M^2} - M$$

WE ARE NOT INTERESTED IN THE LARGE NEGATIVE ENERGY!

we throw away this solution to change the physics to the effective field theory

we can get a $\mathcal{L}$ that gives us only the small positive energy solution perturbatively by eliminating higher powers of E using a momentum expansion — which we can do explicitly by solving the equation of motion perturbatively in the small momentum

$$E = \frac{\vec{p}^{\,2}}{2M} - \frac{E^2}{2M} \to \frac{\vec{p}^{\,2}}{2M} - \frac{1}{2M}\left(\frac{\vec{p}^{\,2}}{2M} - \frac{E^2}{2M}\right)^2 \to \cdots$$

Another equivalent way to get $\mathcal{L}$ is to redefine the field

$$\tilde{\Phi} = \left(\frac{E + \frac{(\sqrt{\vec{p}^{\,2}+M^2}-M)^2}{2M} - \frac{\vec{p}^{\,2}}{2M}}{E + \frac{E^2}{2M} - \frac{\vec{p}^{\,2}}{2M}} \right)^{1/2} \Phi$$

$$\left(i\partial^0 + \frac{\vec{\partial}\cdot\vec{\partial}}{2M} - \frac{\partial^0\partial^0}{2M}\right)\tilde{\Phi} = 0 \quad \text{or} \quad E + \frac{E^2}{2M} - \frac{\vec{p}^{\,2}}{2M} = 0$$

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throwing it away changes the physics to the effective field theory

$$\tilde{\Phi} = \left(\frac{E + \frac{(\sqrt{\vec{p}^{\,2}+M^2}-M)^2}{2M} - \frac{\vec{p}^{\,2}}{2M}}{E + \frac{E^2}{2M} - \frac{\vec{p}^{\,2}}{2M}}\right)^{1/2} \Phi = \left(1 + \frac{E}{4M} + \frac{3E^2 - 4\vec{p}^{\,2}}{32M^2} + \cdots\right)\Phi$$

$$\tilde{\Phi}^\dagger\left(E + \frac{E^2}{2M} - \frac{\vec{p}^{\,2}}{2M}\right)\tilde{\Phi} = \Phi^\dagger\left(E + \frac{(\sqrt{\vec{p}^{\,2} + M^2} - M)^2}{2M} - \frac{\vec{p}^{\,2}}{2M}\right)\Phi$$

or expanding and returning to position space

$$\mathcal{L} = \Phi^\dagger\left(i\partial^0 + \frac{\vec{\partial}\cdot\vec{\partial}}{2M} + \frac{(\vec{\partial}\cdot\vec{\partial})^2}{8M^3} + \cdots\right)\Phi$$

just choose your fields properly to avoid all higher powers of ∂^0

$$\mathcal{L} = \Phi^\dagger \left(i\partial^0 + \frac{\vec{\partial}\cdot\vec{\partial}}{2M} + \frac{(\vec{\partial}\cdot\vec{\partial})^2}{8M^3} + \cdots \right) \Phi$$

just choose your fields properly to avoid all higher powers of ∂^0

this works even in the presence of interactions! — terms with a $\mathcal{D}^0$ acting on Φ or $\Phi^\dagger$ can be eliminated by a redefinition of Φ order by order in the derivative expansion — using the equation of motion perturbatively to remove $\mathcal{D}^0$

so for example a term like $iA\left(\mathcal{D}^j\mathcal{D}^j\mathcal{D}^0 + \mathcal{D}^0\mathcal{D}^j\mathcal{D}^j\right)$ can be eliminated by field redefinition

while terms like $2iA\,\mathcal{D}^j\mathcal{D}^0\mathcal{D}^j$ can be replaced by $iA\,[[\mathcal{D}^j,\mathcal{D}^0],\mathcal{D}^j] = A\,[\mathcal{D}^j, t_\alpha v_\alpha^{j0}]$

$$\mathcal{L} = \Phi^\dagger \left(i\mathcal{D}^0 + \frac{\mathcal{D}^j\mathcal{D}^j}{2M} + a\frac{t_\alpha[\mathcal{D}^j, v_\alpha^{0j}]}{\Lambda^2} + \frac{(\mathcal{D}^j\mathcal{D}^j)^2}{8M^3} + \cdots \right) \Phi$$

because there is only one $\mathcal{D}^0$ the quantum mechanics is simple

$$\Pi = \frac{\partial\mathcal{L}}{\partial\partial^0\Phi} = i\Phi^\dagger \Rightarrow [\Phi_j(t,\vec{x}), \Phi_k^\dagger(t,\vec{x}\,')] = \delta_{jk}\,\delta^3(\vec{x}-\vec{x}\,')$$

so that Φ and $\Phi^\dagger$ are annihilation and creation opeators resp simple first step focus on the vector current — we will come back to the axial vector — and look at a world with only the u and d quarks — just isospin

$$\mathcal{L} = (i\bar{q}\,\not{D}\;q - \bar{q}\,M_q\,q) - \frac{1}{4}G_a^{\mu\nu}\,G_{a\mu\nu} - \bar{q}\,\gamma_\mu\,t_\alpha v_\alpha^\mu\,q \qquad \frac{\partial\mathcal{L}}{\partial v_\alpha^\mu} = -\bar{q}\,\gamma^\mu\,t_\alpha\,q$$

quark field q	$\mathcal{D}^\mu = D^\mu + it_\alpha v_\alpha^\mu$	$igT_aG_a^{\mu\nu} + it_\alpha v_\alpha^{\mu\nu} = [\mathcal{D}^\mu,\,\mathcal{D}^\nu]$
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T_a 3×3 traceless
t_α 2×2 traceless $=\sigma_\alpha/2$

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M_q diagonal in flavor space u, d

v_α^μ is a classical gauge field! — classical flavor gauge symmetry in addition to the quantum color gauge symmetry

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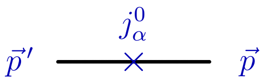
coefficient of $-v_\alpha^\mu$ is the current $j_\alpha^\mu = \bar{q}\,\gamma^\mu\,t_\alpha\,q$ in the EFT!

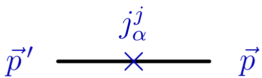
$$\mathcal{L} = \Phi^\dagger \left(i\mathcal{D}^0 + \frac{\mathcal{D}^j\mathcal{D}^j}{2M} + a\frac{t_\alpha[\mathcal{D}^j, v_\alpha^{0j}]}{\Lambda^2} + \frac{(\mathcal{D}^j\mathcal{D}^j)^2}{8M^3} + \cdots \right) \Phi$$

where in the EFT $\quad i\mathcal{D}^\mu = i\partial^\mu - t_\alpha^{\mathcal{I}} v_\alpha^\mu \quad$ and $\quad v_\alpha^{\mu\nu} = \partial^\mu v_\alpha^\nu - \partial^\nu v_\alpha^\mu - \epsilon_{\alpha\beta\gamma} v_\beta^\mu v_\gamma^\nu$



$$t_\alpha^I \left(1 - a\frac{(\vec{p} - \vec{p}\,')^2}{\Lambda^2} + \cdots \right)$$

dynamical scale Λ



$$t_\alpha^I \left(\frac{\vec{p} + \vec{p}\,'}{2M} + \cdots \right)$$

"kinematic" scale M

I haven't really tried to get signs right here — life is too short

Ward identities are automatically implemented by gauge invariance

isospin breaking — two effects (slightly interrelated as we will see)

EM interactions — include photons in effective theory

$m_u \neq m_d$ — the method of spurions

simple first step focus on the vector current — we will come back to the axial vector — and look at a world with only the u and d quarks — just isospin

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spurion method — treat M_q and eQ as variable classical fields

$M_q \to \mathcal{U}M_q\mathcal{U}^\dagger$

$eQ \to \mathcal{U}eQ\mathcal{U}^\dagger$

put **small** M_q and eQ as fields in EFT some technical but important details

$$M_q = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \qquad Q = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix}$$

both transform like a combination of a singlet and the 3rd component of an isospin triplet

$$M_q = \bar{m}\, I + \delta m\, \frac{\sigma_3}{2} = \bar{m}\, I + \delta m\, t_3$$
$$eQ = \frac{e}{6}\, I + e\frac{\sigma_3}{2} = \frac{e}{6}\, I + e\, t_3$$

where $\bar{m} = (m_u + m_d)/2$ is the average mass and $\delta m = m_u - m_d$ is the mass difference

the I terms do not break the isospin symmetry so we won't be very interested in them (at least for now)

in the EFT $\quad \delta m\, t_3 \propto \delta m\, t_3^{\mathcal{I}} \quad$ with an unknown coefficient depending on the dynamics

In EFT, no matter how the field Φ ransforms under isospin, we can always write a term like

$$\mathrm{tr}(M_q t_a)\, \Phi^\dagger t_a^{\mathcal{I}} \Phi = \delta m\, \Phi^\dagger t_3^{\mathcal{I}} \Phi \quad \text{or} \quad \mathrm{tr}(eQ t_a)\, \Phi^\dagger t_a^{\mathcal{I}} \Phi = e\, \Phi^\dagger t_3^{\mathcal{I}} \Phi$$

what about second order

in the EFT 2nd order symmetry breaking $\propto (\delta m)^2 \, (t_3^{\mathcal{I}})^2$ with an unknown coefficient depending on the dynamics

$$\mathrm{tr}(M_q t_a)\,\mathrm{tr}(M_q t_b)\,\Phi^\dagger t_a^{\mathcal{I}} t_b^{\mathcal{I}} \Phi = \delta m^2\,\Phi^\dagger (t_3^{\mathcal{I}})^2 \Phi$$

isospin is particularly simple because the product of two irreducible representations never contains any irreducible representation more than once — but the moral is general — just put the symmetry breaking terms in the effective theory and build the most general terms you can

$$\mathcal{L} = \Phi^\dagger \left(i\mathcal{D}^0 + \frac{\mathcal{D}^j\mathcal{D}^j}{2M} + a\frac{t_\alpha[\mathcal{D}^j, v_\alpha^{0j}]}{\Lambda^2} + \frac{(\mathcal{D}^j\mathcal{D}^j)^2}{8M^3} + \cdots \right) \Phi$$

include symmetry breaking from mass term

$$\mathcal{L} = \Phi^\dagger \left(i\mathcal{D}^0 + b\,\delta m\, t_3^{\mathcal{I}} + c\,\frac{\delta m}{\Lambda}\,\left\{ i\mathcal{D}^0, t_3^{\mathcal{I}} \right\} + \cdots \right) \Phi$$

the b terms gives mass splittings within the isospin multiplet — in our world the d is heavier than the u so $\delta m < 0$ — this effect is just counting the number of slightly heavier d quarks

the interesting thing is that the c term gives no symmetry breaking contribution to the current — can eliminate it by field redefnition (like all $\mathcal{D}^0$ terms)

$$\Phi \to \left(1 - c\, t_3^{\mathcal{I}}\, \frac{\delta m}{\Lambda} \right) \Phi$$

if you just calculate it without field redefinition you will find that the δm in the wave function renormalization cancels the δm in the coefficient of v_α^μ

this is actually not very important for isospin, because as we will see, there are other effects — but for $SU(3)$ it is important, and is called the Ademollo-Gatto theorem

EM contributions include the coupling to the photon we get by putting $Q\,A^\mu$ in the covariant derivative — and of course that means that the electromagnetic current is related to the currents coupled to v_α^μ

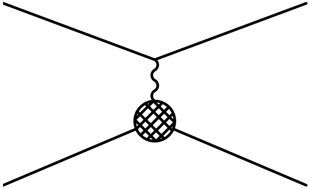
historically interesting because Feynman + Gell-Mann used this in 1958 to relate weak vector currents in semileptonic decays

$$\bar{q}\gamma^\mu(t_1 \pm i t_2)q$$

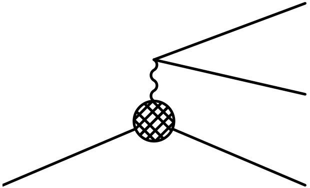
to the isovector part of the hadron electromagnetic current

$$\bar{q}\gamma^\mu t_3 q$$

these currents can be measured! both in scattering and in decays

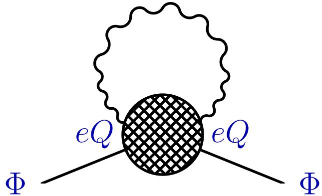


lepton-hadron scattering



hadron semi-leptonic decay

of course, quarks were unknown at the time — this was before Gell-Mann had understood $SU(3)$ — so the connection that Feynman + Gell-Mann found was an important clue to the stucture of the strong interactions as well as the weak interactions also quantum effects from photons



the long distance part of graphs like this are present in the low energy theory — but as we have seen in our discussion of matching — the high energy part appears as a new local term in the EFT $\propto \alpha$ — the joke is that $\alpha\Lambda \approx\approx \delta m$ so isospin breakings have similar sizes — as far as we know this is just a bizarre accident

proportional to two factors of eQ because the photons hook onto quarks somewhere inside the blob

important because each factor of eQ contains a piece that transforms like the neutral component of an isospin triplet $\rightarrow$ two factors of $t_3^{\mathcal{I}}$

thus there are two $t_3^{\mathcal{I}}$s for each photon line so one photon can do the work of two δms

$$\mathcal{L} = \Phi^\dagger \left(i\mathcal{D}^0 + a\,\delta m\, t_3^{\mathcal{I}} + b\,\alpha\,\Lambda\,(t_3^{\mathcal{I}})^2 + c\,\alpha\, t_3^{\mathcal{I}}\, i\mathcal{D}^0\, t_3^{\mathcal{I}} + \cdots \right) \Phi$$

$$\mathcal{L} = \Phi^\dagger \left(i\mathcal{D}^0 + a\,\delta m\, t_3^{\mathcal{I}} + b\,\alpha\,\Lambda\,(t_3^{\mathcal{I}})^2 + c\,\alpha\, t_3^{\mathcal{I}}\, i\mathcal{D}^0\, t_3^{\mathcal{I}} + \cdots \right) \Phi$$

a and b are mass splitting terms — only two parameters — not very interesting for small $t^{\mathcal{I}}$ — but very important for large multiplets

c term is the leading term that contributes to symmetry breaking in the current

this is less complicated than it looks — ignore the mass terms (which don't affect the current) and use the magic of field redefinition

$$\mathcal{L} = \Phi^\dagger \left(i\mathcal{D}^0 + c\,\alpha\, t_3^{\mathcal{I}}\, i\mathcal{D}^0\, t_3^{\mathcal{I}} + \cdots \right) \Phi$$

$$\mathcal{D}^\mu = \partial^\mu + i\, t_\alpha^{\mathcal{I}}\, v_\alpha^\mu \qquad \Phi = \left(1 - c\,\alpha\,(t_3^{\mathcal{I}})^2/2\right)\tilde{\Phi} \quad \rightarrow$$

$$\tilde{\Phi}^\dagger \left(1 - c\,\alpha\,(t_3^{\mathcal{I}})^2/2\right) \left(i\mathcal{D}^0 + c\,\alpha\, t_3^{\mathcal{I}}\, i\mathcal{D}^0\, t_3^{\mathcal{I}} + \cdots \right) \left(1 - c\,\alpha\,(t_3^{\mathcal{I}})^2/2\right)\tilde{\Phi}$$

$$= \tilde{\Phi}^\dagger \left(i\mathcal{D}^0 - c\,\alpha\,[t_3^{\mathcal{I}},[t_3^{\mathcal{I}}, i\mathcal{D}^0]]/2 + \cdots \right)\tilde{\Phi}$$

$$= \tilde{\Phi}^\dagger \left(i\partial^0 - t_\alpha^{\mathcal{I}}\, v_\alpha^0 - c\,\alpha\,[t_3^{\mathcal{I}},[t_3^{\mathcal{I}}, i\partial^0 - t_\alpha^{\mathcal{I}}\, v_\alpha^0]]/2 + \cdots \right)\tilde{\Phi}$$

$$= \tilde{\Phi}^\dagger \left(i\partial^0 - t_\alpha^{\mathcal{I}}\, v_\alpha^0 + c\,\alpha\,(t_1^{\mathcal{I}}\, v_1^0 + t_2^{\mathcal{I}}\, v_2^0)/2 + \cdots \right)\tilde{\Phi}$$

changes the normalization of the charged current but not the neutral current the moral is that we know a great deal about the vector isospin currents from first principles

the leading terms at low energy and momentum transfer are completely determined by the symmetry — not just the form, but the normalization

dynamics appears in the form of higher dimension operators and it is highly constrained by the isospin symmetry

dynamics also appears in the symmetry breaking which can be analyzed in perturbation theory in the symmetry breaking parameters

this is how we know V_{ud} in the KM matrix — superallowed β-decays are decays from one member of an isospin multiplet from another so this picture applies — if the states have spin 0 (which is all we have discussed so far) the axial vector current cannot contribute because of Lorenz invariance and parity — you cannot build any interactions of a single axial vector source

the stable baryons, V_{us} and $SU(3)$

three extensions — spin 1/2 — $SU(2) \to SU(3)$ — axial vector current contribute

we will look at the first two of these first — we will have to wait until we see how to deal with $SU(3) \times SU(3)$ symmetry — but it is interesting to see how the vector currents work the eightfold way –

1. Right view
2. Right intention
3. Right speech
4. Right action
5. Right livelihood
6. Right effort
7. Right mindfulness
8. Right concentration

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

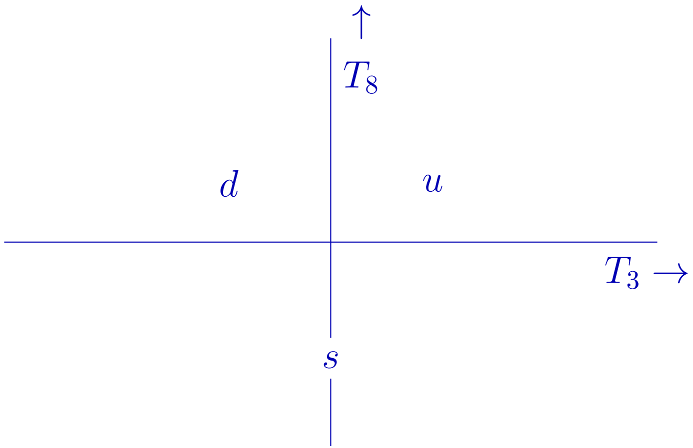
$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

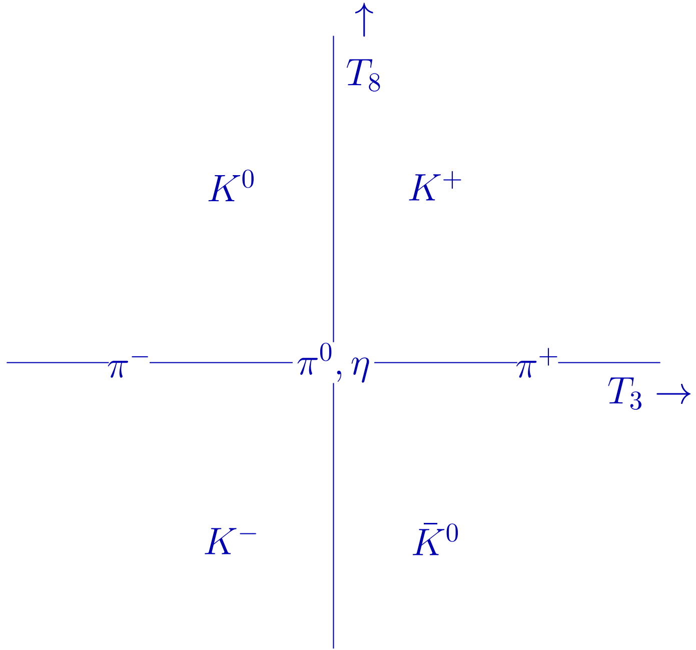
Pauli matrices embedded in first two rows and columns

$$T_a = \lambda_a/2 \quad \text{so} \quad \mathrm{tr}(T_a T_b) = \frac{1}{2}\delta_{ab} \quad [T_a, T_b] = i\, f_{abc} T_c$$

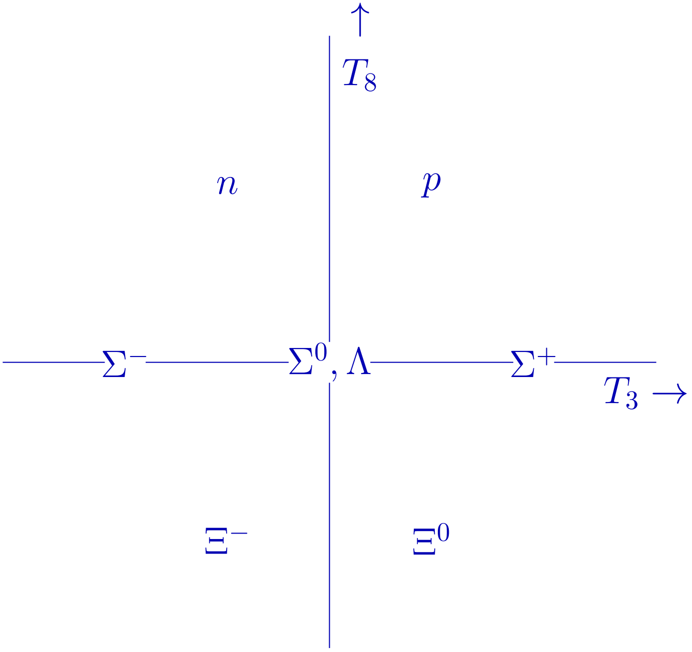


now we would say that

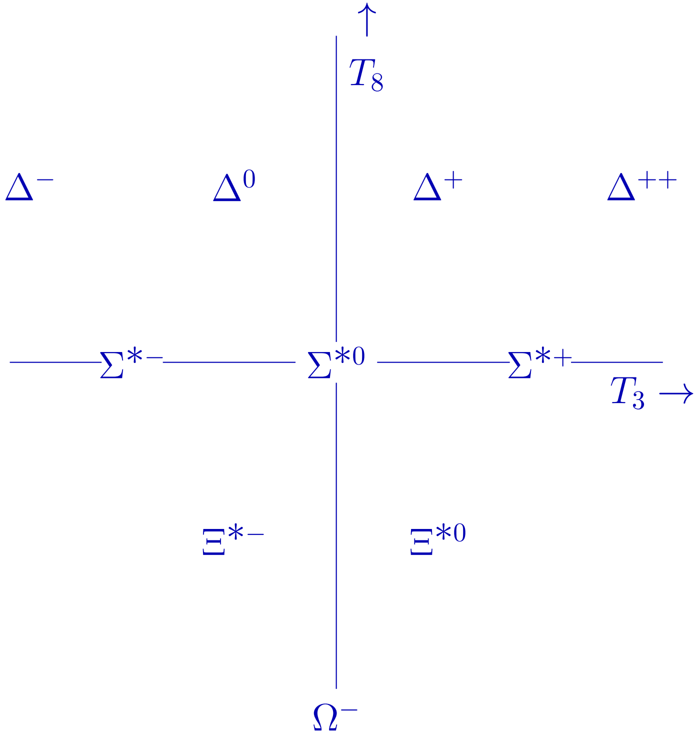
$$T_8 = \frac{1}{2\sqrt{3}}(N_u + N_d - 2N_s)$$



T_8
K^0
K^+
π^-
π^0, η
π^+
T_3
K^-
$\bar{K}^0$



T8
n
p
Σ−
Σ0, Λ
Σ+
T3
Ξ−
Ξ0



T_8
Δ^-
Δ^0
Δ^+
Δ^{++}
Σ^{*-}
Σ^{*0}
Σ^{*+}
T_3
Ξ^{*-}
Ξ^{*0}
Ω^-

the light baryon octet

$$B^i_j = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & \frac{-\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & \frac{-2\Lambda}{\sqrt{6}} \end{pmatrix}_{ij}$$

$B \to U B U^\dagger$ where U is a special unitary 3×3 matrix $\Rightarrow$ generators act by commutation

$$U = 1 + i\epsilon_a T_a + \cdots \quad B \to B + i\epsilon_a [T_a, B] \quad \delta B = +i\epsilon_a [T_a, B]$$

the matrix form is particularly convenient for explicit calculations in simple case, because it is easier to do matrix multiplication than to carry around 8×8 matrices — but it can get confusing and difficult to see which indices are contracted with which — so you should be able to go back and forth from one notation to the other — $B_a \propto \mathrm{tr}(\lambda_a\, B)$

use B to describe the low-energy baryon — matrix structure gets the flavor right — what about spin?

just ordinary angular momentum because we have broken Lorentz invariance by going to the frame in which the baryon is at rest — so B is a Pauli spinor and a flavor matrix

$$\mathcal{L} = \mathrm{tr}\left(B^\dagger i \mathcal{D}^0(B) + i\,\frac{f}{\Lambda}\,B^\dagger \sigma_j \epsilon_{jk\ell}[T_a v_a^{k\ell}, B] + \frac{d}{\Lambda}\,B^\dagger \sigma_j \epsilon_{jk\ell}\{T_a v_a^{k\ell}, B\} + \cdots \right)$$

$$\text{where} \quad \mathcal{D}^\mu(O) = \partial^\mu O + i v_\alpha^\mu [T_\alpha, O]$$

symmetry breaking

$$M_q = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}$$

$$M_q = \frac{m_u + m_d + m_s}{3} I + (m_u - m_d)\, T_3 + \frac{m_u + m_d - 2m_s}{\sqrt{3}}\, T_8$$

treat symmetry breaking perturbatively — use our isospin analysis within isospin multiplets — ignore isospin breaking and Q when looking at $SU(3)$ breaking effect — here the Ademollo-Gatto theorem is really important the light baryon octet mass terms

$$B^i_j = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & \frac{-\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & \frac{-2\Lambda}{\sqrt{6}} \end{pmatrix}_{ij}$$

$$\mathrm{tr}\left(m_0' B^\dagger B + a' B^\dagger T_8 B + b' B^\dagger B T_8\right) = \mathrm{tr}\left(m_0 B^\dagger B + a B^\dagger T_s B + b B^\dagger B T_s\right)$$

where $T_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$m_0\left(N^\dagger N + \Xi^\dagger \Xi + \Lambda^\dagger \Lambda + \Sigma^\dagger \Sigma\right)$$
$$+a\left(\Xi^\dagger \Xi + 2\Lambda^\dagger \Lambda / 3\right) + b\left(N^\dagger N + 2\Lambda^\dagger \Lambda/3\right)$$

$$m_\Sigma = m_0 \qquad m_\Xi = m_0 + a \qquad m_N = m_0 + b \qquad m_\Lambda = m_0 + \frac{2}{3}(a+b)$$

$$2\left(m_N + m_\Xi\right) = 3m_\Lambda + m_\Sigma$$

Gell-Mann — Okubo formula

renormalization of currents - first order in M_q

$$\mathcal{L} = \operatorname{tr}\Big(B^\dagger i\mathcal{D}^0(B) + B^\dagger i\mathcal{D}^0(a_1 M_q B + a_2 B M_q) + (a_1 B^\dagger M_q + a_2 M_q B^\dagger) i\mathcal{D}^0(B) + \cdots \Big)$$

where $\quad \mathcal{D}^\mu(O) = \partial^\mu O + i v^\mu_\alpha [T_\alpha, O]$ avoid $\mathcal{D}^\mu(M_q)$ † confusing

the a_1 and a_2 terms can be removed by field redefinition — if you don't remove them it doesn't matter - you just have to do the calculation right — ∂^0 term determines the normalizations of the states — this is what field redefinition is all about renormalization of currents - second order in M_q

$$\mathcal{L} = \mathrm{tr}\Big(B^\dagger i\mathcal{D}^0(B)\Big) + a_{11}\mathrm{tr}\Big(B^\dagger M_q i\mathcal{D}^0(M_q B)\Big) + a_{22}\mathrm{tr}\Big(M_q B^\dagger i\mathcal{D}^0(B M_q)\Big)$$

$$+a_{12}\mathrm{tr}\Big(B^\dagger M_q i\mathcal{D}^0(B M_q) + M_q B^\dagger i\mathcal{D}^0(M_q B)\Big) + \cdots$$

where $\mathcal{D}^\mu(O) = \partial^\mu O + i v^\mu_\alpha [T_\alpha, O]$

avoid $\mathcal{D}^\mu(M_q)$
† confusing

now it is not so obvious what terms can be removed by field redefinition — but again, if you don't remove them it doesn't matter - you just have to do the calculation right — ∂^0 term determines the normalizations of the states

Exercise 4. I like to think about worlds slightly different from ours. So imagine a world in which the u, d and s quarks are just like they are in our world, but the c quark is lighter, and almost degenerate with the s quark. In this world, there would be two approximate $SU(2)$ symmetries, the usual isospin symmetry under which (u, d) transforms as a doublet, and "isospin2" under which (c, s) transforms as a doublet. And these two $SU(2)$ would fit into an approximate $SU(4)$ symmetry, which is the analog of Gell-Mann's $SU(3)$ in this world. Find the $SU(4)$ representations of the light spin $1/2$ and spin $3/2$ baryons in this world and explain how they decompose into representations of the two $SU(2)$s. I am not assigning this because it is a little more involved, but if you are bored, it is really fun to use the analog of the Gell-Mann — Okubo formula to calculate the masses in this toy world. They can all be expressed very simply in terms of the masses in our world.

the physics of spontaneously broken continuous symmetry — a mechanical analogy

transverse waves in
a string described
by wave function $\psi(x)$

invariance under translations $x \to x + c$
normal modes $\psi(x+c) \propto \psi(x)$
$\Rightarrow$ normal modes e^{ikx}

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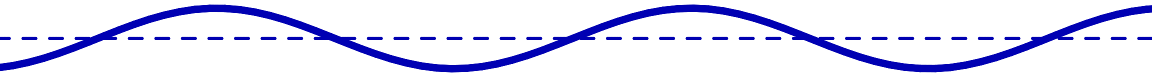

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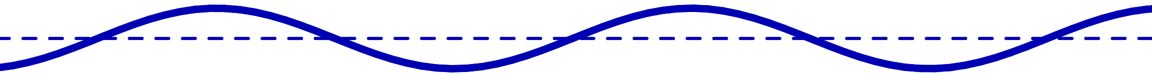

dispersion relation $\omega^2(k^2) \propto$
restoring force for the
k mode from mechanical
properties of the string
(tension, stiffness, etc)

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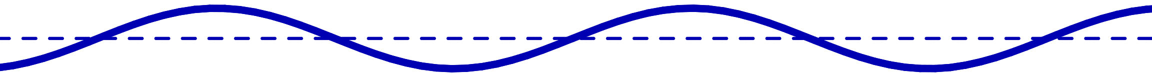

dispersion relation $\omega^2(k^2) \propto$
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properties of the string
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invariance under translations $y \to y + d$
— normal modes already fixed
symmetry spontaneously broken
tells you about the dispersion
relation $\omega^2 \to 0$ and $k^2 \to 0$

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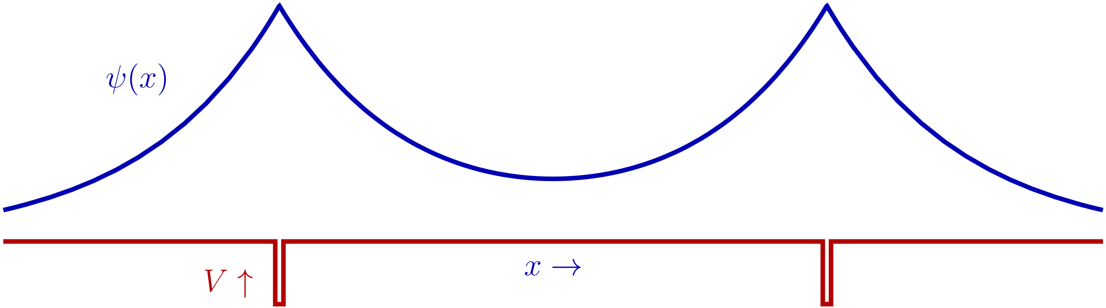
— normal modes already fixed
symmetry spontaneously broken
tells you about the dispersion
relation $\omega^2 \to 0$ as $k^2 \to 0$

$k \to 0$ gets closer and closer to a translation of the whole string
the spontaneously broken symmetry guarantees that there is no
restoring force as $k \to 0$ — but notice that the system never gets
to $k = 0$ — that would require motion of the whole infinite system

I like this example because I think it survives the onslaught of quantum mechanics!

normally quantum mechanics destroys spontaneous symmetry breaking — even if a classical system has degenerate ground states that break symmetry — like the double well potential — the quantum mechanical ground state is symmetric



the reason is superposition and the uncertainty principle — the states spread out around the classical minima because of the uncertainty principle — then the wave functions around the classical minima interfere constructively and the symmetric state has the lowest energy — a very general phenomenon — continuous symmetry also — like a particle moving freely on a circle simplest field theory model with spontaneously broken continuous symmetry

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi$$

invariant under
$\phi \to \phi + c$

The conserved Noether current is $\partial^0 \phi$, so we would normally expect

$$Q = \int \partial^0 \phi(\vec{r}, t)\, d^3 r \quad \text{and} \quad U_c\,|0\rangle = |0\rangle$$

to be a conserved charge generating the transformation.
But this cannot be because

$$U_c = e^{icQ} \quad \text{would satisfy} \quad U_c\,\phi\,U_c{}^\dagger = \phi + c$$

and

$$\langle 0|\phi|0\rangle = \langle 0|U_c\,\phi\,U_c{}^\dagger|0\rangle = \langle 0|(\phi + c)|0\rangle = \langle 0|\phi|0\rangle + c$$

impossible
for $c \neq 0$

the trouble is that Q is not very well defined — so consider a smeared version of Q

$$Q_\lambda(t) \equiv \int e^{-|\vec{y}|^2/2\lambda^2}\, \partial^0 \phi(\vec{y}, t)\, d^3y$$

$Q_\lambda(t)$ goes to Q as λ goes to ∞

Exercise 5. Calculate $e^{icQ_\lambda(t)}\,\phi(\vec{x}, t)\, e^{-icQ_\lambda(t)}$.

If $|0\rangle$ is a vacuum state satisfying $\langle 0|\phi|0\rangle = 0$, then $Q_\lambda(t)\,|0\rangle$ is a well-defined normalized state. But if you calculate

$$f(\lambda) \equiv \langle 0|e^{icQ_\lambda(t)}|0\rangle$$

you find that $f(\lambda) = e^{-c^2\lambda^2/32\pi^2}$ so $f(\lambda) \to 0$ as $\lambda \to \infty$.

So the state $e^{icQ_\lambda(t)}\,|0\rangle$ that wants to become the "transformed vacuum" as $\lambda \to \infty$ is actually orthogonal to the vacuum in the limit. It is easy to show that it also becomes orthogonal to any state obtained by acting on $|0\rangle$ by fields in a finite region of space. So it is an equivalent vacuum, but it is completely disconnected from all the physical states built on the original vacuum. So the transformation is taking us to a completely equivalent, but completely disconnected Hilbert space

continuously ∞ # of equivalent spaces labeled by c — supersection rules the key, clearly, is the particles which transform **inhomogeneously** under the symmetry — they have to be massless as the ϕ is trivially in our toy theory — but the problem is precisely the "global" nature of the symmetry — this mess happened because we insisted on thinking about a symmetry that is spread out over all of infinite space — the transformed vacuum is built out of "particles" whose wave function is similarly spread out, and this has negligible overlap — this saves SSB from the ravages of QM — and it means that what the symmetry tells us is quite different from what happens in Gell-Mann's $SU(3)$ — we are not interested in Q — be we are interested in Q_λ for finite λ because even though the limit $\lambda \to \infty$ is seriously diseased — the approach to the limit provides very interesting information — SSB relates states with different numbers of low momentum particles — this is an absolute set-up for the kind of momentum expansion that we use in EFT — and in fact, the clearest formulation of the effective field theory paradigm arose from the attempts of Steve Weinberg and others to make sense of spontaneously broken global symmetries — the fields like ϕ in our stupid little example are massless in the symmetry limit and must ALWAYS be included in the EFT while EVERYTHING ELSE should be left out — and the symmetry tells us interesting things about the massless fields because they ARE the transformations of the vacuum in a limited region. These are the Goldstone bosons.

But in QCD, the axial $SU(3)$ currents are like the Noether current $\partial^0\phi$ in our simple field theory example. If the quarks masses vanish, the axial $SU(3)$ currents are conserved, but corresponding charges do not exist on the Hilbert space of physical states. The symmetry is spontaneously broken from $SU(3) \times SU(3)$ down to Gell-Mann's $SU(3)$. With zero u, d, s quark masses and $\alpha = G_F = 0$, the light pseudo-scalars would become massless Goldstone bosons. Their interactions at very low energy would be completely determined by the symmetry because the fields that create them correspond to local symmetry transformations of the vacuum.

In the stupid little example, there is no high energy theory. The Goldstone bosons are the fundamental free fields.

The situation is very different for $SU(3) \times SU(3) \to SU(3)$ and it is fun to understand why in detail.

Quick discussion of the Goldstone theorem in perturbative QFT — consider some real scalar fields ϕ

$$\mathcal{L}(\phi) = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)$$

ϕ is some multiplet of spinless fields

$V(\phi)$ and $\mathcal{L}(\phi)$ invariant under symmetry generated by T_a

$$\delta\phi = i\epsilon_a T_a \phi$$

where $T_a = -T_a^* = -T_a^T$ because ϕs are real

these fields transform homogeneously — SSB arises if these are not the right fields to use at low energies which is what happens if the minimum of $V(\phi)$ is not at $\phi = 0$ — suppose there is a minimum of $V(\phi)$ at $\phi = \mathcal{F} \neq 0$ VEV

for convenience in the analysis define

$$V_{j_1\cdots j_n}(\phi) = \frac{\partial^n}{\partial\phi_{j_1}\ldots\partial\phi_{j_n}} V(\phi)$$

the condition that $\mathcal{F}$ be an extremum of $V(\phi)$ as

$\mathcal{F}$ is an extremum if $V_j(\mathcal{F}) = 0$ and $\mathcal{F}$ is a minimum if $V_{jk}(\mathcal{F}) \geq 0$

perturbing around $\mathcal{F}$ shifted fields $\phi'=\phi-\mathcal{F}$ create particles

$$V(\phi) = V(\mathcal{F}) + \phi'_j V_j(\mathcal{F}) + \frac{1}{2}\phi'_j\phi'_k V_{jk}(\mathcal{F}) + \cdots$$

perturbing around $\mathcal{F}$
shifted fields
$\phi'=\phi-\mathcal{F}$ create particles

$$V(\phi) = V(\mathcal{F}) + \phi'_j V_j(\mathcal{F}) + \frac{1}{2}\phi'_j\phi'_k V_{jk}(\mathcal{F}) + \cdots$$

(the term $\phi'_j V_j(\mathcal{F})$ is crossed out)

$V_{jk}(\mathcal{F})$ is the meson mass-squared matrix — there are no tachyons in the free theory about which we are perturbing

$$\delta\phi = i\epsilon_a T_a \phi \Rightarrow \delta\phi' = i\epsilon_a T_a \phi' + i\epsilon_a T_a \mathcal{F}$$

because $\mathcal{F}$ doesn't change

$T_a\mathcal{F}$ is a set of vectors in field space

$$\kappa_{ab} \equiv \mathcal{F}^T T_a T_b \mathcal{F}$$

is a real, symmetric and positive
assume we've diagonalized

two kinds of generators — "unbroken" $S_a \mid S_a\mathcal{F} = 0$ — "broken" $X_a \mid X_a\mathcal{F} \neq 0$ — the shifted fields ϕ' transform homogeneously under S_a and inhomogenously under X_a — S_a form a subalgebra because

$$S_a\mathcal{F} = 0 \text{ and } S_b\mathcal{F} = 0 \Rightarrow [S_a, S_b]\mathcal{F} = 0$$

so S_a generate the "unbroken subalgebra"

$$V(\phi+\delta\phi)-V(\phi)=iV_k(\phi)\epsilon_a[T_a]_{kl}\phi_l=0$$

now differentiate and set $\phi=\mathcal{F}$

$$V_{jk}(\mathcal{F})\epsilon_a[T_a]_{kl}\mathcal{F}_l+V_k(\mathcal{F})\epsilon_a[T_a]_{kj}=0\Rightarrow V_{jk}(\mathcal{F})[X_a]_{kl}\mathcal{F}_l=0$$

the nonzero vectors we get by acting on the VEV $\mathcal{F}$ with the broken generators X_a are eigenvectors of the mass matrix with eigenvalue 0 — massless particles — this is the Goldstone theorem — $\mathcal{F}$ describes the "vacuum" — $X_a\mathcal{F}$ is the change we make to try to "transform" the vacuum — but instead of getting a new vacuum, we get massless fields $\mathcal{F}^T X_a\phi'$ that look like transformed vacuum in some limited region of of space — these are the Goldstone bosons

$$\mathcal{F}^T X_a\phi'$$

example — the linear σ-model 2×2 matrix of spinless fields Σ — could couple to chiral fermions $g\, q_L \Sigma q_R + \text{h.c.}$

$$\Sigma = \tau_2 \Sigma^\dagger \tau_2 \Rightarrow \Sigma = \sigma + i\vec{\tau} \cdot \vec{\pi}$$

where σ and $\vec{\pi}$ are real fields and τ_a are the Pauli matrices — quaternions

transformation of Σ under "chiral" $SU(2)_L \times SU(2)_R$

$$\delta\Sigma = i\epsilon_L^a T_a^L \Sigma - i\Sigma\epsilon_R^a T_a^R$$

where $T_a^L = T_a^R = \tau_a/2$

or we can exponentiate the infinitesimal transformations

$$\Sigma \to L\Sigma R^\dagger \qquad L = \exp\left(il^a T_a^L\right) \quad R = \exp\left(ir^a T_a^R\right)$$

leaves invariant the quaternion "absolute value"

$$\Sigma\Sigma^\dagger = \Sigma^\dagger\Sigma = \sigma^2 + \pi_a^2$$

the magic of Pauli matrices

$$\mathcal{L} = \frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma + \frac{1}{2}\partial^\mu\pi_a\partial_\mu\pi_a - \frac{\lambda}{4}\left(\sigma^2 + \pi_a^2 - f^2\right)^2$$

equivalent formulation with real fields

$$\phi \equiv \begin{pmatrix} \sigma \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} \qquad \mathcal{L} = \frac{1}{2}\partial^\mu\phi^T\partial_\mu\phi - \frac{\lambda}{4}\left(\phi^T\phi - f^2\right)^2$$

$SU(2) \times SU(2)$ algebra $\Leftrightarrow$ $SO(4)$ algebra

with real fields
we can apply the
Goldstone analysis

$$\phi \equiv \begin{pmatrix} \sigma \\ \vec{\pi} \end{pmatrix} \qquad \mathcal{L} = \frac{1}{2}\partial^\mu \phi^T \partial_\mu \phi - \frac{\lambda}{4}\left(\phi^T\phi - f^2\right)^2$$

$$\delta\Sigma = i\epsilon_L^a T_a^L \Sigma - i\Sigma\epsilon_R^a T_a^R$$

where
$$T_a^L = T_a^R = \tau_a/2$$

use $(\vec{\tau}\cdot\vec{a})\,(\vec{\tau}\cdot\vec{b})$
$$= \vec{a}\cdot\vec{b} + i\vec{\tau}\cdot(\vec{a}\times\vec{b})$$

$$\delta_L\Sigma = \delta_L(\sigma + i\vec{\tau}\cdot\vec{\pi}) = i\vec{\epsilon}_L\cdot\frac{\vec{\tau}}{2}(\sigma + i\vec{\tau}\cdot\vec{\pi}) = -\vec{\epsilon}_L\cdot\frac{\vec{\pi}}{2} + i\vec{\tau}\cdot\left(\vec{\epsilon}_L\frac{\sigma}{2} - \vec{\epsilon}_L\times\frac{\vec{\pi}}{2}\right)$$

$$\delta_L\sigma = -\vec{\epsilon}_L\cdot\frac{\vec{\pi}}{2} \qquad \delta_L\vec{\pi} = \vec{\epsilon}_L\frac{\sigma}{2} - \vec{\epsilon}_L\times\frac{\vec{\pi}}{2} \quad \text{or} \quad i\vec{\epsilon}_L\cdot\vec{T}^L = \begin{pmatrix} 0 & -\vec{\epsilon}_L/2 \\ \vec{\epsilon}_L/2 & -\vec{\epsilon}_L/2\times \end{pmatrix}$$

$$\delta_R\Sigma = \delta_R(\sigma + i\vec{\tau}\cdot\vec{\pi}) = -i(\sigma + i\vec{\tau}\cdot\vec{\pi})\,\vec{\epsilon}_R\cdot\frac{\vec{\tau}}{2} = \vec{\epsilon}_R\cdot\frac{\vec{\pi}}{2} + i\vec{\tau}\cdot\left(-\vec{\epsilon}_R\frac{\sigma}{2} - \vec{\epsilon}_R\times\frac{\vec{\pi}}{2}\right)$$

$$\delta_R\sigma = \vec{\epsilon}_R\cdot\frac{\vec{\pi}}{2} \qquad \delta_R\vec{\pi} = -\vec{\epsilon}_R\frac{\sigma}{2} - \vec{\epsilon}_R\times\frac{\vec{\pi}}{2} \quad \text{or} \quad i\vec{\epsilon}_R\cdot\vec{T}^R = \begin{pmatrix} 0 & \vec{\epsilon}_R/2 \\ -\vec{\epsilon}_R/2 & -\vec{\epsilon}_R/2\times \end{pmatrix}$$

$$\mathcal{F} = \begin{pmatrix} f \\ 0 \end{pmatrix} \qquad \vec{\epsilon}_L\cdot\vec{T}^L\,\mathcal{F} = \begin{pmatrix} 0 \\ \vec{\epsilon}_L/2 \end{pmatrix} \qquad \vec{\epsilon}_R\cdot\vec{T}^R\,\mathcal{F} = \begin{pmatrix} 0 \\ -\vec{\epsilon}_R/2 \end{pmatrix}$$

with real fields we can apply the Goldstone analysis

$$\phi \equiv \begin{pmatrix} \sigma \\ \vec{\pi} \end{pmatrix} \qquad \mathcal{L} = \frac{1}{2}\partial^\mu \phi^T \partial_\mu \phi - \frac{\lambda}{4}\left(\phi^T\phi - f^2\right)^2$$

$$\delta_L \sigma = -\vec{\epsilon}_L \cdot \frac{\vec{\pi}}{2} \qquad \delta_L \vec{\pi} = \vec{\epsilon}_L \frac{\sigma}{2} - \vec{\epsilon}_L \times \frac{\vec{\pi}}{2} \quad \text{or} \quad i\vec{\epsilon}_L \cdot \vec{T}^L = \begin{pmatrix} 0 & -\vec{\epsilon}_L/2 \\ \vec{\epsilon}_L/2 & -\vec{\epsilon}_L/2\times \end{pmatrix}$$

$$\delta_R \sigma = \vec{\epsilon}_R \cdot \frac{\vec{\pi}}{2} \qquad \delta_R \vec{\pi} = -\vec{\epsilon}_R \frac{\sigma}{2} - \vec{\epsilon}_R \times \frac{\vec{\pi}}{2} \quad \text{or} \quad i\vec{\epsilon}_R \cdot \vec{T}^R = \begin{pmatrix} 0 & \vec{\epsilon}_R/2 \\ -\vec{\epsilon}_R/2 & -\vec{\epsilon}_R/2\times \end{pmatrix}$$

$$\mathcal{F} = \begin{pmatrix} f \\ 0 \end{pmatrix} \qquad \vec{\epsilon}_L \cdot \vec{T}^L \mathcal{F} = \begin{pmatrix} 0 \\ \vec{\epsilon}_L/2 \end{pmatrix} \qquad \vec{\epsilon}_R \cdot \vec{T}^R \mathcal{F} = \begin{pmatrix} 0 \\ -\vec{\epsilon}_R/2 \end{pmatrix}$$

$$\vec{S} = \vec{T}^V \equiv \vec{T}^L + \vec{T}^R \qquad \vec{\epsilon}_V \cdot \vec{S} = \vec{\epsilon}_V \cdot (\vec{T}^L + \vec{T}^R)\,\mathcal{F}$$

$$= \vec{\epsilon}_V \cdot \vec{T}^L \mathcal{F} + \vec{\epsilon}_V \cdot \vec{T}^R \mathcal{F} = \begin{pmatrix} 0 \\ \vec{\epsilon}_V/2 \end{pmatrix} + \begin{pmatrix} 0 \\ -\vec{\epsilon}_V/2 \end{pmatrix} = 0$$

vector or “diagonal” symmetry is unbroken

$$\vec{X} = \vec{T}^A \equiv \vec{T}^L - \vec{T}^R \qquad i\vec{\epsilon}_A \cdot \vec{X}\,\mathcal{F} = \begin{pmatrix} 0 \\ \vec{\epsilon}_A f/2 \end{pmatrix}$$

chiral symmetry is spontaneiously broken

the shifted fields are

$$\phi' = \phi = \mathcal{F} \qquad \begin{matrix} \sigma' = \sigma - f \\ \pi' = \pi \end{matrix}$$

the Goldstone bosons are

$$\mathcal{F}^T \vec{X} \phi' \propto \vec{\pi}$$

with real fields
we can apply the
Goldstone analysis

$$\phi \equiv \begin{pmatrix} \sigma \\ \vec{\pi} \end{pmatrix} \qquad \mathcal{L} = \frac{1}{2}\partial^\mu \phi^T \partial_\mu \phi - \frac{\lambda}{4}\left(\phi^T\phi - f^2\right)^2$$

$$\mathcal{F} = \begin{pmatrix} f \\ 0 \end{pmatrix} \qquad \vec{\epsilon}_L \cdot \vec{T}^L\,\mathcal{F} = \begin{pmatrix} 0 \\ \vec{\epsilon}_L/2 \end{pmatrix} \qquad \vec{\epsilon}_R \cdot \vec{T}^R\,\mathcal{F} = \begin{pmatrix} 0 \\ -\vec{\epsilon}_R/2 \end{pmatrix}$$

$$\vec{S} = \vec{T}^V \equiv \vec{T}^L + \vec{T}^R \qquad \vec{\epsilon}_V \cdot \vec{S} = \vec{\epsilon}_V \cdot (\vec{T}^L + \vec{T}^R)\,\mathcal{F}$$

$$= \vec{\epsilon}_V \cdot \vec{T}^L\,\mathcal{F} + \vec{\epsilon}_V \cdot \vec{T}^R\,\mathcal{F} = \begin{pmatrix} 0 \\ \vec{\epsilon}_V/2 \end{pmatrix} + \begin{pmatrix} 0 \\ -\vec{\epsilon}_V/2 \end{pmatrix} = 0$$

vector or “diagonal”
symmetry is unbroken

$$\vec{X} = \vec{T}^A \equiv \vec{T}^L - \vec{T}^R \qquad i\vec{\epsilon}_A \cdot \vec{X}\,\mathcal{F} = \begin{pmatrix} 0 \\ \vec{\epsilon}_A f/2 \end{pmatrix}$$

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$$\phi' = \phi = \mathcal{F} \qquad \begin{matrix} \sigma' = \sigma - f \\ \pi' = \pi \end{matrix}$$

the Goldstone
bosons are

$$\mathcal{F}^T \vec{X} \phi' \propto \vec{\pi}$$

$$\mathcal{L} = \frac{1}{2}\partial^\mu \sigma' \partial_\mu \sigma' + \frac{1}{2}\partial^\mu \vec{\pi} \cdot \partial_\mu \vec{\pi} - \frac{\lambda}{4}\left((\sigma' + f)^2 + \pi_a \pi_a - f^2\right)^2$$

$$= \frac{1}{2}\partial^\mu \sigma' \partial_\mu \sigma' + \frac{1}{2}\partial^\mu \vec{\pi} \cdot \partial_\mu \vec{\pi} - \frac{\lambda}{4}\left(\sigma'^2 + \pi_a \pi_a + 2f\sigma'\right)^2$$

no $\vec{\pi}$
mass

σ-model deceptively simple because $SU(2) \times SU(2)$ algebra $\Leftrightarrow SO(4)$

3×3 matrix Φ
$SU(3) \times SU(3)$ symmetry $\quad \Phi \to L\Phi R^\dagger \quad \begin{array}{c} L = \exp\left(il^a T_a^L\right) \\ R = \exp\left(ir^a T_a^R\right) \end{array} \quad \mathcal{L} =$

$$\mathrm{tr}\left(\partial_\mu \Phi^\dagger \partial^\mu \Phi - \frac{\tilde\lambda_1}{2}\Phi\Phi^\dagger \mathrm{tr}(\Phi\Phi^\dagger) - \frac{\tilde\lambda_2}{2}\Phi\Phi^\dagger\Phi\Phi^\dagger + m^2\Phi\Phi^\dagger\right) + \frac{\kappa}{3}(\det\Phi + \text{h.c.})$$

$$\mathrm{cof}\Phi_{ij} = \frac{1}{2}\epsilon_{ii_1i_2}\epsilon_{jj_1j_2}\Phi_{i_1j_1}\Phi_{i_2j_2} \qquad \det\Phi = \frac{1}{3}\sum_{ij}\Phi_{ij}\mathrm{cof}\Phi_{ij}$$

$$\det\Phi = \frac{1}{3}\mathrm{tr}\left(\Phi^T \mathrm{cof}\Phi\right)$$

invariance of $\det\Phi \Rightarrow$ $\quad \mathrm{cof}\Phi \to L^* \mathrm{cof}\Phi R^T$

$$\mathcal{L} = \mathrm{tr}\left(\partial_\mu\Phi^\dagger\partial^\mu\Phi\right) - \frac{\lambda_1}{2}\left(\mathrm{tr}(\Phi\Phi^\dagger) - \tfrac{3f^2}{4}\right)^2 - \frac{\lambda_2}{2}\mathrm{tr}\left(\left(\Phi\Phi^\dagger - \tfrac{f^2}{4}\right)^2\right)$$

$$-\frac{\lambda_3}{2}\mathrm{tr}\left(\mathrm{cof}\Phi - \tfrac{f}{2}\Phi^*\right)\left(\mathrm{cof}\Phi^\dagger - \tfrac{f}{2}\Phi^T\right) \quad \rightsquigarrow \quad \langle\Phi\rangle = fI/2$$

$\langle\Phi\rangle = fI/2$ breaks
$SU(3) \times SU(3) \to SU(3)$

Exercise 6. Show that the traceless $\Phi - \Phi^\dagger$ are massless fields and other fields are massive

the physics of spontaneously broken continuous symmetry — a mechanical analogy

transverse waves in a string described by wave function $\psi(x)$

invariance under translations $x \to x + c$
normal modes $\psi(x+c) \propto \psi(x)$
$\Rightarrow$ normal modes e^{ikx}

dispersion relation $\omega^2(k^2) \propto$ restoring force for the k mode from mechanical properties of the string (tension, stiffness, etc)

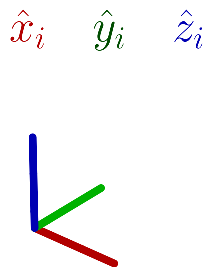
invariance under translations $y \to y + d$
— normal modes already fixed
symmetry spontaneously broken
tells you about the dispersion relation $\omega^2 \to 0$ as $k^2 \to 0$

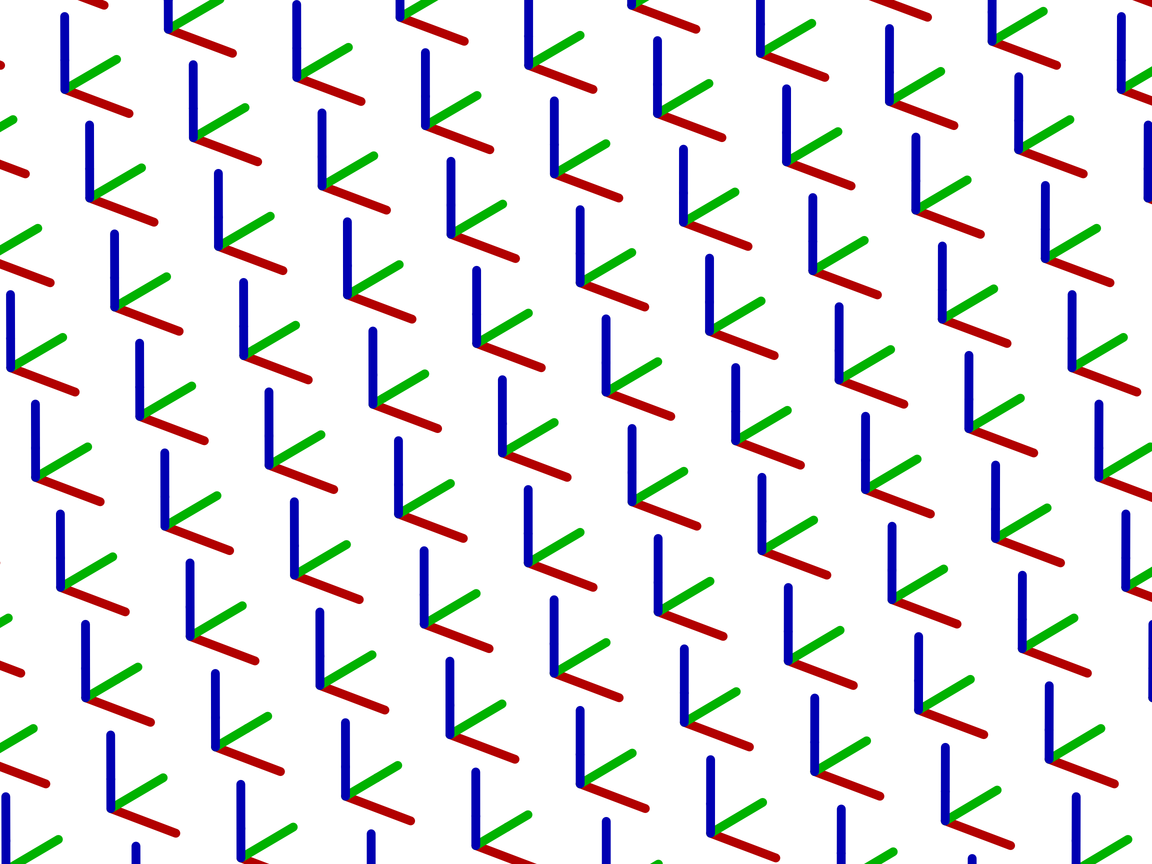
$k \to 0$ gets closer and closer to a translation of the whole string
the spontaneously broken symmetry guarantees that there is no restoring force as $k \to 0$ — but notice that the system never gets to $k = 0$ — that would require motion of the whole infinite system

These perturbative models have spontaneously broken symmetry, but the simple linear transformation laws for the fields obscure the fact that the low energy interactions are completely determined by the symmetry.

Let's try to visualize what is happening with a toy model of $SU(2) \times SU(2) \to SU(2)$ — $SU(2)$ is nice because an $SU(2)$ transformation (at least a small one) looks like a rotation in 3D space, which can be specified by what it does to the orientation of a set or coordinate axes, $\hat{x}_i$, $\hat{y}_i$ $\hat{z}_i$ in some internal space.

$SU(2) \sim SO(3)$ specify vacuum by orientation in isospin space at each point in ordinary space.





what happens when a Goldstone boson wave passes through?

We will send a wave with this shape by a small square region

gb-b0.flv — looking at it from “above” the wave passes through

gb-a2.flv — nothing much happens — vacuum twists and untwists

gb-b1.flv — two waves one after another

gb-b2.flv — more interesting with two overlapping waves

gb-a3.flv — still nothing

gb-a4.flv — more interesting — these waves interacted

3×3 matrix Φ
$SU(3)\times SU(3)$ symmetry

$$\Phi \to L\Phi R^\dagger \qquad L=\exp\left(il^aT^L_a\right) \quad R=\exp\left(ir^aT^R_a\right)$$

$$\mathcal{L}=\operatorname{tr}\left(\partial_\mu\Phi^\dagger\partial^\mu\Phi\right)-\frac{\lambda_1}{2}\left(\operatorname{tr}(\Phi\Phi^\dagger)-\tfrac{3f^2}{4}\right)^2-\frac{\lambda_2}{2}\operatorname{tr}\left(\left(\Phi\Phi^\dagger-\tfrac{f^2}{4}\right)^2\right)$$

$$-\frac{\lambda_3}{2}\operatorname{tr}\left(\operatorname{cof}\Phi-\tfrac{f}{2}\Phi^*\right)\left(\operatorname{cof}\Phi^\dagger-\tfrac{f}{2}\Phi^T\right) \quad \rightsquigarrow \quad \langle\Phi\rangle=fI/2$$

$\langle\Phi\rangle=fI/2$ breaks $SU(3)\times SU(3)$
$\to SU(3)\to$ Goldstone bosons

$l=r\Rightarrow L=R=u \quad \Phi\to u\Phi u^\dagger$
$l=-r\Rightarrow L=R^\dagger=\xi \quad \Phi\to\xi\Phi\xi$

describe the GBs by a chiral transformation!

$$\Phi=\xi e^{i\phi/f}(h+fI/2)\xi \quad \text{where} \quad h=h^\dagger$$

how do ξ, h and ϕ transform?

$$\Phi\to L\Phi R^\dagger=\xi' e^{i\phi'/f}(h'+fI/2)\xi'$$

$\phi'=\phi$ because $\det L=\det R=1$

$\operatorname{tr}\left(\left(\Phi^\dagger\Phi\right)^n\right)$ is invariant so the eigenvalues of h don't change $\Rightarrow h'=uhu^\dagger$ (where $u(L,R,\xi)$)

$$L\xi(h+fI/2)\xi R^\dagger=\xi' u(h+fI/2)u^\dagger\xi' \qquad \xi'=L\xi u^\dagger=u\xi R^\dagger$$

nonlinear transformation

homogeneous on $h\to uhu^\dagger$
crazy on $\xi\to L\xi u^\dagger=u\xi R^\dagger$

$$\xi^2\to L\xi^2R^\dagger$$

3×3 matrix Φ
$SU(3)\times SU(3)$ symmetry

$$\Phi \to L\Phi R^\dagger \qquad L=\exp\left(il^aT^L_a\right) \quad R=\exp\left(ir^aT^R_a\right)$$

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$$-\frac{\lambda_3}{2}\operatorname{tr}\left(\operatorname{cof}\Phi - \tfrac{f}{2}\Phi^*\right)\left(\operatorname{cof}\Phi^\dagger - \tfrac{f}{2}\Phi^T\right) \quad \rightsquigarrow \quad \langle\Phi\rangle = fI/2$$

$\langle\Phi\rangle = fI/2$ breaks $SU(3)\times SU(3)$
$\to SU(3) \to$ Goldstone bosons

$l=r \Rightarrow L=R=u \quad \Phi\to u\Phi u^\dagger$
$l=-r \Rightarrow L=R^\dagger=\xi \quad \Phi\to \xi\Phi\xi$

describe the GBs by a chiral transformation!

$$\Phi = \xi\, e^{i\phi/f}\,(h+fI/2)\,\xi \quad \text{where} \quad h=h^\dagger$$

no ξ dependence
in the potential

$\left(\operatorname{tr}(\Phi\Phi^\dagger) - \frac{3f^2}{4}\right)^2$	$\operatorname{tr}\left(\left(\Phi\Phi^\dagger - \frac{f^2}{4}\right)^2\right)$	$\operatorname{tr}\left(\operatorname{cof}\Phi - \frac{f}{2}\Phi^*\right)\left(\operatorname{cof}\Phi^\dagger - \frac{f}{2}\Phi^T\right)$
mass to $\operatorname{tr}h$ $SO(18)$	mass to h chiral $U(1)$	mass to ϕ, h just $SU(3)\times SU(3)$

at low energies
only GBs survive

$$\mathcal{L} = \frac{f^2}{4}\operatorname{tr}\left(\partial_\mu U^\dagger\partial^\mu U\right) \quad \text{where} \quad U=\xi^2$$

very low energy physics completely determined by symmetry and f

$$\mathcal{L} = (i\bar{q}\,\not{D}\,q - \bar{q}\,M_q\,q) - \frac{1}{4}G_a^{\mu\nu}\,G_{a\mu\nu} - \bar{q}\,\gamma^\mu\,(v_\mu + a_\mu\gamma_5)\,q$$

quark field q vector in 3D color space + 3D flavor space

$D^\mu = \partial^\mu + igT_aG_a^\mu$ is "covariant derivative"

$igT_aG_a^{\mu\nu} = [D^\mu,\, D^\nu]$ $G_a^{\mu\nu}$ is "gluon field–strength"

T_a 3×3 traceless Hermitian matrices in color space

$$\mathrm{tr}(T_aT_b) = \frac{1}{2}\delta_{ab}$$

M_q diagonal in flavor space u, d, s

color gauge symmetry

$$T_aG_a^\mu \to UT_aG_a^\mu U^\dagger - \tfrac{i}{g}U\partial^\mu U^\dagger$$

$$q \to Uq \qquad G^{\mu\nu} \to UG^{\mu\nu}U^\dagger$$

T_a's are "color" charges like EM charge in QED binds quarks and antiquarks into color–neutral combinations like photon exchange binds charged particles into electrically neutral atoms — color–neutral combinations are

$\bar{q}q$ mesons and $\epsilon_{jkl}q_jq_kq_l$ baryons

$v^\mu = v_\alpha^\mu t_\alpha$ and $a^\mu = a_\alpha^\mu t_\alpha$ sources for the "vector" and "axial vector" currents

where t_α are 3×3 hermitian flavor generators

$$\mathcal{L} = i\bar{q}\,\not{D}\,q - \frac{1}{4}G_a^{\mu\nu}\,G_{a\mu\nu} - \bar{q}\gamma^\mu(v_\mu + a_\mu\gamma_5)q - \bar{q}(s + ip\gamma_5)q$$

$$= i\bar{q}\,\not{D}\,q - \frac{1}{4}G_a^{\mu\nu}\,G_{a\mu\nu} - \bar{q}_L\gamma^\mu\ell_\mu q_L - \bar{q}_R\gamma^\mu r_\mu q_R - \bar{q}_R(s + ip)q_L - \bar{q}_L(s - ip)q_R$$

r_μ, ℓ_μ, s and p are classical Hermitian 3×3 matrix fields ($\mathrm{tr}\ell_\mu = \mathrm{tr}r_\mu = 0$) — **just assuming (at this point) no chiral $U(1)$ symmetry (det term)???**

$$r_\mu = v_\mu - a_\mu \qquad \ell_\mu = v_\mu + a_\mu \qquad s \to M_q$$

r_μ and ℓ_μ fields are classical gauge fields for the $SU(3)_R$ and $SU(3)_L$ symmetries, and also the sources for the corresponding Noether currents

we want to build a low energy theory with the same symmetries —

$$\begin{aligned} q_R &\to R\,q_R \\ q_L &\to L\,q_L \\ r_\mu &\to R\,r_\mu\,R^\dagger - iR\,\partial_\mu\,R^\dagger \\ \ell_\mu &\to L\,\ell_\mu\,L^\dagger - iL\,\partial_\mu\,L^\dagger \\ s + ip &\to R\,(s + ip)\,L^\dagger \end{aligned}$$

GB field U

$$\begin{aligned} U &\to L\,U\,R^\dagger \\ r_\mu &\to R\,r_\mu\,R^\dagger - iR\,\partial_\mu\,R^\dagger \\ \ell_\mu &\to L\,\ell_\mu\,L^\dagger - iL\,\partial_\mu\,L^\dagger \\ s + ip &\to R\,(s + ip)\,L^\dagger \end{aligned}$$

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GB field U

$$\begin{aligned} U &\to L\, U\, R^\dagger \\ r_\mu &\to R\, r_\mu\, R^\dagger - iR\,\partial_\mu\, R^\dagger \\ \ell_\mu &\to L\,\ell_\mu\, L^\dagger - iL\,\partial_\mu\, L^\dagger \\ s+ip &\to R\,(s+ip)\,L^\dagger \end{aligned}$$

low energy theory effective $\mathcal{L}$ depends on U and classical sources

$$\begin{aligned} D^\mu U &= \partial^\mu U + i\ell^\mu U - iU r^\mu \\ D^\mu U^\dagger &= \partial^\mu U^\dagger + i r^\mu U^\dagger - iU^\dagger \ell^\mu \\ D^\mu U &\to L\, D^\mu U\, R^\dagger \\ D^\mu U^\dagger &\to R\, D^\mu U^\dagger\, L^\dagger \end{aligned}$$

QCD also invariant under parity

$$\begin{aligned} \vec{r} &\leftrightarrow -\vec{r} \\ U &\leftrightarrow U^\dagger \\ \ell^\mu &\leftrightarrow r^\mu \\ s &\to s \\ p &\to -p \end{aligned}$$

most general effective Lagrangian consistent with the symmetries — only useful because we can expand in powers of derivatives + sources and truncate the expansion after a small number of terms

leading term for $s = p = 0$

$$\mathcal{L}(U, r, \ell, s, p) = \frac{f^2}{4}\mathrm{tr}\left(D^\mu U^\dagger D_\mu U\right)$$

most general effective Lagrangian consistent with the symmetries — only useful because we can expand in powers of derivatives sources and truncate the expansion after a small number of terms

leading term for $s = p = 0$

$$\mathcal{L}(U, r, \ell, s, p) = \frac{f^2}{4} \mathrm{tr}\left(D^\mu U^\dagger D_\mu U\right)$$

$$U = \exp[2i\Pi/f]$$

expand in powers of Π

$$\Pi = \pi_a T_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K^0} & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}$$

$$\pi^- = \pi^{+\dagger}$$
$$K^- = K^{+\dagger}$$
$$\overline{K^0} = K^{0\dagger}$$

expanding derivative term in Π

$$\frac{f^2}{4} \mathrm{tr}\left(D^\mu U^\dagger D_\mu U\right) = \frac{1}{2}\partial_\mu \pi_a \partial^\mu \pi_a$$

$$= \frac{1}{2}\left(\partial_\mu \pi^0 \partial^\mu \pi^0 + \partial_\mu \eta \partial^\mu \eta\right) + \partial_\mu \pi^+ \partial^\mu \pi^- + \partial_\mu K^+ \partial^\mu K^- + \partial_\mu K^0 \partial^\mu \overline{K^0}$$

properly normalized KE for meson fields

$$U = \exp[2i\Pi/f]$$ expand in powers of Π

$$\Pi = \pi_a T_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K^0} & -\frac{2}{\sqrt{6}}\eta \end{pmatrix} \qquad \begin{aligned} \pi^- &= {\pi^+}^\dagger \\ K^- &= {K^+}^\dagger \\ \overline{K^0} &= {K^0}^\dagger \end{aligned}$$

pure vector $SU(3)$ transformation $L = R = u$

$$U \to U' = uUu^\dagger \Rightarrow \Pi \to \Pi' = u\Pi u^\dagger$$ mesons transform like $SU(3)$ octet

pure chiral transformation $L = R^\dagger = c$

$$U' = e^{ic}Ue^{ic}$$

take c to be infinitesimal and write Π' as a power series in c and Π

$$(1 + 2i\Pi'/f + \cdots) = (1 + ic + \cdots)(1 + 2i\Pi/f + \cdots)(1 + ic + \cdots).$$

$$\Pi' = \Pi + f\,c + \cdots \qquad \text{or} \qquad \pi'_a = \pi_a + f c_a + \cdots$$

terms are odd in π and c because of parity — c and π change signs under $L \leftrightarrow R$
inhomogeneous term a signal of spontaneous symmetry breakdown

$$\frac{1}{4}\mathrm{tr}\left(\partial^\mu U^\dagger \partial_\mu U\right) \quad \text{with} \quad U = \exp[2i\Pi/f]$$

explore nonlinear terms — for any matrix M the derivative of e^M

$$\partial^\mu e^M = \int_0^1 \, e^{(1-s)M}\, \partial^\mu M\, e^{sM}\, ds \quad M^\dagger \partial^\mu e^M = \int_0^1 \, e^{-sM}\, \partial^\mu M\, e^{sM}\, ds$$

$$\frac{1}{4}\mathrm{tr}\left(\partial^\mu U^\dagger \partial_\mu U\right) = \frac{1}{4}\mathrm{tr}\left(\partial^\mu U^\dagger U U^\dagger \partial_\mu U\right) = -\frac{1}{4}\mathrm{tr}\left(U^\dagger \partial^\mu U\, U^\dagger \partial_\mu U\right)$$

$$U^\dagger \partial^\mu U = \frac{2i}{f}\int_0^1 \, e^{-2is\Pi/f}\, \partial^\mu \Pi\, e^{2is\Pi/f}\, ds$$

$$= \frac{2i}{f}\int_0^1 \, \left(\partial^\mu \Pi - \frac{2is}{f}[\Pi, \partial^\mu \Pi] - \frac{2s^2}{f^2}[\Pi, [\Pi, \partial^\mu \Pi]] + \cdots\right)\, ds$$

because GBs are rotations of the vacuum, the only reason they interact at all in leading order is the non-Abelian nature of the symmetry — introduces nonlinearity in the scattering of classical GB waves we want to build a low energy theory with the same symmetries —

	GB field U
$q_R \to R\, q_R$	$U \to L\, U\, R^\dagger$
$q_L \to L\, q_L$	
$r_\mu \to R\, r_\mu\, R^\dagger - iR\,\partial_\mu\, R^\dagger$	$r_\mu \to R\, r_\mu\, R^\dagger - iR\,\partial_\mu\, R^\dagger$
$\ell_\mu \to L\, \ell_\mu\, L^\dagger - iL\,\partial_\mu\, L^\dagger$	$\ell_\mu \to L\, \ell_\mu\, L^\dagger - iL\,\partial_\mu\, L^\dagger$
$s + ip \equiv \mathcal{M} \to R\, \mathcal{M}\, L^\dagger$	$\mathcal{M} \to R\, \mathcal{M}\, L^\dagger$

low energy theory effective $\mathcal{L}$ depends on U and classical sources

$$D^\mu U = \partial^\mu U + i\ell^\mu U - iU r^\mu$$
$$D^\mu U^\dagger = \partial^\mu U^\dagger + ir^\mu U^\dagger - iU^\dagger \ell^\mu$$
$$D^\mu U \to L\, D^\mu U\, R^\dagger$$
$$D^\mu U^\dagger \to R\, D^\mu U^\dagger\, L^\dagger$$

QCD also invariant under parity

$$\vec{r} \leftrightarrow -\vec{r}$$
$$U \leftrightarrow U^\dagger$$
$$\ell^\mu \leftrightarrow r^\mu$$
$$\mathcal{M} \leftrightarrow \mathcal{M}^\dagger$$

most general effective Lagrangian consistent with the symmetries — only useful because we can expand in powers of derivatives sources and truncate the expansion after a small number of terms

leading term for $\mathcal{M} = 0$

$$\mathcal{L}(U, r, \ell, s, p) = \frac{f^2}{4}\mathrm{tr}\left(D^\mu U^\dagger D_\mu U\right)$$

including $\mathcal{M}$ — parameter μ (mass)

$$\mathcal{L} = f^2 \left(\frac{1}{4} \mathrm{tr}\left(D^\mu U^\dagger D_\mu U\right) + \mathrm{tr}\left(U \mu \mathcal{M} + \text{h.c.}\right) + \cdots \right)$$

$$\Pi = \pi_a T_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K^0} & -\frac{2}{\sqrt{6}}\eta \end{pmatrix} \qquad \begin{matrix} K^- = {K^+}^\dagger \\ \overline{K^0} = {K^0}^\dagger \end{matrix}$$

$U = \exp(2i\Pi/f)$ and $D^\mu U = \partial^\mu U + i\ell^\mu U - iUr^\mu$ (gives the currents) and

symmetry breaking by $\mathcal{M} = M$

$\bar{q}_L M q_R + \text{h.c.}$ gives “GB” masses

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

for μ and M real, the linear term cancels and the quadratic term is

$$-2\mathrm{tr}(\mu M \Pi^2),$$

which corresponds to a mass term (like baryons but only one term)

$$4\mathrm{tr}(\mu M \Pi^2)$$

for the Π — evaluate these masses in the limit of isospin invariance, ignoring weak and electromagnetic interactions and setting $m_u = m_d = m$

$$m_\pi^2 = 4\mathrm{tr}\left(\mu M \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = 2\mu m$$

$$m_K^2 = 4\mathrm{tr}\left(\mu M \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}\right) = \mu(m + m_s)$$

$$m_\eta^2 = 4\mathrm{tr}\left(\mu M \begin{pmatrix} \frac{1}{12} & 0 & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}\right) = \frac{2\mu}{3}(m + 2m_s)$$

The m^2 determined in this way satisfy the Gell-Mann Okubo relation

$$3m_\eta^2 + m_\pi^2 = 4m_K^2$$

but here we are using the momentum expansion rather than expanding in powers of the symmetry breaking — this depends on the $SU(3)$ symmetric part of M and makes sense even for $m_\pi \to 0$ — and explains why GMO for mesons works much better for m^2 than for m

notice that M gives a potential $-\frac{f^2}{2}\mathrm{tr}\left(U^\dagger \mu M\right) - \frac{f^2}{2}\mathrm{tr}\left(U \mu M\right)$ that is minimized for $U = I$ — until we turned on M all U were equally good vacua if we don't assume $m_u = m_d$

$$m^2_{\pi^0} = m^2_{\pi^\pm} = 4\mathrm{tr}\left(\mu M \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \mu(m_u + m_d)$$

$$m^2_{K^\pm} = 4\mathrm{tr}\left(\mu M \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}\right) = \mu(m_u + m_s)$$

$$m^2_{K^0} = 4\mathrm{tr}\left(\mu M \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}\right) = \mu(m_d + m_s)$$

$$m^2_{\eta} = 4\mathrm{tr}\left(\mu M \begin{pmatrix} \frac{1}{12} & 0 & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}\right) = \frac{\mu}{3}(m_u + m_d + 4m_s)$$

$$m^2_{\eta-\pi} = 4\mathrm{tr}\left(\mu M \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & -\frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \frac{2\mu}{\sqrt{3}}(m_u - m_d)$$

the small $\eta - \pi^0$ mixing term is higher order in isospin breaking and very small

$$m_{\pi^0}^2 = m_{\pi^\pm}^2 = 4\mathrm{tr}\left(\mu M \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \mu(m_u + m_d)$$

$$m_{K^\pm}^2 = 4\mathrm{tr}\left(\mu M \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}\right) = \mu(m_u + m_s)$$

$$m_{K^0}^2 = 4\mathrm{tr}\left(\mu M \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}\right) = \mu(m_d + m_s)$$

$$m_\eta^2 = 4\mathrm{tr}\left(\mu M \begin{pmatrix} \frac{1}{12} & 0 & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}\right) = \frac{\mu}{3}\left(m_u + m_d + 4m_s\right)$$

quark masses do not split $m_{\pi^\pm}$ from m_{π^0} to this order - both proportional to $m_u + m_d$ (the splitting is a $\Delta I = 2$ effect)

but EM interactions do — Q has L and R part — photon loops $\rightsquigarrow$ $\propto \mathrm{tr}(UQU^\dagger Q)$ $\rightsquigarrow \Delta m_{\pi^\pm}^2 = \Delta m_{K^\pm}^2$ others 0 $\to m_q$ ratios including $\mathcal{M}$ and Q

$$\mathcal{L} = f^2 \left(\frac{1}{4} \mathrm{tr}\left(D^\mu U^\dagger D_\mu U \right) + \frac{f^2}{2} \mathrm{tr}\left(U \mu \mathcal{M} + \text{h.c.} \right) + \kappa \mathrm{tr}(UQU^\dagger Q) + \cdots \right)$$

$$\Pi = \pi_a T_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K^0} & -\frac{2}{\sqrt{6}}\eta \end{pmatrix} \qquad \begin{matrix} K^- = K^{+\dagger} \\ \overline{K^0} = K^{0\dagger} \end{matrix}$$

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symmetry breaking by $s = M$

$\bar{q}_L M q_R + \text{h.c.}$ gives "GB" masses

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

ℓ_μ is the source the LH octet currents

$$if^2 \frac{1}{4} \mathrm{tr}\Big(U^\dagger \ell_\mu \partial^\mu U - (\partial^\mu U^\dagger) \ell_\mu U \Big) = -if^2 \frac{1}{2} \mathrm{tr}\Big(\ell_\mu U \partial^\mu U^\dagger \Big)$$

$$= f \int_0^1 \mathrm{tr} \left(\ell_\mu \left(-\partial^\mu \Pi - \frac{2is}{f} [\Pi, \partial^\mu \Pi] + \frac{2s^2}{f^2} [\Pi, [\Pi, \partial^\mu \Pi]] + \cdots \right) \right) ds$$

the general case - coset construction - and how $\rightsquigarrow SU(3) \times SU(3)$

G generated by T_a elements $g = e^{i\gamma_a T^a}$
broken to H generated by S_a

$T_a \rightsquigarrow$ T_a^L elements $L = e^{i\ell_a T^a}$
T_a^R elements $R = e^{ir_a T^a}$

unbroken generators $S_a \rightsquigarrow T_a^L = T_a^R = T_a$ broken generators $X_a \rightsquigarrow T_a^L = -T_a^R = T_a$

Goldstone bosons are the group elements associated with X_a

GBs $\xi = e^{i\pi_a X^a} \rightsquigarrow L = \xi = e^{i\pi_a T_a}$ and $R = \xi^\dagger = e^{-i\pi_a T_a}$

general transformation can be uniquely decomposed into a product of a broken transformation (GB) and an unbroken transformation

$$e^{ix_a X^a}\, e^{is_a S^a} \rightsquigarrow L = e^{ix_a T^a}\, e^{is_a T^a} \text{ and } R = e^{-ix_a T^a}\, e^{is_a T^a}$$

$T \to S + X$ determines how the GBs transform

$$e^{i\gamma_a T_a}\xi = e^{ix_a X^a} e^{is_a S_a} \equiv \xi'\, u(\gamma, \xi) \rightsquigarrow \begin{array}{c} L\xi = \xi' u(L,R,\xi) \\ R\xi^\dagger = \xi'^\dagger u(L,R,\xi) \end{array}$$

Goldstone bosons in the Coset space G/H $\rightsquigarrow \dfrac{SU(3) \times SU(3)}{SU(3)}$

back to the heavy baryon effective theory using matrix B to describe the low-energy baryon — matrix structure gets the flavor right —

spin is just ordinary angular momentum because we have broken Lorentz invariance by going to the frame in which the baryon is at rest — so B is a Pauli spinor and a flavor matrix

$$\mathcal{L} = \mathrm{tr}\left(\overline{B}\mathcal{D}^0(B) + i\,\frac{f}{\Lambda}\,\overline{B}\sigma_j\epsilon_{jk\ell}[T_a v_a^{k\ell}, B] + \frac{d}{\Lambda}\,\overline{B}\sigma_j\epsilon_{jk\ell}\{T_a v_a^{k\ell}, B\} + \cdots \right)$$

$$\text{where} \quad \mathcal{D}^\mu(O) = \partial^\mu O + i v_\alpha^\mu [T_\alpha, O]$$

how do the baryons transform under chiral $SU(3) \times SU(3)$? the answer is that they don't! baryons and all the other QCD bound states live in the vacuum defined by $SU(3) \times SU(3) \to SU(3)$ — we know that it doesn't make sense to make a global rotation of this vacuum and thus it only makes sense to ask how the baryons transform under the unbroken Gell-Mann's $SU(3)$ — and this we know — they are an octet — but this is enough to allow us to construct an $SU(3) \times SU(3)$ invariant theory using the Goldstone Boson fields ξ which transform under a general L and R like

$$\xi \to \xi' = L\xi u(L, R, \xi)^\dagger = u(L, R, \xi)\xi R^\dagger$$

$$\xi \to \xi' = L\xi u(L,R,\xi)^\dagger = u(L,R,\xi)\xi R^\dagger$$

$u(L,R,\xi)$ is the vacuum preserving part of the transformation — all the rest goes into transforming the Πs which are the physically sensible local rotations of the physical vacuum — thus under a general L and R, the baryon transformation law is nonlinear and involves ξ

$$B \to B' = u(L,R,\xi)\, B\, u(L,R,\xi)^\dagger$$

to build an invariant $\mathcal{L}$, we have to deal with derivatives — a little crazy because we now have 3 "local" symmetries — L and R because we demand classical gauge invariance with the sources as gauge field — and u which depends on space and time through its dependence on L, R and ξ — build two octets out of the GB fields ξ with simple transformation laws under u

This beautiful construction comes from two classic papers —CWZ and CCWZ

$$\xi \to \xi' = L\xi u(L,R,\xi)^\dagger = u(L,R,\xi)\xi R^\dagger$$

vector field $\quad V^\mu = -\frac{i}{2}\left(\xi^\dagger(\partial^\mu + i\ell^\mu)\xi + \xi(\partial^\mu + ir^\mu)\xi^\dagger\right)$

transforms like a gauge field for the local u transformation

$$V^\mu \to u(L,R,\xi)V^\mu u(L,R,\xi)^\dagger - i\,u(L,R,\xi)\partial^\mu u(L,R,\xi)^\dagger$$

axial vector field $\quad A^\mu = -\frac{i}{2}\left(\xi^\dagger(\partial^\mu + i\ell^\mu)\xi - \xi(\partial^\mu + ir^\mu)\xi^\dagger\right)$

transforms like an $SU(3)$ octet field (note $\mathrm{tr}V^\mu = \mathrm{tr}A^\mu = 0$)

$$A^\mu \to u(L,R,\xi)A^\mu u(L,R,\xi)^\dagger$$

with V^μ we can make an improved covariant derivative with chiral symmetry

$$\mathcal{D}^\mu(B) = \partial^\mu B + i[V^\mu, B] \to u(L,R,\xi)\mathcal{D}^\mu(B)\,u(L,R,\xi)^\dagger$$

the leading derivative terms in $\mathcal{L}$ are

$$\mathcal{L} = \mathrm{tr}\left(B^\dagger\mathcal{D}^0(B) + d\,B^\dagger\vec{\sigma}\cdot\{\vec{A},B\} + if\,B^\dagger\vec{\sigma}\cdot[\vec{A},B] + \cdots\right)$$

$$\mathcal{L} = \mathrm{tr}\left(B^\dagger \mathcal{D}^0(B) + d\, B^\dagger \vec{\sigma} \cdot \{\vec{A}, B\} + if\, B^\dagger \vec{\sigma} \cdot [\vec{A}, B] + \cdots \right)$$

$$\mathcal{D}^\mu(B) = \partial^\mu B + i[V^\mu, B] \qquad \xi = e^{i\Pi/f}$$

$$V^\mu = -\frac{i}{2}\left(\xi^\dagger(\partial^\mu + i\ell^\mu)\xi + \xi(\partial^\mu + ir^\mu)\xi^\dagger\right) = \frac{\ell^\mu + r^\mu}{2} + \cdots$$

$$A^\mu = -\frac{i}{2}\left(\xi^\dagger(\partial^\mu + i\ell^\mu)\xi - \xi(\partial^\mu + ir^\mu)\xi^\dagger\right) = \frac{\ell^\mu - r^\mu}{2} + \partial^\mu \Pi/f + \cdots$$

the $\mathcal{D}^0$ term is what we saw before — it determines the couplings of the vector current

$$\gamma^0 = \tau_3 \quad \gamma^j = i\tau_1\sigma^j \quad \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \tau^2 \quad \gamma^0\gamma^j\gamma^5 = -\sigma^j$$

the d and f terms determine both the baryon matrix element of the axial vector current (some of which can be measured in the semileptonic weak interactions) and the GB couplings to the baryons — for the chiral $SU(2)$ currents between proton and neutron states only the f term contributes and the matrix element of the axial vector current in β decay is related to the pion coupling to nucleons — Goldberger-Treiman relation