Model #1 — Electrodynamics with a light pseudoscalar, ϕ, mass μ

$$\mathcal{L} = \bar{\psi}\,(i\,\not{\partial} - e\,\not{A} - m)\,\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 - i f\,\phi\,\bar{\psi}\gamma_5\psi$$

renormalizable quantum field theory —

all couplings have zero or positive mass dimension

no couplings are missing

most general couplings for this set of fields with zero or positive dimension and consistent with symmetries — Lorentz invariance, EM gauge invariance and parity — complete world except perhaps for Landau poles at very high energy, and the lack of gravity

we will use this simple world to introduce the idea of an effective quantum field theory at low energies — assuming $m \gg \mu$

this will be one of our central themes

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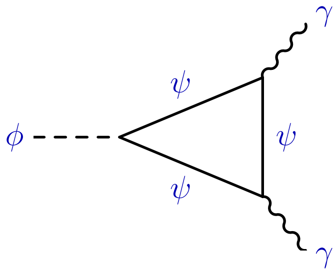
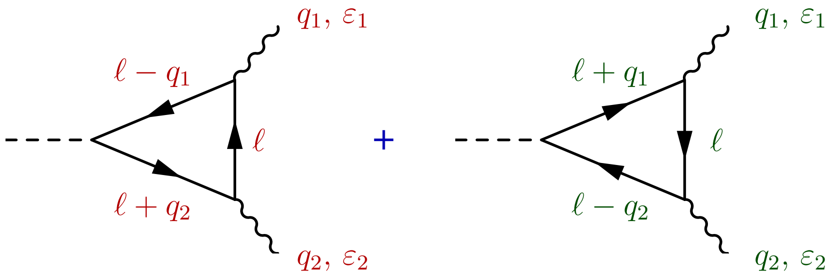
ϕ has no electric charge — but its Yukawa coupling to the charged electron allows quantum effects to induce the decay $\phi \to \gamma\gamma$ through loop diagrams



This is a famous Feynman diagram going back to Jack Steinberger's calculation of $\pi \to \gamma\gamma$ in the late '40s. In particular we will be interested in the limit $\mu \ll m$, which has something very important to teach us. But first let's just start calculate the amplitude from this diagram!

Actually there are two diagrams:



$$-f(-ie)^2\,\varepsilon_{1\mu_1}\varepsilon_{2\mu_2}\int \mathrm{Tr}\Big(\gamma^{\mu_1}S(\ell-q_1)\gamma_5\,S(\ell+q_2)\gamma^{\mu_2}S(\ell)\Big)\,\frac{d^4\ell}{(2\pi)^4}$$

where

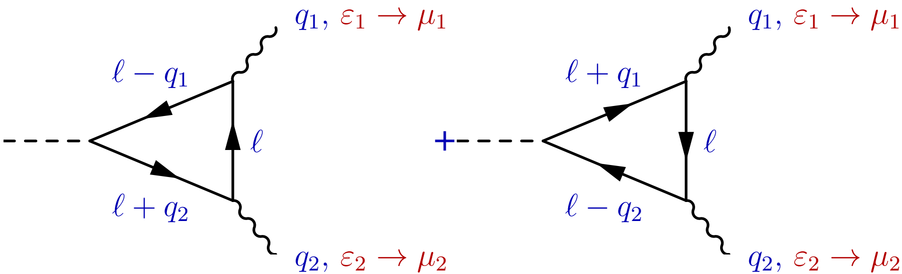
$$S(\ell)\equiv\frac{i(\not\ell+m)}{\ell^2-m^2+i\epsilon}$$

Tr on Dirac indices — order matters — minus sign from fermi statistics

$$-f(-ie)^2\,\varepsilon_{1\mu_1}\varepsilon_{2\mu_2}\int \mathrm{Tr}\Big(\gamma^{\mu_2}S(\ell-q_2)\gamma_5\,S(\ell+q_1)\gamma^{\mu_1}S(\ell)\Big)\,\frac{d^4\ell}{(2\pi)^4}$$

note that the two diagrams are related by Bose statistics — $\{q_1,\varepsilon_1\}\leftrightarrow\{q_2,\varepsilon_2\}$

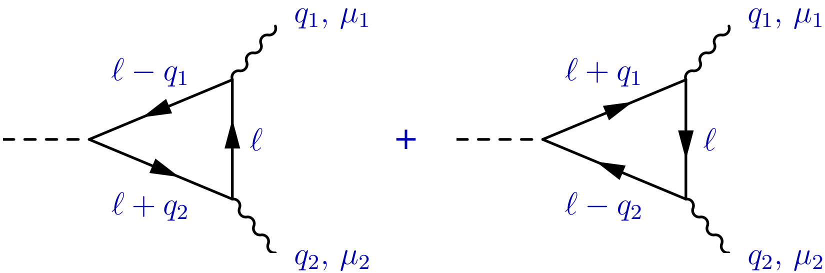
Useful to remove the εs



$$
\begin{aligned}
&M^{\mu_1\mu_2}(q_1, q_2) = \\
&-f(-ie)^2 \int \mathrm{Tr}\Big(\gamma^{\mu_1} S(\ell - q_1)\gamma_5\, S(\ell + q_2)\gamma^{\mu_2} S(\ell)\Big)\, \frac{d^4\ell}{(2\pi)^4} \\
&-f(-ie)^2 \int \mathrm{Tr}\Big(\gamma^{\mu_2} S(\ell - q_2)\gamma_5\, S(\ell + q_1)\gamma^{\mu_1} S(\ell)\Big)\, \frac{d^4\ell}{(2\pi)^4}
\end{aligned}
$$

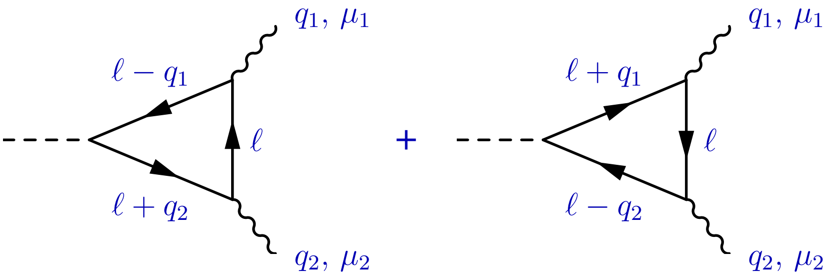
note that the two diagrams are related by Bose statistics — $\{q_1, \mu_1\} \leftrightarrow \{q_2, \mu_2\}$

There is another way to understand $M^{\mu_1\mu_2}$ —



$$M^{\mu_1\mu_2}(q_1, q_2) =$$
$$-f(-ie)^2 \int \mathrm{Tr}\Big(\gamma^{\mu_1} S(\ell - q_1)\gamma_5\, S(\ell + q_2)\gamma^{\mu_2} S(\ell)\Big)\, \frac{d^4\ell}{(2\pi)^4}$$
$$-f(-ie)^2 \int \mathrm{Tr}\Big(\gamma^{\mu_2} S(\ell - q_2)\gamma_5\, S(\ell + q_1)\gamma^{\mu_1} S(\ell)\Big)\, \frac{d^4\ell}{(2\pi)^4}$$
$$\propto \int \langle 0|\, T\, j^{\mu_1}(x_1)\, j^{\mu_2}(x_2)\, |\phi\rangle\ e^{iq_1x_1}\, e^{iq_2x_2}\, d^4x_1\, d^4x_2$$

Note that current conservation implies $q_{\mu_1} M^{\mu_1\mu_2}(q_1, q_2) = 0$ and this is automatic for $M^{\mu_1\mu_2}(q_1, q_2) \propto \epsilon^{\mu_1\mu_2\alpha\beta} q_{1\alpha} q_{2\beta}$



$$
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&-f(-ie)^2 \int \mathrm{Tr}\Big(\gamma^{\mu_2} S(\ell - q_2)\gamma_5\, S(\ell + q_1)\gamma^{\mu_1} S(\ell)\Big)\,\frac{d^4\ell}{(2\pi)^4}
\end{aligned}
$$

looks like integrands go like $1/\ell^3$ for large ℓ — potential divergence
but Lorentz invariance and parity

$$\Rightarrow M^{\mu_1\mu_2}(q_1,q_2) \propto \epsilon^{\mu_1\mu_2\alpha\beta} q_{1\alpha} q_{2\beta}$$

after the two qs are extracted, the integral is finite

$$M^{\mu_1\mu_2}(q_1,q_2) = -8m\,f\,e^2 \int \frac{\epsilon^{\mu_1\mu_2\nu_1\nu_2}\, q_{1\nu_1} q_{2\nu_2}}{\Big[(\ell-q_1)^2 - m^2\Big]\Big[(\ell+q_2)^2 - m^2\Big]\Big[\ell^2 - m^2\Big]} \frac{d^4\ell}{(2\pi)^4}$$

$$= -8m\,f\,e^2 \epsilon^{\mu_1\mu_2\nu_1\nu_2}\, q_{1\nu_1} q_{2\nu_2}\, I(q_1,q_2)$$

$$I(q_1,q_2) = \int \frac{1}{\Big[(\ell-q_1)^2 - m^2 + i\epsilon\Big]\Big[(\ell+q_2)^2 - m^2 + i\epsilon\Big]\Big[\ell^2 - m^2 + i\epsilon\Big]} \frac{d^4\ell}{(2\pi)^4}$$

What do we know about $I(q_1,q_2)$?

It is Lorentz invariant - depending only on q_1^2, q_2^2 and $(q_1 q_2) = q_{1\mu} q_2^\mu$

if q_1^2, q_2^2 and $(q_1 q_2)$ are large, it should scale like $1/q^2$

if q_1^2, q_2^2 and $(q_1 q_2) \ll m^2$ it should go to constant over m^2

we can calculate it exactly by using the standard trick to combine denominators

$$\frac{1}{ABC} = 2\int_0^1 \int_0^{1-\alpha} \frac{1}{\Big(\alpha A + \beta B + (1-\alpha-\beta)C\Big)^3}\, d\beta\, d\alpha$$

$$I(q_1, q_2) = 2\int_0^1 \int_0^{1-\alpha} \int \frac{1}{\left(k^2 + \alpha(1-\alpha)q_1^2 + \beta(1-\beta)q_2^2 + 2\alpha\beta(q_1 q_2) - m^2 + i\epsilon\right)^3} \frac{d^4k}{(2\pi)^4}\, d\beta\, d\alpha$$

so we need $\displaystyle \int \frac{1}{(k^2 - M^2 + i\epsilon)^3} \frac{d^4k}{(2\pi)^4}$

where

$$M^2 = m^2 - \alpha(1-\alpha)q_1^2 - \beta(1-\beta)q_2^2 - 2\alpha\beta(q_1 q_2)$$

Wick rotate to Euclidean space by deforming the k^0 integration for $M^2 > 0$

$$k^0 \to i\, k^4$$

$$\int \frac{1}{(k^2 - M^2 + i\epsilon)^3} \frac{dk^0\, d^3k}{(2\pi)^4} \to \int \frac{i}{(-k^2 - M^2)^3} \frac{d^3k\, dk^4}{(2\pi)^4}$$

$$\overbrace{\int \frac{1}{(k^2 - M^2 + i\epsilon)^3} \frac{d^4k}{(2\pi)^4}}^{\text{Minkowski}} = \overbrace{\int \frac{-i}{(k^2 + M^2)^3} \frac{d^4k}{(2\pi)^4}}^{\text{Euclidean}}$$

Euclidean space integral is straightforward

$$\int \frac{-i}{(k^2+M^2)^3}\frac{d^4k}{(2\pi)^4} = \int_0^\infty \frac{-i}{(k^2+M^2)^3}\frac{2\pi^2\,k^3\,dk}{(2\pi)^4}$$

$$= \int_0^\infty \frac{-i}{(k^2+M^2)^3}\frac{k^2\,dk^2}{16\pi^2} = \frac{-i}{32\pi^2\,M^2}$$

$$I(q_1,q_2) = 2\int_0^1\int_0^{1-\alpha}\int \frac{1}{(k^2-M^2+i\epsilon)^3}\frac{d^4k}{(2\pi)^4}\,d\beta\,d\alpha$$

where

$$M^2 = m^2 - \alpha(1-\alpha)q_1^2 - \beta(1-\beta)q_2^2 - 2\alpha\beta(q_1q_2)$$

$$I(q_1,q_2) = \frac{-i}{16\pi^2}\int_0^1\int_0^{1-\alpha}\frac{d\beta\,d\alpha}{m^2-\alpha(1-\alpha)q_1^2-\beta(1-\beta)q_2^2-2\alpha\beta(q_1q_2)}$$

the parameter integral is still messy - easier for an "on-shell" ϕ and "on-shell" photons, $q_1^2 = q_2^2 = 0$ and $= (q_1+q_2)^2 = 2(q_1q_2) = \mu^2$.

$$32i\pi^2m^2I \equiv K(\mu/m) = 2\int_0^1\int_0^{1-\alpha}\frac{m^2}{m^2-\alpha\beta\mu^2-i\epsilon}\,d\beta\,d\alpha$$

Mathematica knows how to do this one!

$$K(\mu/m) = 2\int_0^1 \int_0^{1-\alpha} \frac{m^2}{m^2 - \alpha\beta\mu^2 - i\epsilon}\, d\beta\, d\alpha$$

$$= 2\int_0^1 \int_0^{1-\alpha} \frac{1}{1 - \alpha\beta y^2 - i\epsilon}\, d\beta\, d\alpha \quad \text{where } y = \mu/m$$

$$= \frac{2}{y^2}\left(\mathrm{Li}_2\left(y^2/2 - i\sqrt{y^2 - y^4/4}\right) + \mathrm{Li}_2\left(y^2/2 + i\sqrt{y^2 - y^4/4}\right)\right)$$

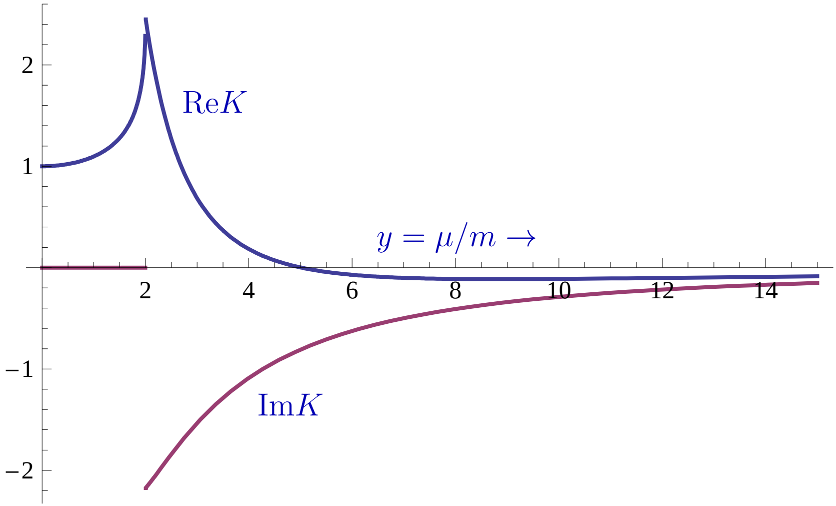
where Li_n is the PolyLog function

$$\mathrm{Li}_n(z) \equiv \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

notice that

$$\mathrm{Li}_1(z) \equiv \sum_{k=1}^{\infty} \frac{z^k}{k} = -\ln(1 - z)$$

$$32i\pi^2 m^2 I \equiv K(y) = 2\int_0^1 \int_0^{1-\alpha} \frac{1}{1-\alpha\beta y^2 - i\epsilon}\, d\beta\, d\alpha$$



This is the answer - but not the whole story - easy in the limit $\mu \ll m$

$$K(y) = 1 + \frac{y^2}{12} + \frac{y^4}{90} + \cdots$$

series useful only for $y \ll 1$

Explore the Taylor expansion — it's more than just a handy approximation

$$\mathcal{L} = \bar{\psi}\,(i\,\partial\!\!\!/ - e\,A\!\!\!/\ - m)\,\psi + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 - i f\,\phi\,\bar{\psi}\gamma_5\psi$$

Renormalizable QFT — completely describes a consistent (if boring) world

What does this world look like if $\mu \ll m$ and at energies and momenta very small compared to m? Only ϕs and γs — not enough energy to produce electrons and positrons. We can always calculate the relevant amplitudes and as we have seen, even though there are no couplings between ϕ and γ in $\mathcal{L}$, quantum loops intoduce interactions between them. And furthermore, we immediately notice that things are simpler for $p, \mu \ll m$. Let's go back to the messy integral that we didn't want to do.

$$I(q_1, q_2) = \frac{-i}{16\pi^2}\int_0^1\int_0^{1-\alpha}\frac{d\beta\,d\alpha}{m^2 - \alpha(1-\alpha)q_1^2 - \beta(1-\beta)q_2^2 - 2\alpha\beta(q_1 q_2)}$$

Now that we are thinking about approximating things for small momenta, you can see that things simplify even off the photon mass shell, for $q_1^2, q_2^2 \neq 0$, but still $\ll m^2$ — to first approximation we can neglect all the momentum dependence in the numerator —

$$I(q_1, q_2) \to \frac{-i}{16\pi^2}\int_0^1\int_0^{1-\alpha}\frac{d\beta\,d\alpha}{m^2} = \frac{-i}{32\pi^2 m^2}$$

So we have an approximate, but very simple result for $M^{\mu_1\mu_2}$ that is valid off shell. Putting all the factors back in from

$$M^{\mu_1\mu_2}(q_1, q_2) = -8m\, f\, e^2 \int \frac{\epsilon^{\mu_1\mu_2\nu_1\nu_2}\, q_{1\nu_1} q_{2\nu_2}}{\left[(\ell - q_1)^2 - m^2\right]\left[(\ell + q_2)^2 - m^2\right]\left[\ell^2 - m^2\right]} \frac{d^4\ell}{(2\pi)^4}$$

$$M^{\mu_1\mu_2}(q_1, q_2) \to i\frac{f\, e^2}{4\pi^2 m}\epsilon^{\mu_1\mu_2\nu_1\nu_2}\, q_{1\nu_1} q_{2\nu_2}$$

Polynomial in momentum — this could arise from a **local** term in $\mathcal{L}$!

$$-\frac{f\, e^2}{32\pi^2 m}\phi\, \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$$

where $F^{\mu\nu}$ is the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Only polynomials in q can arise in this way - no more complicated functions. **The Taylor series has not only made things simpler, but also because the result is local, it has allowed us to interpret the effect as a new term in $\mathcal{L}$.**

$$M^{\mu_1\mu_2}(q_1,q_2) \to i\frac{f\,e^2}{4\pi^2 m}\epsilon^{\mu_1\mu_2\nu_1\nu_2}\,q_{1\nu_1}q_{2\nu_2}$$

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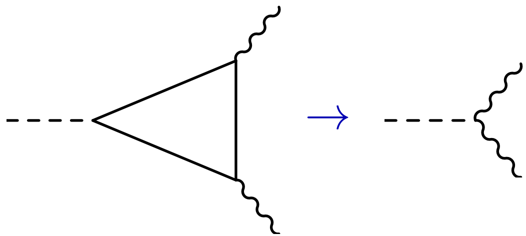
$\phi \to \gamma\gamma$



Higher order terms in the Taylor expansion in powers of q/m would be equivalent to terms with more derivatives.

Moral: Feynman graphs involving heavy particles can give effects at low energies that look like the effects of new terms in the Lagrangian.

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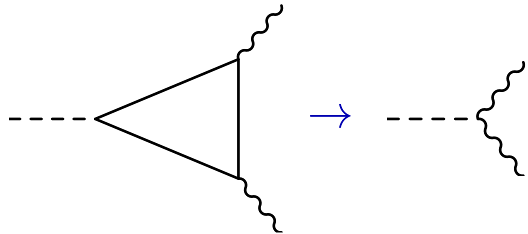
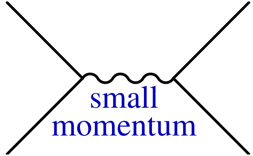
$\phi \to \gamma\gamma$



But this is NOT a new term in the same Lagrangian!!! That would be double counting. This term only makes sense as a term in a Lagrangian of an "effective low energy field theory" from which the heavy fermions have been removed. This is an example of the process of "matching" the physics from a high energy theory to a low energy theory. This is the key to the idea of effective field theory, and I want to examine it in some detail.

This happens all the time — but in field theory it is complicated!!!

W and Z exchange
weak interactions



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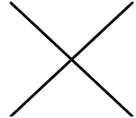
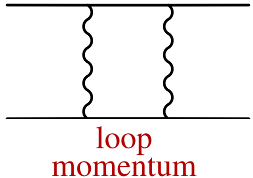


CP Violation
second order
weak interactions



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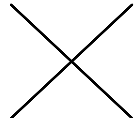
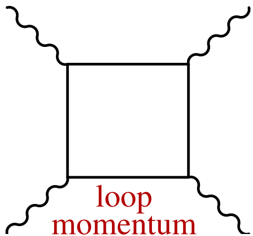


light by light
scattering



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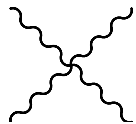


How can this MATCHING possibly work??? Convergent versus divergent diagrams? Counterterms? IR divergences???

But in spite of the complications of loops, it does work — I want to begin my series by showing explicitly how it works in perturbation theory in the simplest case of matching at the scale of a heavy particle. All effects of heavy particles at low energies small compared to their masses can be MATCHED into parameters in a new Lagrangian — **this is the "effective theory" of this model for scales small compared to m**

— NOT a renormalizable theory — terms $\propto 1/m^k$

— NOT a complete theory — accurate only for $p \ll m$

— BUT perfectly consistent in its domain of validity and it is **useful** to think about this theory on its own — without discussing the heavy particles at al

— The last statement is the interesting one. Like any nonrenormalizable theory, the effective theory requires an ∞ of counter terms and therefore an ∞ number of terms in $\mathcal{L}$ — not just $\phi\,\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ but terms with more derivatives and more (and fewer) Fs and ϕs — anything allowed by Lorentz invariance and parity will be there with a nonzero coefficient — so what good is it?

— Each coefficient can be calculated in terms of e, m, μ, λ and f — at least in perturbation theory. And as we have seen in the example of $\phi\,\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$, it is sometimes much easier to calculate these coefficients than the full amplitude, because Taylor expanding the denominators may lead to simple polynomial integrations at least for one loop diagrams.

— But still you can't calculate an infinite number of them so what good is it?

— Fortunately, only a finite number of terms are needed to calculate any process to a given accuracy!

$$
\begin{array}{ll}
\text{dimension} \leq 4 & \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{\mu^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4 \\
\text{dimension} = 5 & \frac{\alpha_5}{m} \phi \, \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \quad \left[\alpha_5 = -\frac{f e^3}{32\pi^2} \right] \\
\text{dimension} = 6 & \frac{\alpha_{6,1}}{m^2} \phi^2 \, F_{\mu\nu} F_{\mu\nu} + \frac{\alpha_{6,2}}{m^2} \phi^6 + \cdots \\
\cdots & \cdots \\
\text{dimension} = n & \frac{\alpha_{n,1}}{m^{n-4}} \cdots \\
\cdots & \cdots
\end{array}
$$

the effects of terms in $\mathcal{L}$ of dimension n are supressed AT LEAST by

$$\frac{p^k \mu^{n-4-k}}{m^{n-4}}$$

where p is the typical momentum in the process. The α coefficients may also be small because they involve small couplings from the original theory. This $1/m^{n-4}$ supression is on top of that.

dimension ≤ 4	$\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$	$p, \mu \ll m$
dimension $= 5$	$\frac{\alpha_5}{m}\phi\,\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$	$\downarrow$
dimension $= 6$	$\frac{\alpha_{6,1}}{m^2}\phi^2\,F_{\mu\nu}F_{\mu\nu} + \frac{\alpha_{6,2}}{m^2}\phi^6 + \cdots$	$p, \mu < m$
$\ldots$	$\ldots$	$\downarrow$
dimension $= n$	$\frac{\alpha_{n,1}}{m^{n-4}}\cdots$	$p, \mu \approx m$
$\ldots$	$\ldots$	$\downarrow$

At very low energies only the lowest dimension terms are important and the theory is nearly renormalizable.

As energy increases, more and more terms become important

As energy approaches m, all the infinite number of terms become important and the theory becomes useless.

This is the life cycle of an effective theory and it mirrors the chronology of exploration in particle physics.

Very little of this depends on knowing the theory at high energies.

So maybe this is a way of making sense of nonrenormalizable theories another model — much simpler example, but with two separate scales — 4 complex scalars ϕ_j for $j = 1$ to 4

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$$

$$\mathcal{L}_0 = \sum_{j=1}^{4} \left(\partial_\mu \phi_j \partial^\mu \phi_j^* - m_j^2\, \phi_j \phi_j^* - \frac{\lambda_j}{4} (\phi_j \phi_j^*)^2 \right) - \sum_{j=1}^{3} \sum_{\substack{k= \\ j+1}}^{4} \lambda_{jk} (\phi_j \phi_j^*)(\phi_k \phi_k^*)$$

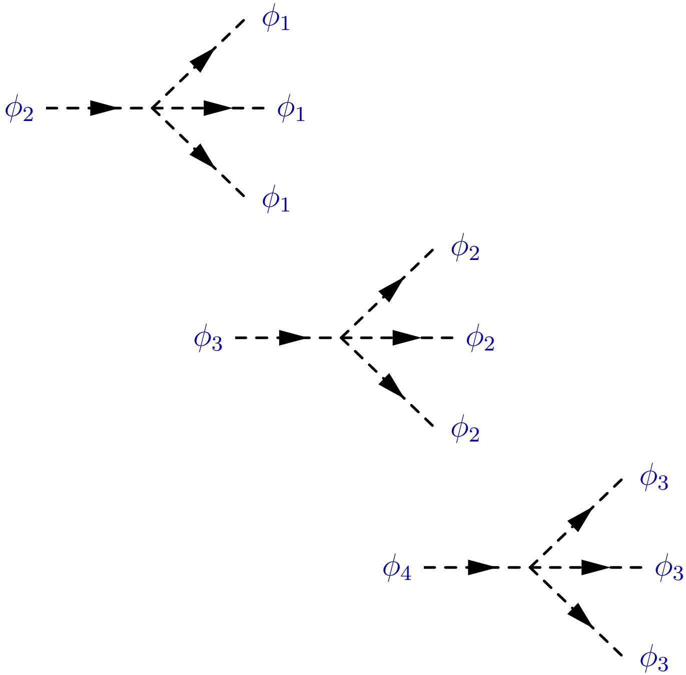
$$\mathcal{L}_1 = -\frac{\kappa_{12}}{6} \phi_1^3 \phi_2^* - \frac{\kappa_{23}}{6} \phi_2^3 \phi_3^* - \frac{\kappa_{34}}{6} \phi_3^3 \phi_4^* + \text{h.c.}$$

$$m_1, m_4 \ll m_2 \ll m_3$$

$\mathcal{L}_0$ invariant under $U(1)^4$
$\mathcal{L}_1$ invariant under a global $U(1)$ symmetry of the form

$$\phi_1 \to e^{i\theta} \phi_1 \quad \phi_2 \to e^{3i\theta} \phi_2 \quad \phi_3 \to e^{9i\theta} \phi_3 \quad \phi_4 \to e^{27i\theta} \phi_4$$

Exercise 1. Show that these are the most general interactions consistent with the symmetry and with dimension ≤ 4 (which implies that this is a renormalizable theory).



ϕ_1
ϕ_2
ϕ_1
ϕ_1
ϕ_2
ϕ_3
ϕ_2
ϕ_2
ϕ_3
ϕ_4
ϕ_3
ϕ_3

$$m_1, m_4 \ll m_2 \ll m_3$$

original renormalizable theory

ϕ_1, ϕ_2, ϕ_3, ϕ_4

m_3

eliminate ϕ_3
and match

ϕ_1, ϕ_2, ϕ_4

$\uparrow$
p

m_2

eliminate ϕ_2
and match

ϕ_1, ϕ_4

m_1, m_4

eliminate ϕ_1, ϕ_4
and match

nobody home - no particles - no physics

$$m_1, m_4 \ll m_2 \ll m_3$$

original renormalizable theory

ϕ_1, ϕ_2, ϕ_3, ϕ_4

m_3

eliminate ϕ_3
and match

ϕ_1, ϕ_2, ϕ_4

$\uparrow$
p

m_2

eliminate ϕ_2
and match

ϕ_1, ϕ_4

m_1, m_4

eliminate ϕ_1, ϕ_4
and match

nobody home - no particles - no physics

$m_1, m_4 \ll m_2 \ll m_3$

original renormalizable theory

ϕ_1, ϕ_2, ϕ_3, ϕ_4

eliminate ϕ_3
and match

m_3

ϕ_1, ϕ_2, ϕ_4

$\uparrow$
p

m_2

eliminate ϕ_2
and match

ϕ_1, ϕ_4

eliminate ϕ_1, ϕ_4
and match

m_1, m_4

nobody home - no particles - no physics

$$m_1, m_4 \ll m_2 \ll m_3$$

original renormalizable theory

ϕ_1, ϕ_2, ϕ_3, ϕ_4

m_3

eliminate ϕ_3
and match

ϕ_1, ϕ_2, ϕ_4

$\uparrow$
p

m_2

eliminate ϕ_2
and match

ϕ_1, ϕ_4

m_1, m_4

eliminate ϕ_1, ϕ_4
and match

nobody home - no particles - no physics

$$m_1, m_4 \ll m_2 \ll m_3$$

original renormalizable theory

$\phi_1,\, \phi_2,\, \phi_3,\, \phi_4$

m_3

eliminate ϕ_3
and match

$\phi_1,\, \phi_2,\, \phi_4$

$\uparrow$
p

m_2

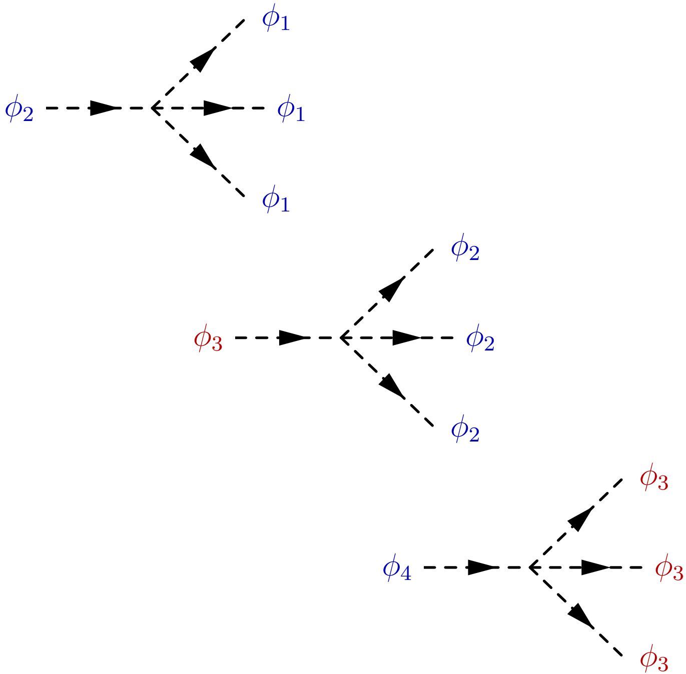
eliminate ϕ_2
and match

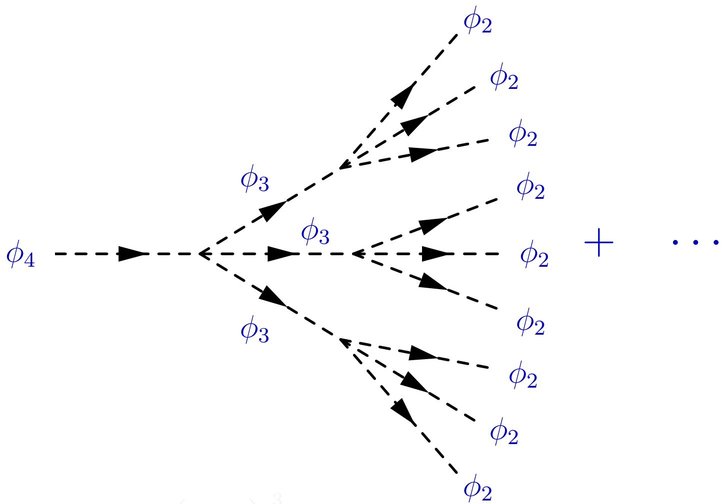
$\phi_1,\, \phi_4$

m_1, m_4

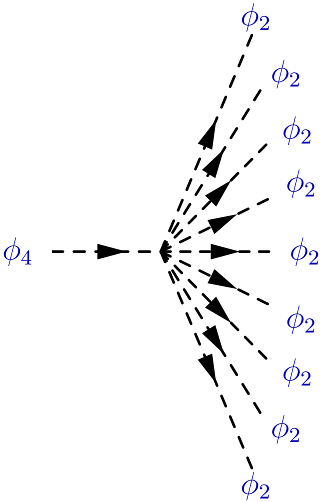
eliminate $\phi_1,\, \phi_4$
and match

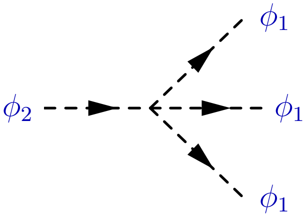
nobody home - no particles - no physics

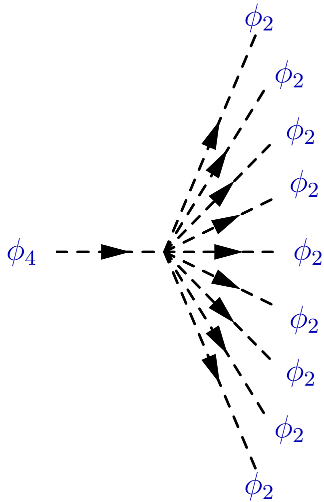

ϕ_1
ϕ_2
ϕ_1
ϕ_1
ϕ_2
ϕ_3
ϕ_2
ϕ_2
ϕ_3
ϕ_4
ϕ_3
ϕ_3



$$\frac{\kappa_{34}}{6}\left(\frac{\kappa_{23}}{6m_3^2}\right)^3 \times 9!$$


ϕ_4
ϕ_2
ϕ_2
ϕ_2
ϕ_2
ϕ_2
ϕ_2
ϕ_2
ϕ_2
ϕ_2


ϕ_2
ϕ_1
ϕ_1
ϕ_1


ϕ_4
ϕ_2
ϕ_2
ϕ_2
ϕ_2
ϕ_2
ϕ_2
ϕ_2
ϕ_2
ϕ_2

$$m_1, m_4 \ll m_2 \ll m_3$$

original renormalizable theory

ϕ_1, ϕ_2, ϕ_3, ϕ_4

m_3

eliminate ϕ_3
and match

$$-\frac{\kappa_{12}}{6}\phi_1^3\phi_2^* + \alpha_{2,4}\,\phi_2^9\phi_4^* + \cdots$$

$\uparrow$
p

m_2

eliminate ϕ_2
and match

$$-\alpha_{1,4}\,\phi_1^{27}\phi_4^* + \cdots$$

m_1, m_4

eliminate ϕ_1, ϕ_4
and match

nobody home - no particles - no physics

$$m_1, m_4 \ll m_2 \ll m_3$$

original renormalizable theory

ϕ_1, ϕ_2, ϕ_3, ϕ_4

m_3

eliminate ϕ_3
and match

$$-\frac{\kappa_{12}}{6}\phi_1^3\phi_2^* + \alpha_{2,4}\,\phi_2^9\phi_4^* + \cdots$$

$\uparrow$
p

m_2

eliminate ϕ_2
and match

$$-\alpha_{1,4}\,\phi_1^{27}\phi_4^* + \cdots$$

m_1, m_4

eliminate ϕ_1, ϕ_4
and match

nobody home - no particles - no physics

$$m_1, m_4 \ll m_2 \ll m_3$$

original renormalizable theory

ϕ_1, ϕ_2, ϕ_3, ϕ_4

eliminate ϕ_3
and match

m_3

$$-\frac{\kappa_{12}}{6}\phi_1^3\phi_2^* + \alpha_{2,4}\,\phi_2^9\phi_4^* + \cdots$$

$\uparrow$
p

m_2

eliminate ϕ_2
and match

$$-\alpha_{1,4}\,\phi_1^{27}\phi_4^* + \cdots$$

eliminate ϕ_1, ϕ_4
and match

m_1, m_4

nobody home - no particles - no physics

Exercise 2. Find $\alpha_{2,4}$ and $\alpha_{1,4}$.

This example is really obvious, because these terms in the effective theory could be obtained just by "integrating out" the heavy fields — that is (in the functional language) doing the functional integral over the heavy fields leaving the light fields unchanged. One of the things I want to show you is that this is not the general situation (even when it makes sense at all — which it doesn't in situations in which the light fields are not present in the high energy theory). **"Integrating out" is not the whole story of effective field theory.**

But effetive field theory, properly defined without making simplistic assumptions about how it works, is both powerful and inevitable..

Renormalizable QFT in n dimensions

interactions terms have dimension n or less

Effective QFT with scale μ in n dimensions

no constraint on dimension of interaction terms
BUT coefficient μ_k of terms of dimension $k > n$ must satisfy

$$\mu_k \lesssim \frac{1}{\mu^{k-n}}$$

so that the effects of nonrenormalizable interactions are suppressed by powers $(p/\mu)^{k-n}$

$$m_1, m_4 \ll m_2 \ll m_3$$

original renormalizable theory

$$\phi_1,\ \phi_2,\ \phi_3,\ \phi_4$$

m_3

eliminate ϕ_3
and match

$$-\frac{\kappa_{12}}{6}\phi_1^3\phi_2^* + \alpha_{2,4}\,\phi_2^9\phi_4^* + \cdots$$

$\uparrow$
p

m_2

eliminate ϕ_2
and match

$$-\alpha_{1,4}\,\phi_1^{27}\phi_4^* + \cdots$$

m_1, m_4

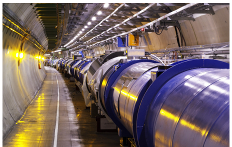
eliminate ϕ_1, ϕ_4
and match

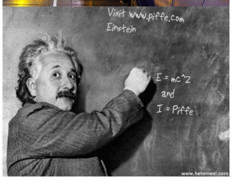
nobody home - no particles - no physics

$e \gtrsim 4\pi$ ☹ the theory is not useful at this energy

$$e A_\mu \bar{\psi} \gamma^\mu \psi$$

$e \ll 4\pi$ ☺ perturbation theory works





energy E
+ theory

$\kappa \ll 1/E$ ☺ perturbation theory works

$$\kappa\, \phi\, \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$$

$\kappa \gtrsim 1/E$ ☹ the theory is not useful at this energy

1PI graphs, background fields, and all that — (good reference is 33-42 of "INTRODUCTION TO THE BACKGROUND FIELD METHOD" Vol. 813 (1982) ACTA PHYSICA POLONICA No 1-2, By L. F. Abbott)

$$Z[s] = e^{iW[s]} = \langle 0 \mid 0 \rangle_s = \int e^{i\int(\mathcal{L}(\phi)+s\phi)} [\delta\phi]$$

$Z[s]$ is the generating function for the Green functions

$W[s]$ is the generating function for the connected Green functions

$W[0]$ all connected Feynman graphs with no external lines

$W'[0]$ all connected Feynman graphs with one external line

$W''[0]$ all connected Feynman graphs with two external lines

$W^{(3)}[0]$ all connected Feynman graphs with three external lines

$\ldots$

$$Z[s] = e^{iW[s]} = \langle 0 \mid 0 \rangle_s = \int e^{i\int (\mathcal{L}(\phi) + s\phi)}\,[\delta\phi]$$

$W[s]$ is the generating function for the connected Green functions.
$\int s\phi$ is short for

$$\sum_\alpha \int s_\alpha(x)\,\phi_\alpha(x)$$

but we will usually drop the α and the (x) and just remember that these objects are "vectors" in position space and internal space
$\Gamma[\Phi]$, the generating functional for 1PI graphs — $\Gamma[\Phi]$ is obtained by making a Legendre transformation on $W[s]$ (familiar from thermodynamics and stat mech I suspect)

$$\Gamma[\Phi] = W[s] - \int s\,\Phi \quad \text{where} \quad \Phi = \frac{\delta W}{\delta s}$$

is the "classical field" corresponding to the quantum field ϕ — note

$$\frac{\delta}{\delta\Phi}\left\{\Gamma[\Phi] = W[s] - \int s\,\Phi\right\} \Rightarrow s = -\frac{\delta\Gamma}{\delta\Phi}$$

we want this to be 0 for the theory without sources — determines $\langle\phi\rangle$

$\Gamma[\Phi]$ is sometimes called the "effective action" — but this terminology would be confusing, because the "effective Lagrangian" is going to mean something slightly different to us, so I will just call it $\Gamma[\Phi]$

$$Z[s] = e^{iW[s]} = \langle 0 \mid 0 \rangle_s = \int e^{i\int(\mathcal{L}(\phi)+s\phi)}\,[\delta\phi]$$

$$\Gamma[\Phi] = W[s] - \int s\,\Phi \quad \text{where} \quad \Phi = \frac{\delta W}{\delta s}$$

Example - free scalar field $\quad \mathcal{L}(\phi) = -\frac{1}{2}\phi(\partial^2 + m^2)\phi$

$$e^{iW[s]} = \int e^{i\int\left(-\frac{1}{2}\phi(\partial^2+m^2)\phi+s\phi\right)}\,[\delta\phi]$$

$$= e^{i\int\left(\frac{1}{2}s\frac{1}{\partial^2+m^2}s\right)} \int e^{i\int\left(-\frac{1}{2}\tilde{\phi}(\partial^2+m^2)\tilde{\phi}\right)}\,[\delta\tilde{\phi}]$$

$$\text{where} \quad \tilde{\phi} = \phi - \frac{1}{\partial^2 + m^2}s$$

$$e^{iW[s]} = e^{ik}e^{i\int\left(\frac{1}{2}s\frac{1}{\partial^2+m^2}s\right)}$$

$$Z[s] = e^{iW[s]} = \langle 0 \mid 0 \rangle_s = \int e^{i\int (\mathcal{L}(\phi)+s\phi)} \, [\delta\phi]$$

$$\Gamma[\Phi] = W[s] - \int s\,\Phi \quad \textbf{where} \quad \Phi = \frac{\delta W}{\delta s}$$

$$\mathcal{L}(\phi) = -\frac{1}{2}\phi(\partial^2 + m^2)\phi$$

$$e^{iW[s]} = e^{ik} e^{i\int \left(\frac{1}{2}s\frac{1}{\partial^2+m^2}s\right)}$$

$$W[s] = \int \left(\frac{1}{2}s\frac{1}{\partial^2+m^2}s\right) + k$$

$$\Phi = \frac{\delta W}{\delta s} = \frac{1}{\partial^2+m^2}s \Rightarrow s = (\partial^2+m^2)\Phi$$

$$\Gamma[\Phi] = W[s] - \int s\,\Phi = \int \left(-\frac{1}{2}\Phi(\partial^2+m^2)\Phi\right) + k$$

For a quadratic $\mathcal{L}$, up to an additive constant, $\Gamma[\Phi] = \int \mathcal{L}(\Phi)$

Symmetry $\phi \to \tilde{\phi} = U\phi$ where $\quad \mathcal{L}(U\phi) = \mathcal{L}(\phi)$

$$Z[s] = e^{iW[s]} = \langle 0 \mid 0 \rangle_s = \int e^{i\int(\mathcal{L}(\phi)+s\phi)}\,[\delta\phi]$$

$$= \int e^{i\int(\mathcal{L}(U\phi)+s\phi)}\,[\delta\phi] = \int e^{i\int(\mathcal{L}(U\phi)+sU^{-1}U\phi)}\,[\delta\phi]$$

$$= \int e^{i\int(\mathcal{L}(\tilde{\phi})+sU^{-1}\tilde{\phi})}\,[\delta(U^{-1}\tilde{\phi})]$$

then if $[\delta(U^{-1}\tilde{\phi})] = [\delta\tilde{\phi}]$, $Z[s] = Z[sU^{-1}]$ and $W[s] = W[sU^{-1}]$

$$\Gamma[U\Phi] = W[s] - \int s\,U\Phi$$

$$= W[\tilde{s}U^{-1}] - \int \tilde{s}U^{-1}\,U\Phi = W[\tilde{s}] - \int \tilde{s}\Phi = \Gamma[\Phi]$$

thus Γ inherits the symmetries of $\mathcal{L}$ if the functional integral is also symmetric (anomalies?)

$$\Gamma[\Phi] = W[s] - \int s\,\Phi \quad \text{where} \quad \Phi = \frac{\delta W}{\delta s} \quad \Rightarrow \quad s = -\frac{\delta \Gamma}{\delta \Phi}$$

$$\frac{\delta \Phi}{\delta s} = \frac{\delta^2 W}{\delta s \delta s} = i\,D$$

is a "matrix" in position space and interal indices (α) - it is the full interacting propagator in the presence of the source — now look at

$$I = \frac{\delta \Phi}{\delta \Phi} = \frac{\delta s}{\delta \Phi}\frac{\delta \Phi}{\delta s} = -\frac{\delta^2 \Gamma}{\delta \Phi \delta \Phi}\,\frac{\delta^2 W}{\delta s \delta s}$$

$$\Rightarrow \frac{\delta s}{\delta \Phi} = -\frac{\delta^2 \Gamma}{\delta \Phi \delta \Phi} = \frac{1}{iD}$$

is the inverse propagator and it means that

$$\frac{\delta}{\delta \Phi} = \frac{\delta s}{\delta \Phi}\,\frac{\delta}{\delta s} = \frac{1}{iD}\,\frac{\delta}{\delta s}$$

so differentiating with respect to the classical field is like differentiating with respect to the source, adding an external line, except the line gets amputated
thus $\Gamma[\Phi]$ is a generating functional for amputated diagrams
next let's see why it actually generates the 1PI diagrams — proof later

three facts

$$\frac{\delta^2\Gamma}{\delta\Phi_1\delta\Phi_2} = -[iD^{-1}]_{12} \qquad \frac{\delta^2 W}{\delta s_1\delta s_2} = [iD]_{12} \qquad \frac{\delta}{\delta\Phi_1} = [iD^{-1}]_{11'}\frac{\delta}{\delta s_{1'}}$$

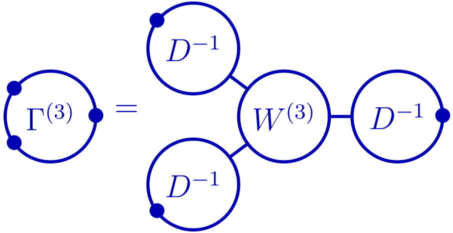
$$\frac{\delta^3\Gamma}{\delta\Phi_1\delta\Phi_2\delta\Phi_3} = \frac{\delta}{\delta\Phi_3}\frac{\delta^2\Gamma}{\delta\Phi_1\delta\Phi_2}$$

$$= \frac{\delta}{\delta\Phi_3}\left(-[iD^{-1}]_{12}\right) = -[iD^{-1}]_{33'}\frac{\delta}{\delta s_{3'}}[iD^{-1}]_{12}$$

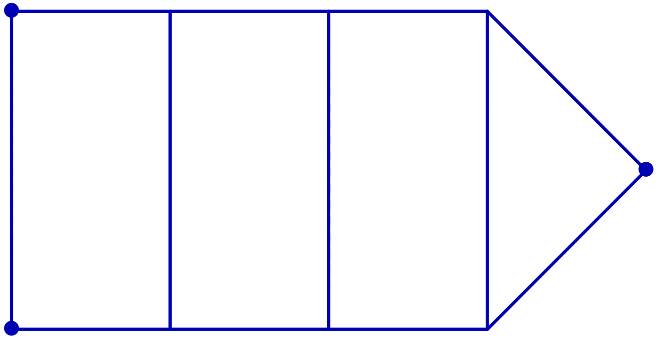
$$= [iD^{-1}]_{11'}[iD^{-1}]_{22'}[iD^{-1}]_{33'}\frac{\delta}{\delta s_{3'}}[iD]_{1'2'}$$

$$= [iD^{-1}]_{11'}[iD^{-1}]_{22'}[iD^{-1}]_{33'}\frac{\delta^3 W}{\delta s_{1'}\delta s_{2'}\delta s_{3'}}$$

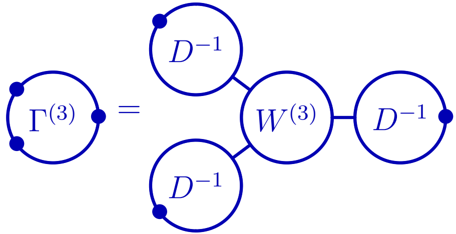
term in $\Gamma[\Phi]$ with three Φs is the amputated connected 3-point function — the full 3-point vertex in the interacting theory including sources —



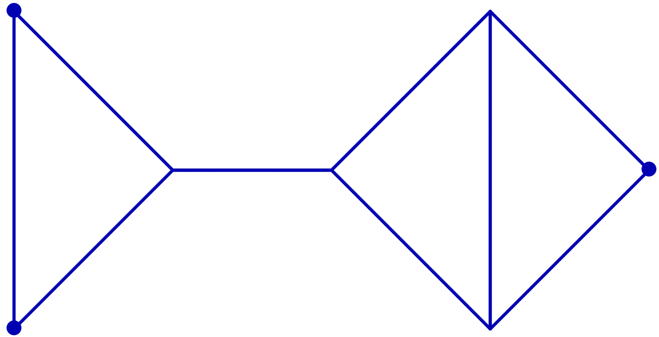
so for example



is a graph that could contribute to $\Gamma^{(3)}$



but



is not because the right side is part of a propagator that gets amputated

three facts

$$\frac{\delta^2\Gamma}{\delta\Phi_1\delta\Phi_2} = -[iD^{-1}]_{12} \qquad \frac{\delta^2 W}{\delta s_1\delta s_2} = [iD]_{12} \qquad \frac{\delta}{\delta\Phi_1} = [iD^{-1}]_{11'}\frac{\delta}{\delta s_{1'}}$$

$$\frac{\delta^3\Gamma}{\delta\Phi_1\delta\Phi_2\delta\Phi_3} = [iD^{-1}]_{11'}[iD^{-1}]_{22'}[iD^{-1}]_{33'}\frac{\delta^3 W}{\delta s_{1'}\delta s_{2'}\delta s_{3'}}$$

term in $\Gamma[\Phi]$ with three Φs is amputated connected 3-point function

$$\frac{\delta^4\Gamma}{\delta\Phi_1\delta\Phi_2\delta\Phi_3\delta\Phi_4} = [iD^{-1}]_{44'}\frac{\delta}{\delta s_{4'}}\left([iD^{-1}]_{11'}[iD^{-1}]_{22'}[iD^{-1}]_{33'}\frac{\delta^3 W}{\delta s_{1'}\delta s_{2'}\delta s_{3'}}\right)$$

the $s_{4'}$ differentiation can act in one of 4 places

three facts

$$\frac{\delta^2\Gamma}{\delta\Phi_1\delta\Phi_2} = -[iD^{-1}]_{12} \qquad \frac{\delta^2 W}{\delta s_1\delta s_2} = [iD]_{12} \qquad \frac{\delta}{\delta\Phi_1} = [iD^{-1}]_{11'}\frac{\delta}{\delta s_{1'}}$$

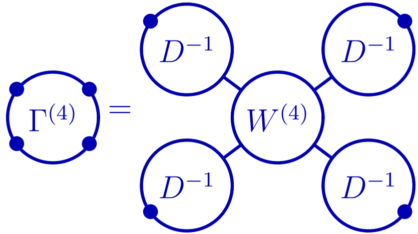
$$\frac{\delta^3\Gamma}{\delta\Phi_1\delta\Phi_2\delta\Phi_3} = [iD^{-1}]_{11'}[iD^{-1}]_{22'}[iD^{-1}]_{33'}\frac{\delta^3 W}{\delta s_{1'}\delta s_{2'}\delta s_{3'}}$$

term in $\Gamma[\Phi]$ with three Φs is amputated connected 3-point function

$$\frac{\delta^4\Gamma}{\delta\Phi_1\delta\Phi_2\delta\Phi_3\delta\Phi_4} = [iD^{-1}]_{44'}\frac{\delta}{\delta s_{4'}}\left([iD^{-1}]_{11'}[iD^{-1}]_{22'}[iD^{-1}]_{33'}\frac{\delta^3 W}{\delta s_{1'}\delta s_{2'}\delta s_{3'}}\right)$$

if the $s_{4'}$ differentiation acts on the $\delta^3 W/\delta s_{1'}\delta s_{2'}\delta s_{3'}$ we get an amputated connected 4-point function

$$[iD^{-1}]_{11'}[iD^{-1}]_{22'}[iD^{-1}]_{33'}[iD^{-1}]_{44'}\frac{\delta^4 W}{\delta s_{1'}\delta s_{2'}\delta s_{3'}\delta s_{4'}}$$


$\Gamma^{(4)}$
=
D^{-1}
D^{-1}
$W^{(4)}$
D^{-1}
D^{-1}

three facts

$$\frac{\delta^2\Gamma}{\delta\Phi_1\delta\Phi_2} = -[iD^{-1}]_{12} \qquad \frac{\delta^2 W}{\delta s_1 \delta s_2} = [iD]_{12} \qquad \frac{\delta}{\delta\Phi_1} = [iD^{-1}]_{11'}\frac{\delta}{\delta s_{1'}}$$

$$\frac{\delta^3\Gamma}{\delta\Phi_1\delta\Phi_2\delta\Phi_3} = [iD^{-1}]_{11'}[iD^{-1}]_{22'}[iD^{-1}]_{33'}\frac{\delta^3 W}{\delta s_{1'}\delta s_{2'}\delta s_{3'}}$$

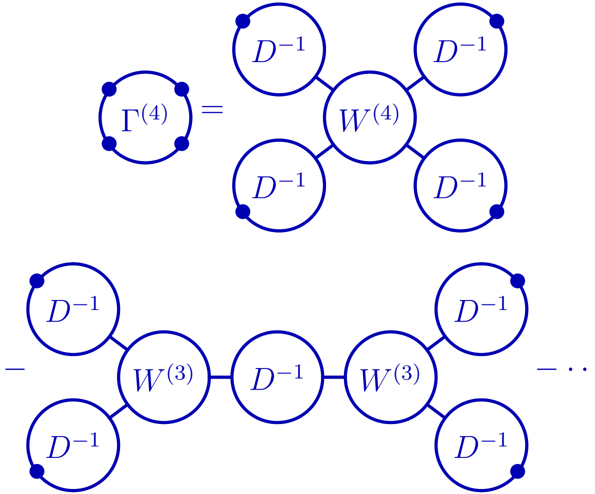
term in $\Gamma[\Phi]$ with three Φs is amputated connected 3-point function

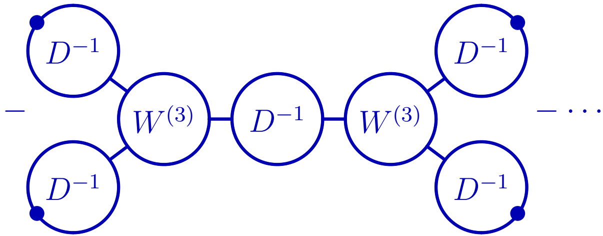
$$\frac{\delta^4\Gamma}{\delta\Phi_1\delta\Phi_2\delta\Phi_3\delta\Phi_4} = [iD^{-1}]_{44'}\frac{\delta}{\delta s_{4'}}\left([iD^{-1}]_{11'}[iD^{-1}]_{22'}[iD^{-1}]_{33'}\frac{\delta^3 W}{\delta s_{1'}\delta s_{2'}\delta s_{3'}}\right)$$

if the $s_{4'}$ differentiation acts on the $[iD^{-1}]_{33'}$ (say) we get

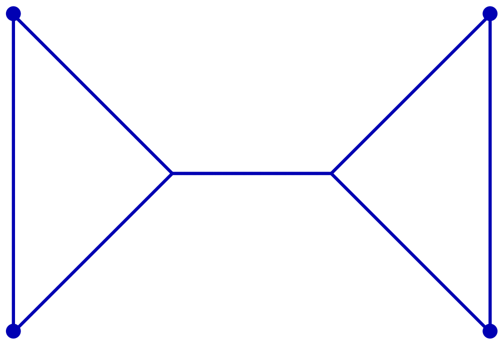
$$-[iD^{-1}]_{11'}[iD^{-1}]_{22'}\frac{\delta^3 W}{\delta s_{1'}\delta s_{2'}\delta s_5}[iD^{-1}]_{55'}\frac{\delta^3 W}{\delta s_{3'}\delta s_{4'}\delta s_{5'}}[iD^{-1}]_{33'}[iD^{-1}]_{44'}$$

internal $[iD^{-1}]_{55'}$ — incomplete amputation — what this (and two cross terms) do is to subtract out all the diagrams that fall apart into two 3-point vertices connected by a propagator


$\Gamma^{(4)}$
$=$
D^{-1}
D^{-1}
$W^{(4)}$
D^{-1}
D^{-1}
$-$
D^{-1}
D^{-1}
$W^{(3)}$
D^{-1}
$W^{(3)}$
D^{-1}
D^{-1}
$- \cdots$



so something like this does not contribute to $\Gamma^{(4)}$



three facts

$$\frac{\delta^2 \Gamma}{\delta \Phi_1 \delta \Phi_2} = -[iD^{-1}]_{12} \qquad \frac{\delta^2 W}{\delta s_1 \delta s_2} = [iD]_{12} \qquad \frac{\delta}{\delta \Phi_1} = [iD^{-1}]_{11'} \frac{\delta}{\delta s_{1'}}$$

this keeps happening — because each new Φ derivative can act on the external and internal D^{-1}s as well as the vertices, all internal propagators get removed in $\Gamma^{(n)}$

$\Rightarrow \Gamma^{(n)}$ consists only of graphs that cannot be divided in two by cutting any single graph — these are called "one-particle irreducible" or "1PI" graphs

1Pi has all the physcs — $[\Gamma^2]^{-1}$ gives the propagators that describe the long-distance physics of particles moving from here to there in space-time — $\Gamma^{(n)}$ for $n > 2$ describes the vertices that describe interactions in regions of space-time — skeleton expansion

The loop expansion — put the factors of $\hbar$ in our Feynman graphs

$$Z[s] = e^{iW[s]} = \langle 0 \mid 0 \rangle_s = \int e^{i\int(\hbar^{-1}\mathcal{L}(\phi)+s\phi)}\,[\delta\phi]$$

vertices $\propto \hbar^{-1}$ — propagators $\propto \hbar$

for amputated Feynman diagrams factors of $\hbar \leftrightarrow$ loop integrations —

$$\begin{array}{lcccc} \text{internal line} & \to & d^4\ell & \propto & \hbar \\ \text{vertex} & \to & \delta^4(\ell) & \propto & \hbar^{-1} \end{array} \qquad \hbar^{\text{\# of lines-\# of vertices}}$$

$$\begin{array}{c}\text{\# of}\\ d^4\ell\end{array} \quad - \quad \left(\begin{array}{c}\text{\# of}\\ \delta^4(\ell)\end{array} - 1\right) \quad = \quad \begin{array}{c}\text{\# of}\\ \text{remaining}\\ d^4\ell\end{array}$$

$$\begin{array}{c}\text{\# of}\\ \text{internal}\\ \text{lines}\end{array} \quad - \quad \left(\begin{array}{c}\text{\# of}\\ \text{vertices}\end{array} - 1\right) \quad = \quad \begin{array}{c}\text{\# of}\\ \text{loops}\end{array}$$

$$\hbar^{\text{\# of loops}-1} \qquad \Gamma[\Phi] = \frac{1}{\hbar}\sum_{\substack{j=\text{\# of}\\ \text{loops}}} \hbar^j\,\Gamma_j[\Phi] \qquad \Gamma_0[\Phi]?$$

$\Gamma[\Phi]$ is the generating function for the sum of 1PI diagram

$\Gamma[0]$ is the sum of all 1PI graphs with no external lines

$\Gamma'[0]$ is the sum of all 1PI graphs with one external line

$\Gamma''[0]$ is the sum of all 1PI graphs with two external lines

$\Gamma^{(3)}[0]$ is the sum of all 1PI graphs with three external lines

$\ldots$

you might think that $\Gamma[0]$ and $\Gamma'[0]$ would not be very interesting — but in fact they are very important —

$\Gamma[0]$ are sometimes refered to as “vacuum graphs”

$\Gamma'[0]$ are sometimes refered to as “tadpole graphs” background field method (not background field gauge)

$$e^{i\tilde{W}[s,\phi_b]} = \int e^{i\int(\mathcal{L}(\phi+\phi_b)+s\phi)}\,[\delta\phi] = e^{i(W[s]-\int s\phi_b)}$$

ϕ_b is a "background field" — define classical field $\tilde{\Phi}$ in the presence of the background field ϕ_b

$$\tilde{\Gamma}[\tilde{\Phi},\phi_b] = \tilde{W}[s,\phi_b] - \int s\,\tilde{\Phi} \quad \text{where} \quad \tilde{\Phi} \equiv \frac{\delta\tilde{W}}{\delta s} = \frac{\delta W}{\delta s} - \phi_b = \Phi - \phi_b$$

$$\tilde{\Gamma}[\tilde{\Phi},\phi_b] = W[s] - \int s\phi_b - \int s(\Phi-\phi_b) = \Gamma[\Phi] = \Gamma[\tilde{\Phi}+\phi_b]$$

shifting the quantum field by adding the background field gives the same Γ of a shifted classical field!!!! — now look at vacuum graphs in the presence of ϕ_b

$$\tilde{\Gamma}[0,\phi_b] = \Gamma[\phi_b]$$

vacuum graphs in the shifted theory as a function of the background field ϕ_b give you $\Gamma[\phi_b]$

Exercise 3. Consider a field theory of a single real scalar field with mass m and coupling $\lambda\,\phi^4/4!$. Think about $\Gamma(\Phi)$ for a constant classical field Φ and calculate the coefficent of Φ^6 in two ways:

1. Calculate the 1PI 6-point function with conventional Feynman diagrams; and

2. Use the background field method.

background field method makes it easy to see that $\Gamma[\Phi]$ generates 1PIs

$$e^{i\tilde{W}[s,\phi_b]} = \int e^{i\int(\mathcal{L}(\phi+\phi_b)+s\phi)}\,[\delta\phi] = e^{i(W[s]-\int s\phi_b)}$$

$$\tilde{\Gamma}[\tilde{\Phi},\phi_b] = \tilde{W}[s,\phi_b] - \int s\,\tilde{\Phi} \quad \text{where} \quad \tilde{\Phi} \equiv \frac{\delta\tilde{W}}{\delta s} = \frac{\delta W}{\delta s} - \phi_b = \Phi - \phi_b$$

$$\tilde{\Gamma}[\tilde{\Phi},\phi_b] = W[s] - \int s\phi_b - \int s(\Phi-\phi_b) = \Gamma[\Phi] = \Gamma[\tilde{\Phi}+\phi_b]$$

$$\tilde{\Gamma}[0,\phi_b] = \Gamma[\phi_b]$$

$\Gamma[\phi_b]$ graphs are $1PI$ if $\tilde{\Gamma}[0,\phi_b]$ vacuum graphs are 1PI

$$\tilde{\Gamma}[0,\phi_b] = \tilde{W}[s,\phi_b] \quad \text{evaluated where} \quad \frac{\delta\tilde{W}}{\delta s} = 0$$

$\delta\tilde{W}/\delta s = 0$ means that we have chosen s so that the sum of diagrams with a single line coming out vanishes — this means we can throw out all the 1-particle reducible diagrams, because we know their contribution vanishes when we sum over all diagrams connected to the single line background fields - classical fields - and the loop expansion

$$e^{i\tilde{W}[s,\phi_b]} = \int e^{i\int(\mathcal{L}(\phi+\phi_b)+s\phi)}\,[\delta\phi] = e^{i(W[s]-\int s\phi_b)}$$

$$\tilde{\Gamma}[\tilde{\Phi},\phi_b] = \Gamma[\tilde{\Phi}+\phi_b] \qquad \tilde{\Gamma}[0,\phi_b] = \Gamma[\phi_b]$$

look at Γ in the loop expansion $\Gamma[\Phi] = \sum_{j=\#\text{ of loops}} \Gamma_j[\Phi]$

0-loops: these are just tree graphs

— but tree graphs are not 1PI if they have any propagators! thus only the bare vertices contribute

— but this is just $\mathcal{L}$ — this is easiest to see in the background field method where the vacuum contribution — the diagram with no external lines — is just the value of $\int \mathcal{L}(\phi+\phi_b)$ for $\phi = 0$ — that is $\int \mathcal{L}(\phi_b)$

$$\Gamma_0[\Phi] = \int \mathcal{L}(\Phi)$$

this will allow us systematically to match onto the effective theory

What does this have to do with effective field theory?

what happens at the boundary between two quantum field theories?

compare a high energy theory that has some heavy stuff and some light stuff with a low energy theory that describes only the light stuff — expand $\mathcal{L}_\ell$ in powers of $\hbar$ and compare powers

how do we choose the parameters in the low energy theory so that the low energy theory gives us the same physics at low energies as the high energy theory?

because we are only interested in the low-energy physics, we only need sources for the light degrees of freedom — what happens when we construct $\Gamma[\Phi]$?

internal heavy particle lines are not canceled - 1-light-particle irreducible (1LPI) graphs

high energy theory	low energy theory
$\hbar^{-1}\mathcal{L}_h(\phi_\ell,\phi_h) \quad e^{iW_h[s]} =$	$\sum_{j=0}^{\infty}\hbar^{j-1}\mathcal{L}_{\ell j}(\phi_\ell) \quad e^{iW_\ell[s]} =$
$\int e^{i\int(\mathcal{L}_h(\phi_\ell,\phi_h)+s\phi_\ell)}\,[\delta\phi_\ell\,\delta\phi_h]$	$\int e^{i\int(\mathcal{L}_\ell(\phi_\ell)+s\phi_\ell)}\,[\delta\phi_\ell]$
$\Gamma_h[\Phi_\ell] = W_h[s] - \int s\,\Phi_\ell$	$\Gamma_\ell[\Phi_\ell] = W_\ell[s] - \int s\,\Phi_\ell$
$\Phi_\ell = \frac{\delta W_h}{\delta s} \quad \to$ 1LPI graphs	$\Phi_\ell = \frac{\delta W_\ell}{\delta s} \quad \to$ 1PI graphs
light and heavy internal lines only light external lines	only light internal lines only light external lines

how do we choose the parameters in $\mathcal{L}_\ell$ to get the same physics at low energies — what does this mean precisely?

$$\hbar^{-1}\mathcal{L}_h(\phi_\ell,\phi_h) \to e^{iW_h[s]}$$

$$\Gamma_h[\Phi_\ell] = W_h[s] - \int s\,\Phi_\ell$$

$$\Phi_\ell = \frac{\delta W_h}{\delta s} \quad \to \text{1LPI graphs}$$

light and heavy internal lines

0-loops: heavy particle trees

$$\sum_{j=0}^{\infty} \hbar^{j-1}\mathcal{L}_{\ell j}(\phi_\ell) \to e^{iW_\ell[s]}$$

$$\Gamma_\ell[\Phi_\ell] = W_\ell[s] - \int s\,\Phi_\ell$$

$$\Phi_\ell = \frac{\delta W_\ell}{\delta s} \quad \to \text{1PI graphs}$$

only light internal lines

0-loops: $\Gamma_{\ell 0}[\Phi_\ell] = \int \mathcal{L}_{\ell 0}(\Phi_\ell)$

0-loops: we can calculate Γ_{h0} by summing over all heavy particle trees with external (amputated) light particle lines —

because all internal lines are heavy, we can Taylor expand each diagram in powers of momenta over the heavy particle masses —

this gives a series of local terms with increasing dimension (because of more power of p which become derivatives) —

we identify the terms in this series with the terms in $\int \mathcal{L}_{\ell 0}(\Phi_\ell)$

$$\hbar^{-1}\mathcal{L}_h(\phi_\ell,\phi_h) \to e^{iW_h[s]}$$

$$\sum_{j=0}^{\infty}\hbar^{j-1}\mathcal{L}_{\ell j}(\phi_\ell) \to e^{iW_\ell[s]}$$

$$\Gamma_h[\Phi_\ell] = W_h[s] - \int s\,\Phi_\ell$$

$$\Gamma_\ell[\Phi_\ell] = W_\ell[s] - \int s\,\Phi_\ell$$

$$\Phi_\ell = \frac{\delta W_h}{\delta s} \quad \to \text{1LPI graphs}$$

light and heavy internal lines

$$\Phi_\ell = \frac{\delta W_\ell}{\delta s} \quad \to \text{1PI graphs}$$

only light internal lines

0-loops: heavy particle trees

0-loops: $\Gamma_{\ell 0}[\Phi_\ell] = \int \mathcal{L}_{\ell 0}(\Phi_\ell)$

1-loop - $\hbar^0$: 1 loop attached to heavy particle trees

"1-loop" - $\hbar^0$: a light particle loop with only $\mathcal{L}_{\ell 0}$ vertices **or** $\int \mathcal{L}_{\ell 1}$

2-loops - $\hbar^1$: 2 loops attached to heavy particle trees

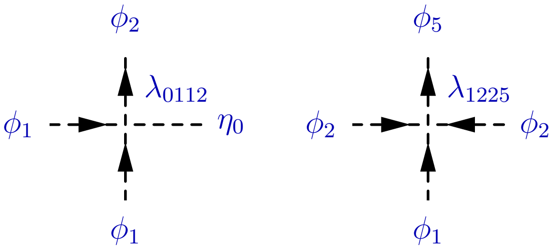
"2-loops" - $\hbar^1$: 2 light particle loops with only $\mathcal{L}_{\ell 0}$ vertices **or** one light particle loop with one $\mathcal{L}_{\ell 1}$ **or** $\int \mathcal{L}_{\ell 2}$

$\cdots$

$\cdots$

set equal order by order in $\hbar$

another scalar example — just showing "non-compulsory" couplings



η_0 a real scalar field — what are the symmetries?

take $m_0, m_1, m_5 \ll m_2$ and look at effective theory below m_2

$$\hbar^{-1}\mathcal{L}_h(\phi_\ell, \phi_h) \to e^{iW_h[s]}$$

$$\Gamma_h[\Phi_\ell] = W_h[s] - \int s\,\Phi_\ell$$

$$\Phi_\ell = \frac{\delta W_h}{\delta s} \quad \to \text{1LPI graphs}$$

light and heavy internal lines

0-loops: heavy particle trees

1-loop - $\hbar^0$: 1 loop attached
to heavy particle trees

2-loops - $\hbar^1$: 2 loops attached
to heavy particle trees

. . .

$$\sum_{j=0}^{\infty} \hbar^{j-1}\mathcal{L}_{\ell j}(\phi_\ell) \to e^{iW_\ell[s]}$$

$$\Gamma_\ell[\Phi_\ell] = W_\ell[s] - \int s\,\Phi_\ell$$

$$\Phi_\ell = \frac{\delta W_\ell}{\delta s} \quad \to \text{1PI graphs}$$

only light internal lines

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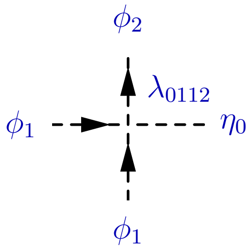
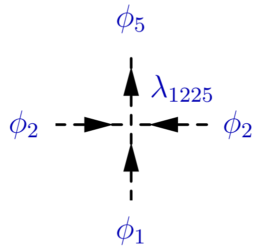
. . .

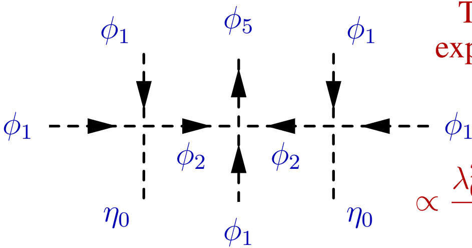
set equal order by order in $\hbar$

another scalar example — just showing “non-compulsory” couplings



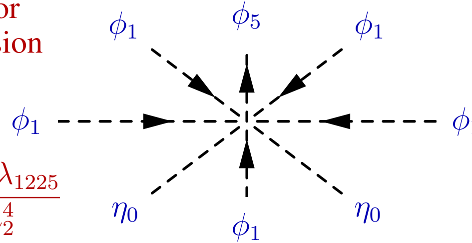


take $m_0, m_1, m_5 \ll m_2$ and look at effective theory below m_2



Taylor expansion
$\longrightarrow$

$$\propto \frac{\lambda_{0112}^2 \lambda_{1225}}{m_2^4}$$



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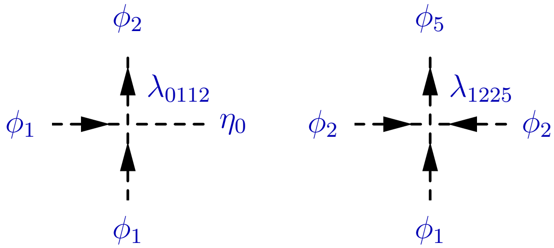

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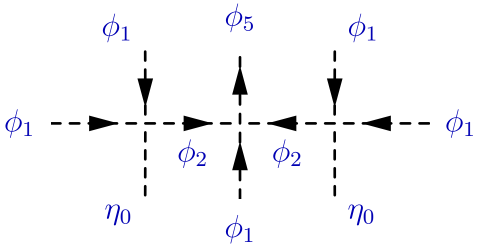

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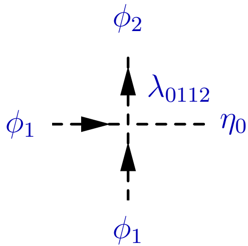

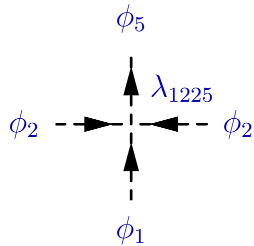

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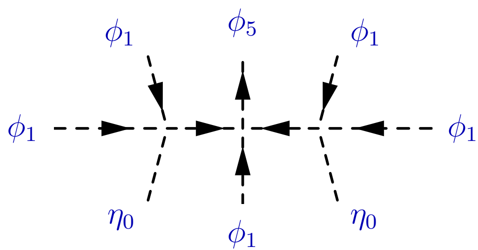

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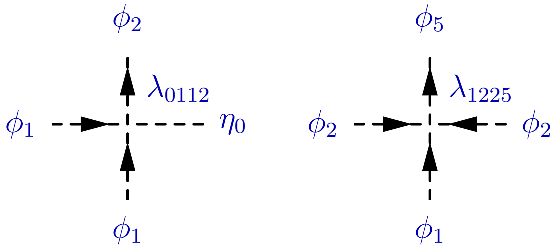

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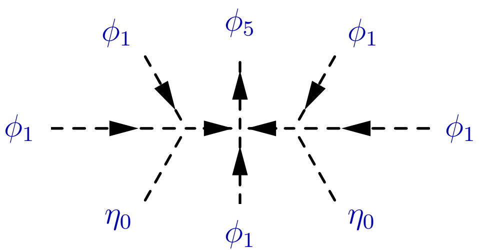

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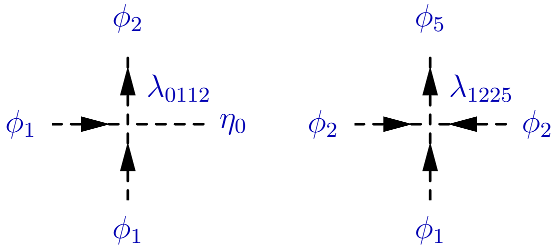

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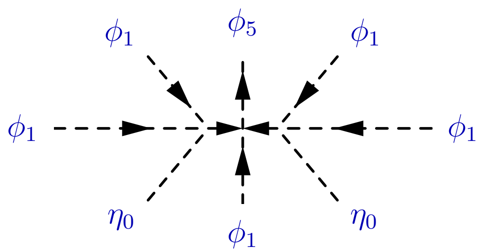

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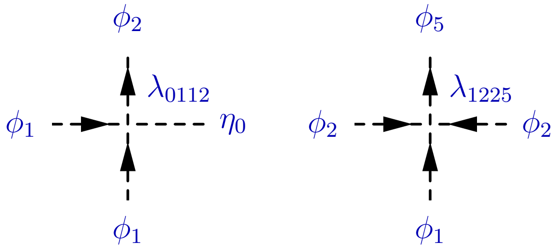

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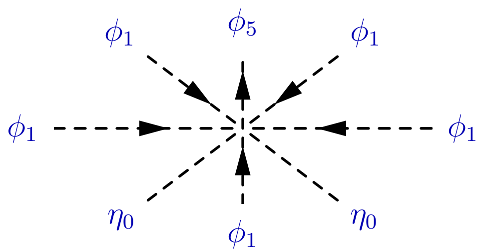

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light and heavy internal lines

0-loops: heavy particle trees

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$\cdots$

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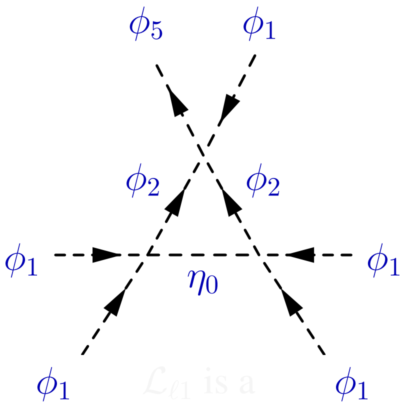

$\mathcal{L}_{\ell 1}$ is a difference between 1-loop h and ℓ graphs

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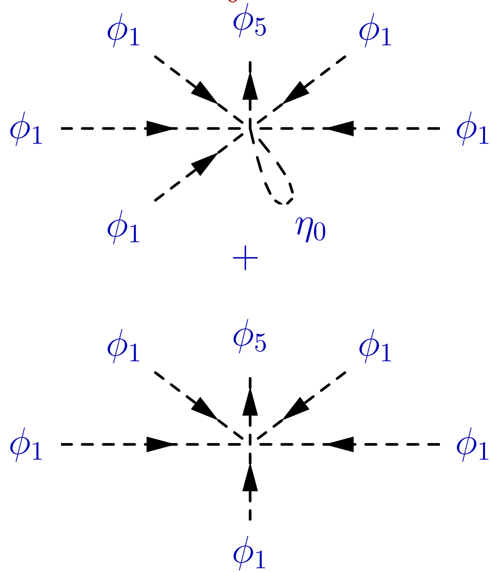


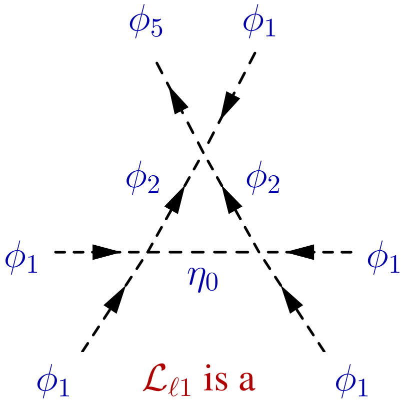
Potential IR divergences from the η_0 propagator

$$\int \frac{1}{q^2 + m_0^2 + i\epsilon} \frac{d^4 q}{(2\pi)^4}$$

UV divergent and IR finite, but the m_0^2 dependence is problematic - derivatives wrto m_0^2 have IR divergences as $m_0 \to 0$

But the matching term $\mathcal{L}_{\ell 1}$ is a difference of h and ℓ diagrams **with the same small q behavior by construction** so the result is IR finite — this will turn out to be a big advantage, because it means that we can use dimensional regularization and minimal subtraction even though this scheme (I will argue later) changes the physics both in the UV (which we want) and in the IR (which could be dangerous).

1-loop - $\hbar^0$: 1 loop attached to heavy particle trees



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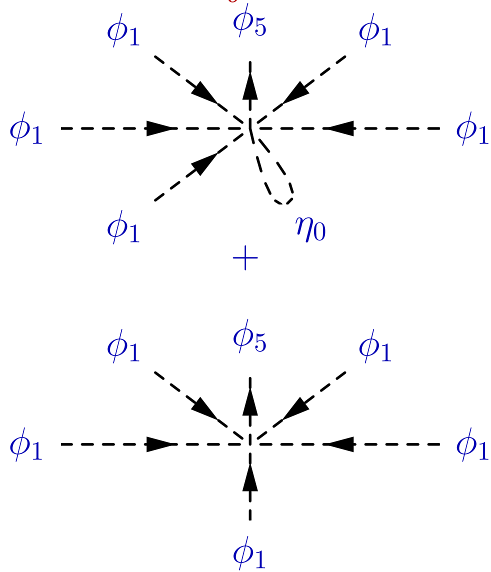

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But the matching term $\mathcal{L}_{\ell 1}$ is a **difference of h and ℓ diagrams with the same small q behavior by construction** so the result is IR finite and it can be explanded in powers of m_0 safely — this will turn out to be a big advantage, because it means that we can use dimensional regularization and minimal subtraction even though this scheme (I will argue later) changes the physics both in the UV (which we want) and in the IR (which could be dangerous).

1973 —- mass indendent renormalization schemes — Weinberg and 't Hooft–Veltman

great advantages to eliminating UV divergences from the zero-mass theory — treating the masses just like coupling constants — leads to simple, homogeneous renormalization group equations

liberates us from the notion that parameters in the Lagragian are necessarily "physical"

dimensional regularization and minimal subtraction or $\overline{MS}$ is a great example

I will focus on $\overline{MS}$ — 40 years ago, 't Hooft and Veltman showed mathematically that it works to eliminate UV divergences, and we all use it, but there are subtleties that are easy to forget.

I will try to explain more physically **why** it works, **when** it works and why this very convenient tool really **requires effective field theory to be of much use.**

formal review of dimensional regularization (DR) - begin with the calculation of the n-dimensional volume $\Omega(n)$ such that $d^n k = \Omega(n) k^{n-1} dk$

$$\pi^{n/2} = \left(\int_{-\infty}^{\infty} e^{-k^2}\, dk \right)^n = \int e^{-k^2}\, d^n k = \Omega(n) \int_0^{\infty} e^{-k^2}\, k^{n-1}\, dk$$

$$= \frac{1}{2}\Omega(n) \int_0^{\infty} (k^2)^{\frac{n}{2}-1}\, e^{-k^2}\, d(k^2) = \frac{1}{2}\Omega(n)\, \Gamma\left(\frac{n}{2}\right)$$

$$\Rightarrow \Omega(n) = \frac{2\,\pi^{n/2}}{\Gamma(n/2)}$$

this form can be extended to fractional dimensions

now compute generic Feynman graph in κ dimensions (in Euclidean space):

$$I(\alpha, \beta, \kappa) \equiv \int \frac{(k^2)^{\beta}}{(k^2 + A^2)^{\alpha}}\, \frac{d^{\kappa} k}{(2\pi)^{\kappa}}$$

where A^2 is a polynomial in momenta, masses and Feynman parameters

$$= \frac{\Omega(\kappa)}{(2\pi)^{\kappa}} \int_0^{\infty} \frac{k^{\kappa+2\beta}}{(k^2 + A^2)^{\alpha}}\, \frac{dk}{k} = \frac{\Omega(\kappa)}{(2\pi)^{\kappa}}\, (A^2)^{\beta-\alpha+\kappa/2} \int_0^{\infty} \frac{y^{\kappa+2\beta}}{(1 + y^2)^{\alpha}}\, \frac{dy}{y}$$

$$I_{\alpha,\beta,\kappa}(A^2) = \int \frac{(k^2)^\beta}{(k^2+A^2)^\alpha}\,\frac{d^\kappa k}{(2\pi)^\kappa} = \frac{\Omega(\kappa)}{(2\pi)^\kappa}\int_0^\infty \frac{k^{\kappa+2\beta}}{(k^2+A^2)^\alpha}\,\frac{dk}{k}$$

$$= \frac{\Omega(\kappa)}{(2\pi)^\kappa}\,(A^2)^{\beta-\alpha+\kappa/2}\int_0^\infty \frac{y^{\kappa+2\beta}}{(1+y^2)^\alpha}\,\frac{dy}{y}$$

$$x = \frac{y^2}{1+y^2}\,, \quad y^2 = \frac{x}{1-x}\,, \quad 1+y^2 = \frac{1}{1-x}\,,$$

$$2\ln y = \ln x - \ln(1-x)\,,$$

$$2\,\frac{dy}{y} = dx\left(\frac{1}{x}+\frac{1}{1-x}\right) = \frac{dx}{x(1-x)}\,.$$

$$I_{\alpha,\beta,\kappa}(A^2) = \frac{\Omega(\kappa)}{(2\pi)^\kappa}\,(A^2)^{\beta-\alpha+\kappa/2}\,\frac{1}{2}\int_0^1 x^{\beta+\kappa/2-1}\,(1-x)^{\alpha-\beta-\kappa/2-1}\,dx$$

$$= \frac{\Omega(\kappa)}{(2\pi)^\kappa}\,(A^2)^{\beta-\alpha+\kappa/2}\,\frac{1}{2}\,\frac{\Gamma(\beta+\kappa/2)\,\Gamma(\alpha-\beta-\kappa/2)}{\Gamma(\alpha)}\,.$$

or

$$\int \frac{(k^2)^\beta}{(k^2+A^2)^\alpha}\,\frac{d^\kappa k}{(2\pi)^\kappa} = \frac{(A^2)^{\beta-\alpha+\kappa/2}}{(4\pi)^{\kappa/2}}\,\frac{\Gamma(\beta+\kappa/2)\,\Gamma(\alpha-\beta-\kappa/2)}{\Gamma(\kappa/2)\,\Gamma(\alpha)}$$

$$\int \frac{(k^2)^\beta}{(k^2+A^2)^\alpha}\,\frac{d^\kappa k}{(2\pi)^\kappa} = \frac{(A^2)^{\beta-\alpha+\kappa/2}}{(4\pi)^{\kappa/2}}\,\frac{\Gamma(\beta+\kappa/2)\,\Gamma(\alpha-\beta-\kappa/2)}{\Gamma(\kappa/2)\,\Gamma(\alpha)}$$

$\kappa = 4-\epsilon$, dividing by $\mu^{-\epsilon}$ to get dimensions right $\rightarrow$

$$\int \frac{(k^2)^\beta}{(k^2+A^2)^\alpha}\,\frac{d^{4-\epsilon} k}{(2\pi)^{4-\epsilon}\mu^{-\epsilon}} = \frac{(A^2)^{2+\beta-\alpha-\epsilon/2}}{(4\pi)^{2-\epsilon/2}\mu^{-\epsilon}}\,\frac{\Gamma(\beta+2-\epsilon/2)\,\Gamma(\alpha-\beta-2+\epsilon/2)}{\Gamma(2-\epsilon/2)\,\Gamma(\alpha)}$$

The important thing I want to focus is the completely trivial way that the dimensional parameter μ is introduced, completely independent of the details of masses and momenta. Calculationally, this is a huge advantage.

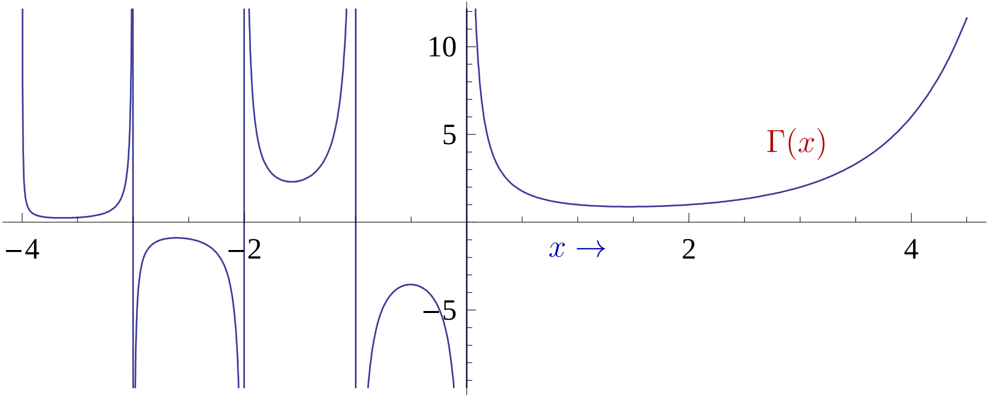
But what on earth does this mean physically???

defined by analytic continuation from regions in ϵ where the calculation is well-defined

why is this a sensible thing to do?

$$\int \frac{(k^2)^\beta}{(k^2+A^2)^\alpha} \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}\mu^{-\epsilon}} = \frac{(A^2)^{2+\beta-\alpha-\epsilon/2}}{(4\pi)^{2-\epsilon/2}\mu^{-\epsilon}} \frac{\Gamma(\beta+2-\epsilon/2)\,\Gamma(\alpha-\beta-2+\epsilon/2)}{\Gamma(2-\epsilon/2)\,\Gamma(\alpha)}$$

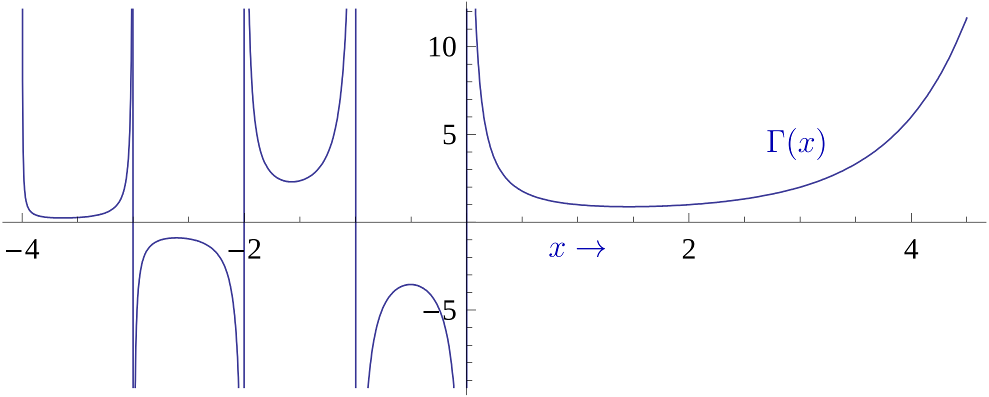
everything is finite and well-defined as $\epsilon \to 0$ except for $\Gamma(\alpha-\beta-2+\epsilon/2)$



poles as $\epsilon \to 0$ for $2+\beta-\alpha$ any nonnegative integer — makes sense — the LHS is a divergent graph for $\epsilon = 0$ — $2+\beta-\alpha = 0 \to$ log divergence — $2+\beta-\alpha = 1 \to$ quadradic divergence — etc

$$\int \frac{(k^2)^\beta}{(k^2+A^2)^\alpha} \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}\mu^{-\epsilon}} = \frac{(A^2)^{2+\beta-\alpha-\epsilon/2}}{(4\pi)^{2-\epsilon/2}\mu^{-\epsilon}} \frac{\Gamma(\beta+2-\epsilon/2)\,\Gamma(\alpha-\beta-2+\epsilon/2)}{\Gamma(2-\epsilon/2)\,\Gamma(\alpha)}$$

everything is finite and well-defined as $\epsilon \to 0$ except for $\Gamma(\alpha-\beta-2+\epsilon/2)$



but the nature of the divergences is a little peculiar — no large scale just powers of A — all divergences are $1/\epsilon$

$$I_{\alpha,\beta,4-\epsilon}(A^2) = \int \frac{(k^2)^\beta}{(k^2+A^2)^\alpha} \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}\mu^{-\epsilon}}$$

$$= \frac{(A^2)^{2+\beta-\alpha-\epsilon/2}}{(4\pi)^{2-\epsilon/2}\mu^{-\epsilon}} \frac{\Gamma(\beta+2-\epsilon/2)\,\Gamma(\alpha-\beta-2+\epsilon/2)}{\Gamma(2-\epsilon/2)\,\Gamma(\alpha)}$$

the integral is convergent for $\alpha > \beta + 2$ and $A^2 > 0$ and then we can just take the limit

$$I_{\alpha,\beta,4}(A^2) = \lim_{\epsilon\to 0} I_{\alpha,\beta,4-\epsilon}(A^2)$$

no regularization or subtraction is required

for $\alpha \leq \beta + 2$ the interval is UV divergent, but there is no large scale — just powers of A with $1/\epsilon$ poles for non-negative integral values of $\beta - \alpha - 2$

MS and $\overline{MS}$

divergences appear in a standard way as poles in ϵ — rather than "renormalzing" them we just throw them away — "subtract" them by adding counterterms

step 1 — calculate with $\epsilon \neq 0$ and expand the result around $\epsilon = 0$ — note that ϵ dependence can arise in many different places from the extension to $4-\epsilon$ dimensions, and all these must be included

step 2 — $\boldsymbol{MS}$ — $1/\epsilon \to 0$ — subtract poles in ϵ

step 2 — $\overline{\boldsymbol{MS}}$ — $1/\epsilon \to \Big(\gamma - \log(4\pi)\Big)/2$ — designed so that simple 1-loop graphs are unchanged

$$\Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)$$

$$\int \frac{1}{(k^2+A^2)^2} \frac{d^{4-\epsilon}k}{\mu^{-\epsilon}\,(2\pi)^{4-\epsilon}} = \left(\frac{A^2}{\mu^2}\right)^{-\epsilon/2} \frac{\Gamma(\epsilon/2)}{(4\pi)^{2-\epsilon/2}}$$

$$= \frac{1}{16\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log\left(\frac{A^2}{\mu^2}\right)\right) + \mathcal{O}(\epsilon) \to \frac{1}{16\pi^2}\log\left(\frac{\mu^2}{A^2}\right)$$

exactly what we would get in a momentum-space cut-off with $\Lambda \to \mu$

for $\alpha = \beta + 2$ the integral is log divergent — this is where all the physics is — all the other divergences are positive interger powers of A^2 times this — associated with local counterterms.

$$I_{\alpha,\alpha-2,4-\epsilon}(A^2) = \int \frac{(k^2)^{\alpha-2}}{(k^2+A^2)^{\alpha}} \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}\mu^{-\epsilon}}$$

$$= \frac{(A^2)^{-\epsilon/2}}{(4\pi)^{2-\epsilon/2}\mu^{-\epsilon}} \frac{\Gamma(\alpha-\epsilon/2)\,\Gamma(\epsilon/2)}{\Gamma(2-\epsilon/2)\,\Gamma(\alpha)}$$

using $k^2 = (k^2+A^2) - A^2$ we can always write this as

$$I_{\alpha,\alpha-2,4-\epsilon}(A^2) = \int \frac{1}{(k^2+A^2)^2} \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}\mu^{-\epsilon}} + \begin{array}{c}\text{finite}\\ \text{terms}\end{array}$$

$$= \frac{(A^2)^{-\epsilon/2}\,\Gamma(\epsilon/2)}{(4\pi)^{2-\epsilon/2}\mu^{-\epsilon}} + \begin{array}{c}\text{finite}\\ \text{terms}\end{array}$$

let's try to understand what this means physically physical idea of a regularization scheme — a modification of the physics of the theory at short distances that allows us to calculate the quantum corrections

if we modify the physics only at short distances, we expect that all the effects of the regularization can be MATCHED into the parameters of the theory

it is not obvious that DR is a modification of the physics at short distances

$$I_\epsilon = \int \frac{1}{(k_\epsilon^2 + k^2 + A^2)^2} \frac{d^{4-\epsilon}k}{\mu^{-\epsilon}\,(2\pi)^{4-\epsilon}}$$

$$= \int \frac{1}{(k_\epsilon^2 + k^2 + A^2)^2} \frac{d^{-\epsilon}k}{(2\pi\mu)^{-\epsilon}} \frac{d^4k}{(2\pi)^4}$$

explicitly separating out the "extra" $-\epsilon$ dimensions, so that k^2 is the 4 dimensional squared norm of k — do the integral over the $-\epsilon$ extra dimensions (of course this is not the way we would actually calculate the graph – but it will help us to understand what is happening)

$$I_\epsilon = \int \frac{1}{(k_\epsilon^2 + k^2 + A^2)^2} \frac{d^{-\epsilon}k}{(2\pi\mu)^{-\epsilon}} \frac{d^4k}{(2\pi)^4}$$

explicitly separating out the “extra” $-\epsilon$ dimensions, so that k^2 is the 4 dimensional squared norm of k — do the integral over the $-\epsilon$ extra dimensions (this is not the way we would actually calculate — but it will help us to understand what is happening)

$$I_\epsilon = \int \frac{1}{(k^2 + A^2)^2}\, r(\epsilon) \left(\frac{k^2 + A^2}{4\pi\mu^2}\right)^{-\epsilon/2} \frac{d^4k}{(2\pi)^4} \quad \text{where} \quad r(\epsilon) = \frac{\Gamma(2+\epsilon/2)}{\Gamma(2)}$$

multiplicative factor, $r(\epsilon) \to 1$ as $\epsilon \to 0$ — important factor is

$$\rho^{-\epsilon/2} \quad \text{where} \quad \rho = \frac{k^2 + A^2}{4\pi\mu^2}$$

$\rho^{-\epsilon/2} \to 1$ as $\epsilon \to 0$ but depends on k and A

$$\rho^{-\epsilon/2} = e^{-(\epsilon \ln\rho)/2} \quad \Rightarrow \quad \rho^{-\epsilon/2} \approx 1 \quad \text{for} \quad |\ln\rho| \ll \frac{1}{\epsilon}\,.$$

DR does not change the physics for k and A of order μ
different in UV if k OR $A \gg \mu$ — different in IR if k AND $A \ll \mu$
DR changes the physics in the infrared as well as the ultraviolet region

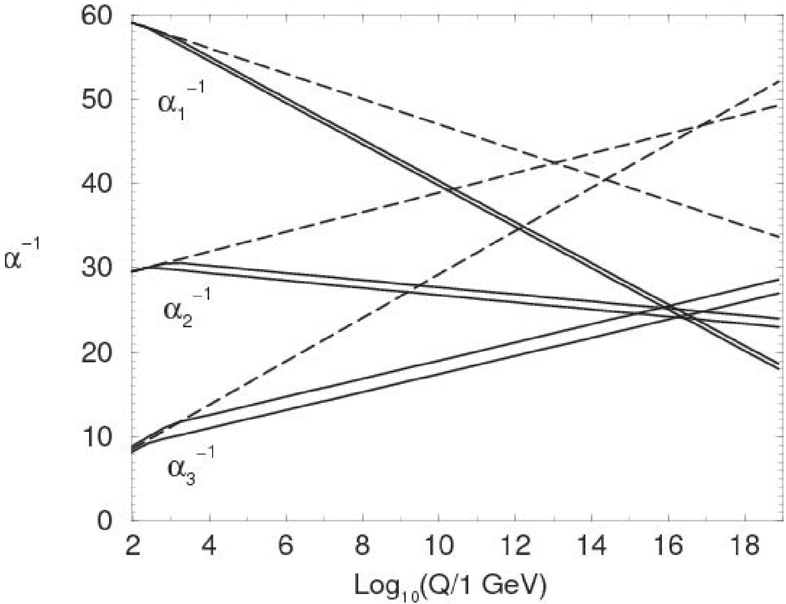
DR changes the physics in the infrared as well as the ultraviolet region

We have argued that this is OK for matching at a boundary between two different effective theories because the matching contributions involve differences between the high energy and low energy theories, from which the IR physics cancels because it is constructed to be in the same in the two theories order by order in the loop expansion.

But how **and why** do we choose μ in either theory? The goal is to make perturbation theory work as well as possible. So choose μ to minimize the logs that appear in calculation of physical quantities.

This is hopeless if there are heavy particles with different masses in the theory because different heavy particle loops will give different logs and you cannot eliminate all of them by choosing μ. It is nice that the renormalization group equations in $\overline{MS}$ are simple, but without effective theory they don't do any good because there is no way to choose μ that really helps. If you want to use the simple renormalization group you get with $\overline{MS}$, you must eliminate heavy particles as you go down in energy and match onto a succession of new effective theories at each threshold. You can do this because each matching makes sense in spite of the $\overline{MS}$s issues with IR divergences.

Of course, this argument is close to my heart, because it is the basis of coupling constant unification in GUTS.

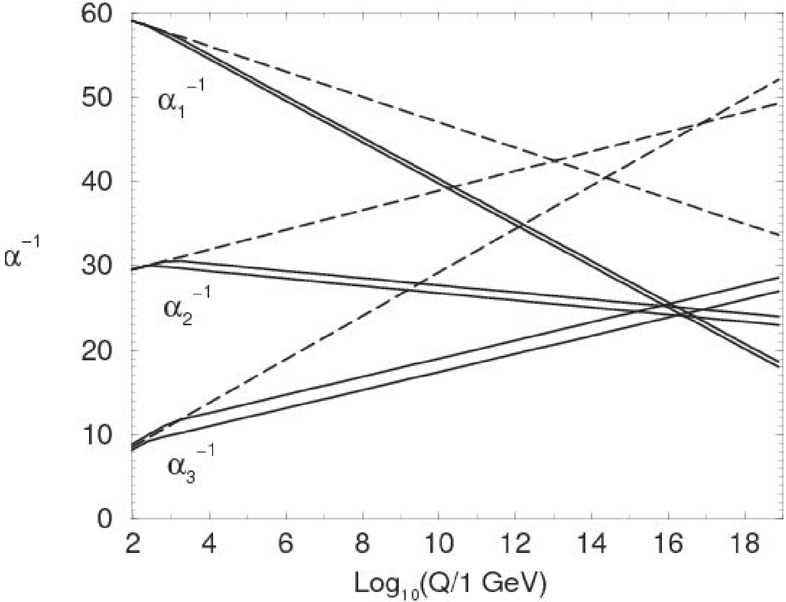


If we want to use a mass-independent scheme we are forced to replace our single theory in an on-mass-shell scheme with a whole tower of effective field theories — then the large logs can be controlled

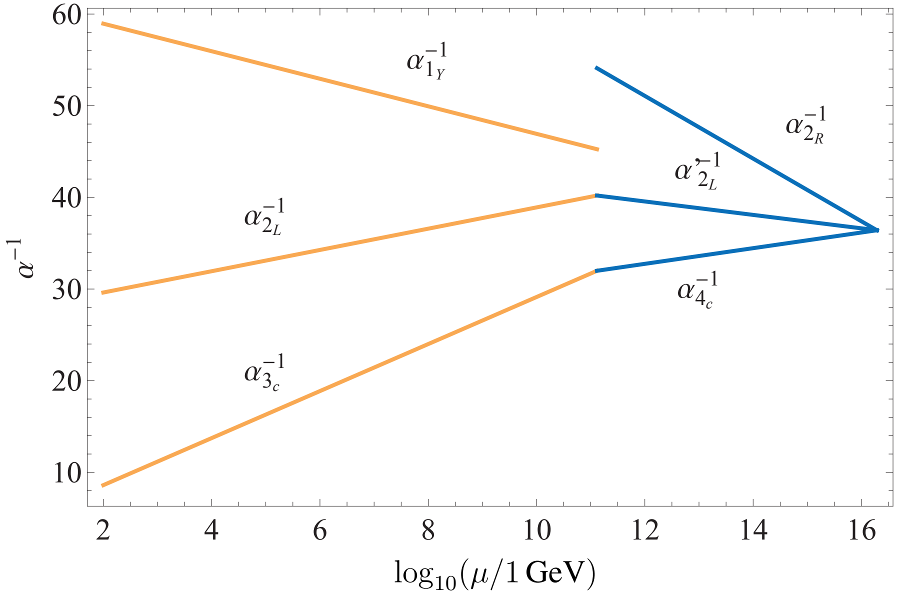
the extreme perturbative effective field picture theory replaces a single theory with a tower of effective theories depending on the continuous scale μ — at the mass of each type of particle (or other new physics with a mass scale) there is a boundary where we switch from one effective theory to another, **matching** the couplings in the two theories to get the same physics on the boundary — between the boundaries the couplings (and masses) of the theory evolve or "**run**" with μ — matching and running are the basic tools we use in working with effective field theory — going down in μ we can calculate — this gets less useful when the scales are not well-separate, as often happens in our world — but it is still the best way to think about QFT

Of course, what we really want to do to understand the world at short distances is to go UP in μ — this is an imperfect process of trying to find clues in the low energy theory to the physics of the boundary and above

It is worth emphasizing that this depends on the unknown details of physics between a TeV and GUT scale. Somewhat robust because complete multiplets leading order do not affect the rate at which the inverse couplings approach one another.



Altarelli *et al.* A non supersymmetric SO(10) grand unified model for all the physics below M_{GUT}



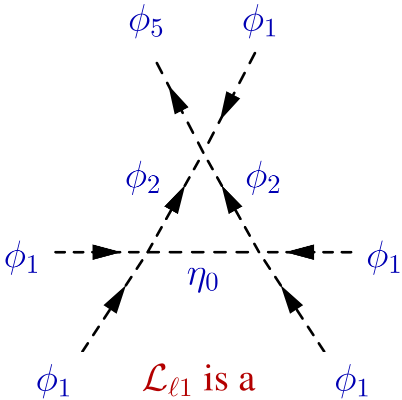
Morals

Something has to be left out to go from a high energy theory to a lower energy effective theory — otherwise you are double counting. Sometimes in the literature, people ignore this obvious fact and use the term "effective theory" to describe the full theory in a restricted energy range, and this is not very helpful. You understand an effective theory only when you really understand what is left out to get it from the high energy theory above it.

In constructing the Lagrangian of the effective theory, you are not just integrating something out. You are matching!

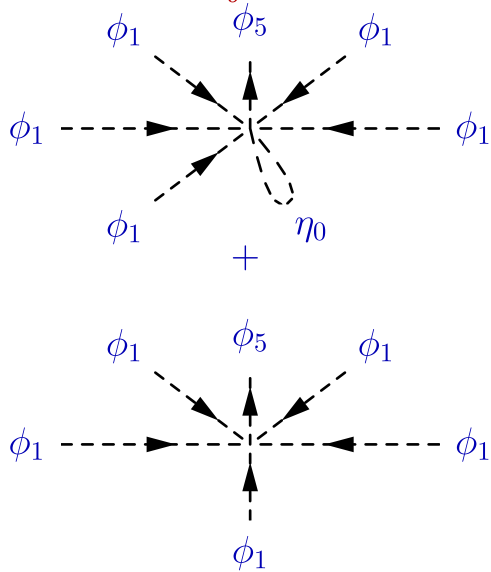
You can see the difference in one of the examples we discussed earlier which shows that you can't just do the functional integral over heavy fields.

1-loop - $\hbar^0$: 1 loop attached to heavy particle trees



$\mathcal{L}_{\ell 1}$ is a difference between 1-loop h and ℓ graphs

"1-loop" - $\hbar^0$: a light particle loop with only $\mathcal{L}_{\ell 0}$ vertices **or** $\int \mathcal{L}_{\ell 1}$



There is a closely related issue. Within the general framework of the effective field theory idea, there are two rather different approaches, which I will call the Wilsonian approach, and the continuum effective field theory approach. It is the second of these that I will discuss in detail here. But I should start by explaining why I think that they are different. I will argue that the two take a very different approach to renormalization.

In Wilson effective theory, the fundamental question is **How does the full theory change as you integrate out high momentum modes and look at it at larger distances?** This question fits in nicely with a physical renormalization scheme such as momentum space subtraction and physical renormalization.

In what I call continuum effective field theory, the question is **How do we modify the theory to allow the use of a mass independent scheme and still get the physics right?** The idea is to put in **by hand** as much as possible of the dependence on distance scale. The more of the physics of distance scale that is put in by hand, the easier it becomes to extract the physics that your really care about.

Many years ago, my wonderful former colleague Sidney Coleman once asked me: **What's wrong with form factors? What's wrong with just integrating out heavy particles and large momentum modes,** *ala* **Wilson and using the resulting nonlocal theory as your interaction?**

The answer of course is **There nothing wrong with it — but this is not an effective field theory calculation. It is just a way of doing the full theory calculation. and so you do not get any of the advantages of an EFT calculation.**

The fact that one of the world's greatest field-theorists would ask such a question convinced me that the idea of continuum effective field theory was not universally understood then. I think that people are still sometimes confused.

The advantages that we have seen of continuum EFT include

1. Concentration on relevant physics: It allows us to deal just with the particles that we actually know about, and interactions that we already see, and postpone speculation about higher energies.

2. Consistency with a mass independent renormalization scheme: It allows us to use a convenient scheme like $\overline{MS}$ and still get the physics right. This leads to simpler, more transparent calculations.

3. Dealing efficiently with IR divergences: As I have shown you, the effective theory calculations can be organized to explicitly avoid infrared divergences. In particular, calculations of matching corrections are automatically infrared finite.

4. The EFT stucture incorporates important nonperturbative information automatically (as in coupling-constant running).

Quadratic divergences and fine tuning.

One thing that is missing in continuum EFT is quadratic divergences. Mass independent schemes don't have quadratic divergences for the obvious reason that power-law dependence on a cut-off mass is not mass independent.

It is not obvious what this means for arguments that depend on cancellation of quadratic divergences. Personally, I believe that fine-tuning IS an issue, I am just not sure that it can always be quantified in this simplistic way.