Scattering Amplitudes and Geometry

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Geometry

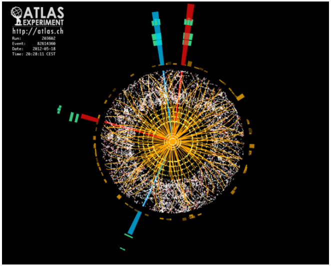


Figure: A scattering process at the LHC.

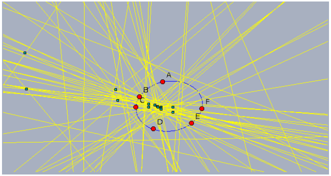


Figure: Pascal lines for six points on a conic.

I will discuss scattering amplitudes in *planar* $\mathcal{N} = 4$ super-Yang-Mills theory.

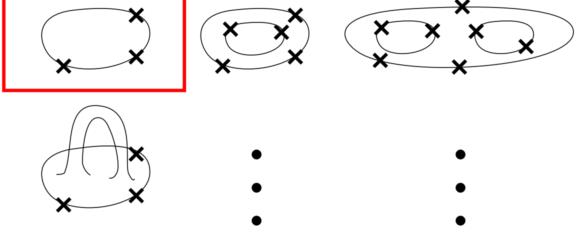


Figure: The topological classification of color flow.

The planar $\mathcal{N} = 4$ theory shows signs of integrability.

Why scattering amplitudes?

In a conformal field theory one usually computes correlation functions of conformal operators. Such quantities manifest all the symmetries of the conformal theory.

Historically [Bern, Dixon, Dunbar, Kosower] scattering amplitudes in $\mathcal{N} = 4$ have been computed as a preparation for computing scattering amplitudes in more complicated but more phenomenologically relevant gauge theories.

Scattering amplitudes are kinematically simpler because of the on-shell conditions $k_i^2 = 0$. Also they are easier to compute (using unitarity).

However, scattering amplitudes of massless particles (like in $\mathcal{N} = 4$ SYM) are IR divergent. After regularizing the IR divergences the result is *not* invariant under superconformal transformations anymore.

Surprising development [Drummond, Henn, Korchemsky, Sokatchev]: scattering amplitudes in the limit $N \to \infty$ with $g_{YM}^2 N = \text{const}$ have *more* symmetry than correlation functions. The extra symmetry is *dual superconformal symmetry*.

Statement of the problem

The ℓ-loop scattering amplitude in $\mathcal{N} = 4$ super-Yang-Mills contains transcendental functions of transcendentality 2ℓ.
We would like to write down the answers for scattering/Wilson-loop in planar $\mathcal{N} = 4$ SYM as explicitly as possible. (We can write the answer in terms of integrals, but can we do better?)
Unfortunately, we are not able to do this in full generality because of the poor mathematical understanding of the transcendental functions involved. Nevertheless, a lot of the complexity of the answer seems to stem *not* from the complicated transcendental functions, but from their *rational* arguments.

Iterated integrals

Many of the functions appearing in the $\mathcal{N} = 4$ scattering amplitudes can be written in terms of iterated integrals. For example,

$$\mathrm{Li}_n(z) = \int_0^z dt \frac{\mathrm{Li}_{n-1}(t)}{t},$$
$$\mathrm{Li}_1(z) = -\ln(1 - z).$$

Therefore the Li_n functions can be written as an n-fold iterated integral

$$\mathrm{Li}_n(z) = -\int_0^z d\ln t_1 \int_0^{t_1} d\ln t_2 \dots \int_0^{t_{n-1}} d\ln(1 - t_n).$$

Transcendentality

In general, we define a function of k variables to be of transcendentality n if it can be written as an n-fold iterated integral along a path in $\mathbb{C}^k$

$$T_n(z) = \int^z d\ln R_1 \int^{t_1} d\ln R_2 \ldots \int^{t_{n-1}} d\ln R_n,$$

where R_i are rational functions with rational coefficients. Transcendentality is additive: the transcendentality of a product of transcendental functions is the sum of transcendentalities of terms. Numerical constants also have transcendentality n if they are obtained from functions of transcendentality n evaluated at rational points. For example, rational numbers have transcendentality zero, π^{2n} has transcendentality $2n$ since, up to a rational number it is $\zeta_{2n} = \mathrm{Li}_{2n}(1)$.

Symbols

For a function which can be written as

$$T_n(z) = \int^z d\ln R_1 \int^{t_1} d\ln R_2 \ldots \int^{t_{n-1}} d\ln R_n,$$

let us focus on the rational fractions R_i and group them in a string ordered from the innermost integral to the outermost

$$R_n \otimes R_{n-1} \otimes \ldots \otimes R_1.$$

This is called the *symbol* of T_n.
If we know the derivatives of the function T_n, we can compute the rational fractions R_i recursively

$$dT_n(z) = d\ln R_1(z) \int^z d\ln R_2 \ldots .$$

Then $d^2 = 0$ imposes *integrability constraints* on the rational fractions R_i.

Properties

Every transcendental function has a unique symbol, up to the following equivalences

$$\ldots \otimes RR' \otimes \ldots = \ldots \otimes R \otimes \ldots + \ldots \otimes R' \otimes \ldots ,$$
$$\ldots \otimes cR \otimes \ldots = \ldots \otimes R \otimes \ldots ,$$

for $c \in \mathbb{Q}$ and R, R' rational fractions.
If two transcendental functions have the same symbol, then they differ only by terms which are transcendental constants times transcendental functions (lower *functional* transcendentality). This can be used to prove identities between polylogarithms.

Li_2 example

We have

$$\mathcal{S}(\mathrm{Li}_n(x)) = -(1-x)\otimes x.$$

Define a sequence a_n recursively by $a_{n+1} = \frac{1-a_n}{a_{n-1}}$. This is *periodic* with periodicity 5. Now compute the following combination

$$\mathcal{S}\left(\sum_{n=1}^{5}\mathrm{Li}_2(a_n)\right) = \sum_{n=1}^{5} -(1-a_n)\otimes a_n = -\sum_{n=1}^{5}(a_{n-1}a_{n+1})\otimes a_n =$$

$$-\sum_{n=1}^{5}(a_{n-1}\otimes a_n + a_n\otimes a_{n-1}) = \mathcal{S}\left(-\sum_{n=1}^{5}\ln a_{n-1}\ln a_n\right).$$

The left-over "subleading functional transcendentality part" turns out to be a constant, $\zeta(2) = \frac{\pi^2}{6}$, so we get

$$\sum_{n=1}^{5}(\mathrm{Li}_2(a_n) + \ln a_{n-1}\ln a_n) = \frac{\pi^2}{6}.$$

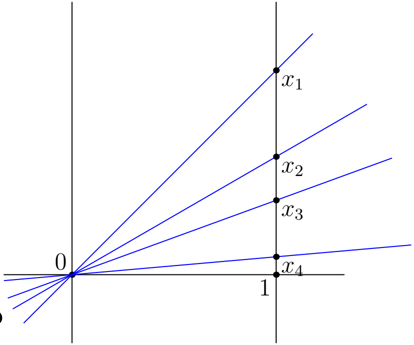
So the symbol is a *canonical* way of representing transcendental functions (we can rewrite the transcendental functions by using identities, but the symbol remains unchanged).
However, the symbol does not contain information about the subleading functional transcendentality or the position of the branch cuts (the branch points are at the positions of the zeros or poles of the rational functions R_i).
Nevertheless, using analyticity, integrability of the symbol and some physical input (collinear limits, OPE, Regge limits), one can get very far.

Conformal ratios on a line

Take $x_1, \ldots, x_4 \in \mathbb{C}$. The conformal group $SL(2)$ acts on these points by $x_i \to \frac{ax_i+b}{cx_i+d}$, with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2)$.

The points $x_1, \ldots, x_4$ can be identified with lines in $\mathbb{C}^2$. The group $SL(2)$ acts linearly on the lines passing through the origin, but non-linearly (see above) on the intersection coordinates $x_1, \ldots, x_4$.

This construction also suggests a compactification, by adding a point at infinity (corresponding to a line parallel to the y axis.

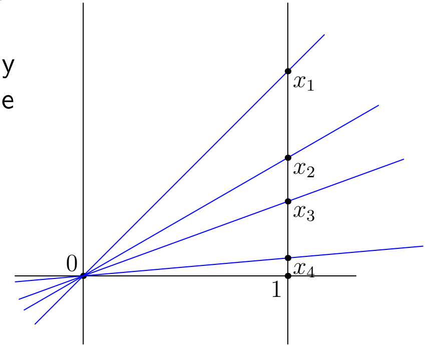


Conformal transformations in Minkowski space

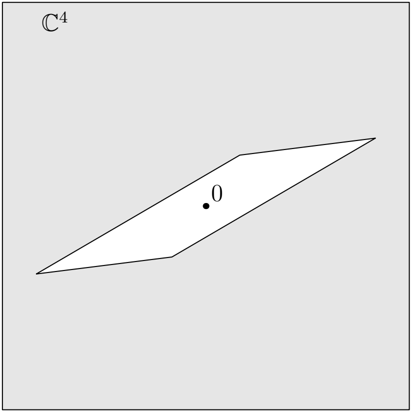
The case of four-dimensional Minkowski space is similar to the case of the line. This time the (complexified) conformal group is $SL(2)$, which acts on these points by $x \to (ax+b)(cx+d)^{-1}$, where x, a, b, c, d are 2×2 matrices and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(4)$.

The points x in Minkowski space can be identified with planes in $\mathbb{C}^4$. The group $SL(4)$ acts linearly on the planes passing through the origin, but non-linearly (see above) on the intersection coordinates x.

In this case the compactification is more complicated. It is the Grassmannian space $G_2(4)$ of two-planes in four dimensions.



Momentum twistors

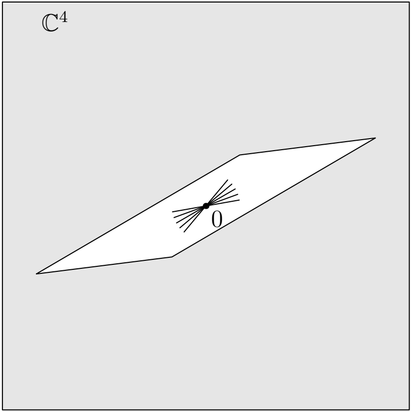


Complexified compactified Minkowski space $\widetilde{CM} \cong G_2(4)$. In some coordinate patch (a top cell) we can describe points in $G_2(4)$ by a 2×4 matrix

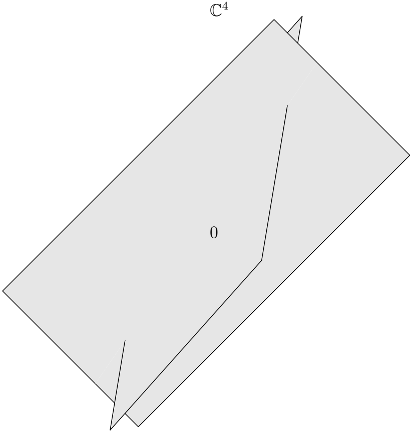
$$\mathcal{M} = \begin{pmatrix} 1 & 0 & x_{11} & x_{12} \\ 0 & 1 & x_{21} & x_{22} \end{pmatrix}.$$

Then the equation of the plane is $\mathbb{C}^2 \ni \lambda \mapsto \lambda^T \mathcal{M} \in \mathbb{C}^4$.

Momentum twistors



Projecting to $\mathbb{CP}^3$ a point in $G_2(4)$ corresponds to a projective line in $\mathbb{CP}^3$.



Two null separated points correspond to two planes intersecting in a line. Upon projection, this becomes a point in $\mathbb{CP}^3$.

Configurations of points

Space-time kinematics to "momentum twistor" space:
momenta $\{k_i\}_{i=1,\ldots,n}$ with $k_i^2 = 0 \rightarrow$
dual space $\{x_i\}_{i=1,\ldots,n}$ with $x_i - x_{i+1} = k_i \rightarrow$
two-planes in $\mathbb{C}^4$ intersecting pairwise $\rightarrow$
n points in $\mathbb{CP}^3$ with a cyclic ordering.
We denote by $\mathrm{Conf}_n(\mathbb{CP}^k)$ the space of configurations of n points in $\mathbb{CP}^k$ with a cyclic ordering. Therefore, the kinematics can be described as configurations of points in $\mathbb{CP}^3$. There is also a supersymmetric extension as configurations of points in $\mathbb{CP}^{3|4}$.

Projective geometry

From now on, we will express all the kinematics in twistor space. Since this is a projective space we are studying a projective geometry.
Projective geometry is in some sense simpler than the more familiar Euclidean geometry.

- no notion of length or angle
- no parallelism; all the lines in a plane intersect
- no notion of "between".

A special feature of projective geometry is that every statement has a dual. In two dimensions points are dualized to lines and lines are dualized to points. This duality is related to the parity transformation in $\mathcal{N} = 4$.

Spinors and twistors

We have seen that we can represent the kinematics in terms of the points $Z_i \in \mathbb{CP}^3$. The conformal group is $SL(4)$ and it preserves volumes. Therefore, $\langle i, j, k, l \rangle = \epsilon^{abcd} Z_{ia} Z_{jb} Z_{kc} Z_{ld}$ are invariant. It is also easy to show that

$$x_{ij}^2 = (x_i - x_j)^2 = \frac{\langle i, i+1, j, j+1 \rangle}{\langle i, i+1 \rangle \langle j, j+1 \rangle},$$

where $\langle ab \rangle = \langle Iab \rangle$ and I is rank two and antisymmetric. It breaks the conformal symmetry by singling out a point at infinity.

Cross-ratios

Cross-ratios can be easily written in twistor space.

$$u_{ij} \equiv \frac{x_{i,j+1}^2 x_{i+1,j}^2}{x_{ij}^2 x_{i+1,j+1}^2} = \frac{\langle i, i+1, j+1, j+2\rangle\langle i+1, i+2, j, j+1\rangle}{\langle i, i+1, j, j+1\rangle\langle i+1, i+2, j+1, j+2\rangle}.$$

Using the Plücker identity

$$\langle abxy\rangle\langle abzt\rangle + \langle abxz\rangle\langle abty\rangle + \langle abxt\rangle\langle abyz\rangle = 0,$$

we can write, for example,

$$1 - u_{ij} = \frac{\langle i, i+1, i+2, j+1\rangle\langle i+1, j, j+1, j+2\rangle}{\langle i, i+1, j, j+1\rangle\langle i+1, i+2, j+1, j+2\rangle}.$$

If we drop the common twistors $i+1$ and $j+1$ from all the four-brackets we get relations between cross-ratios as simple as in $2D$ CFT.

Example of application

Del Duca, Duhr and Smirnov computed the the non-trivial part of the 6-point MHV scattering amplitude (AKA the remainder function) in terms of a special class of transcendental functions called Goncharov polylogs.

This result is complicated, but its symbol is much simpler. Due to the dual conformal symmetry the result depends only on three cross-ratios

$$u_1 = \frac{s_{12}s_{45}}{s_{123}s_{345}}, \quad u_2 = \frac{s_{23}s_{56}}{s_{234}s_{123}}, \quad u_3 = \frac{s_{34}s_{61}}{s_{345}s_{234}}.$$

It turns out that the entries of the symbol are algebraic, not rational.

There is one expression appearing with a square root

$$\sqrt{\Delta} = \sqrt{(u_1 + u_2 + u_3 - 1)^2 - 4u_1 u_2 u_3}.$$

But when transforming to twistor space we obtain

$$\sqrt{\Delta} = \pm \frac{\langle 1234\rangle\langle 3456\rangle\langle 5612\rangle - \langle 2345\rangle\langle 4561\rangle\langle 6123\rangle}{\langle 1245\rangle\langle 2356\rangle\langle 3461\rangle},$$

so all the square roots disappear. In the end, the symbol can be written as

$$\langle \cdot,\cdot,\cdot,\cdot\rangle \otimes \langle \cdot,\cdot,\cdot,\cdot\rangle \otimes \langle \cdot,\cdot,\cdot,\cdot\rangle \otimes \langle \cdot,\cdot,\cdot,\cdot\rangle,$$

where the four-brackets are of special form $\langle i, i+1, j, j+1\rangle$ and $\langle i, j-1, j, j+1\rangle$.

Integration

Since the symbol is simple, we can hope to find a simple function with that symbol. This is the problem of "integrating" a symbol, which is in general non-trivial.

But for six-point two-loop MHV this turns out to be easy, because only classical polylogs are necessary. The final result is (Goncharov, Spradlin, CV, Volovich):

$$\frac{1}{2}\,\mathrm{Li}_4\left(-\frac{\langle 1456\rangle\langle 2356\rangle}{\langle 1256\rangle\langle 3456\rangle}\right)+\frac{1}{2}\,\mathrm{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right)$$
$$-\frac{1}{16}\,\mathrm{Li}_4\left(\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 4561\rangle}\right)+\text{cyclic permutations}+$$
$$\text{products of lower transcendentality functions.}$$

Beyond six-point MHV

If we go to higher-point amplitudes two complications arise

- the answers are not expressible in terms of classical polylogarithms
- more complicated entries appear in the symbol, like $\langle ab(ijk) \cap (lmn) \rangle \equiv \langle aijk \rangle \langle blmn \rangle - \langle bijk \rangle \langle almn \rangle$.

So we have two questions

- What kind of entries can appear in the symbol?
- What kind of polylogarithms appear in the answer?

The second question is hard. We can not answer it fully, but instead we answer it by partially integrating, up to terms of transcendentality two and three.

Cluster algebras

We find that the arguments appearing in the partial integration of the symbol are (minus) cluster $\mathcal{X}$ coordinates for various cluster algebras.

Cluster algebras are commutative algebras

1. constructed from distinguished generators (called *cluster variables*)
2. which are grouped into non-disjoint sets of constant cardinality (called *clusters*),
3. which are constructed recursively by an operation called *mutation* from an initial cluster.

The number of variables in a cluster is called the rank of the cluster algebra.

Example of cluster algebra

The A_2 cluster algebra is defined by the following data:

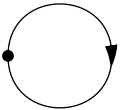
- cluster variables: $x_m, \quad m \in \mathbb{Z}$
- clusters: $\{x_m, x_{m+1}\}$
- initial cluster: $\{x_1, x_2\}$
- rank: 2
- exchange relations: $x_{m-1}x_{m+1} = 1 + x_m$
- mutation: $\{x_{m-1}, x_m\} \to \{x_m, x_{m+1}\}$.

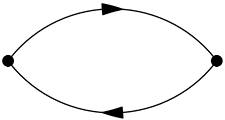
This algebra has appeared before in the five-term dilogarithm identity, with $x_m \to -a_m$!

Quivers

These cluster algebras are defined by a quiver: a finite oriented graph without loops (arrows with the same origin and target) and two-cycles (pairs of arrows going in opposite directions between two vertices).

For example,





are forbidden.

Cluster algebras of geometric type

To each vertex i we associate cluster $\mathcal{A}$ coordinates x_i. We also define a skew-symmetric matrix

$$b_{ij} = (\#\text{arrows } i \to j) - (\#\text{arrows } j \to i).$$

Since only one of the terms above is nonvanishing, $b_{ij} = -b_{ji}$.
A mutation at vertex k is obtained by applying the following operations on the initial quiver:

- for each path $i \to k \to j$ we add an arrow $i \to j$
- reverse all the arrows on the edges incident with k
- remove all the two-cycles that may have formed.

It is an involution; when applied twice in succession we obtain the initial cluster.

Mutation of cluster $\mathcal{A}$ coordinates

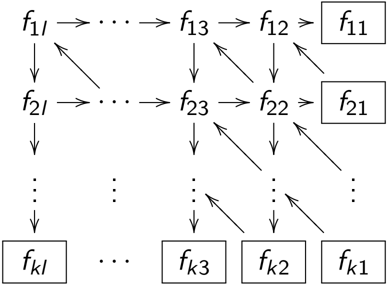
The mutation at k changes x_k to x'_k defined by

$$a_k a'_k = \prod_{i|b_{ik}>0} a_i^{b_{ik}} + \prod_{i|b_{ik}<0} a_i^{-b_{ik}},$$

and leaves the other cluster variables unchanged. (An empty product is set to one.)
Example: the A_2 cluster algebra can be expressed by a quiver $a_1 \to a_2$. Then, a mutation at a_1 replaces it by $a'_1 = \frac{1+a_2}{a_1} \equiv a_3$ and reverses the arrow. A mutation at a_2 replaces it by $a'_2 = \frac{1+a_1}{a_2} \equiv a_5$ and reverses the arrow.

Grassmannian cluster algebras

According to [Gekhtman, Shapiro, Vainshtein], the initial quiver for the $G_k(n)$ cluster algebra is given by[1]



where

$$f_{ij} = \begin{cases} \frac{\langle i+1,\ldots,k,k+j,\ldots,i+j+k-1\rangle}{\langle 1,\ldots,k\rangle}, & i \leq l-j+1, \\ \frac{\langle 1,\ldots,i+j-l-1,i+1,\ldots,k,k+j,\ldots,n\rangle}{\langle 1,\ldots,k\rangle}, & i > l-j+1 \end{cases}.$$

[1] Here we are presented a flipped version of the quiver and with the arrows reversed with respect to the quivers of that ref.

A duality for configurations of points

Let us assume that the n points in $\mathbb{CP}^k$ are in a generic position (no three points on a line, no four points on a plane, etc.).
Let us group the homogeneous coordinates of the n points in a $(k+1) \times n$ matrix. By the action of the $PSL(k+1)$ group and rescaling of the coordinates we can put this matrix in a special form

$$(\mathbf{1}_{k+1}, Y_{k+1,n-k-1})\,.$$

From this we can make a $(n-k-1) \times n$ matrix

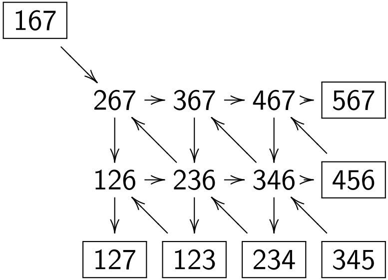
$$\left(\mathbf{1}_{n-k-1}, (Y^T)_{n-k-1,k+1}\right),$$

which can be obtained by the action of $PSL(n-k-1)$ on an arbitrary $(n-k-1) \times n$ matrix.
Therefore, we have an isomorphism
$\mathrm{Conf}_n(\mathbb{CP}^k) \cong \mathrm{Conf}_n(\mathbb{CP}^{n-k-2})$.

$G_3(7)$ or $\mathrm{Conf}_7(\mathbb{CP}^2) \cong \mathrm{Conf}_7(\mathbb{CP}^3)$

The cluster $\mathcal{X}$ coordinates are defined by

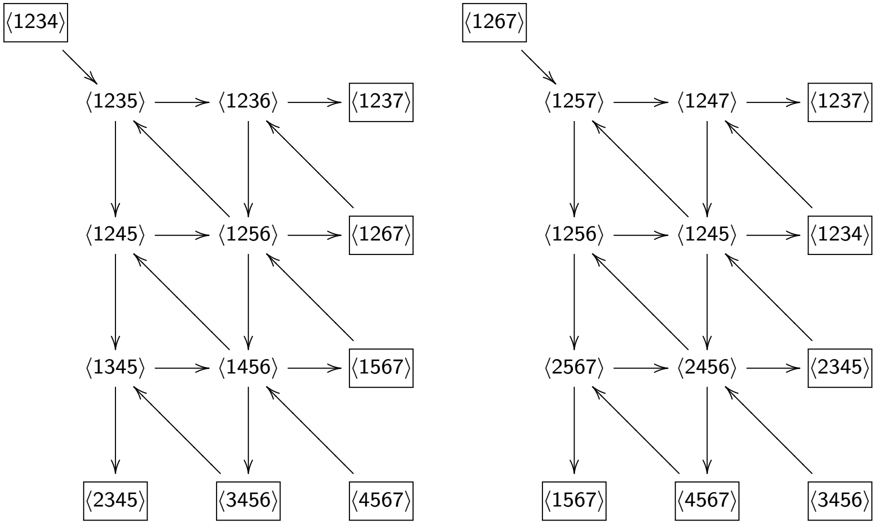
$$x_i = \prod_j a_j^{b_{ij}}.$$



Goncharov's triple ratio is a cluster $\mathcal{X}$ coordinate

$$\frac{\langle 345\rangle\langle 236\rangle\langle 467\rangle}{\langle 234\rangle\langle 367\rangle\langle 456\rangle}.$$

Parity transformation



Finite and infinite cluster algebras

A_n	B_n, C_n	D_n	E_6	E_7	E_8	F_4	G_2
$\frac{1}{n+2}\binom{2n+2}{n+1}$	$\binom{2n}{n}$	$\frac{3n-2}{n}\binom{2n-2}{n-1}$	833	4160	25080	105	8

Table: The number of clusters for cluster algebras of finite type.

The only cluster algebras with a finite number of clusters arise from $G_2(n)$ and $G_3(6)$, $G_3(7)$ and $G_3(8)$. Starting at eight-point the cluster algebras are of *infinite* type!
$\mathrm{Conf}_{n+3}(\mathbb{CP}^1) \cong A_n$, $\mathrm{Conf}_7(\mathbb{CP}^2) \cong D_4$, $\mathrm{Conf}_7(\mathbb{CP}^2) \cong E_6$,
$\mathrm{Conf}_8(\mathbb{CP}^2) \cong E_8$.

Seven-point results

The $\Lambda^2 B_2$ part of the seven-point two-loop MHV remainder function is

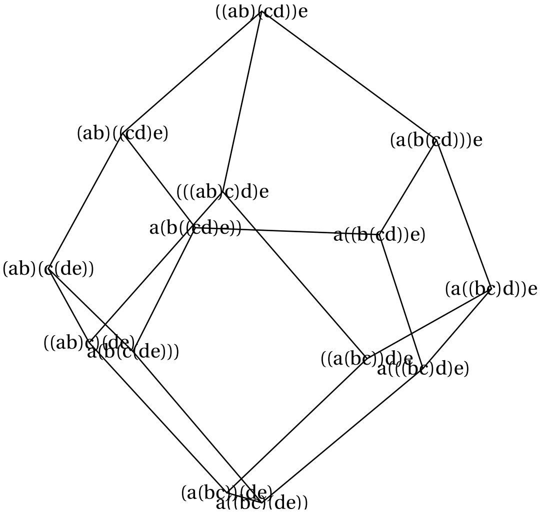
$$
\begin{aligned}
&-\left\{-\frac{\langle 2\times 3,4\times 6,7\times 1\rangle}{\langle 167\rangle\langle 234\rangle}\right\}_2 \wedge \left\{-\frac{\langle 7\times 1,2\times 3,4\times 5\rangle}{\langle 127\rangle\langle 345\rangle}\right\}_2 \\
&\quad -\left\{-\frac{\langle 2\times 3,4\times 6,7\times 1\rangle}{\langle 167\rangle\langle 234\rangle}\right\}_2 \wedge \left\{-\frac{\langle 234\rangle\langle 456\rangle}{\langle 246\rangle\langle 345\rangle}\right\}_2 \\
&\quad -\left\{-\frac{\langle 2\times 3,4\times 6,7\times 1\rangle}{\langle 167\rangle\langle 234\rangle}\right\}_2 \wedge \left\{-\frac{\langle 146\rangle\langle 567\rangle}{\langle 167\rangle\langle 456\rangle}\right\}_2 \\
&-\left\{-\frac{\langle 2\times 3,4\times 6,7\times 1\rangle}{\langle 167\rangle\langle 234\rangle}\right\}_2 \wedge \left\{-\frac{\langle 5\times 6,7\times 1,2\times 3\rangle}{\langle 123\rangle\langle 567\rangle}\right\}_2 + \cdots
\end{aligned}
$$

In four-bracket language the entries with cross-products correspond to composite four-brackets $\langle ij(klm)\cap(npq)\rangle$.
Is there anything special about these terms?

Poisson structure

The entries of B_2 elements are minus cluster coordinates. But more surprisingly, the terms $\{x\}_2 \wedge \{y\}_2$ are such that a certain Poisson bracket vanishes: $\{\ln x, \ln y\}_{PB} = 0$.
It is sufficient to define the Poisson bracket of two $\mathcal{X}$ in the same cluster. We take $\{\ln x_i, \ln x_j\}_{PB} = b_{ij}$. This definition is compatible with mutations in the sense that $\{\ln x'_i, \ln x'_j\}_{PB} = b'_{ij}$ where the primed variables are obtained by mutation.
We also computed the $B_2 \wedge B_2$ of the n-point two-loop MHV amplitudes and checked that all the Poisson brackets of $\{x\}_2 \wedge \{y\}_2$ terms vanish.
For NMHV, we also get simple answers, ± 1.

The associahedron



An interesting byproduct

We have found the first 40-term trilogarithm identity of cluster type:

$$\left\{-\frac{\langle 125\rangle\langle 134\rangle}{\langle 123\rangle\langle 145\rangle}\right\}_3+\left\{-\frac{\langle 126\rangle\langle 145\rangle}{\langle 124\rangle\langle 156\rangle}\right\}_3+\left\{-\frac{\langle 126\rangle\langle 145\rangle\langle 234\rangle}{\langle 123\rangle\langle 146\rangle\langle 245\rangle}\right\}_3+$$
$$\frac{1}{3}\left\{-\frac{\langle 136\rangle\langle 145\rangle\langle 235\rangle}{\langle 123\rangle\langle 156\rangle\langle 345\rangle}\right\}_3+\text{signed dihedral permutations}=0.$$

It is possible to associate $\{x\}_3 \to \text{function}(x)$ such that the identity is satisfied. Mathematicians use

$$\mathrm{L}_3(z) := \Re\Big(\mathrm{Li}_3(z)-\mathrm{Li}_2(z)\log|z|-\frac{1}{3}\log^2|z|\log(1-z)\Big),\quad z\in\mathbb{C},$$

which satisfy "clean" functional equations. However, these functions are only real analytic, not complex analytic. We can find functions which are complex analytic instead. For Li_2 we have

$$L_2(x^+,x^-)=\mathrm{Li}_2(x^+)+\mathrm{Li}_2(x^-)-\frac{1}{2}\log(x^+x^-)(\mathrm{Li}_1(x^+)+\mathrm{Li}_1(x^-))-$$
$$\frac{1}{2}\log\frac{x^+}{x^-}(\mathrm{Li}_1(x^+)-\mathrm{Li}_1(x^-))-\frac{1}{6}\log^2\frac{x^+}{x^-}.$$

Conclusions

- The notion of symbol of a transcendental function is useful in understanding and simplifying scattering amplitudes.
- Cluster coordinates seem to play an important role, but the interplay with supersymmetry is not completely understood.
- Transcendentality four functions are poorly understood mathematically, but explicit answers arising in physics can help to build and guide mathematical intuition.
- Geometrical thinking is a very powerful guide.

Thank you!