Yang-Baxter operators and scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory

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LAPTH

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Based on work in collaboration with
S. Derkachov and R. Kirschner
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Outline

- Basic facts about scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory and their hidden symmetries. Dual super conformal symmetry and Yangian symmetry (Drinfeld's formulation)
- Basic constructions of Quantum Inverse Scattering Method. Spin chain. Yangian symmetry
- R-operator construction of Yangian invariants
- Cyclic symmetry, Inverse Soft Limit, link integral representation
- On-shell diagrams and R-operator factorization
- Super-twistor variables
- Conclusions

Scattering amplitudes in a gauge theory

Color decomposition. $SU(N_c)$

$$M(\{p_i, a_i, h_i\}) = \epsilon^{\mu_1}_{h_1} \cdots \epsilon^{\mu_n}_{h_n} \langle A_{\mu_1}(p_1) \cdots A_{\mu_n}(p_n) \rangle = \mathrm{Tr}\,(T^{a_1} \cdots T^{a_n})\, M(\{p_i, h_i\}) + \text{Bose}$$

$$p^{\alpha\dot\alpha} = (\sigma_\mu)^{\alpha\dot\alpha} p^\mu = \begin{pmatrix} p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & p^0-p^3 \end{pmatrix}\ ;$$

massless particle $\quad p^{\alpha\dot\alpha} = \lambda^\alpha \tilde\lambda^{\dot\alpha}\ ;\ \langle ij \rangle = \epsilon_{\alpha\beta} \lambda_i^\alpha \lambda_j^\beta$

Parke-Taylor amplitude

$$\mathrm{MHV}_n = \frac{\langle ij \rangle^4 \delta^4(p)}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle} \qquad 1^+ 2^+ \cdots i^- \cdots j^- \cdots n^+$$

$\mathcal{N} = 4$ SYM. PCT self-conjugate multiplet. η^A $(A = 1, \cdots, 4)$ Grassmann variable

$$\Phi(\eta) = G_+ + \text{fermions} + \text{scalars} + (\eta)^4 G_-$$

$$\mathrm{MHV}_n = \frac{\delta^4(p)\delta^8(q)}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

supercharge $q^{\alpha A} = \lambda^\alpha \eta^A$

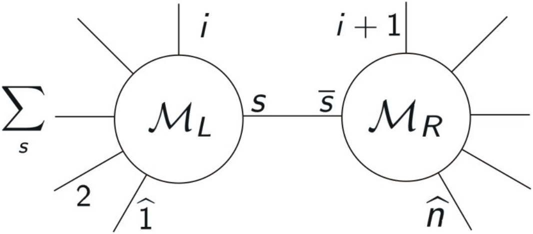
General superamplitude

$$M_n = \mathrm{MHV}_n \left[\mathcal{P}_{0,n} + \mathcal{P}_{4,n} + \cdots + \mathcal{P}_{4n-16,n} \right]$$

BCFW recursion for tree amplitudes

$M = \delta^4(p)\,\mathcal{M}$, $\mathcal{M}$ – rational function

$$\mathcal{M}_n = \sum_{L,R} \int \mathrm{d}^4\eta\, \mathcal{M}_L\big(\hat{1}(z_*), \cdots, i, -P_{1\cdots i}(z_*)\big) \frac{1}{P^2_{1\cdots i}} \mathcal{M}_R\big(P_{1\cdots i}(z_*), i{+}1, \cdots, \hat{n}(z_*)\big)$$



$$\lambda_1(z) = \lambda_1 - z\lambda_n\,, \quad \tilde{\lambda}_n(z) = \tilde{\lambda}_n + z\tilde{\lambda}_1\,, \quad \eta_n(z) = \eta_n + z\eta_1$$

$$P_{1\cdots i} = p_1 + p_2 + \cdots + p_i\,, \quad P_{1\cdots i}(z) = P_{1\cdots i} - z\lambda_n\tilde{\lambda}_1\,, \quad z_* = \frac{P^2_{\mathbf{1}\cdots i}}{\lambda_n^\alpha (P_{\mathbf{1}\cdots i})_{\alpha\dot{\alpha}} \tilde{\lambda}_{\mathbf{1}}^{\dot{\alpha}}}$$

Building blocks

 $= \text{MHV}_3 = \dfrac{\delta^4(p)\delta^8(q)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$;  $= \text{anti-MHV}_3 = \dfrac{\delta^4(p)\delta^4([12]\eta_3 + \text{cycl})}{[12][23][31]}$

Hidden symmetry

- $psu(2,2|4)$. Superconformal symmetry (**Lagrangian** origin). Generators – sum of **local** differential operators acting on the space $\{\lambda_i, \tilde{\lambda}_i, \eta_i\}$

$$J_a = \sum_{i=1}^{n} J_{i,a}$$

- Dual superconformal symmetry (**Dynamical**). Dual coordinates x, θ

$$x_i - x_{i+1} = p_i \quad ; \quad p_1 + \cdots + p_n = 0 \leftrightarrow x_1 \equiv x_{n+1}$$

Covariance of tree amplitudes

$$\text{inversion} \quad x^\mu \to -\frac{x^\mu}{x^2} \qquad M_n \to (x_1^2 \cdots x_n^2)\, M_n$$

$psu(2,2|4)$ – infinitesimal form. Generators – sum of **local** differential operators on the space $\{x_i, \theta_i, \lambda_i, \tilde{\lambda}_i, \eta_i\} \longrightarrow$ Sum of **bilocal** differential operators on the space $\{\lambda_i, \tilde{\lambda}_i, \eta_i\}$

$$J_a^{(1)} = f_a{}^{bc} \sum_{i<k} J_{i,b} J_{k,c}$$

- Closure of two algebras $\longrightarrow$ Yangian algebra $Y(psu(2,2|4))$. Infinite-dimensional

$$[\,J_a\,,\,J_b\,] = f_{ab}{}^c J_c \qquad [\,J_a\,,\,J_b^{(1)}\,] = f_{ab}{}^c J_c^{(1)} \qquad + \quad \text{Serre relations} \quad + \quad \text{Higher levels}$$

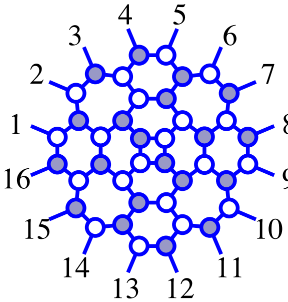
Why Quantum Inverse Scattering Method ?

Why spin chain ?

Yangian symmetry

Basic idea of QISM: Complicated nonlocal objects (like Hamiltonian and higher integrals of motion) are constructed from simple local building blocks according to simple local rule $\rightarrow$ dynamic is integrable

BCFW: Construct superamplitudes from 3-point amplitudes  and 



Basic constructions of QISM. $g\ell(N|M)$ spin chain

- Dynamical variables. $\mathbf{x} = (x_a)_{a=1}^N$ and $\mathbf{p} = (p_a)_{a=1}^N$

$$\left[\, x_a \,,\, p_b \,\right] = -\delta_{ab} \qquad \text{Heisenberg pairs}$$

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$\#$ fermionic $= M$

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- **Jordan-Schwinger** type representation of symmetry algebra $g\ell(N|M)$.
  Non-compact representation

$$J_{ab} = x_a p_b$$

- L-operator. u – spectral parameter. Matrix $(N+M)\times(N+M)$

$$\Big[\mathrm{L}(u)\Big]_{ab} = u\,\delta_{ab} + x_a\,p_b =$$

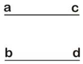
- Fundamental commutation relation (**FCR**)

$$\mathcal{R}_{ab,ef}(u-v)\,\Big[\mathrm{L}(u)\Big]_{ec}\,\Big[\mathrm{L}(v)\Big]_{fd} = \Big[\mathrm{L}(v)\Big]_{bf}\,\Big[\mathrm{L}(u)\Big]_{ae}\,\mathcal{R}_{ef,cd}(u-v)$$

\# quantum spaces $= 1$, \# auxiliary spaces $= 2$

Yang's $\mathcal{R}$-matrix $(N+M)^2\times(N+M)^2$

$$\mathcal{R}_{ab,cd}(u) = u + \mathrm{P} = u$$



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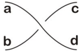


- The spin chain – quantum-mechanical system with many degrees of freedom. # sites $= n$

$$(\mathbf{x}_1, \cdots, \mathbf{x}_n) \text{ and } (\mathbf{p}_1, \cdots, \mathbf{p}_n)$$

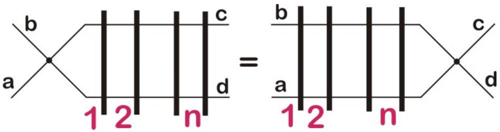
- Homogeneous monodromy matrix

$$\left[\mathrm{T}(u)\right]_{ac} = \left[\mathrm{L}_1(u)\right]_{ab_1} \left[\mathrm{L}_2(u)\right]_{b_1 b_2} \cdots \left[\mathrm{L}_n(u)\right]_{b_{n-1} c} =$$

quantum spaces $= n$, # auxiliary spaces $= 1$

- Co-multiplication property

$$\mathcal{R}_{ab,ef}(u-v) \left[\mathrm{T}(u)\right]_{ec} \left[\mathrm{T}(v)\right]_{fd} = \left[\mathrm{T}(v)\right]_{bf} \left[\mathrm{T}(u)\right]_{ae} \mathcal{R}_{ef,cd}(u-v)$$



Yangian algebra

$\mathrm{T}(u)$ – "generating function" – polynomial in spectral parameter u

$$\Big[\mathrm{T}(u)\Big]_{ab} = \sum_{m=0}^{n} u^{n-m+1} J_{ab}^{(m)}$$

FCR is equivalent to a set of commutation relations for generators $J_{ab}^{(m)}$

$$J_{ab}^{(0)} = \sum_{1\leq i\leq n} x_{a,i}\, p_{b,i} \quad , \quad J_{ab}^{(1)} = \sum_{1\leq i<j\leq n} x_{a,i}\, p_{c,i}\, x_{c,j}\, p_{b,j}$$

Yangian symmetry condition (M – Yangian invariant)

$$\Big[\mathrm{T}(u)\Big]_{ab} M(\mathbf{x}_1, \cdots, \mathbf{x}_n) = C\, \delta_{ab}\, M(\mathbf{x}_1, \cdots, \mathbf{x}_n)$$

Main result: Solving the eigenvalue problem for the monodromy matrix formulated in appropriate variables we recover crucial constructions for SYM scattering amplitudes: link integral representation, Inverse Soft Limit, on-shell diagrams, R-invariants.

R-operator (bilocal)

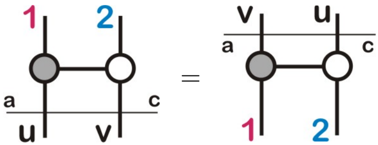
We use R-operator as the main building block in the construction of scattering amplitudes (Yangian invariants)

Defined by RLL-relation (intertwining relation)

$$\mathrm{R}_{12}(u-v)\left[\mathrm{L}_1(u)\right]_{ab}\left[\mathrm{L}_2(v)\right]_{bc}=\left[\mathrm{L}_1(v)\right]_{ab}\left[\mathrm{L}_2(u)\right]_{bc}\mathrm{R}_{12}(u-v)$$

$\#$ quantum spaces $=2$, $\#$ auxiliary spaces $=1$

$$\mathrm{R}_{12}(u)=(\mathbf{p}_1\cdot\mathbf{x}_2)^{-u}$$



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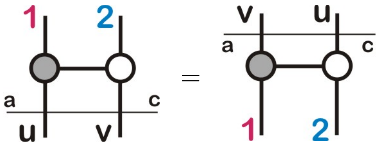
$\#$ quantum spaces $= 2$, $\#$ auxiliary spaces $= 1$

$$\mathrm{R}_{12}(u) = (\mathbf{p}_1 \cdot \mathbf{x}_2)^{-u} = \frac{\Gamma(1-u)}{2\pi i}\int_{\mathcal{C}} \frac{\mathrm{d}z}{z^{1-u}}\, e^{-z(\mathbf{p}_1\cdot\mathbf{x}_2)}$$



$s\ell(4|4)$ symmetry algebra

$$p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha} \;;\; \eta_A \;(A{=}1,\cdots,4)$$

Spin chain dynamical variables = **Spinor-helicity** variables for asymptotic states of scattering particles

$$\mathbf{x} \to (\lambda_\alpha, \quad \partial_{\tilde\lambda_{\dot\alpha}}, \quad \partial_{\eta_A}) \quad ; \quad \mathbf{p} \to (\partial_{\lambda_\alpha}, \quad -\tilde\lambda_{\dot\alpha}, \quad -\eta_A)$$

Reproduce BCFW-shift applying R-operator to arbitrary function

$$\left[\mathrm{R}_{ij}(u)F\right](\lambda_i, \tilde\lambda_i, \eta_i|\lambda_j, \tilde\lambda_j, \eta_j) = \int \frac{\mathrm{d}\mathbf{z}}{\mathbf{z}^{1-u}} F(\lambda_i - \mathbf{z}\lambda_j, \tilde\lambda_i, \eta_i|\lambda_j, \tilde\lambda_j + \mathbf{z}\tilde\lambda_i, \eta_j + \mathbf{z}\eta_i)$$

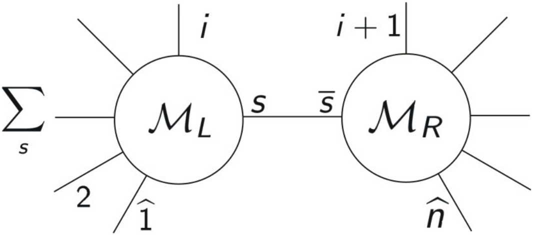
BCFW-recursion and BCFW-bridge

$$M_n = \mathrm{R}_{1n} \int \mathrm{d}^4\eta_0 \, \mathrm{d}^4 P_0 \, \delta({P_0}^2) \, M_L(\eta_1 , \lambda_1 , \tilde\lambda_1 ; \eta_0 , -P_0) \, M_R(\eta_n , \lambda_n , \tilde\lambda_n ; \eta_0 , P_0)$$

Yangian symmetry of scattering amplitudes $\Leftrightarrow$ Eigenvalue relation for monodromy matrix

BCFW recursion for tree amplitudes

$$\mathcal{M}_n = \sum_{L,R} \int \mathrm{d}^4\eta \, \mathcal{M}_L\big(\hat{1}(z_*), \cdots, i, -P_{1\cdots i}(z_*)\big) \frac{1}{P^2_{1\cdots i}} \mathcal{M}_R\big(P_{1\cdots i}(z_*), i{+}1, \cdots, \hat{n}(z_*)\big)$$



$$\lambda_1 \to \lambda_1 - \mathbf{z}\lambda_n \,, \quad \tilde{\lambda}_n \to \tilde{\lambda}_n + \mathbf{z}\tilde{\lambda}_1 \,, \quad \eta_n \to \eta_n + \mathbf{z}\eta_1$$

We start with the simplest solution of eigenvalue problem – **Basic state**

(Not entangled degrees of freedom)

$$\Omega_{k,n} = \cdots \delta^2(\lambda_i)\cdots \underbrace{\delta^2(\tilde{\lambda}_j)\delta^4(\eta_j)\cdots\delta^2(\tilde{\lambda}_l)\delta^4(\eta_l)}_{k}\cdots\delta^2(\lambda_m)\cdots$$

n particles (n-site monodromy), Grassmann degree $4k$

- 3-particle anti-MHV. $\Omega_{1,3} = \delta^2(\lambda_1)\,\delta^2(\lambda_2)\,\delta^2(\tilde{\lambda}_3)\delta^4(\eta_3)$

$$\mathrm{L}_1(u)\mathrm{L}_2(u)\mathrm{L}_3(u)\,\Omega_{1,3} = u(u-1)^2\,\Omega_{1,3}$$

Introduce interaction in integrable way (entangling degrees of freedom)

$$\mathrm{R}_{12}\,\mathrm{R}_{23}\,\Omega_{1,3} = \text{anti-MHV}_3 =$$ 

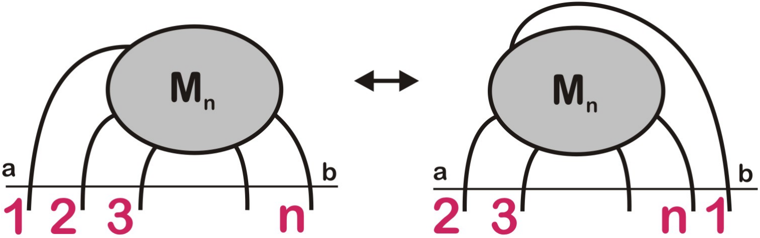
- 3-particle MHV. $\Omega_{2,3} = \delta^2(\lambda_1)\,\delta^2(\tilde{\lambda}_2)\delta^4(\eta_2)\,\delta^2(\tilde{\lambda}_3)\delta^4(\eta_3)$

$$\mathrm{R}_{23}\,\mathrm{R}_{12}\,\Omega_{2,3} = \text{MHV}_3 =$$  $\Longrightarrow\ \mathrm{T}(u)$  $= u^2(u-1)$ 

- General tree superamplitude of the type $\mathrm{N}^{\mathrm{k}-2}\mathrm{MHV}_n$

$$\mathrm{T}(u)\,M_{k,n} = u^k(u-1)^{n-k}\,M_{k,n}$$

- **Cyclicity** of superamplitudes = cyclicity of the spin chain



$$\mathrm{L}_1(u)\cdots\mathrm{L}_n(u)\,M = C\,M \quad \Leftrightarrow \quad \mathrm{L}_{\sigma_1}(u)\cdots\mathrm{L}_{\sigma_n}(u)\,M = C\,M$$

where $\sigma_1, \cdots, \sigma_n$ is a cyclic permutation of $1, 2, \cdots, n$

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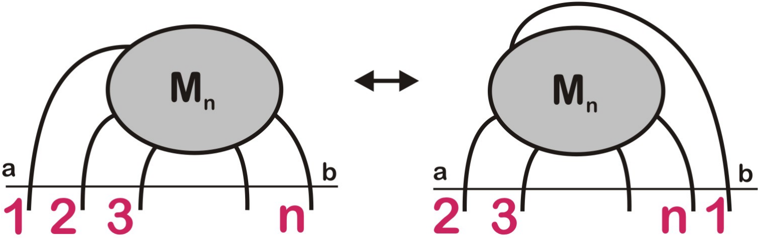

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- **Reflection** of the particle ordering = reflection of the spin chain sites

$$\mathrm{L}_1(u)\cdots\mathrm{L}_n(u)\,M = C\,M \quad \Leftrightarrow \quad \mathrm{L}_n(u')\cdots\mathrm{L}_1(u')\,M = C'\,M$$

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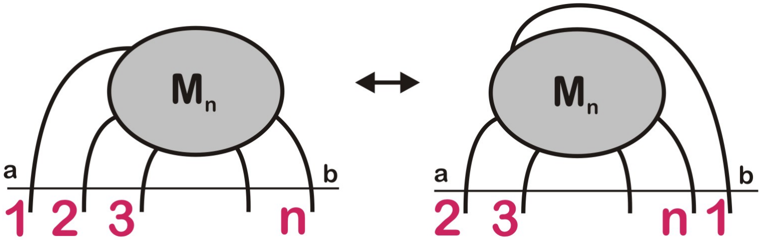

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Inverse Soft Limit. Recursive construction of BCFW terms (Yangian invariants)

- $R_{n1} R_{n\,n-1}\, M_{k,n-1}\, \delta^2(\lambda_n) = M_{k,n} = \mathcal{S}^+(n-1,n,1)\, M_{k,n-1}(1',2,\cdots,(n-1)')$

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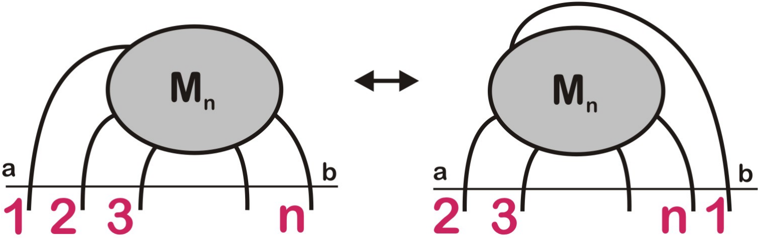

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- $\mathrm{R}_{1n}\mathrm{R}_{n-1\,n}\,M_{k,n-1}\,\delta^2(\tilde\lambda_n)\delta^4(\eta_n) = M_{k+1,n} = \mathcal{S}^-(n-1,n,1)\,M_{k,n-1}(1',2,\cdots,(n-1)')$

Momentum and supercharge conservation

$$\mathrm{L}(u) = \begin{pmatrix} u \cdot 1 + \cdots & -\lambda \otimes \tilde{\lambda} & -\lambda \otimes \eta \\ \multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \end{pmatrix} \xrightarrow{u^{n-1} J^{(1)}} \sum_{i=1}^{n} \lambda_{\alpha} \tilde{\lambda}_{\dot{\alpha}} \, M = 0 \, , \; \sum_{i=1}^{n} \lambda_{\alpha} \eta_{A} \, M = 0$$

$$M \sim \delta^{4|0}\left(\sum_{i=1}^{n} p_i\right) \delta^{0|8}\left(\sum_{i=1}^{n} q_i\right)$$

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4-point amplitude.

$$M_{2,4} = \frac{\delta^4(p)\, \delta^8(q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

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4-point amplitude. Unitary cut

$$M_{2,4} = \frac{[43]\delta^4(p)\delta^8(q)}{\langle 12\rangle\langle 23\rangle\langle 41\rangle} \cdot \frac{1}{(p_3+p_4)^2} \to \frac{[43]\delta^4(p)\delta^8(q)}{\langle 12\rangle\langle 23\rangle\langle 41\rangle} \cdot \delta\left((p_3+p_4)^2\right) = \not{M}_{2,4}$$

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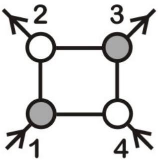
$$\not{M}_{2,4} = \mathrm{R}_{12}\underbrace{\mathrm{R}_{34}\mathrm{R}_{23}\,\Omega_{2,4}}_{\delta^2(\lambda_1)\,\mathrm{MHV}_3}$$

$$M_{2,4} = \mathrm{R}_{14}\underbrace{\mathrm{R}_{12}\mathrm{R}_{34}\mathrm{R}_{23}\,\Omega_{2,4}}_{\not{M}_{2,4}}$$



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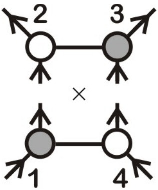


Momentum and supercharge conservation

$$\mathrm{L}(u) = \begin{pmatrix} u\cdot 1 + \cdots & -\lambda\otimes\tilde\lambda & -\lambda\otimes\eta \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \end{pmatrix} \xrightarrow{u^{n-1} J^{(1)}} \sum_{i=1}^{n} \lambda_\alpha \tilde\lambda_{\dot\alpha}\, M = 0\,,\ \sum_{i=1}^{n} \lambda_\alpha \eta_A\, M = 0$$

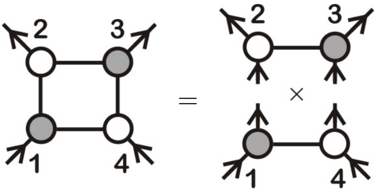
$$M \sim \delta^{4|0}\left(\sum_{i=1}^{n} p_i\right) \delta^{0|8}\left(\sum_{i=1}^{n} q_i\right)$$

4-point amplitude. Unitary cut

$$M_{2,4} = \frac{[43]\delta^4(p)\delta^8(q)}{\langle 12\rangle\langle 23\rangle\langle 41\rangle} \cdot \frac{1}{(p_3+p_4)^2} \to \frac{[43]\delta^4(p)\delta^8(q)}{\langle 12\rangle\langle 23\rangle\langle 41\rangle} \cdot \delta\left((p_3+p_4)^2\right) = \not{M}_{2,4}$$

$$\not{M}_{2,4} = \mathrm{R}_{12}\underbrace{\mathrm{R}_{34}\mathrm{R}_{23}\,\Omega_{2,4}}_{\delta^2(\lambda_1)\ \mathrm{MHV}_3}$$

$$M_{2,4} = \mathrm{R}_{14}\underbrace{\mathrm{R}_{12}\mathrm{R}_{34}\mathrm{R}_{23}\,\Omega_{2,4}}_{\not{M}_{2,4}}$$



Kernels of integral operators

$$\int \not{M}_{2,4} \leftrightarrow \mathrm{R}_{23}$$

$$\int M_{2,4} \leftrightarrow \mathrm{R}_{32}\mathrm{R}_{23}$$

$$\mathrm{T}(u_4, u_1, u_2, u_3)\, M = C\, M \ \leftrightarrow\ \mathrm{L}_2(u_2)\,\mathrm{L}_3(u_3)\,\widehat{M} = C\,\widehat{M}\,\mathrm{L}_2(u_1')\,\mathrm{L}_3(u_4')$$

Yang-Baxter type relation

Super-twistor variables

Super momentum twistors and the general tree superamplitude of the type $\mathrm{N}^{k-2}\mathrm{MHV}$

$$M_{k,n} = \mathrm{MHV}_n \, \mathcal{P}_{4k-8,n}$$

Let us choose another dynamical variables for the spin chain

$$\mathbf{x} \to \mathcal{Z} = (\underbrace{\lambda, \mu}_{\text{twistor } Z} \, , \chi) \quad ; \quad \mathbf{p} \to \partial_{\mathcal{Z}} = (\partial_Z, -\partial_\chi)$$

R-operator

$$\left[\mathrm{R}_{ij}(u)F\right](\mathcal{Z}_i|\mathcal{Z}_j) = \int \frac{\mathrm{d}\mathbf{z}}{\mathbf{z}^{1-u}} \, F(\, \mathcal{Z}_i - \mathbf{z}\mathcal{Z}_j | \, \mathcal{Z}_j \,)$$

Basic state

$$\Omega = \prod_i \delta^{4|4}(\mathcal{Z}_i)$$

For example, the simplest R-invariant

$$[1,2,3,4,5] = \mathrm{R}_{45}\mathrm{R}_{34}\mathrm{R}_{23}\mathrm{R}_{12}\, \delta^{4|4}(\mathcal{Z}_1)$$

Analogue of
Grassmannian construction

The construction is compatible with monodromy eigenvalue relation

$$\mathrm{T}(u)\,[1,2,3,4,5] = u^4(u-1)\,[1,2,3,4,5]$$

Scattering amplitudes in a gauge theory

Color decomposition. $SU(N_c)$

$$M(\{p_i,a_i,h_i\}) = \epsilon^{\mu_1}_{h_1}\cdots\epsilon^{\mu_n}_{h_n}\langle A_{\mu_1}(p_1)\cdots A_{\mu_n}(p_n)\rangle = \mathrm{Tr}\,(T^{a_1}\cdots T^{a_n})\, M(\{p_i,h_i\}) + \text{Bose}$$

$$p^{\alpha\dot\alpha} = (\sigma_\mu)^{\alpha\dot\alpha}p^\mu = \begin{pmatrix} p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & p^0-p^3 \end{pmatrix}\ ;$$ massless particle $$p^{\alpha\dot\alpha} = \lambda^\alpha\tilde\lambda^{\dot\alpha}\ ;\ \langle ij\rangle = \epsilon_{\alpha\beta}\lambda_i^\alpha\lambda_j^\beta$$

Parke-Taylor amplitude

$$\text{MHV}_n = \frac{\langle ij\rangle^4\delta^4(p)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle} \qquad 1^+2^+\cdots i^-\cdots j^-\cdots n^+$$

$\mathcal{N}=4$ SYM. PCT self-conjugate multiplet. η^A $(A=1,\cdots,4)$ Grassmann variable

$$\Phi(\eta) = G_+ + \text{fermions} + \text{scalars} + (\eta)^4 G_-$$

$$\text{MHV}_n = \frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle} \qquad \text{supercharge } q^{\alpha A} = \lambda^\alpha\eta^A$$

General superamplitude

$$M_n = \text{MHV}_n\left[\mathcal{P}_{0,n} + \mathcal{P}_{4,n} + \cdots + \mathcal{P}_{4n-16,n}\right]$$

Conclusions

- Yangian symmetry of super Yang-Mills amplitudes is formulated in terms of an eigenvalue problem involving the monodromy matrix
- The Quantum Inverse Scattering Method on which this approach is based provides convenient tools for calculation and investigation of amplitudes
- Jordan-Schwinger type representations $\longrightarrow$ L-operator $\longrightarrow$ R-operator
- Homogeneous monodromy $\leftrightarrow$ scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory
- Less supersymmetric theory $\longrightarrow$ Inhomogeneous monodromy
- Representations of other types ?