Dimensional transition of weakly-bound three-boson system

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Connecting few- and many-body physics - universal regime

For $a \gg r_0$ Physical observables do not depend on the details of the short-range interaction

Two-body sector $E_2 \approx \frac{\hbar^2}{ma^2}$

Three-body sector (3D) $a \to \pm\infty$ (two-body bound state with $E_2=0$)

Infinite three-body bound states

Efimov states discovered
by Vitaly Efimov in 1970

Energies differ by a factor 515.03
Sizes differ by a factor 22.7

Connecting few- and many-body physics - contact parameters

The two-body contact parameter

$$C \equiv \lim_{k\to\infty} k^4 n(k)$$

- Bosons 1D
Olshanii/Dunjko (2003)
- Two-component Fermi gases
Tan(2008)
- Experimentally verified
Stewart (2010), Kuhnle (2010)

short-range two-body correlations

↕

many-body thermodynamics

Three-body contact parameter

$$n(k) \to \frac{C}{k^4} + f(k)D$$

Functional form depends on dimensionality

Experimentally verified Wild et al. (2012)

Three-body AAB systems in 3D

Spectator functions (zero-range potential)

$$\chi_{AA}(y) = 2\tau_{AA}(y;E_3)\int_0^\infty dx\frac{x}{y}G_1(y,x;E_3)\chi_{AB}(x)$$

$$\chi_{AB}(y) = \tau_{AB}(y;E_3)\int_0^\infty dx\frac{x}{y}\left[G_1(x,y;E_3)\chi_{AA}(x)+\mathcal{A}G_2(y,x;E_3)\chi_{AB}(x)\right]$$

$$\tau_{AA}(y;E_3) \equiv \frac{1}{\pi}\left[\sqrt{E_3+\frac{\mathcal{A}+2}{4\mathcal{A}}y^2}\mp\sqrt{E_{AA}}\right]^{-1}$$

$$\tau_{AB}(y;E_3) \equiv \frac{1}{\pi}\left(\frac{\mathcal{A}+1}{2\mathcal{A}}\right)^{3/2}\left[\sqrt{E_3+\frac{\mathcal{A}+2}{2(\mathcal{A}+1)}y^2}\mp\sqrt{E_{AB}}\right]^{-1}$$

$$G_1(y,x;E_3) \equiv \ln\frac{2\mathcal{A}(E_3+x^2+xy)+y^2(\mathcal{A}+1)}{2\mathcal{A}(E_3+x^2-xy)+y^2(\mathcal{A}+1)} - \ln\frac{2\mathcal{A}(\mu^2+x^2+xy)+y^2(\mathcal{A}+1)}{2\mathcal{A}(\mu^2+x^2-xy)+y^2(\mathcal{A}+1)}$$

$$G_2(y,x;E_3) \equiv \ln\frac{2(\mathcal{A}E_3+xy)+(y^2+x^2)(\mathcal{A}+1)}{2(\mathcal{A}E_3-xy)+(y^2+x^2)(\mathcal{A}+1)} - \ln\frac{2(\mathcal{A}\mu^2+xy)+(y^2+x^2)(\mathcal{A}+1)}{2(\mathcal{A}\mu^2-xy)+(y^2+x^2)(\mathcal{A}+1)}$$

Inserting the ansätze $\chi_{AA}(y) = c_{AA}\, y^{-2+\imath s}$ $\chi_{AB}(y) = c_{AB}\, y^{-2+\imath s}$

and taking the limit $\mu\to 0 \quad E_3 = E_{AA} = E_{AB}\to 0$

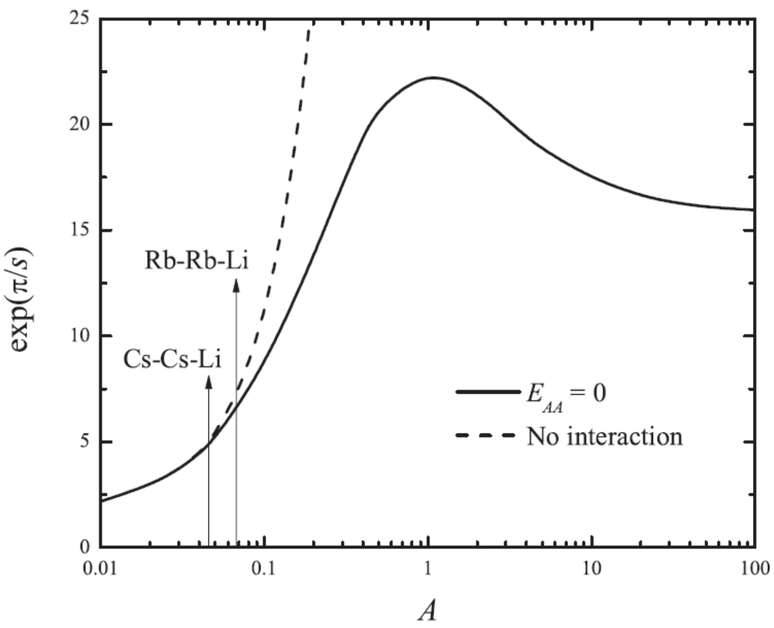


FIG. 1. Scaling parameter s as a function of $\mathcal{A} = m_B/m_A$ for $E_{AA} = 0$ and $E_{AB} = 0$ (resonant interactions) (solid line) and for the situation where $E_{AB} = 0$ but with no interaction between AA (dashed line). The arrows show the corresponding mass ratios for ^{133}Cs-^{133}Cs-^{6}Li and ^{87}Rb-^{87}Rb-^{6}Li.

Asymptotic forms

$$\chi_{AA}(q) = c_{AA} \frac{\sin[s \ln(q/q^*)]}{q^2}$$

$$\chi_{AB}(q) = c_{AB} \frac{\sin[s \ln(q/q^*)]}{q^2}$$

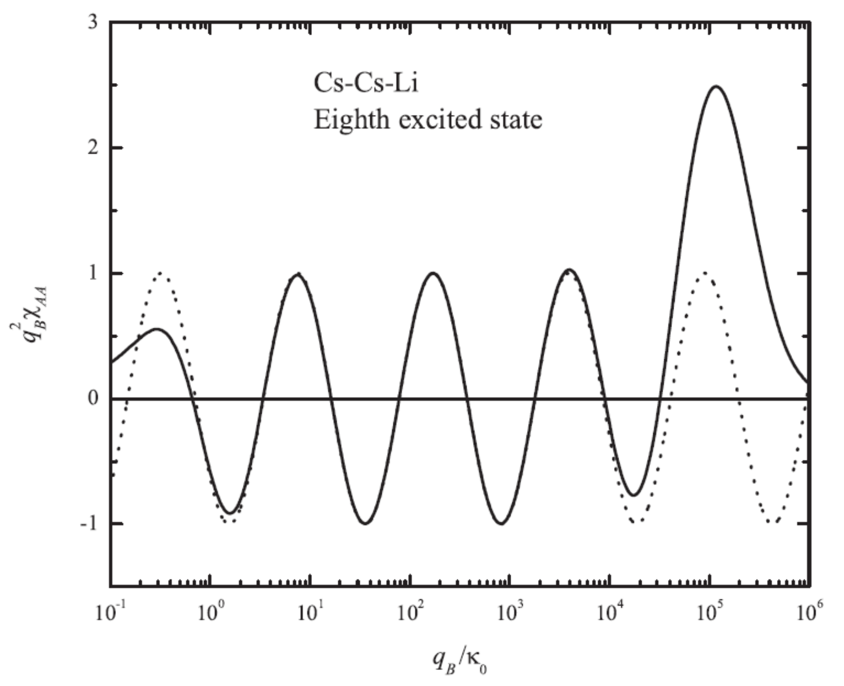
Asymptotic region?

Validity of our approach ($\hbar = m_A = \mu = 1$)

$$\sqrt{E_3} \ll q \ll \mu$$

$$\kappa_0 \equiv \sqrt{E_3}$$

$$1 \ll \frac{q}{\kappa_0} \ll \frac{1}{\kappa_0}$$



3D Momentum distribution

$$\langle \vec{q}_B \vec{p}_B | \Psi \rangle = \frac{\chi_{AA}(q_i) + \chi_{AB}(q_j) + \chi_{AB}(q_k)}{E_3 + H_0} = \frac{\chi_{AA}(q_B) + \chi_{AB}(|\vec{p}_B - \frac{\vec{q}_B}{2}|) + \chi_{AB}(|\vec{p}_B + \frac{\vec{q}_B}{2}|)}{E_3 + H_0}$$

$$n(q_B) = \int d^3 p_B |\langle \vec{q}_B \vec{p}_B | \Psi \rangle|^2$$

$$n(q_B) = \sum_{i=1}^{4} n_i(q_B)$$

$$n_1(q_B) = |\chi_{AA}(q_B)|^2 \int d^3 p_B \frac{1}{(E_3 + p_B^2 + q_B^2 \frac{\mathcal{A}+2}{4\mathcal{A}})^2} = \pi^2 \frac{|\chi_{AA}(q_B)|^2}{\sqrt{E_3 + q_B^2 \frac{\mathcal{A}+2}{4\mathcal{A}}}},$$

$$n_2(q_B) = 2 \int d^3 p_B \frac{\left|\chi_{AB}(|\vec{p}_B - \frac{\vec{q}_B}{2}|)\right|^2}{(E_3 + p_B^2 + q_B^2 \frac{\mathcal{A}+2}{4\mathcal{A}})^2} = 2 \int d^3 q_A \frac{|\chi_{AB}(q_A)|^2}{(E_3 + q_A^2 + \vec{q}_A \cdot \vec{q}_B + q_B^2 \frac{\mathcal{A}+1}{2\mathcal{A}})^2}$$

$$n_3(q_B) = 2\, \chi^*_{AA}(q_B) \int d^3 p_B \frac{\chi_{AB}(|\vec{p}_B - \frac{\vec{q}_B}{2}|)}{(E_3 + p_B^2 + q_B^2 \frac{\mathcal{A}+2}{4\mathcal{A}})^2} + c.c.$$

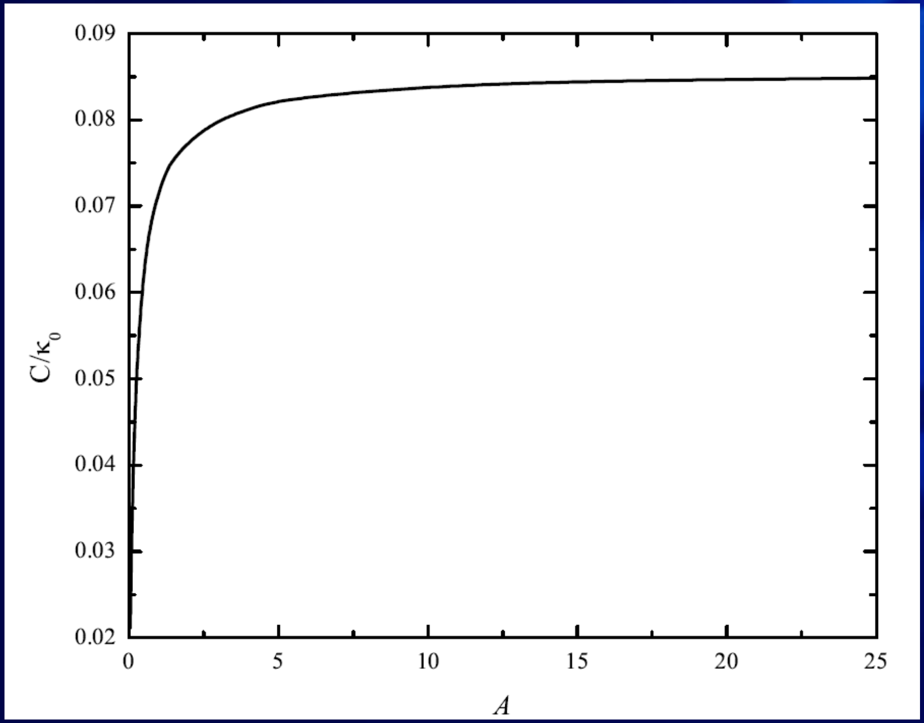
$$n_4(q_B) = \int d^3 p_B \frac{\chi^*_{AB}(|\vec{p}_B - \frac{\vec{q}_B}{2}|)\chi_{AB}(|\vec{p}_B + \frac{\vec{q}_B}{2}|)}{(E_3 + p_B^2 + q_B^2 \frac{\mathcal{A}+2}{4\mathcal{A}})^2} + c.c.$$

The asymptotic limit in n_2 $E_3 \to 0$ gives the leading-order term $\frac{C}{q_B^4}$

$$n_2(q_B) = 2\int d^3q_A \frac{|\chi_{AB}(q_A)|^2}{\left(q_A^2 + \vec{q}_A\cdot\vec{q}_B + q_B^2\,\frac{\mathcal{A}+1}{2\mathcal{A}}\right)^2} = \frac{8\mathcal{A}^2}{q_B^4\,(\mathcal{A}+1)^2}\int d^3q_A\ |\chi_{AB}(q_A)|^2$$

$$+\int d^3q_A\,|\chi_{AB}(q_A)|^2\left[\frac{2}{\left(q_A^2 + \vec{q}_A\cdot\vec{q}_B + q_B^2\,\frac{\mathcal{A}+1}{2\mathcal{A}}\right)^2} - \frac{8\mathcal{A}^2}{(\mathcal{A}+1)^2}\frac{1}{q_B^4}\right]$$

and C is simply given by $C = \frac{8\mathcal{A}^2}{(\mathcal{A}+1)^2}\int d^3q_A\,|\chi_{AB}(q_A)|^2$



	C / κ_0
^{133}Cs-^{133}Cs-^{6}Li	0.0274
^{87}Rb-^{87}Rb-^{6}Li	0.0211
$\mathcal{A} = 1$	0.0715

x $3(2\pi)^3$ = 53.197

"Exact" value: 53.097

Werner and Castin PRA 83, 063614 (2011)

Analysis of subleading terms

Werner and Castin PRA 83, 063614 (2011)

The nonoscillatory term of order q_B^{-5} coming from n_1 to n_4 cancels for A =1. Can we extend this result for other mass ratios?

After averaging out the oscillating part:

$$\langle n_1(q_B)\rangle = \frac{\pi^2}{q_B^5}\ |c_{AA}|^2\ \sqrt{\frac{\mathcal{A}}{\mathcal{A}+2}}$$

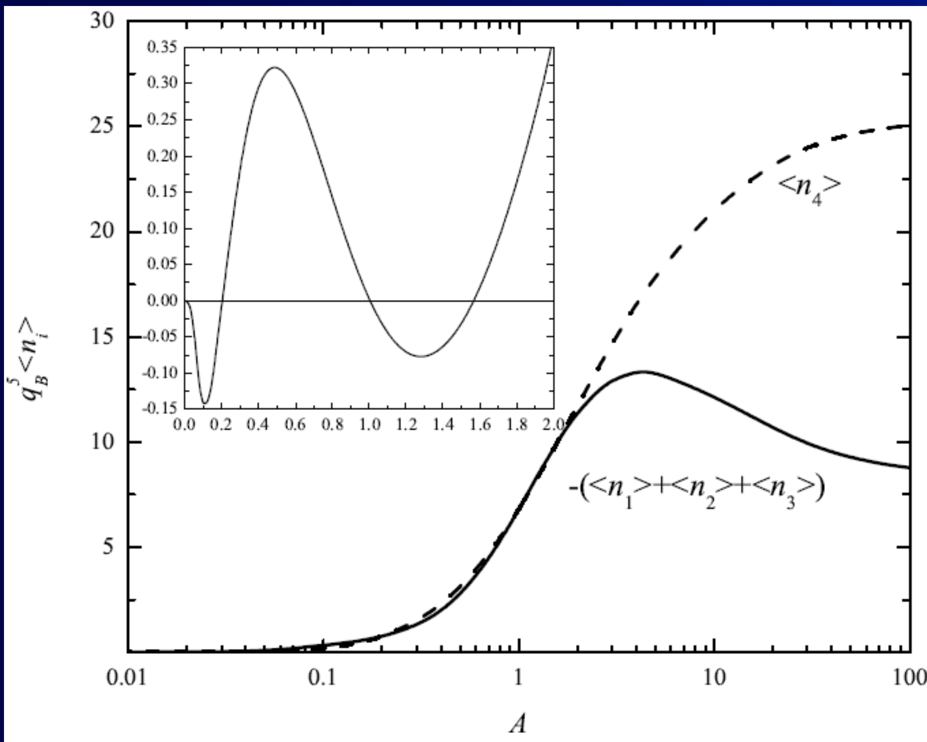
$$\langle n_2(q_B)\rangle = -\frac{8\pi^2\,|c_{AB}|^2}{q_B^5}\frac{\mathcal{A}^3(\mathcal{A}+3)}{(\mathcal{A}+1)^3\sqrt{\mathcal{A}(\mathcal{A}+2)}}$$

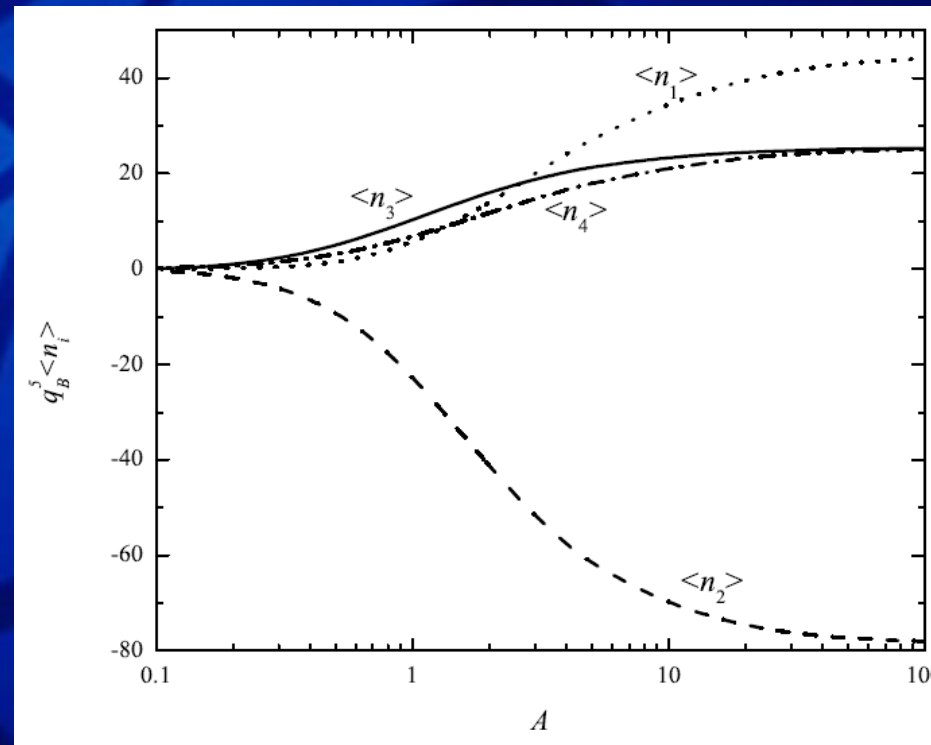
$$\langle n_3(q_B)\rangle = \frac{4\pi^2 c_{AA}\,c_{AB}}{q_B^5\cosh\left(\frac{s\pi}{2}\right)}\left\{\sqrt{\frac{\mathcal{A}}{\mathcal{A}+2}}\cos\left(s\ln\sqrt{\frac{\mathcal{A}+1}{2\mathcal{A}}}\right)\cosh\left[s\left(\frac{\pi}{2}-\theta_3\right)\right]+\sin\left(s\ln\sqrt{\frac{\mathcal{A}+1}{2\mathcal{A}}}\right)\sinh\left[s\left(\frac{\pi}{2}-\theta_3\right)\right]\right\}$$

$$\tan\theta_3 = \sqrt{\frac{\mathcal{A}+2}{\mathcal{A}}}\quad \text{for}\quad 0\ \leq\ \theta_3\ \leq\ \pi/2$$

$$\langle n_4(q_B)\rangle = \frac{8\pi^2|c_{AB}|^2\mathcal{A}^2}{s\, q_B^5\, \cosh\left(\frac{s\pi}{2}\right)}\left\{\sinh\left[s\left(\frac{\pi}{2}-\theta_4\right)\right] - \frac{s\,\mathcal{A}}{\sqrt{\mathcal{A}(\mathcal{A}+2)}(\mathcal{A}+1)}\cosh\left[s\left(\frac{\pi}{2}-\theta_4\right)\right]\right\}$$

$$\tan\theta_4 = \sqrt{\mathcal{A}(\mathcal{A}+2)} \text{ for } 0 \le \theta_4 \le \pi/2$$





The <u>nonoscillatory</u> term of order q_B^{-5} coming from n_1 to n_4 cancels for A = 0.2, 1 and 1.57

What happens if we change to 2D?

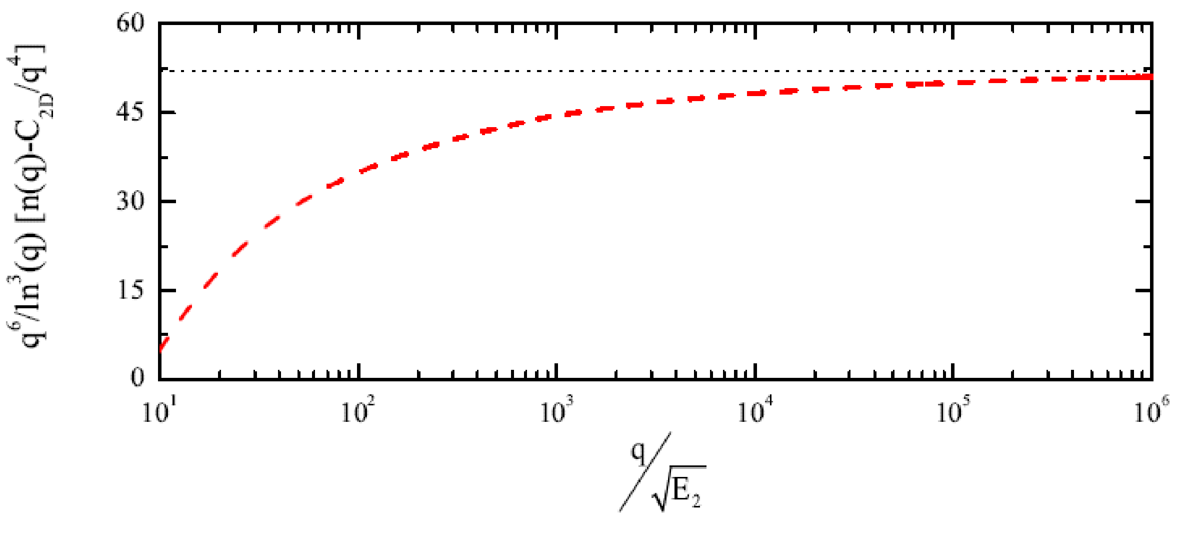
Efimov effect disappears

$$E_3^{(0)} = 16.52 E_2$$

$$E_3^{(1)} = 1.270 E_2$$

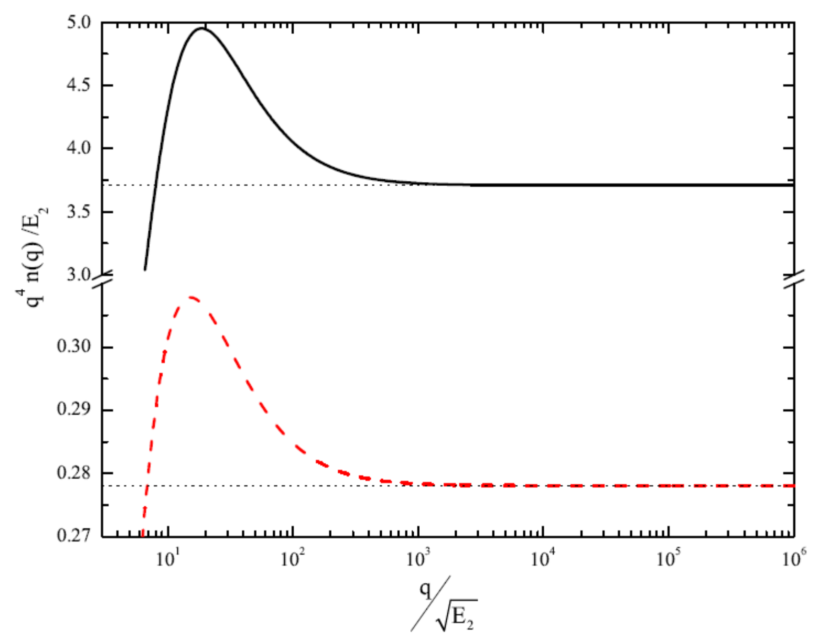
$$n_{3\mathrm{D}}(q) \to \frac{C}{q^4} + \frac{\sin^2[s\ln(q/q_3^*)]}{q^5} D_3 + \ldots$$

$$n_{2\mathrm{D}}(q) \to \frac{C^{(n)}}{q^4} + \frac{\ln^3(q/q_2^*)}{q^6} D_2^{(n)} + \ldots$$



$D_2^{(n)}$ depends weakly on (n)

Leading-order term: two-body contact parameter



$$n_3^0(q) \to \frac{3.71E_2}{q^4}$$

$$n_3^1(q) \to \frac{0.28E_2}{q^4}$$

Dividing $C^{(n)}$ by the "natural scale" E_3

$$\frac{C^{(1)}}{E_3^{(1)}} = \frac{0.28E_2}{1.270E_2} = 0.219 \qquad \frac{C^{(0)}}{E_3^{(0)}} = \frac{3.71E_2}{16.52E_2} = 0.224$$

$$\frac{C}{E_3} = 0.222 \pm 0.003$$

Continuous transition from 3D to 2D

$$\vec{p}$$

$$\vec{p}_\perp = (p_x, p_y)$$

$$p_z = \frac{2\pi n}{L} = \frac{n}{R}$$

$n = 0, \pm 1, \pm 2 \ldots$ and $L = 2\pi R$

$$f(\vec{q}_\perp, n) = -2\,\tau_R \left(E_3 - \frac{3}{4}(q_\perp^2 + \frac{n^2}{R^2}) \right) \sum_m \int \frac{d^2 p_\perp}{R} \left[g_{0R}(E) - g_{0R}(-\mu^2) \right] f(\vec{p}_\perp, m)$$

$$g_{0R}^{-1}(E) = E - q_\perp^2 - p_\perp^2 - \vec{q}_\perp \cdot \vec{p}_\perp - \frac{n^2}{R^2} - \frac{m^2}{R^2} + \frac{n\,m}{R^2}$$

$$\tau_R(E) = -R \left[\pi \ln \left(\frac{\sinh \pi \sqrt{-E} R}{\sinh \pi \sqrt{-E_2} R} \right) \right]^{-1}$$

Continuous transition from 3D to 2D

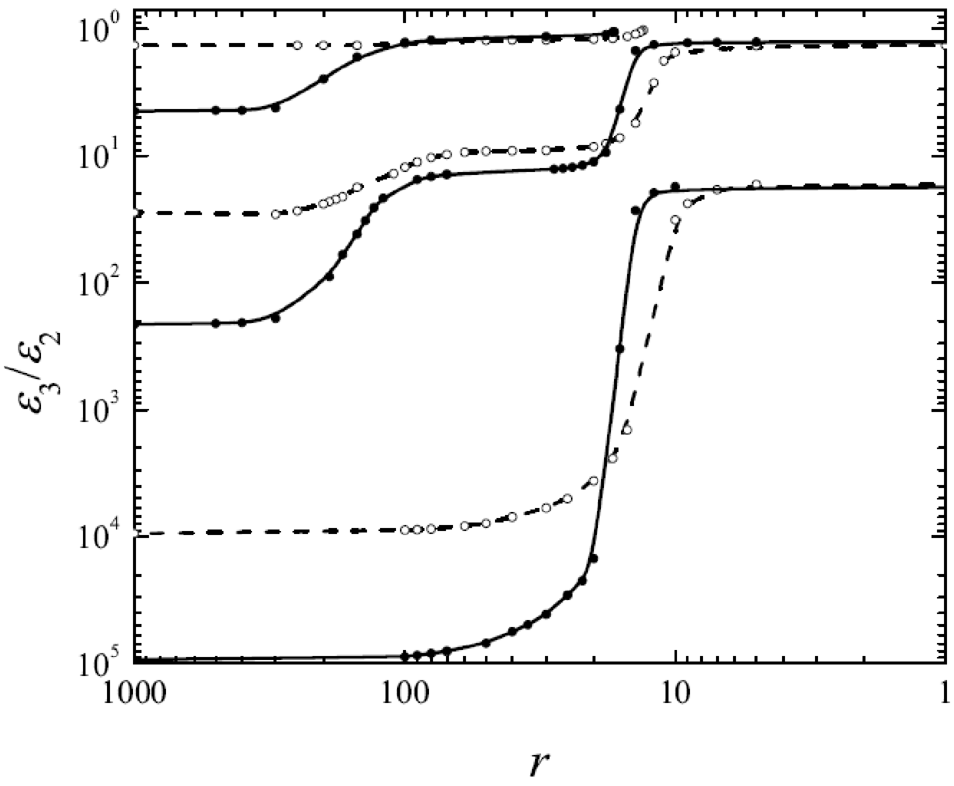
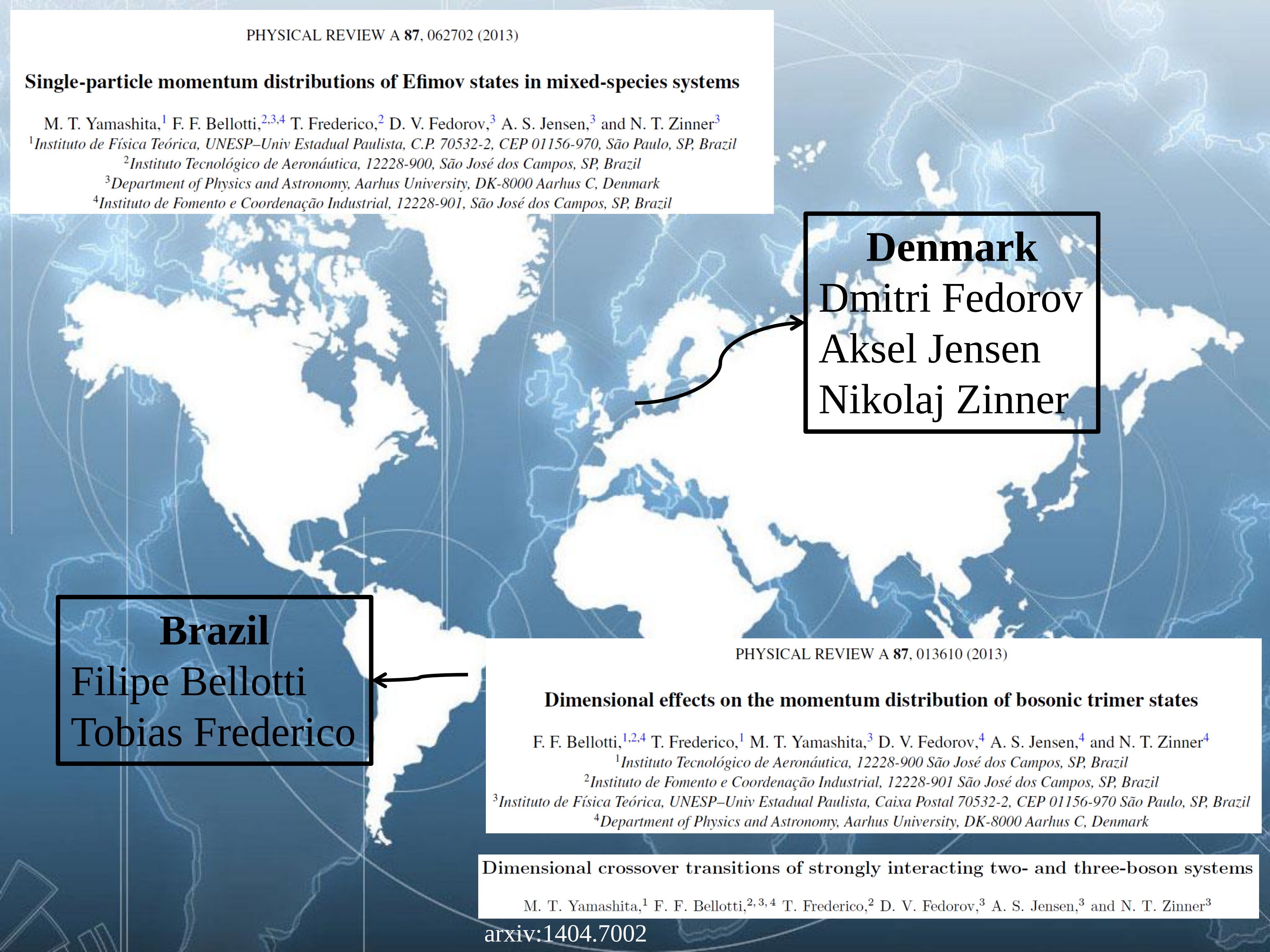


FIG. 2: ϵ_3/ϵ_2 as a function of r, for $\epsilon_2^{3D} = 10^{-7}$ (full circles) and 10^{-6} (empty circles). The solid and dashed lines are guides to the eye. As we approach the 2D limit ($r \to 0$), higher excited states disappear and only the ground and first excited states remain. Note that the values of r and ϵ_3/ϵ_2 increase from right to left and top to bottom respectively.

Conclusions

- Asymptotic momentum distribution

  - 3D: q_B^{-5} nonoscillatory term cancels for *A* = 0.2, 1 and 1.57
  - 2D/3D: Analytical forms for the asymptotic spectator functions and momentum distribution for general *A*
  - Functional form changes drastically when passing from 3D to 2D
  - Considering the three-body energy as scale the two-body contact parameter is universal in 2D
  - "Compactifying" the momenta we can continuously interpolate between 3-2D



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Single-particle momentum distributions of Efimov states in mixed-species systems

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Dimensional effects on the momentum distribution of bosonic trimer states

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Dimensional crossover transitions of strongly interacting two- and three-boson systems

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