Exploring Universality in Efimov Physics

Mario Gattobigio

Santos, Critical Stability 2014

Institut nonlinéaire de Nice
Université Nice Sophia Antipolis
CNRS

Outline

Efimov Physics
- Efimov Effect
- Discrete Scale Invariance

Finite-range Effect
- 3-Body Bound States
- Scattering Length
- Recombination
- Measured energies

N-body Universality
- N-Body States
- Universality

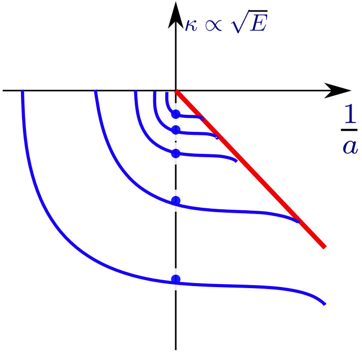
Work in progress...
- ... back to Nuclear Physics

Outline

- Efimov Physics
  - Efimov Effect
  - Discrete Scale Invariance
- Finite-range Effect
  - 3-Body Bound States
  - Scattering Length
  - Recombination
  - Measured energies
- N-body Universality
  - N-Body States
  - Universality
- Work in progress...
  - ... back to Nuclear Physics

Efimov Effect

$$@1/a = 0 \quad \begin{cases} E_3^n \to 0 & n \to \infty \\ E_3^{n+1}/E_3^n \to 1/515 = 1/(22.7)^2 \end{cases}$$

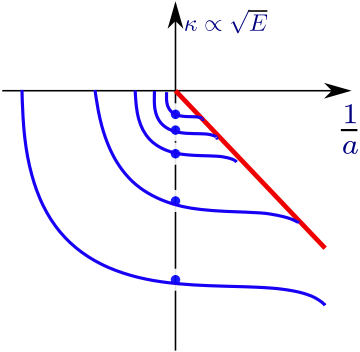


Efimov Effect

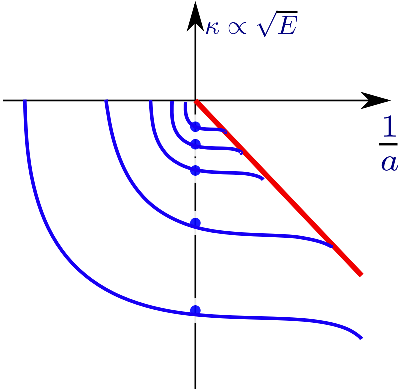
$$@1/a = 0 \quad \begin{cases} E_3^n \to 0 & n \to \infty \\ E_3^{n+1}/E_3^n \to 1/515 = 1/(22.7)^2 \end{cases}$$

Discrete Scale Invariance

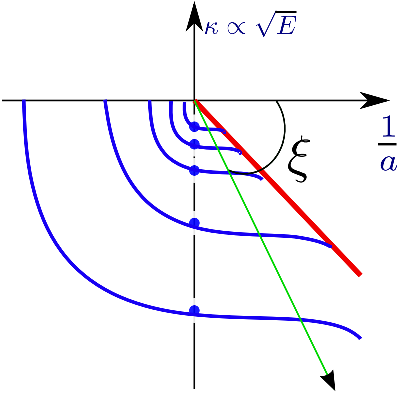
Sornette, Physics Reports 297, 239-270 (1998)



Discrete Scale Invariance



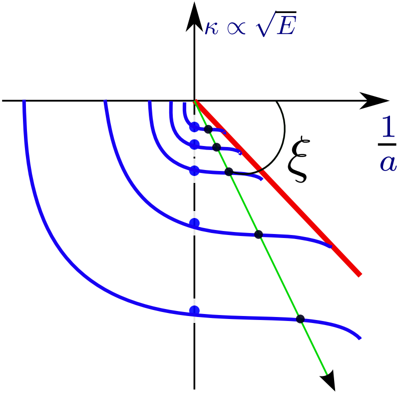
Discrete Scale Invariance



Polar coordinates

$$(H)^2 = (E_3 + E_2)/(\hbar^2/m)$$
$$\tan^2 \xi = E_3/E_2$$

Discrete Scale Invariance

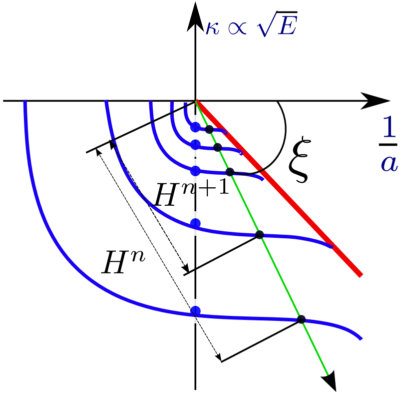


Polar coordinates

$$(H)^2 = (E_3 + E_2)/(\hbar^2/m)$$

$$\tan^2 \xi = E_3/E_2$$

Discrete Scale Invariance



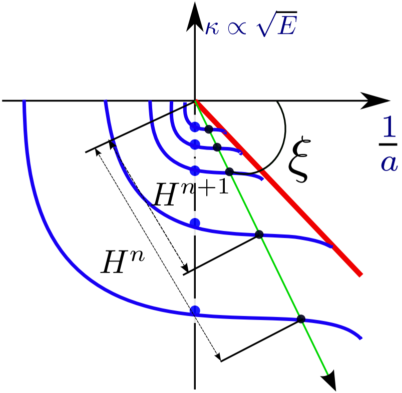
Polar coordinates

$$(H)^2 = (E_3 + E_2)/(\hbar^2/m)$$
$$\tan^2 \xi = E_3/E_2$$

For each ξ

$$H^{n+1}/H^n \to 1/22.7$$

Discrete Scale Invariance



Polar coordinates

$$(H)^2 = (E_3 + E_2)/(\hbar^2/m)$$

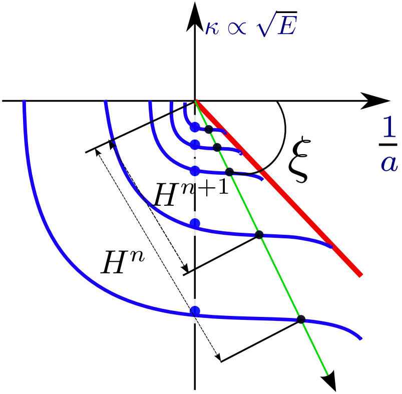
$$\tan^2 \xi = E_3/E_2$$

For each ξ

$$H^{n+1}/H^n \to 1/22.7$$

$$E_3^n + \frac{\hbar^2}{ma^2} = \frac{\hbar^2 \kappa_*^2}{m} e^{-2(n-n^*)\pi/s_0} e^{\Delta(\xi)/s_0}$$

Discrete Scale Invariance



Polar coordinates

$$(H)^2 = (E_3 + E_2)/(\hbar^2/m)$$

$$\tan^2 \xi = E_3/E_2$$

For each ξ

$$H^{n+1}/H^n \to 1/22.7$$

$$E_3^n + \frac{\hbar^2}{ma^2} = \frac{\hbar^2 \kappa_*^2}{m} e^{-2(n-n^*)\pi/s_0} e^{\Delta(\xi)/s_0}$$

$$\begin{cases} E_3^n/(\hbar^2/ma^2) = \tan^2 \xi \\ \kappa_* a = e^{(n-n^*)\pi/s_0} \dfrac{e^{-\Delta(\xi)/2s_0}}{\cos \xi} \end{cases}$$

Discrete Scale Invariance

- DSI $\Rightarrow$ Universal form of observables
  Log-periodic functions (cfr. Sornette)

Discrete Scale Invariance

- DSI $\Rightarrow$ Universal form of observables
  Log-periodic functions (cfr. Sornette)
- Zero-range interaction ($\ell = 0$)

Discrete Scale Invariance

- DSI $\Rightarrow$ Universal form of observables
  Log-periodic functions (cfr. Sornette)
- Zero-range interaction ($\ell = 0$)

Particle-Dimer Scattering Length

$$a_{AD}/a = d_1 + d_2 \tan[s_0 \ln(\kappa_* a) + d_3]$$

- d_1, d_2, d_3 Universal Constants

Discrete Scale Invariance

- DSI $\Rightarrow$ Universal form of observables
  Log-periodic functions (cfr. Sornette)
- Zero-range interaction ($\ell = 0$)

Particle-Dimer Scattering Length

$$a_{AD}/a = d_1 + d_2 \tan[s_0 \ln(\kappa_* a) + d_3]$$

- d_1, d_2, d_3 Universal Constants

Recombination Rate at the threshold

$$K_3 = \frac{128\pi^2(4\pi - 3\sqrt{3})}{\sinh^2(\pi s_0) + \cosh^2(\pi s_0)\cot^2[s_0 \ln(\kappa_* a) + \gamma]} \frac{\hbar a^4}{m},$$

- γ Universal Constant

Outline

Efimov Physics
- Efimov Effect
- Discrete Scale Invariance

Finite-range Effect
- 3-Body Bound States
- Scattering Length
- Recombination
- Measured energies

N-body Universality
- N-Body States
- Universality

Work in progress...
- ... back to Nuclear Physics

Finite-range Calculations

- N-body calculation using Schrödinger Equation

Finite-range Calculations

- N-body calculation using Schrödinger Equation
- Finite-range potential

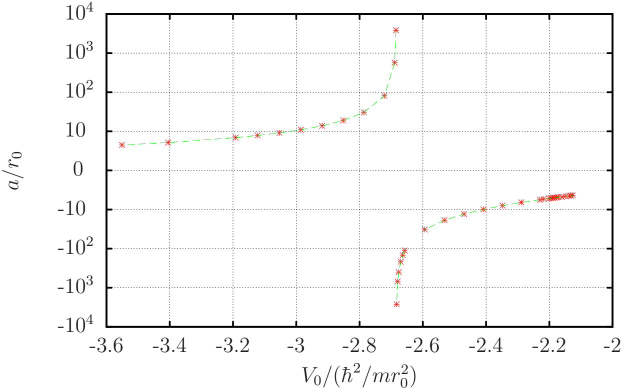
$$V(r) = V_0 \; e^{-r^2/r_0^2}$$

Finite-range Calculations

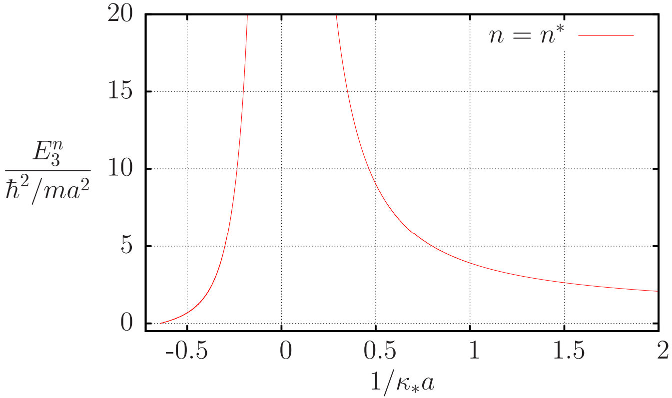
- N-body calculation using Schrödinger Equation
- Finite-range potential

$$V(r) = V_0 \ e^{-r^2/r_0^2}$$

- Tuning of the Scattering Length

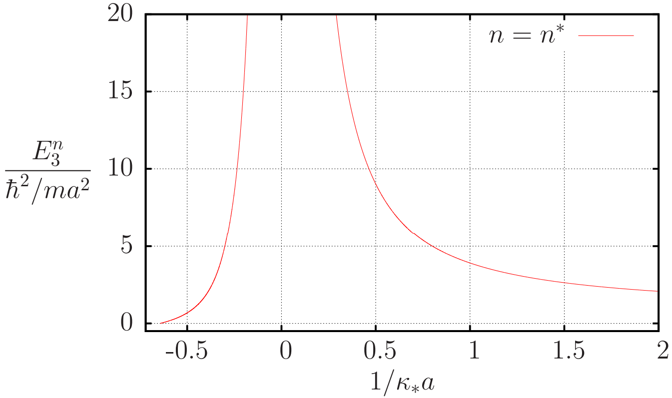


3-Body Bound States



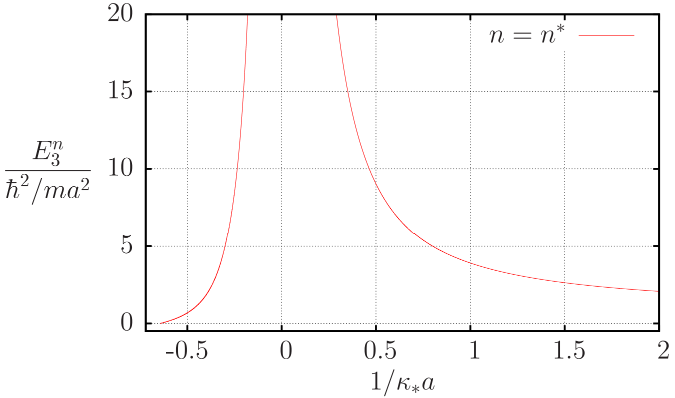
$$\begin{cases} E_3^n/(\hbar^2/ma^2) = \tan^2 \xi \\ \kappa_* a = e^{(n-n^*)\pi/s_0} \dfrac{e^{-\Delta(\xi)/2s_0}}{\cos \xi} \end{cases}$$

3-Body Bound States



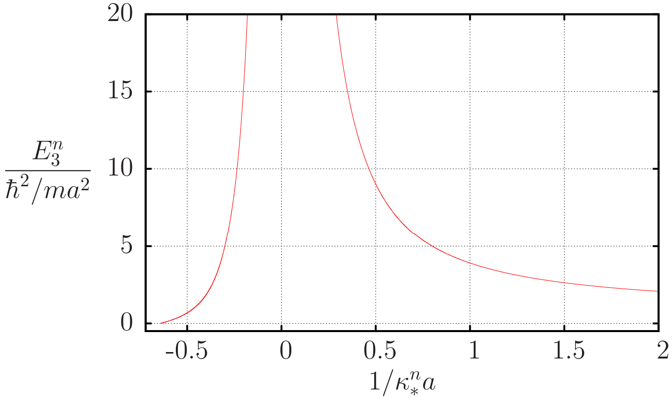
$$\begin{cases} E_3^n/(\hbar^2/ma^2) = \tan^2 \xi \\ \kappa_* e^{-(n-n^*)\pi/s_0}\, a = \dfrac{e^{-\Delta(\xi)/2s_0}}{\cos \xi} \end{cases}$$

3-Body Bound States



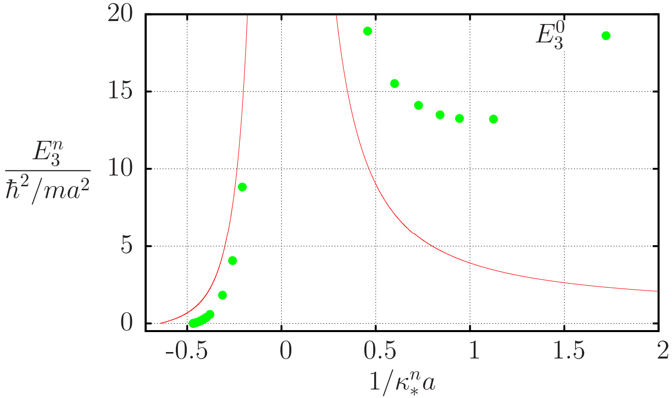
$$\begin{cases} E_3^n/(\hbar^2/ma^2) = \tan^2\xi \\ \kappa_*^n a = \dfrac{e^{-\Delta(\xi)/2s_0}}{\cos\xi} \end{cases}$$

3-Body Bound States



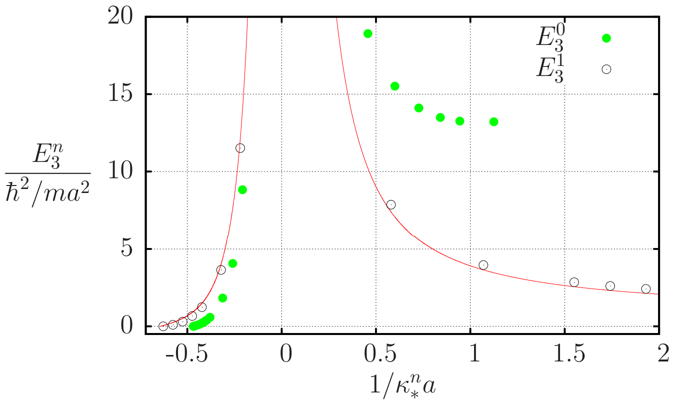
$$\begin{cases} E_3^n/(\hbar^2/ma^2) = \tan^2\xi \\ \kappa_*^n a = \dfrac{e^{-\Delta(\xi)/2s_0}}{\cos\xi} \end{cases}$$

3-Body Bound States



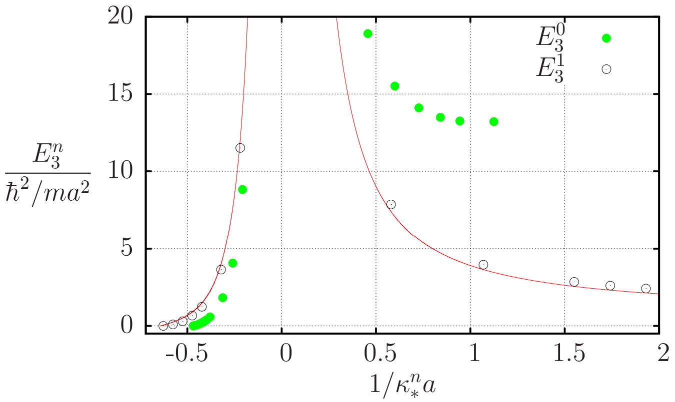
$$\begin{cases} E_3^n/(\hbar^2/ma^2) = \tan^2\xi \\ \kappa_*^n a = \dfrac{e^{-\Delta(\xi)/2s_0}}{\cos\xi} \end{cases}$$

3-Body Bound States



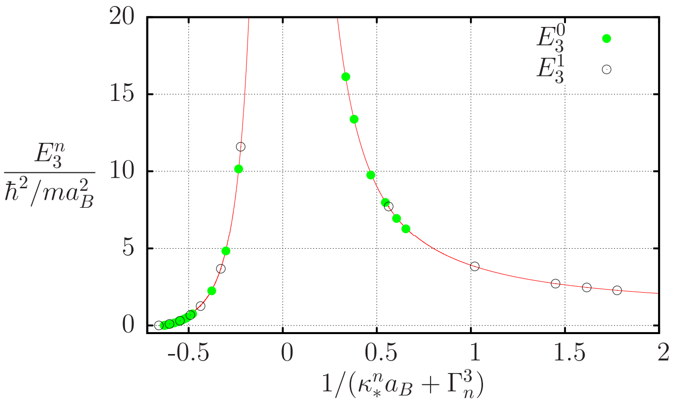
$$\begin{cases} E_3^n/(\hbar^2/ma^2) = \tan^2\xi \\ \kappa_*^n a = \dfrac{e^{-\Delta(\xi)/2s_0}}{\cos\xi} \end{cases}$$

3-Body Bound States



$$\begin{cases} E_3^n/(\hbar^2/m a_B{}^2) = \tan^2 \xi \\ \kappa_*^n a_B + \Gamma_n^3 = \dfrac{e^{-\Delta(\xi)/2s_0}}{\cos \xi} \end{cases} \qquad \frac{\hbar^2}{m a_B{}^2} = \begin{cases} \text{Bound State} & a > 0 \\ \text{Virtual State} & a < 0 \end{cases}$$

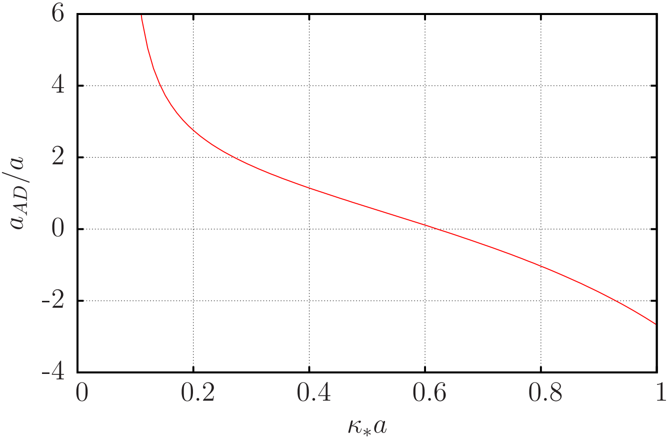
3-Body Bound States



$$\begin{cases} E_3^n/(\hbar^2/m{a_B}^2) = \tan^2 \xi \\ \kappa_*^n a_B + \Gamma_n^3 = \dfrac{e^{-\Delta(\xi)/2s_0}}{\cos \xi} \end{cases}$$

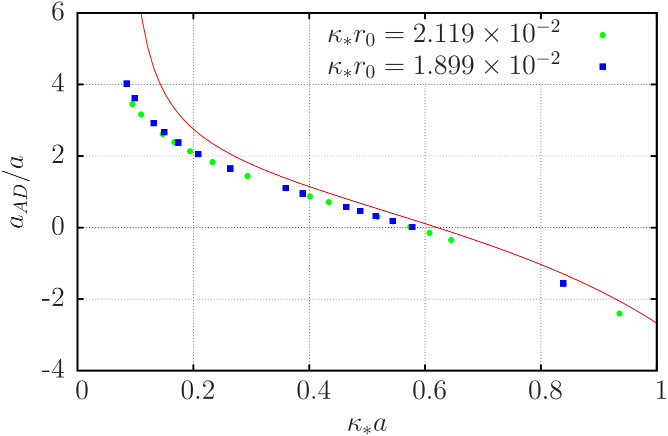
$$\frac{\hbar^2}{m{a_B}^2} = \begin{cases} \text{Bound State} & a > 0 \\ \text{Virtual State} & a < 0 \end{cases}$$

Particle-Dimer Scattering Length



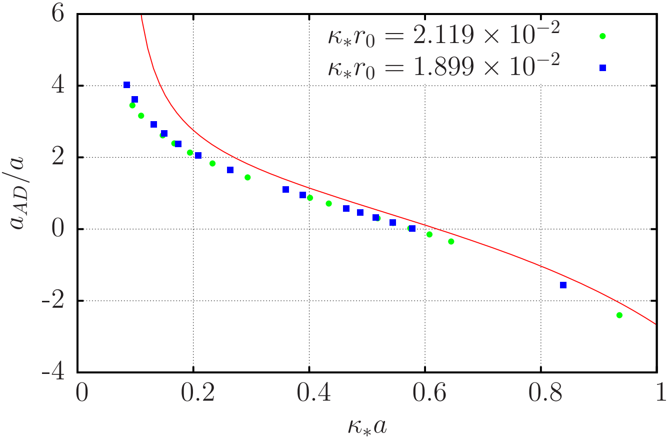
$$a_{AD}/a = d_1 + d_2 \tan[s_0 \ln(\kappa_* a) + d_3]$$

Particle-Dimer Scattering Length



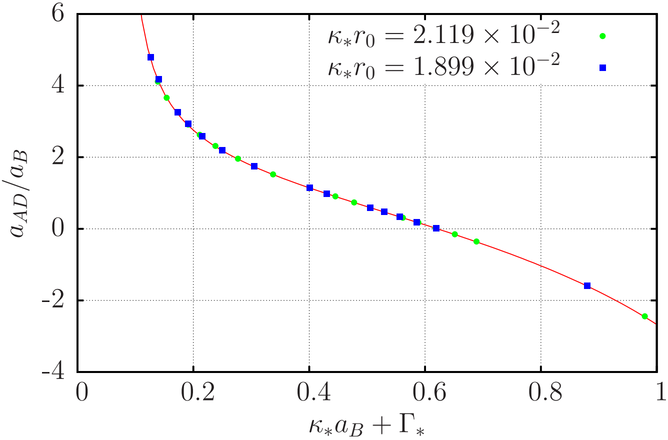
$$a_{AD}/a = d_1 + d_2 \tan[s_0 \ln(\kappa_* a) + d_3]$$

Particle-Dimer Scattering Length



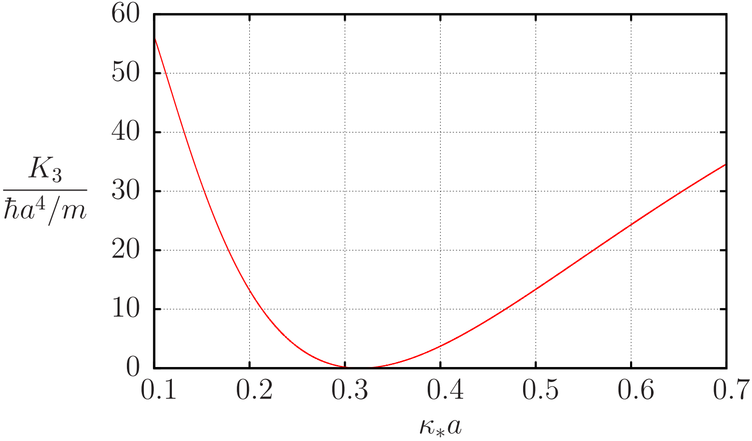
$$a_{AD}/a_B = d_1 + d_2 \tan[s_0 \ln(\kappa_* a_B + \Gamma_*) + d_3]$$

Particle-Dimer Scattering Length



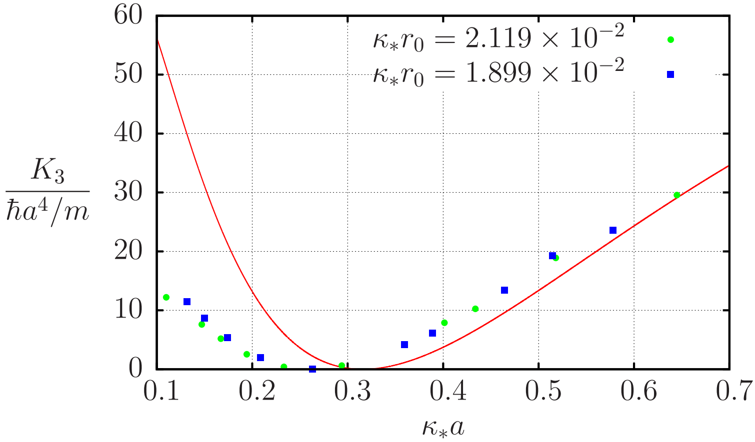
$$a_{AD}/a_B = d_1 + d_2 \tan[s_0 \ln(\kappa_* a_B + \Gamma_*) + d_3]$$

Recombination at the threshold



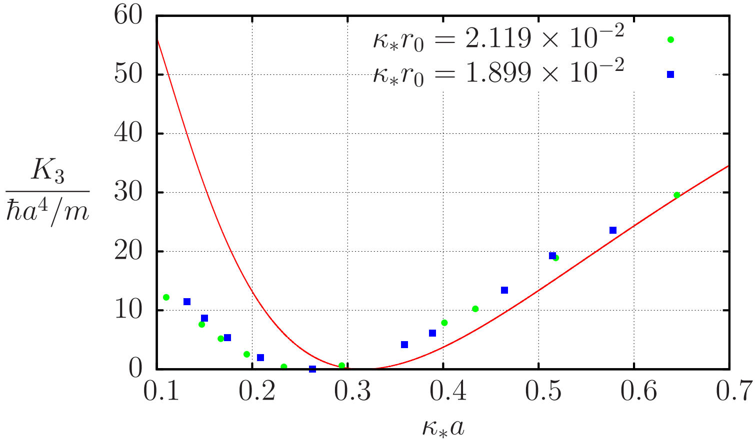
$$\frac{K_3}{\hbar a^4/m} = \frac{128\pi^2(4\pi - 3\sqrt{3})}{\sinh^2(\pi s_0) + \cosh^2(\pi s_0)\cot^2[s_0 \ln(\kappa_* a) + \gamma]}$$

Recombination at the threshold



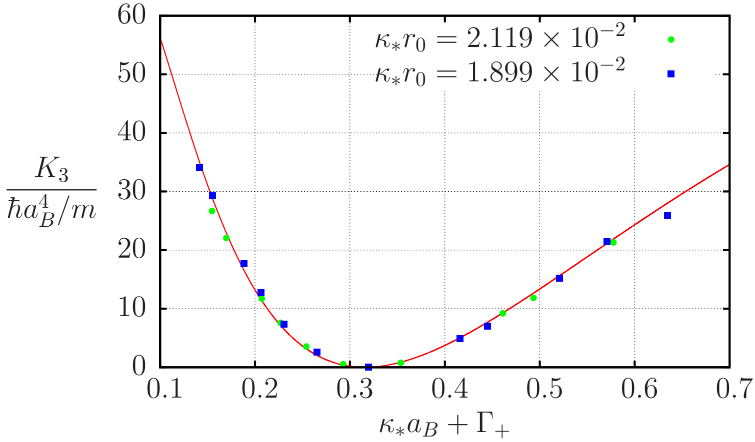
$$\frac{K_3}{\hbar a^4/m} = \frac{128\pi^2(4\pi - 3\sqrt{3})}{\sinh^2(\pi s_0) + \cosh^2(\pi s_0)\cot^2[s_0 \ln(\kappa_* a) + \gamma]}$$

Recombination at the threshold



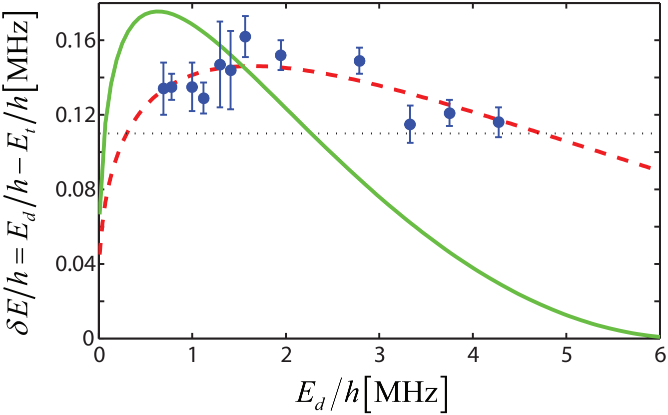
$$\frac{K_3}{\hbar {a_B}^4/m} = \frac{128\pi^2(4\pi - 3\sqrt{3})}{\sinh^2(\pi s_0) + \cosh^2(\pi s_0)\cot^2[s_0 \ln(\kappa_* a_B + \Gamma_+) + \gamma]}$$

Recombination at the threshold



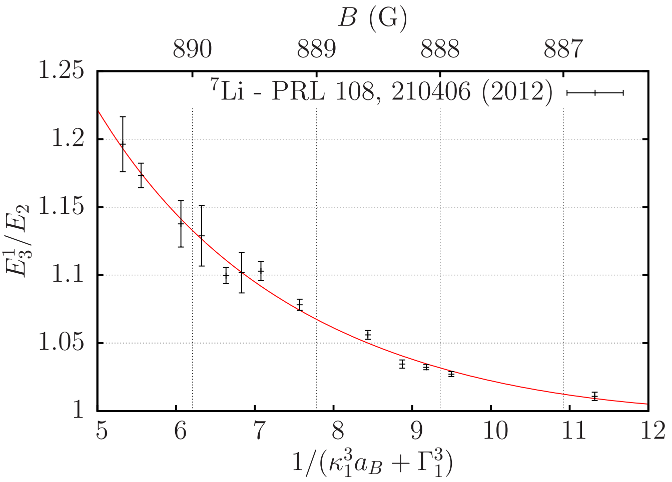
$$\frac{K_3}{\hbar a_B{}^4/m} = \frac{128\pi^2(4\pi - 3\sqrt{3})}{\sinh^2(\pi s_0) + \cosh^2(\pi s_0)\cot^2[s_0 \ln(\kappa_* a_B + \Gamma_+) + \gamma]}$$

Experimental data - Bound states



Olga Machtey, Zav Shotan, Noam Gross, and Lev Khaykovich
Phys. Rev. Lett. 108, 210406 (2012)

Experimental data - Bound states



Olga Machtey, Zav Shotan, Noam Gross, and Lev Khaykovich
Phys. Rev. Lett. 108, 210406 (2012)

$\kappa_1^3 = 1.6061 \times 10^{-4} a_0^{-1}$

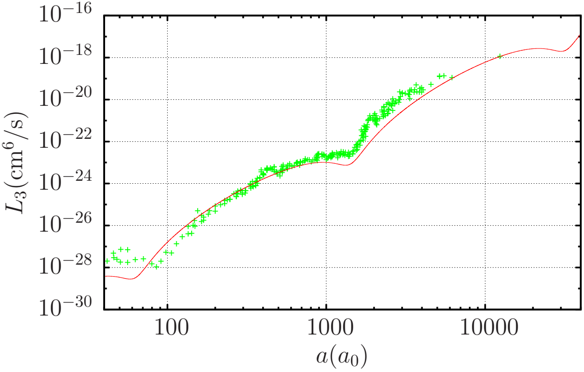
$\Gamma_1^3 = 4.95 \times 10^{-2}$.

Experimental data - Recombination

$$L_3(a) = 3\mathcal{N}\mathcal{C}(a)\hbar a^4/m,$$

$$\mathcal{C}(a) = 67.12\, e^{-2\eta_+}[\sin^2(s_0 \log(a/a^+)) + \sinh^2 \eta_+] + 16.84\,(1 - e^{-4\eta_+})$$

$a^+ = 1420\, a_0$, $\eta_+ = 7.5 \times 10^{-2}$, $\mathcal{N} = 5.5$



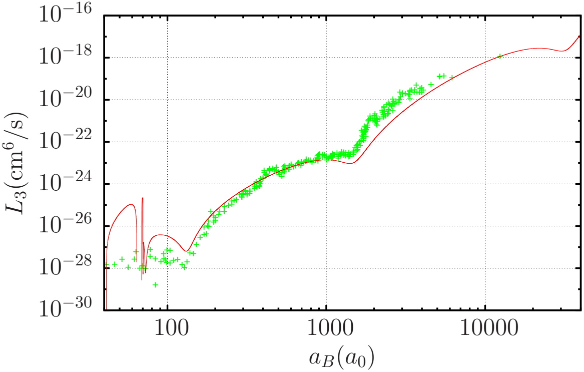
P. Dyke, S. E. Pollack, and R. G. Hulet
Phys. Rev. A 88, 023625 (2013)

Experimental data - Recombination

$$L_3(a) = 3\mathcal{N}\mathcal{C}(a)\hbar a^4/m,$$

$$\mathcal{C}(a_B) = 67.12\, e^{-2\eta_+}[\sin^2(s_0 \log(\kappa_* a_B + \Gamma) + 1.16) + \sinh^2 \eta_+] + 16.84\,(1 - e^{-4\eta_+})$$

$\kappa_* = 2.21 \times 10^{-4} a_0^{-1}$, and $\Gamma = -1.55 \times 10^{-2}$, $\eta_+ = 7.5 \times 10^{-2}$, $\mathcal{N} = 5.5$

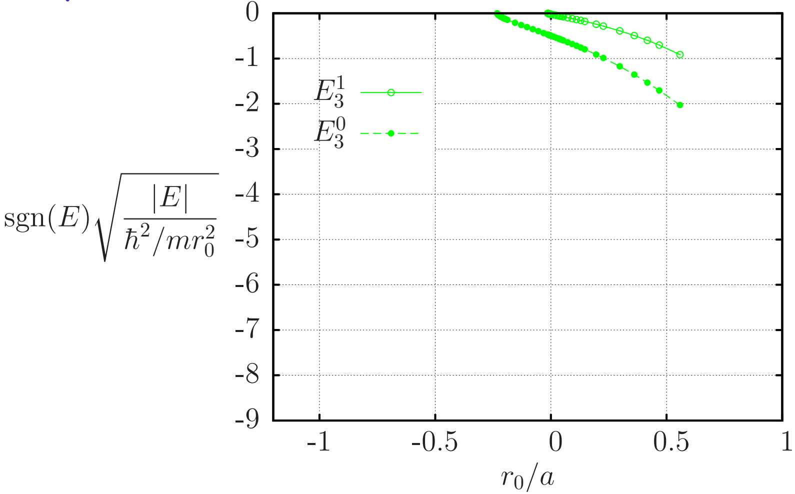


P. Dyke, S. E. Pollack, and R. G. Hulet
Phys. Rev. A 88, 023625 (2013)

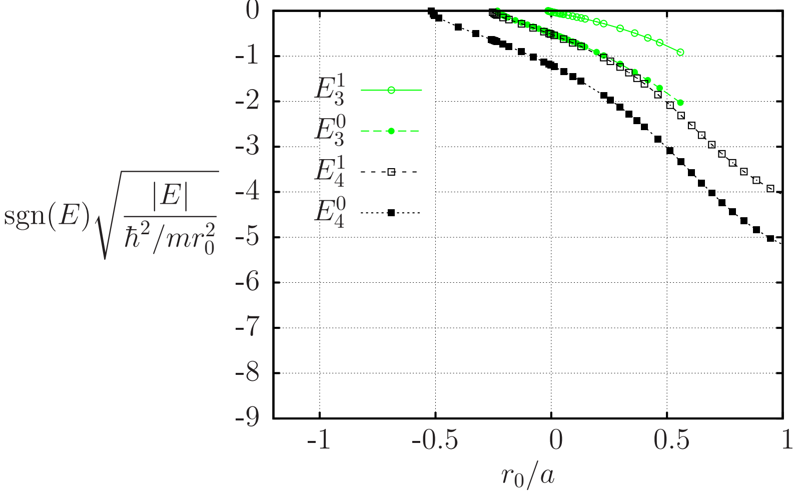
Outline

- Efimov Physics
  - Efimov Effect
  - Discrete Scale Invariance
- Finite-range Effect
  - 3-Body Bound States
  - Scattering Length
  - Recombination
  - Measured energies
- N-body Universality
  - N-Body States
  - Universality
- Work in progress...
  - ... back to Nuclear Physics

N-body Efimov Plot

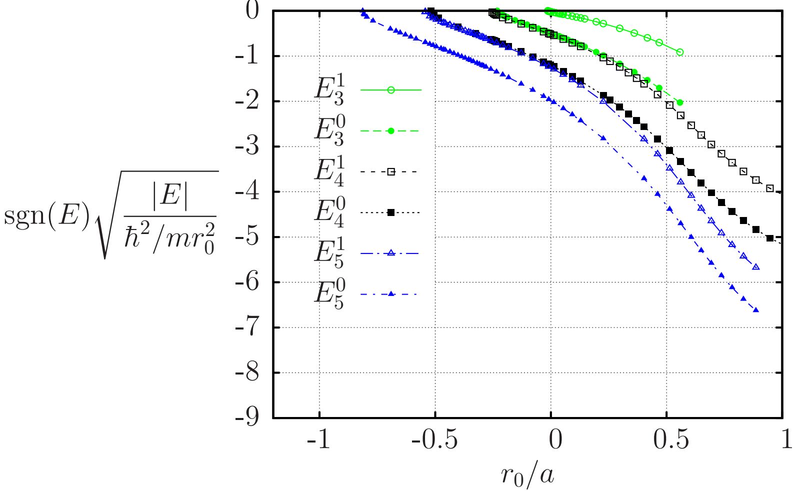


N-body Efimov Plot



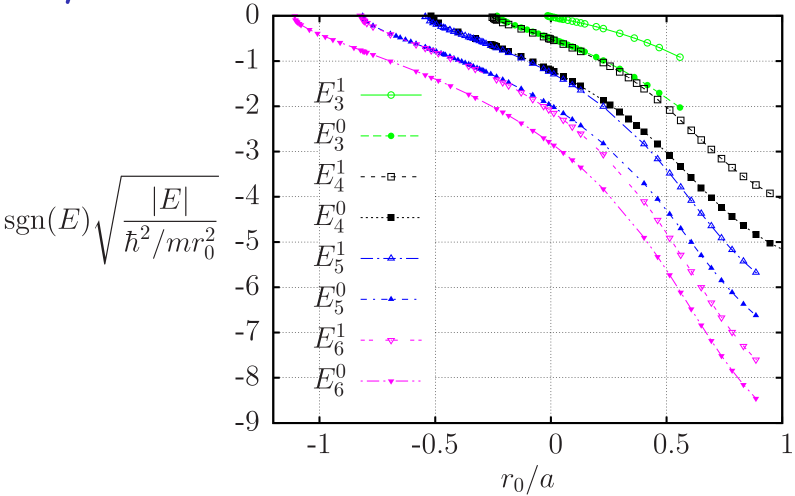
- Two four-body states for each three-body state

N-body Efimov Plot



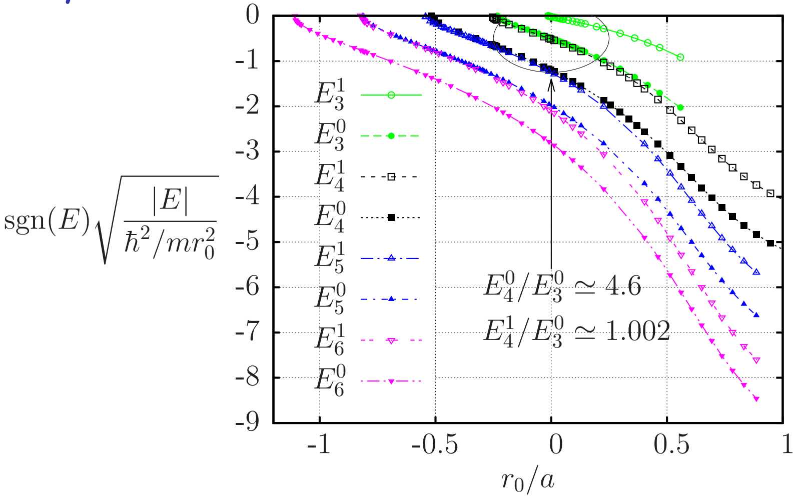
- Two four-body states for each three-body state
- Two five-body states for each four-body state

N-body Efimov Plot



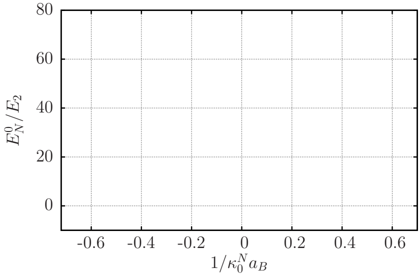
- Two four-body states for each three-body state
- Two five-body states for each four-body state
- Two six-body states for each five-body state

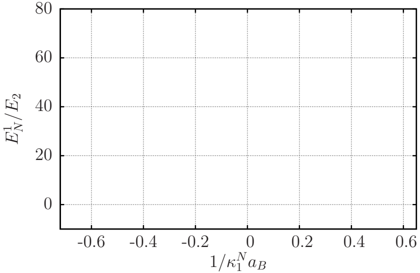
N-body Efimov Plot



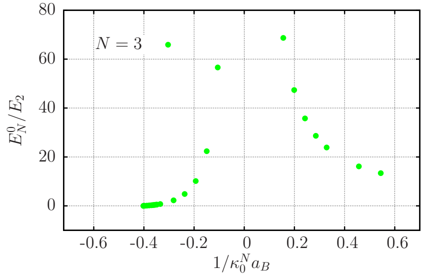
- Two four-body states for each three-body state
- Two five-body states for each four-body state
- Two six-body states for each five-body state

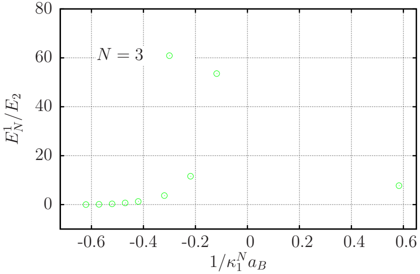
Universality



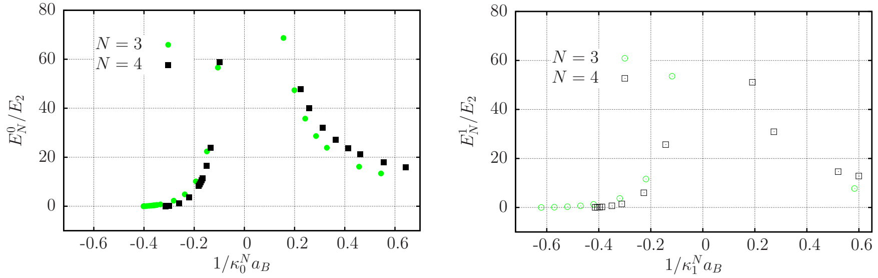


Universality

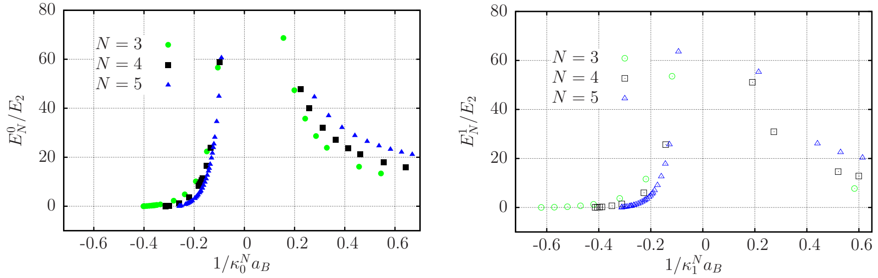




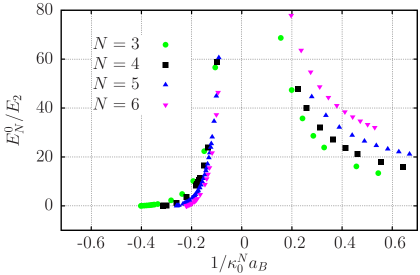
Universality

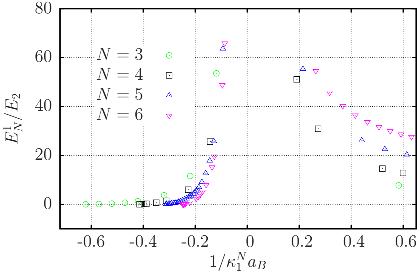


Universality

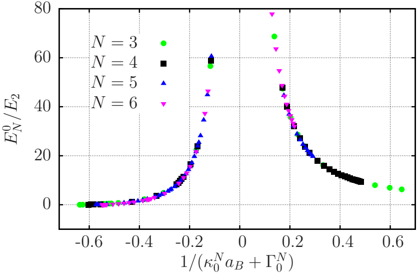


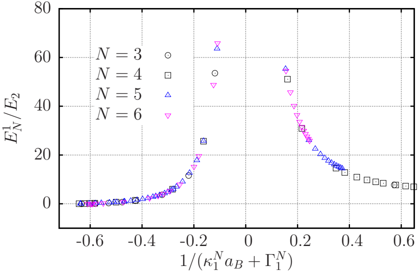
Universality



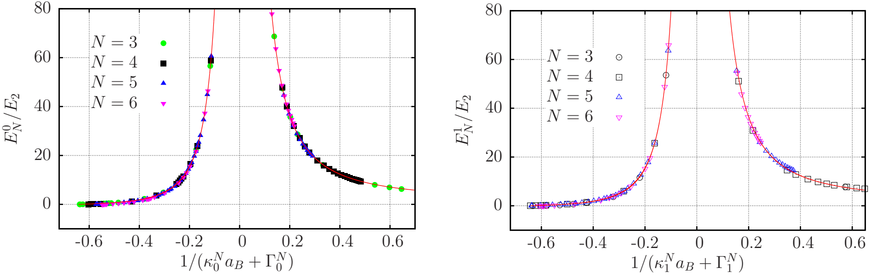


Universality





Universality

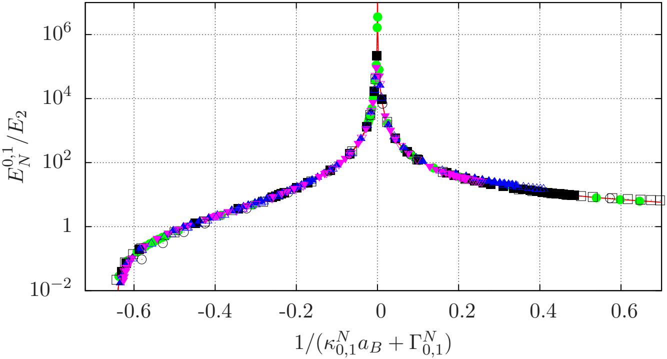


Universal Formula

$$E_N^n/E_2 = \tan^2 \xi$$

$$\kappa_n^N a_B + \Gamma_n^N = \frac{e^{-\Delta(\xi)/2s_0}}{\cos \xi}$$

Universality



Universal Formula

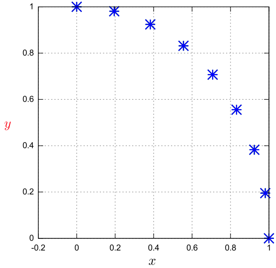
$$E_N^n/E_2 = \tan^2 \xi$$

$$\kappa_n^N a_B + \Gamma_n^N = \frac{e^{-\Delta(\xi)/2s_0}}{\cos \xi}$$

Efimov Straighteners

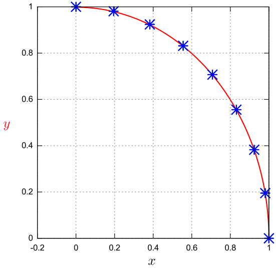
Efimov Straighteners

Data on a Circle



Efimov Straighteners

Data on a Circle

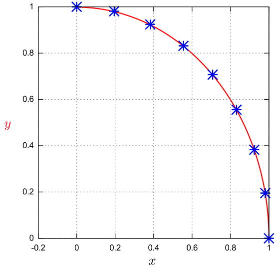


$$y = \sin \xi$$

$$x = \cos \xi$$

Efimov Straighteners

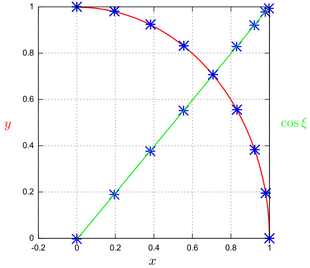
Data on a Circle



$$\begin{array}{l} y = \sin\xi \\ x = \cos\xi \end{array} \Leftrightarrow \begin{array}{l} y/x = \tan\xi \\ x = \cos\xi(x,y) \end{array}$$

Efimov Straighteners

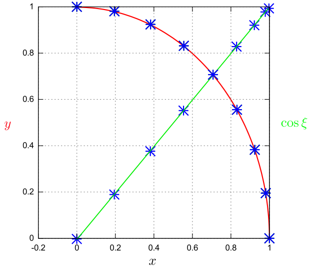
Data on a Circle



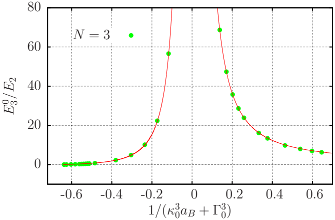
$$y = \sin\xi \qquad\qquad y/x = \tan\xi$$
$$\Leftrightarrow$$
$$x = \cos\xi \qquad\qquad x = \cos\xi(x, y)$$

Efimov Straighteners

Data on a Circle



Data on Efimov curve



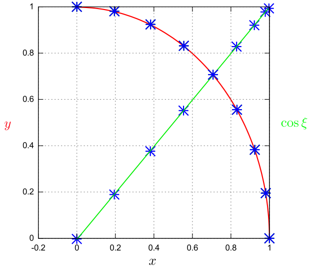
$$y = \sin\xi \quad x = \cos\xi \;\Leftrightarrow\; y/x = \tan\xi \quad x = \cos\xi(x,y)$$

$$E_3^0/E_2 = \tan^2\xi$$

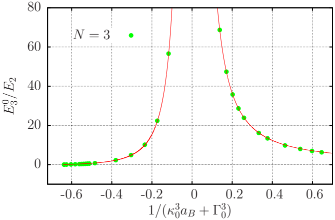
$$\kappa_0^3 a_B + \Gamma_0^3 = \frac{e^{-\Delta(\xi)/2s_0}}{\cos\xi}$$

Efimov Straighteners

Data on a Circle



Data on Efimov curve



$$y = \sin\xi \qquad\qquad y/x = \tan\xi$$
$$\Leftrightarrow$$
$$x = \cos\xi \qquad\qquad x = \cos\xi(x, y)$$

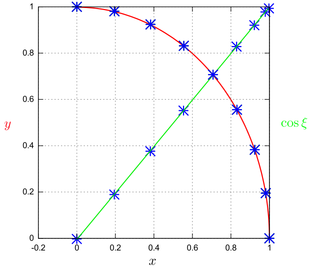
$$E_3^0/E_2 = \tan^2\xi$$

$$\kappa_0^3 a_B + \Gamma_0^3 = \frac{e^{-\Delta(\xi)/2s_0}}{\cos\xi}$$

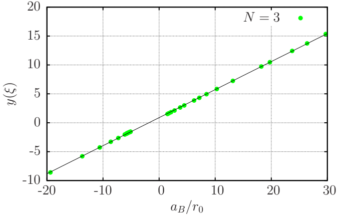
$$y(\xi) \stackrel{\text{def}}{=} \frac{e^{-\Delta(\xi)/2s_0}}{\cos\xi}$$

Efimov Straighteners

Data on a Circle



Data on Efimov curve



$$y = \sin\xi \quad \Leftrightarrow \quad y/x = \tan\xi$$
$$x = \cos\xi \qquad\quad x = \cos\xi(x,y)$$

$$E_3^0/E_2 = \tan^2\xi$$

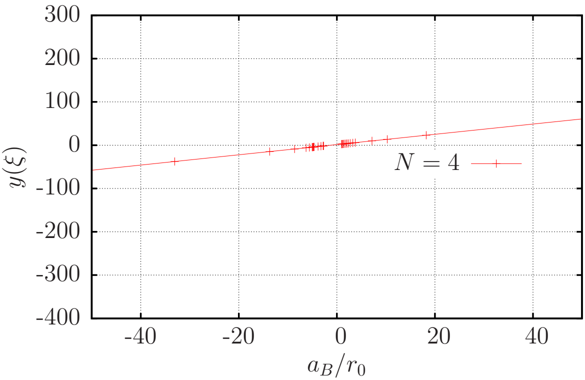
$$\kappa_0^3 a_B + \Gamma_0^3 = \frac{e^{-\Delta(\xi)/2s_0}}{\cos\xi}$$

$$y(\xi) \stackrel{\text{def}}{=} \frac{e^{-\Delta(\xi)/2s_0}}{\cos\xi}$$

Universality up to $N = 16$

$$y(\xi) = \kappa_N a_B + \Gamma_N$$

N	$\kappa_N r_0$	Γ_N
4	1.185	1.475

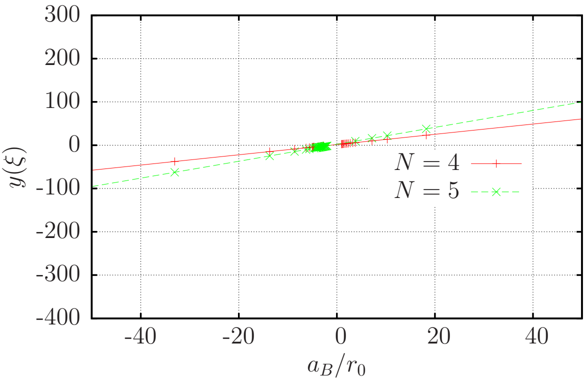


$$V(r) = V_0\ e^{-r^2/r_0^2}$$

Universality up to $N = 16$

$$y(\xi) = \kappa_N a_B + \Gamma_N$$

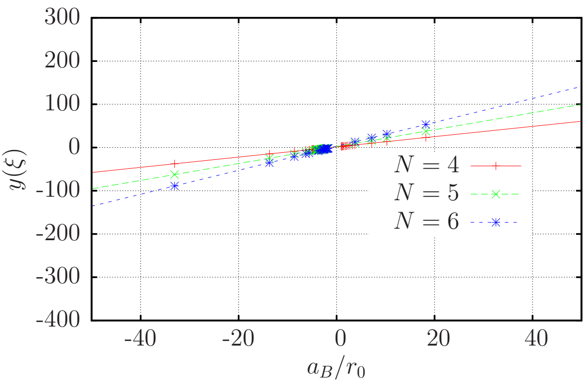
N	$\kappa_N r_0$	Γ_N
4	1.185	1.475
5	1.955	2.128



$$V(r) = V_0 \; e^{-r^2/r_0^2}$$

Universality up to $N = 16$

$$y(\xi) = \kappa_N a_B + \Gamma_N$$

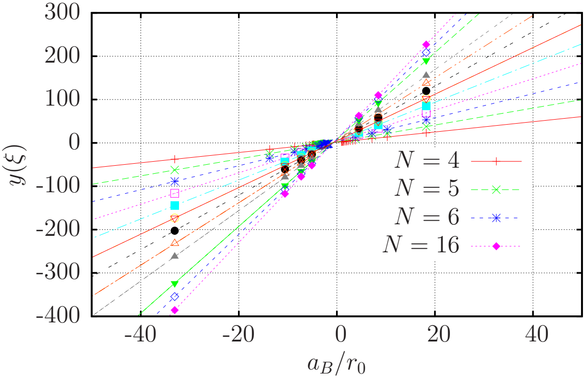


N	$\kappa_N r_0$	Γ_N
4	1.185	1.475
5	1.955	2.128
6	2.770	2.752

$$V(r) = V_0\, e^{-r^2/r_0^2}$$

Universality up to $N = 16$

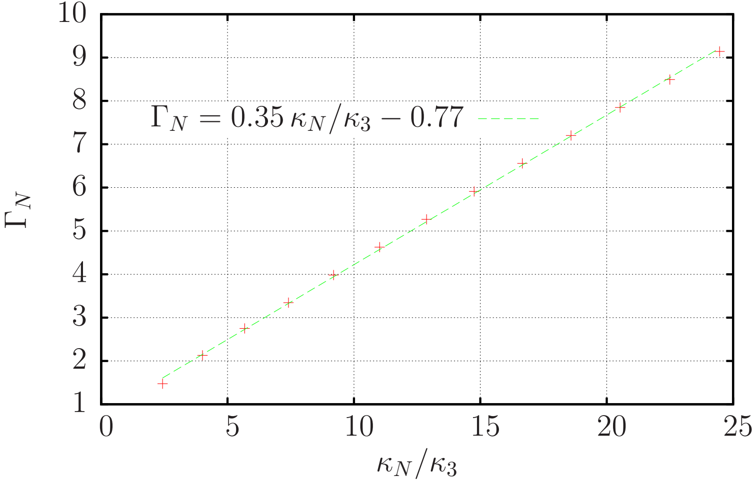
$$y(\xi) = \kappa_N a_B + \Gamma_N$$



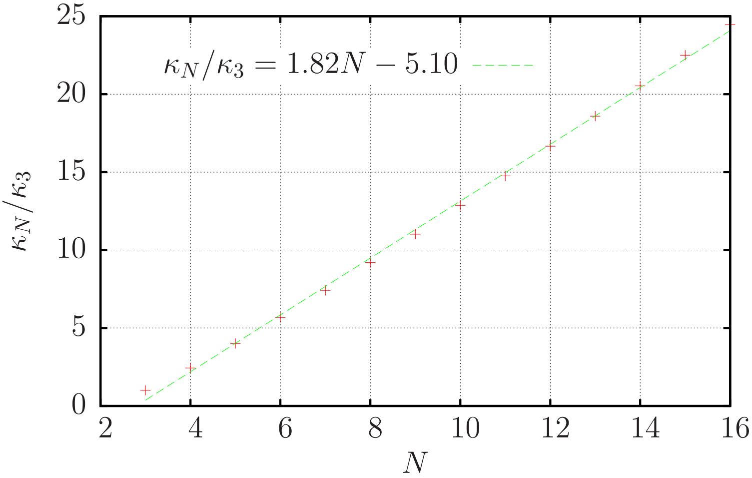
$$V(r) = V_0 \; e^{-r^2/r_0^2}$$

N	$\kappa_N r_0$	Γ_N
4	1.185	1.475
5	1.955	2.128
6	2.770	2.752
7	3.617	3.344
8	4.487	3.983
9	5.377	4.625
10	6.282	5.268
11	7.201	5.912
12	8.131	6.557
13	9.071	7.202
14	10.02	7.848
15	10.98	8.494
16	11.94	9.141

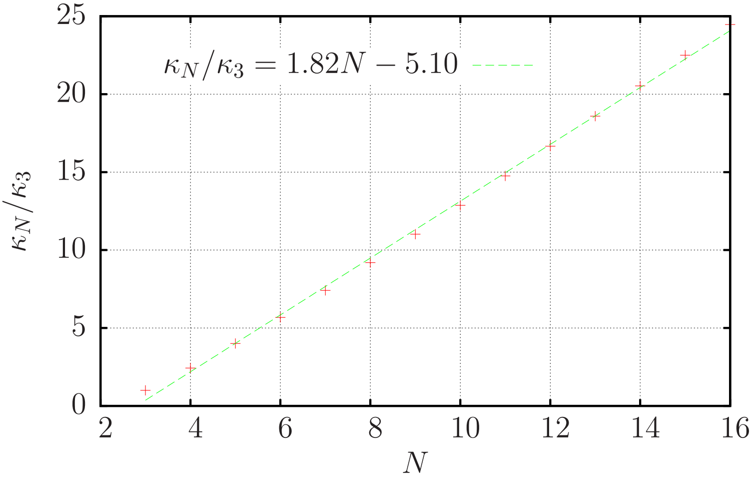
Universality up to $N = 16$



Universality up to $N = 16$

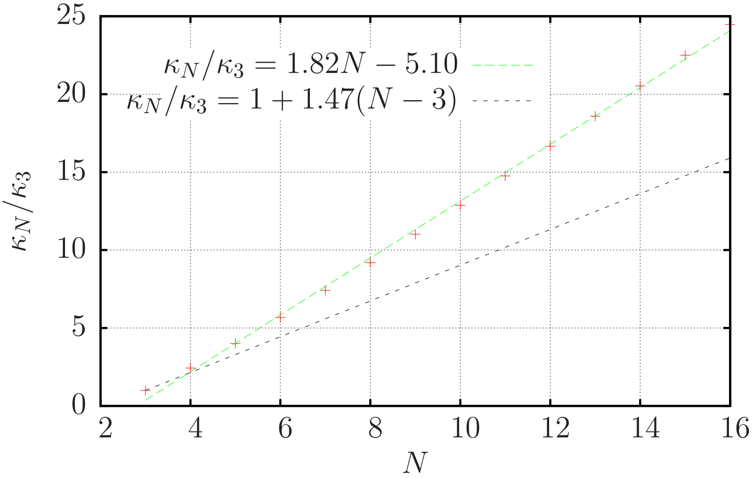


Universality up to $N = 16$



$$\kappa_N/\kappa_3 = 1 + (N-3)(\kappa_4/\kappa_3 - 1)$$

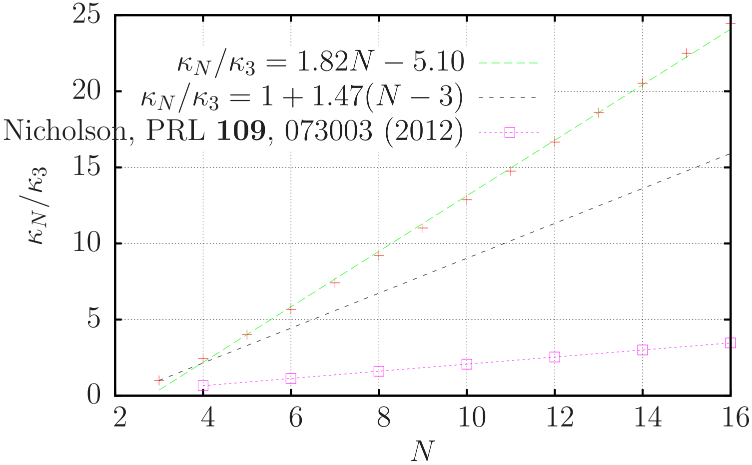
Universality up to $N = 16$



$$\kappa_N/\kappa_3 = 1 + (N - 3)(\kappa_4/\kappa_3 - 1)$$
$$= 1 + 1.147(N - 3)$$

$\kappa_4 = 2.147\kappa_3$ - Deltuva, Few-Body Syst 54, 569 (2013)

Universality up to $N = 16$



$$\kappa_N/\kappa_3 = 1 + (N-3)(\kappa_4/\kappa_3 - 1)$$
$$= 1 + 1.147(N-3)$$

$\kappa_4 = 2.147\kappa_3$ - Deltuva, Few-Body Syst 54, 569 (2013)

Outline

Efimov Physics
- Efimov Effect
- Discrete Scale Invariance

Finite-range Effect
- 3-Body Bound States
- Scattering Length
- Recombination
- Measured energies

N-body Universality
- N-Body States
- Universality

Work in progress...
- ... back to Nuclear Physics

Efimov and Nuclear Physics

$$V(r, \sigma_1 \cdot \sigma_2) = V_0 \, e^{-(r/R_0)^2} \mathcal{P}_0 + V_1 \, e^{-(r/R_1)^2} \mathcal{P}_1$$

- Two control parameter a_0 and a_1

Efimov and Nuclear Physics

$$V(r, \sigma_1 \cdot \sigma_2) = V_0\, e^{-(r/R_0)^2} \mathcal{P}_0 + V_1\, e^{-(r/R_1)^2} \mathcal{P}_1$$

- Two control parameter a_0 and a_1
- How to organize data? We fix $\tan \varphi = a_1 / a_0$

Efimov and Nuclear Physics

$$V(r, \sigma_1 \cdot \sigma_2) = V_0\, e^{-(r/R_0)^2} \mathcal{P}_0 + V_1\, e^{-(r/R_1)^2} \mathcal{P}_1$$

- Two control parameter a_0 and a_1
- How to organize data? We fix $\tan\varphi = a_1/a_0$

$$\tan^2 \xi = E_3/E_2^{(1)}$$

$$y(\xi) = \kappa_* a_B^{(1)} + \Gamma$$

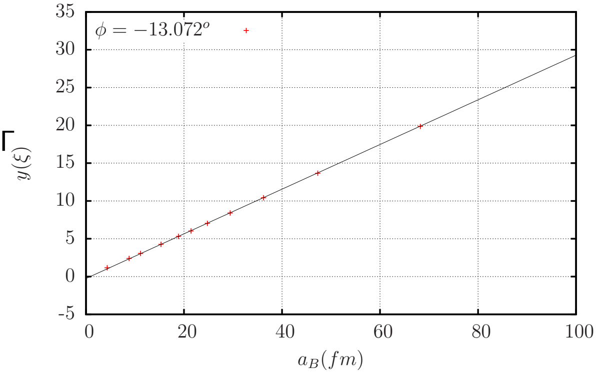
Efimov and Nuclear Physics

$$V(r,\sigma_1\cdot\sigma_2) = V_0\, e^{-(r/R_0)^2}\mathcal{P}_0 + V_1\, e^{-(r/R_1)^2}\mathcal{P}_1$$

- Two control parameter a_0 and a_1
- How to organize data? We fix $\tan\varphi = a_1/a_0$

$$\tan^2\xi = E_3/E_2^{(1)}$$

$$y(\xi) = \kappa_* a_B^{(1)} + \Gamma$$



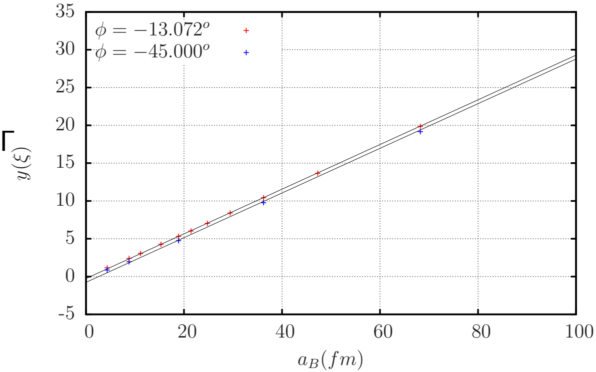
Efimov and Nuclear Physics

$$V(r,\sigma_1\cdot\sigma_2) = V_0\, e^{-(r/R_0)^2}\mathcal{P}_0 + V_1\, e^{-(r/R_1)^2}\mathcal{P}_1$$

- Two control parameter a_0 and a_1
- How to organize data? We fix $\tan\varphi = a_1/a_0$

$$\tan^2\xi = E_3/E_2^{(1)}$$

$$y(\xi) = \kappa_* a_B^{(1)} + \Gamma$$



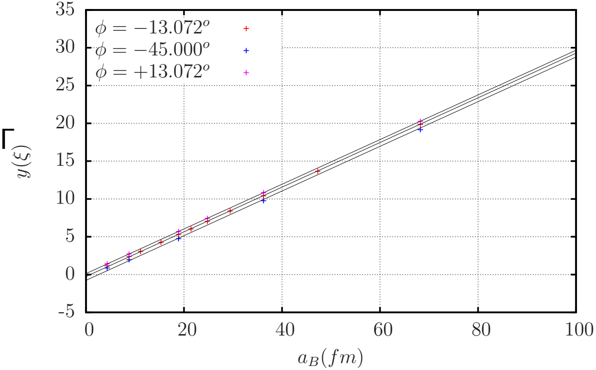
Efimov and Nuclear Physics

$$V(r,\sigma_1\cdot\sigma_2) = V_0\, e^{-(r/R_0)^2}\mathcal{P}_0 + V_1\, e^{-(r/R_1)^2}\mathcal{P}_1$$

- Two control parameter a_0 and a_1
- How to organize data? We fix $\tan\varphi = a_1/a_0$

$$\tan^2\xi = E_3/E_2^{(1)}$$

$$y(\xi) = \kappa_* a_B^{(1)} + \Gamma$$



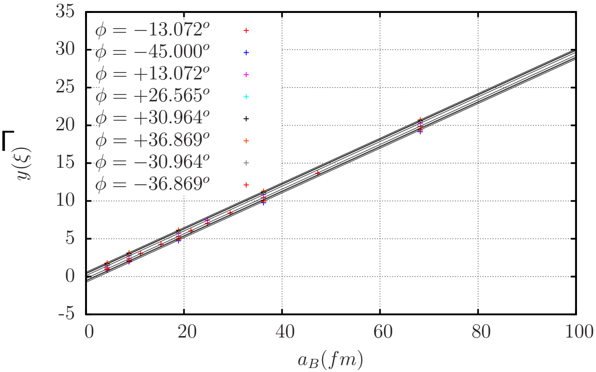
Efimov and Nuclear Physics

$$V(r,\sigma_1\cdot\sigma_2) = V_0\, e^{-(r/R_0)^2}\mathcal{P}_0 + V_1\, e^{-(r/R_1)^2}\mathcal{P}_1$$

- Two control parameter a_0 and a_1
- How to organize data? We fix $\tan\varphi = a_1/a_0$

$$\tan^2\xi = E_3/E_2^{(1)}$$

$$y(\xi) = \kappa_* a_B^{(1)} + \Gamma$$



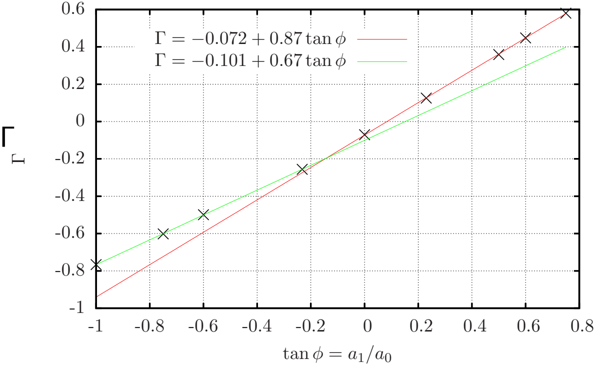
Efimov and Nuclear Physics

$$V(r,\sigma_1\cdot\sigma_2) = V_0\, e^{-(r/R_0)^2}\mathcal{P}_0 + V_1\, e^{-(r/R_1)^2}\mathcal{P}_1$$

- Two control parameter a_0 and a_1
- How to organize data? We fix $\tan\varphi = a_1/a_0$

$$\tan^2\xi = E_3/E_2^{(1)}$$

$$y(\xi) = \kappa_* a_B^{(1)} + \Gamma$$



Efimov and Nuclear Physics

$$V(r, \sigma_1 \cdot \sigma_2) = V_0\, e^{-(r/R_0)^2} \mathcal{P}_0 + V_1\, e^{-(r/R_1)^2} \mathcal{P}_1$$

- Two control parameter a_0 and a_1
- How to organize data? We fix $\tan\varphi = a_1/a_0$
- $y(\xi) = \kappa_* a_B^{(1)} + \alpha + \beta a_1/a_0$

Efimov and Nuclear Physics

$$V(r, \sigma_1 \cdot \sigma_2) = V_0\, e^{-(r/R_0)^2} \mathcal{P}_0 + V_1\, e^{-(r/R_1)^2} \mathcal{P}_1$$

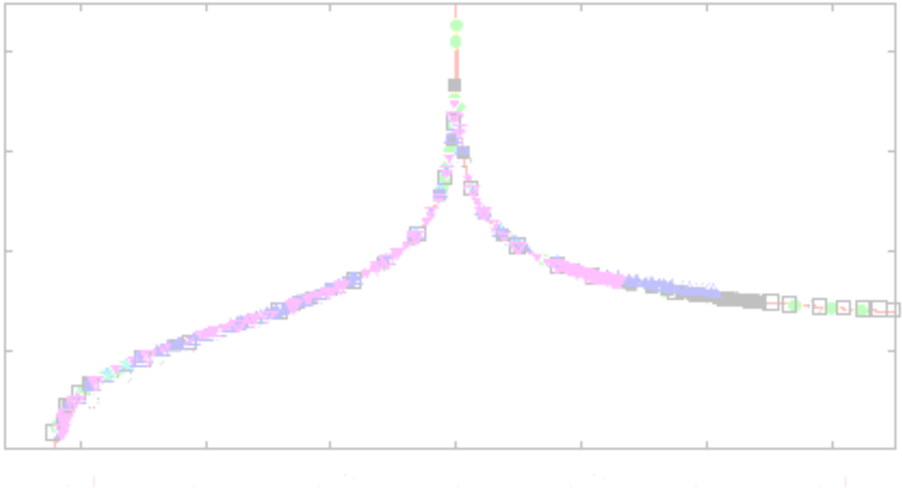
- Two control parameter a_0 and a_1
- How to organize data? We fix $\tan\varphi = a_1/a_0$
- $y(\xi) = \kappa_* a_B^{(1)} + \alpha + \beta a_1/a_0$
- Better way to analyse? Ex. $\tan^2\xi = E_3/(1/a_0^2 + 1/a_1^2)$

Efimov and Nuclear Physics

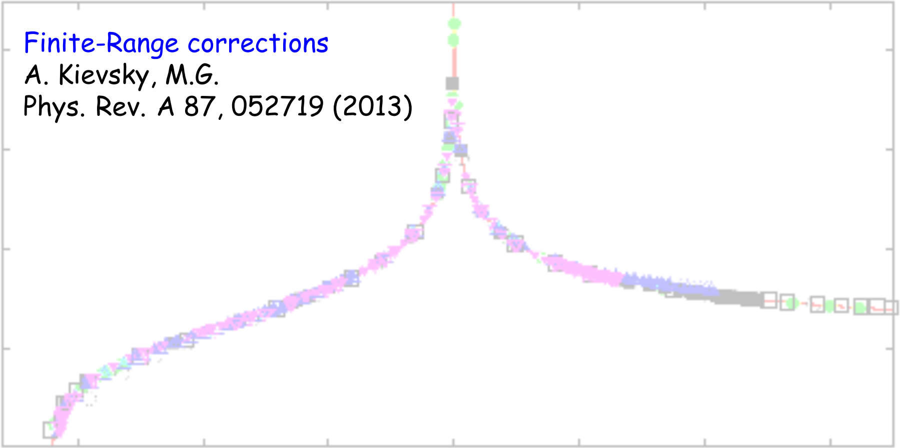
$$V(r,\sigma_1\cdot\sigma_2) = V_0\, e^{-(r/R_0)^2}\mathcal{P}_0 + V_1\, e^{-(r/R_1)^2}\mathcal{P}_1$$

- Two control parameter a_0 and a_1
- How to organize data? We fix $\tan\varphi = a_1/a_0$
- $y(\xi) = \kappa_* a_B^{(1)} + \alpha + \beta a_1/a_0$
- Better way to analyse? Ex. $\tan^2\xi = E_3/(1/a_0^2 + 1/a_1^2)$
- Explore the Nuclear plane $a_1/a_0 = -0.228$
  - Use of a three-body force
  - Look at the light-nuclei spectrum

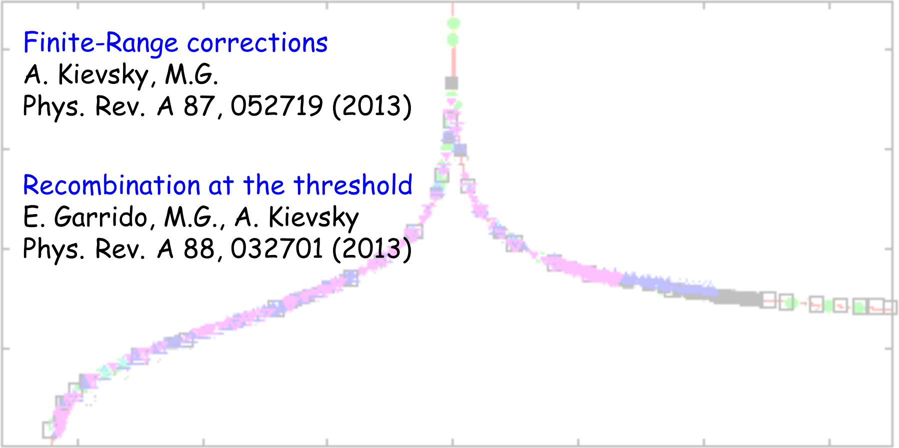
References and Collaborators



References and Collaborators



References and Collaborators



Finite-Range corrections

A. Kievsky, M.G.

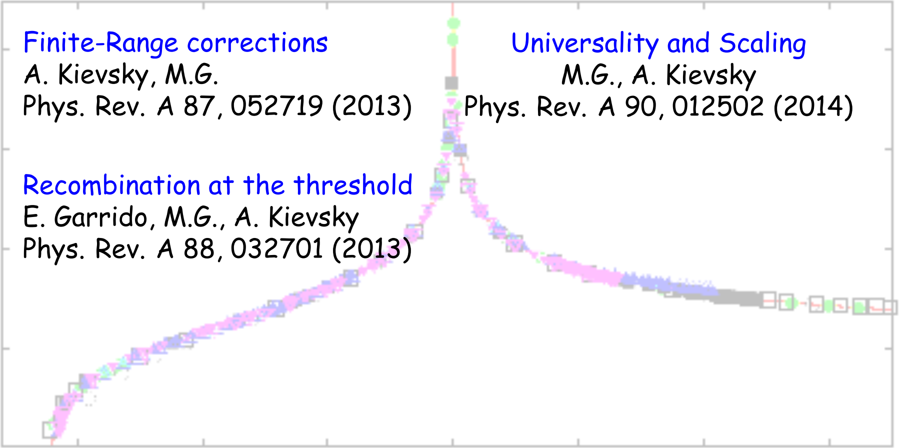
Phys. Rev. A 87, 052719 (2013)

Recombination at the threshold

E. Garrido, M.G., A. Kievsky

Phys. Rev. A 88, 032701 (2013)

References and Collaborators

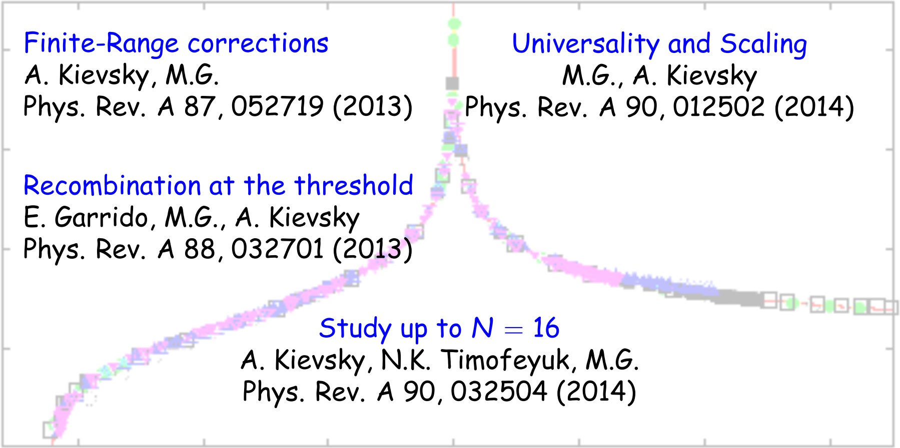


Finite-Range corrections
A. Kievsky, M.G.
Phys. Rev. A 87, 052719 (2013)

Universality and Scaling
M.G., A. Kievsky
Phys. Rev. A 90, 012502 (2014)

Recombination at the threshold
E. Garrido, M.G., A. Kievsky
Phys. Rev. A 88, 032701 (2013)

References and Collaborators



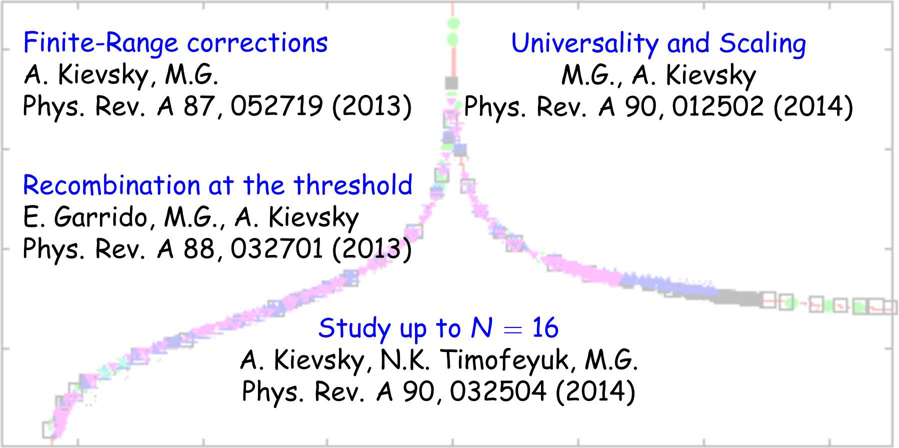
Finite-Range corrections
A. Kievsky, M.G.
Phys. Rev. A 87, 052719 (2013)

Universality and Scaling
M.G., A. Kievsky
Phys. Rev. A 90, 012502 (2014)

Recombination at the threshold
E. Garrido, M.G., A. Kievsky
Phys. Rev. A 88, 032701 (2013)

Study up to $N = 16$
A. Kievsky, N.K. Timofeyuk, M.G.
Phys. Rev. A 90, 032504 (2014)

References and Collaborators



Finite-Range corrections
A. Kievsky, M.G.
Phys. Rev. A 87, 052719 (2013)

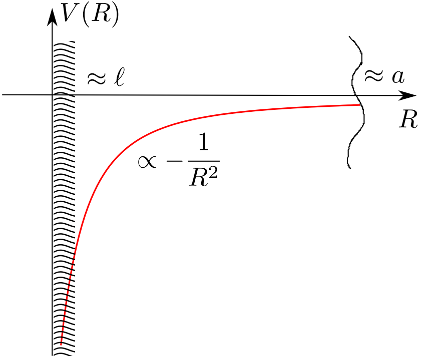
Universality and Scaling
M.G., A. Kievsky
Phys. Rev. A 90, 012502 (2014)

Recombination at the threshold
E. Garrido, M.G., A. Kievsky
Phys. Rev. A 88, 032701 (2013)

Study up to $N = 16$
A. Kievsky, N.K. Timofeyuk, M.G.
Phys. Rev. A 90, 032504 (2014)

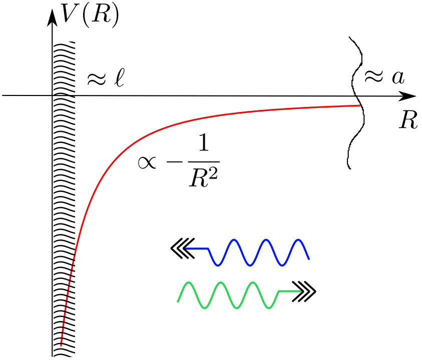
Thanks!

Origin of the Shift



Origin of the Shift

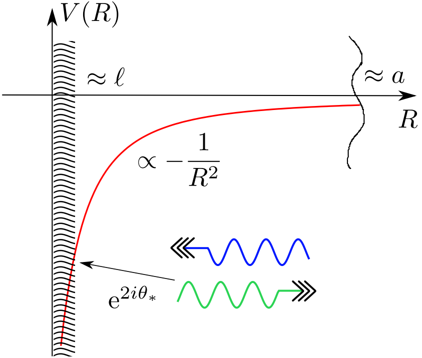
$$\Psi \propto A e^{i s_0 \log(HR)} + B e^{-i s_0 \log(HR)}$$



$$\hbar^2 H^2 / m = E_3 + E_2$$

$$\tan^2(\xi) = E_3 / E_2$$

Origin of the Shift



$$\Psi \propto A e^{i s_0 \log(HR)} + B e^{-i s_0 \log(HR)}$$

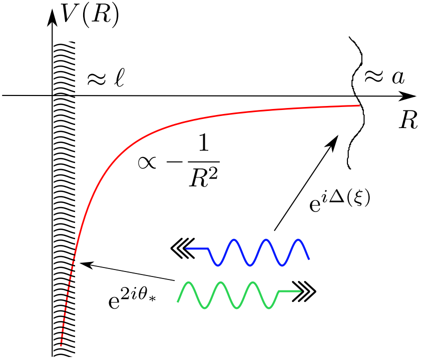
$$R \approx \ell : \quad A = e^{2i\theta_*} B$$

$$\theta_* = -s_0 \log(H/\Lambda_0)$$

$$\hbar^2 H^2/m = E_3 + E_2$$

$$\tan^2(\xi) = E_3/E_2$$

Origin of the Shift



$$\Psi \propto A e^{is_0 \log(HR)} + B e^{-is_0 \log(HR)}$$

$$R \approx \ell: \quad A = e^{2i\theta_*} B$$

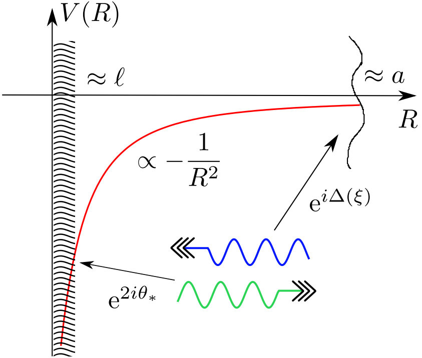
$$\theta_* = -s_0 \log(H/\Lambda_0)$$

$$R \approx a: \quad A = e^{i\Delta(\xi)} B$$

$$\hbar^2 H^2/m = E_3 + E_2$$

$$\tan^2(\xi) = E_3/E_2$$

Origin of the Shift



$$\Psi \propto A e^{is_0 \log(HR)} + B e^{-is_0 \log(HR)}$$

$$R \approx \ell : \quad A = e^{2i\theta_*} B$$

$$\theta_* = -s_0 \log(H/\Lambda_0)$$

$$R \approx a : \quad A = e^{i\Delta(\xi)} B$$

Bound State

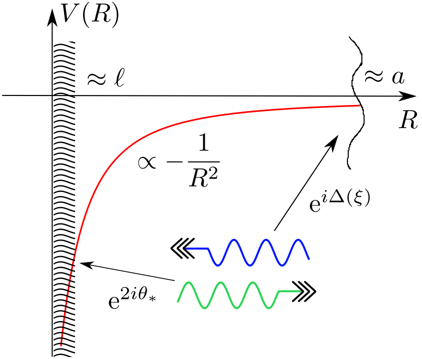
$$2\theta_* + \Delta(\xi) = 2\pi n$$

$$H = \Lambda_0 \, e^{\Delta(\xi)/2s_0} \, e^{-\pi n/s_0}$$

$$\hbar^2 H^2/m = E_3 + E_2$$

$$\tan^2(\xi) = E_3/E_2$$

Origin of the Shift



$$\Psi \propto A e^{is_0 \log(HR)} + B e^{-is_0 \log(HR)}$$

$$R \approx \ell : \quad A = e^{2i\theta_*} B$$

$$\theta_* = -s_0 \log(H/\Lambda_0)$$

$$R \approx a : \quad A = e^{i\Delta(\xi)} B$$

$$\hbar^2 H^2 / m = E_3 + E_2$$

$$\tan^2(\xi) = E_3 / E_2$$

Bound State

$$2\theta_* + \Delta(\xi) = 2\pi n$$

$$H = \Lambda_0 \, e^{\Delta(\xi)/2s_0} \, e^{-\pi n / s_0}$$

$$\Lambda_0 a = e^{2\pi n / s_0} \, e^{-\Delta(\xi)/s_0} / \cos\xi$$

Origin of the Shift

- Bound State

$$\Lambda_0 a = e^{-\Delta(\xi)/s_0} / \cos\xi$$

Origin of the Shift

- Bound State

$$\Lambda_0 a = e^{-\Delta(\xi)/s_0} / \cos\xi$$

- Zero-range parameterization of $\Delta(\xi)$

$$\Lambda_0 = \kappa_*$$

Origin of the Shift

- Bound State

$$\Lambda_0 a = e^{-\Delta(\xi)/s_0} / \cos \xi$$

- Zero-range parameterization of $\Delta(\xi)$

$$\Lambda_0 = \kappa_*$$

- Finite-range case

Origin of the Shift

- Bound State

$$\Lambda_0 a = e^{-\Delta(\xi)/s_0} / \cos \xi$$

- Zero-range parameterization of $\Delta(\xi)$

$$\Lambda_0 = \kappa_*$$

- Finite-range case
  - Parametrization of $\Delta(\xi)$ unchanged

Origin of the Shift

- Bound State

$$\Lambda_0 a = e^{-\Delta(\xi)/s_0} / \cos \xi$$

- Zero-range parameterization of $\Delta(\xi)$

$$\Lambda_0 = \kappa_*$$

- Finite-range case
  - Parametrization of $\Delta(\xi)$ unchanged
  - $V(R)$ changes

Origin of the Shift

- Bound State

$$\Lambda_0 a = e^{-\Delta(\xi)/s_0} / \cos \xi$$

- Zero-range parameterization of $\Delta(\xi)$

$$\Lambda_0 = \kappa_*$$

- Finite-range case
  - Parametrization of $\Delta(\xi)$ unchanged
  - $V(R)$ changes

$$\Lambda_0 = \kappa_*(1 + \mathcal{A}\ell/a)$$

Origin of the Shift

- Bound State

$$\Lambda_0 a = e^{-\Delta(\xi)/s_0} / \cos \xi$$

- Zero-range parameterization of $\Delta(\xi)$

$$\Lambda_0 = \kappa_*$$

- Finite-range case
  - Parametrization of $\Delta(\xi)$ unchanged
  - $V(R)$ changes

$$\Lambda_0 = \kappa_*(1 + \mathcal{A}\ell/a)$$

- Shift ...

$$\kappa_* a + \Gamma = e^{-\Delta(\xi)/s_0} / \cos \xi$$

$$\Gamma = \mathcal{A}\,\kappa_* \ell$$

Universality and Scattering

- Effective Range Function

  Zero-range interaction $(\ell = 0)$

  $$ka \cot \delta = c_1(ka) + c_2(ka) \cot[s_0 \ln(\kappa_* a) + \varphi(ka)]$$

  - $c_1(ka)$, $c_2(ka)$, $\varphi(ka)$ Universal Functions

Universality and Scattering

- Effective Range Function

Zero-range interaction $(\ell = 0)$

$$ka \cot\delta = c_1(ka) + c_2(ka)\cot[s_0 \ln(\kappa_* a) + \varphi(ka)]$$

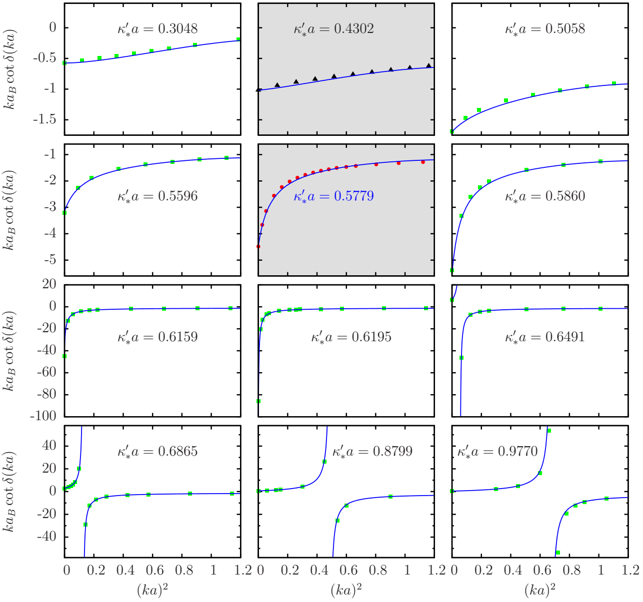
  - $c_1(ka)$, $c_2(ka)$, $\varphi(ka)$ Universal Functions

Finite-range interaction $(\ell \neq 0)$

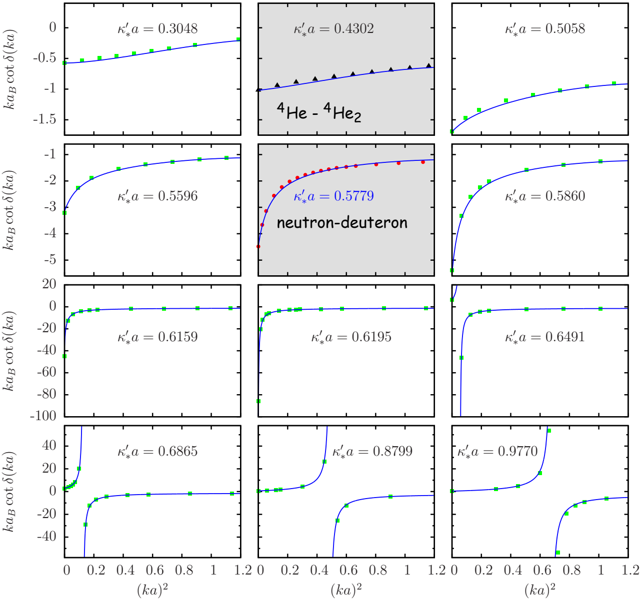
$$k a_B \cot\delta = c_1(ka) + c_2(ka)\cot[s_0 \ln(\kappa'_* a) + \varphi(ka)]$$

$$E_2 = \hbar^2/m {a_B}^2 \quad \text{and} \quad \kappa'_* = \kappa_* + \Gamma/a$$

Universality and Scattering



Universality and Scattering



Efimov and Light Nuclei

0.546 MeV [2] 0^+

A = 2

0.599 MeV [3] 0^+

8.465 MeV [3] 0^+

A = 3

8.562 MeV [4] 0^+

10.406 MeV [3 1] 1^-

30.418 MeV [4] 0^+

A = 4

28.72 MeV [4 1] 0^+

31.72 MeV [5] 0^+

43.03 MeV [4 1] 1^-

68.28 MeV [5] 0^+

A = 5

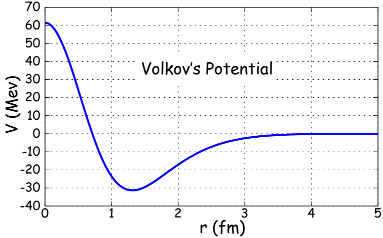
66.49 MeV [4 2] 0^+

70.28 MeV [5 1] 0^+

73.49 MeV [6] 0^+

122.78 MeV [6] 0^+

A = 6



Efimov and Light Nuclei

0.546 MeV 0^+

^2H

0.599 MeV 0^+

8.431 MeV 0^+

^3H

7.725 MeV 0^+

^3He

6.417 MeV $2^-, 0$

6.850 MeV $1^-, 1$

6.965 MeV $0^-, 0$

8.085 MeV $0^+, 0$

28.43 MeV 0^+

^4He

33.02 MeV 0^+

^6He

S-wave potential – only acts when $l_{ij} = 0$