Few-body universality and "super" Efimov effect

Yusuke Nishida (Tokyo Tech)

7th International and Interdisciplinary Workshop
on the Dynamics of Critically Stable
Quantum Few-Body Systems (Critical Stability 2014)

October 12-17 (2014)

Plan of this talk

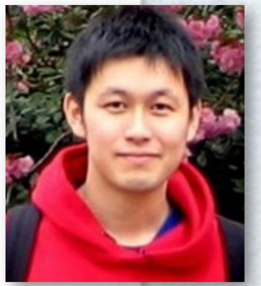
1. Introduction on few-body universality

2. Prediction of super Efimov effect

PRL 110, 235301 (2013) PHYSICAL REVIEW LETTERS week ending 7 JUNE 2013

Super Efimov Effect of Resonantly Interacting Fermions in Two Dimensions

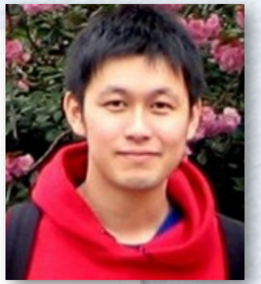
Yusuke Nishida,[1] Sergej Moroz,[2] and Dam Thanh Son[3]
[1]*Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*
[2]*Department of Physics, University of Washington, Seattle, Washington 98195, USA*
[3]*Enrico Fermi Institute, University of Chicago, Chicago, Illinois 60637, USA*
(Received 18 January 2013; published 4 June 2013)



3. Extension to mass imbalance mixtures

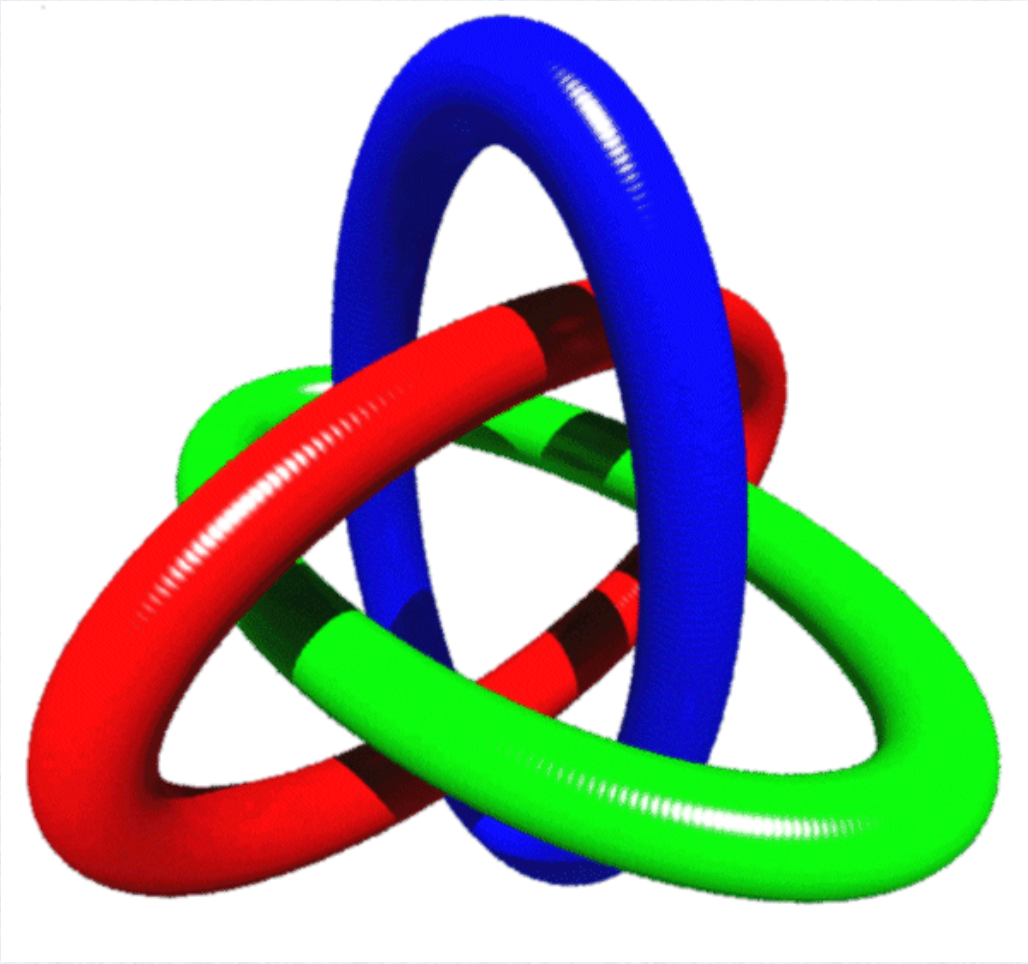
Super Efimov effect for mass imbalanced systems

Sergej Moroz[1] and Yusuke Nishida[2]
[1]*Department of Physics, University of Washington, Seattle, Washington 98195, USA*
[2]*Department of Physics, Tokyo Institute of Technology, Ookayama, Meguro, Tokyo 152-8551, Japan*
(Dated: July 2014)




arXiv:1407.7664

Introduction:
Few-body universality



Few-body universality

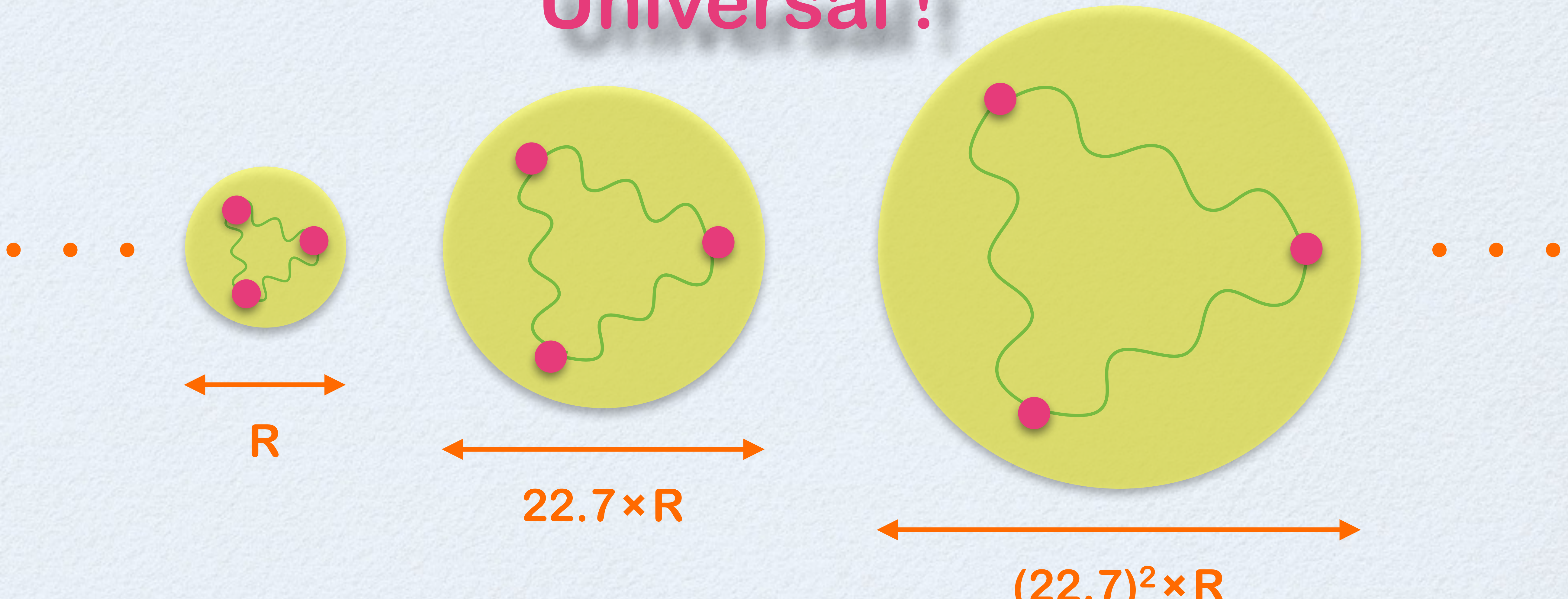
Efimov effect (1970)

- 3 bosons
- 3 dimensions
- s-wave resonance

→ Infinite bound states with exponential scaling

$$E_n \sim e^{-2\pi n}$$

Universal !



Few-body universality

Efimov effect (1970)

- 3 bosons
- 3 dimensions
- s-wave resonance

→ Infinite bound states with exponential scaling

$$E_n \sim e^{-2\pi n}$$

Efimov effect in other systems ?

No, only in 3D with s-wave resonance

	s-wave	p-wave	d-wave
3D	O	x	x
2D	x	x	x
1D	x	x	

Y.N. & S.Tan, Few-Body Syst

Y.N. & D.Lee Phys Rev A

Few-body universality

Efimov effect (1970)

- 3 bosons
- 3 dimensions
- s-wave resonance

→ Infinite bound states with exponential scaling

$$E_n \sim e^{-2\pi n}$$

Different universality in other systems ?

Yes, super Efimov effect in 2D with p-wave !

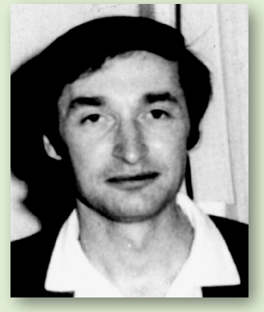
	s-wave	p-wave	d-wave
3D	O	x	x
2D	x	!x!	x
1D	x	x	

Y.N. & S.Tan, Few-Body Syst

Y.N. & D.Lee Phys Rev A

Few-body universality

Efimov effect



- 3 bosons
- 3 dimensions
- s-wave resonance

exponential scaling

$$E_n \sim e^{-2\pi n}$$

Super Efimov effect

New !

- 3 fermions
- 2 dimensions
- p-wave resonance

"doubly" exponential

$$E_n \sim e^{-2e^{3\pi n/4}}$$

PRL **110,** 235301 (2013) PHYSICAL REVIEW LETTERS week ending 7 JUNE 2013

Super Efimov Effect of Resonantly Interacting Fermions in Two Dimensions

Yusuke Nishida,[1] Sergej Moroz,[2] and Dam Thanh Son[3]

[1]*Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*
[2]*Department of Physics, University of Washington, Seattle, Washington 98195, USA*
[3]*Enrico Fermi Institute, University of Chicago, Chicago, Illinois 60637, USA*
(Received 18 January 2013; published 4 June 2013)



Few-body universality

Efimov effect

- 3 bosons
- 3 dimensions
- s-wave resonance

exponential scaling

$$E_n \sim e^{-2\pi n}$$

Super Efimov effect

New !

- 3 fermions
- 2 dimensions
- p-wave resonance

"doubly" exponential

$$E_n \sim e^{-2e^{3\pi n/4}}$$

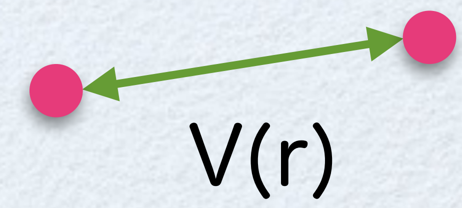
- Low-energy EFT for 2D p-wave scattering
- RG analysis for 3-body & 4-body couplings

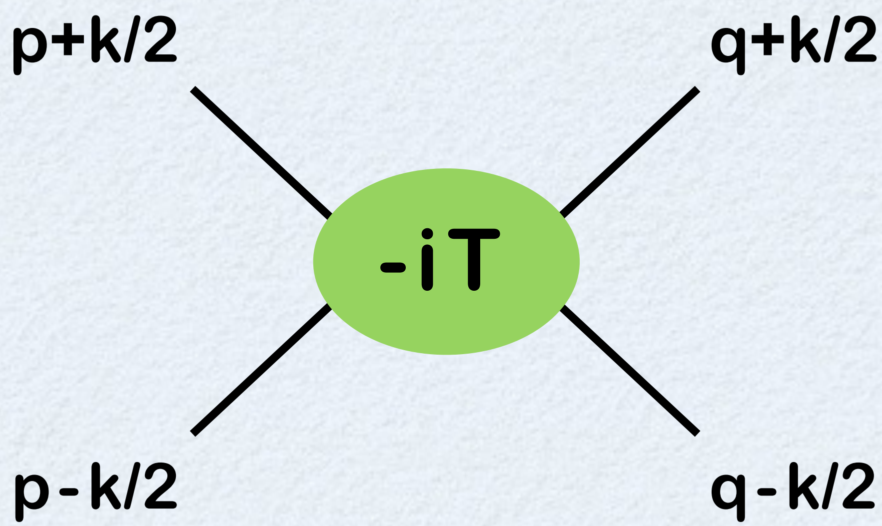
=> Exact spectrum in the low-energy limit !

Prediction:
Super Efimov effect

P-wave scattering in 2D

Two fermions with short-range potential





⇒ Effective range expansion

Cf. H.-W. Hammer & D. Lee
Ann. Phys. 325, 2212 (2010)

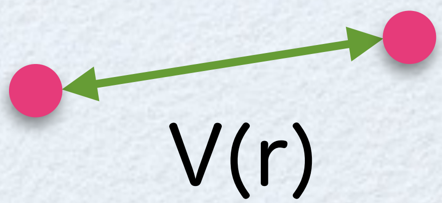
$$-iT = \frac{2i}{m} \frac{\vec{p} \cdot \vec{q}}{-\frac{1}{a} - \frac{m\varepsilon}{\pi} \ln\left(-\frac{\Lambda^2}{m\varepsilon}\right) + \sum_{n=2}^{\infty} c_n (m\varepsilon)^n}$$

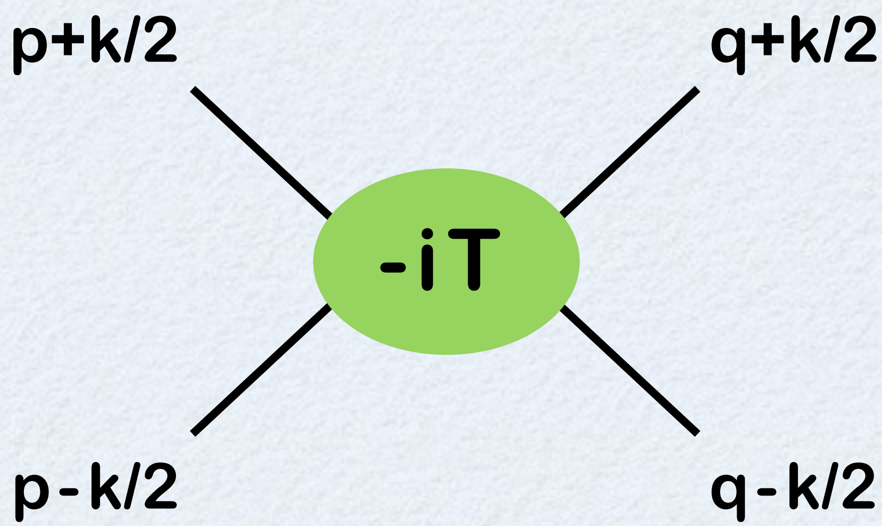
scattering “length” (a) effective “range” (Λ^2)

collision energy $\varepsilon = E - \frac{k^2}{4m} + i0^+$

P-wave scattering in 2D

Two fermions with short-range potential





⇒ Effective range expansion

Cf. H.-W. Hammer & D. Lee
Ann. Phys. 325, 2212 (2010)

$$-iT = \frac{2i}{m}\frac{\vec{p}\cdot\vec{q}}{\cancel{-\frac{1}{a}} - \frac{m\varepsilon}{\pi}\ln\left(-\frac{\Lambda^2}{m\varepsilon}\right) + \sum_{n=2}^{\infty} c_n \cancel{(m\varepsilon)^n}}$$

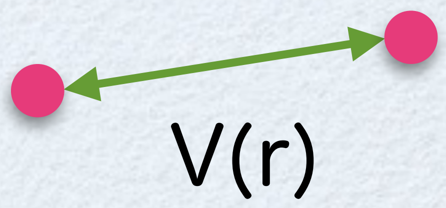
resonance ($a \to \infty$)

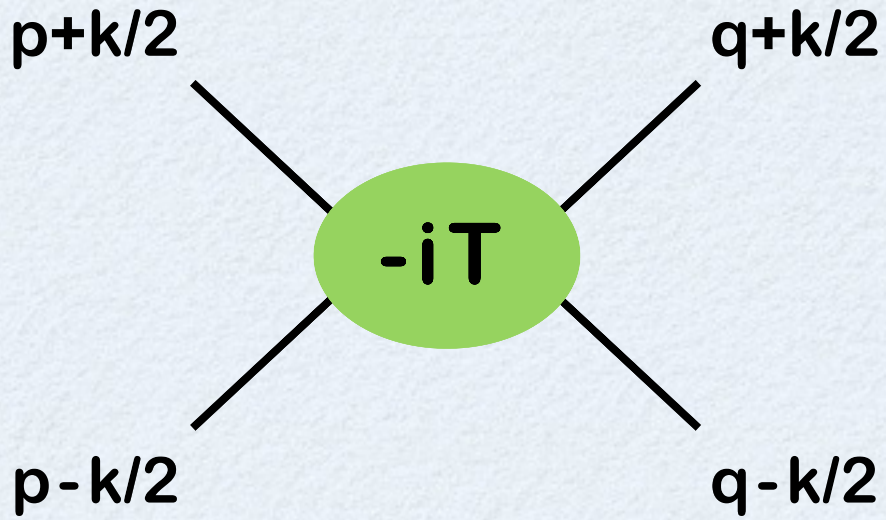
low-energy ($\varepsilon \to 0$)

collision energy $\varepsilon = E - \frac{k^2}{4m} + i0^+$

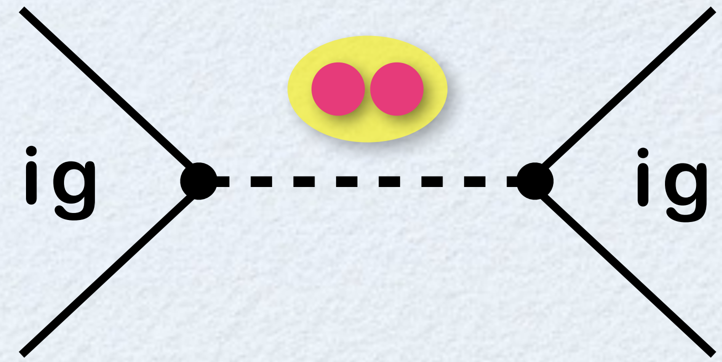
P-wave scattering in 2D

Two fermions with short-range potential





resonance
→
low-energy

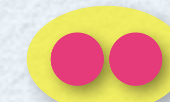


⇒ Effective range expansion

$$-iT \to -\frac{2\pi\, \vec{p}\cdot\vec{q}}{m^2 \ln\left(-\frac{\Lambda^2}{m\varepsilon}\right)} \times \frac{i}{E - \frac{k^2}{4m} + i0^+}$$

= (ig)^2 p·q

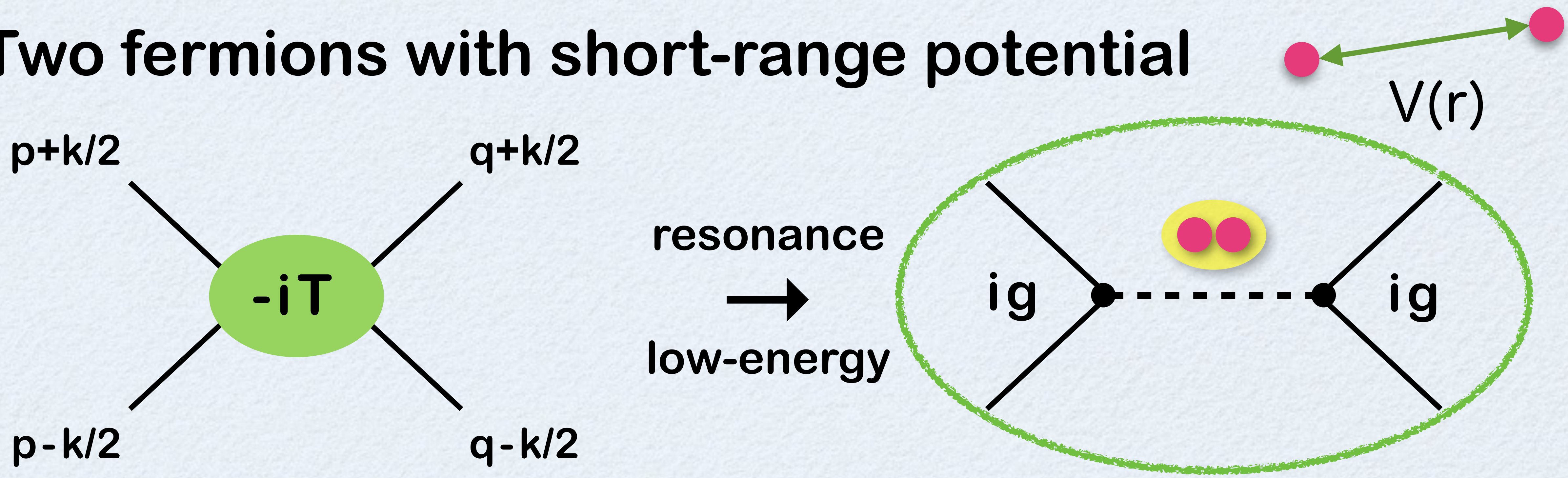
"running" coupling

propagator of dimer 

(logarithmic decrease toward low-energy p/Λ→0)

P-wave scattering in 2D

Two fermions with short-range potential



⇒ Low-energy effective field theory

$$\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} \right) \psi + \sum_\pm \Big[\phi_\pm^\dagger \left(i\partial_t + \frac{\nabla^2}{4m} \right) \phi_\pm$$

$$+ g\, \phi_\pm^\dagger \psi \,(-i)\, (\nabla_x \pm i\nabla_y)\, \psi + \mathrm{h.\,c.} \Big]$$

dimer field $\Phi_\pm$ couples to two fermions ψ

with orbital angular momentum L=±1

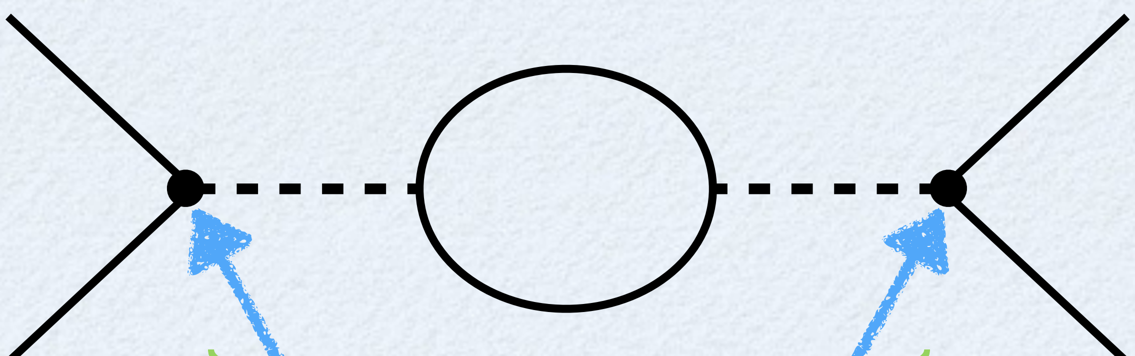
RG in 2-body sector

Low-energy effective field theory

$$\mathcal{L} = \psi^\dagger\left(i\partial_t + \frac{\boldsymbol{\nabla}^2}{2m}\right)\psi + \sum_{\pm}\Bigg[\phi_\pm^\dagger\left(i\partial_t + \frac{\boldsymbol{\nabla}^2}{4m}\right)\phi_\pm$$

$$+\, g\,\phi_\pm^\dagger\psi\,(-i)\,(\nabla_x \pm i\nabla_y)\,\psi + \text{h.c.}\Bigg] + \cdots$$

marginal coupling (g) — irrelevant ($\cdots$)



$$\frac{1 - \frac{g^2}{\pi}\ln\frac{\Lambda}{e^{-s}\Lambda}}{E - \frac{k^2}{4m} + i0^+}$$

($e^{-s}\Lambda < p < \Lambda$ integrated out)

RG equation $\quad \dfrac{dg}{ds} = -\dfrac{g^3}{2\pi}$

$$\Rightarrow \quad g^2(s) = \frac{1}{\frac{1}{g^2(0)} + \frac{s}{\pi}} \to \frac{\pi}{s}$$

logarithmical decrease

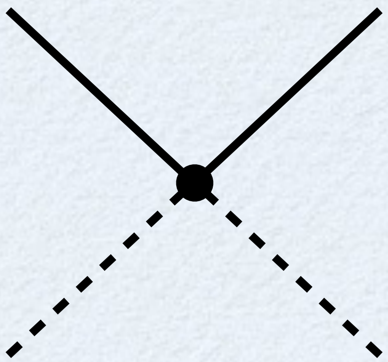
toward low-energy $s \to \infty$

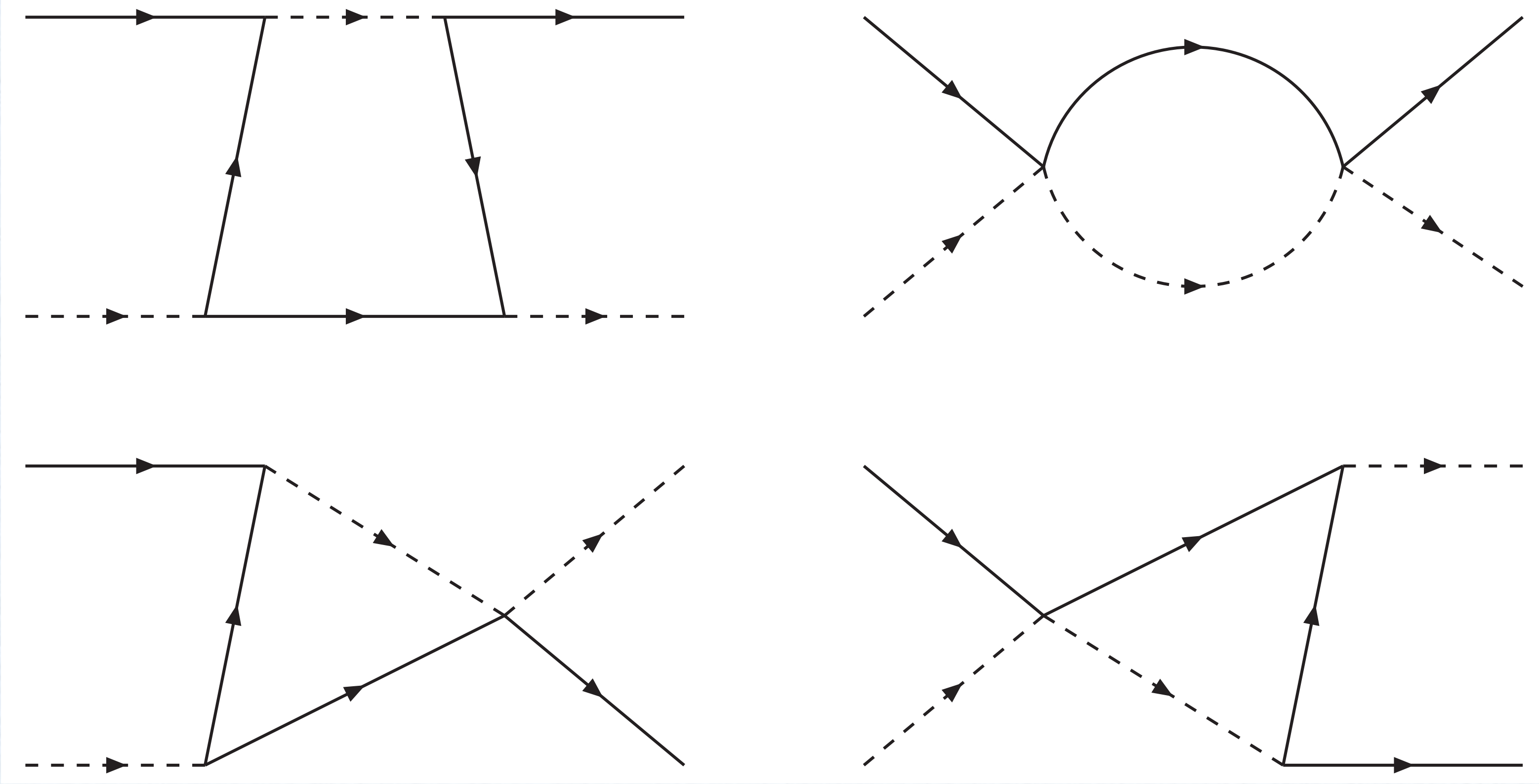
RG in 3-body sector

3-body problem $\Leftrightarrow$ fermion+dimer scattering

$$\mathcal{L}_{3\text{-body}} = v_3 \sum_{a=\pm} \psi^\dagger \phi_a^\dagger \phi_a \psi + \underbrace{\cdots}_{\text{irrelevant}}$$

marginal coupling renormalized by





⇒ RG equation $$\frac{dv_3}{ds} = \frac{16}{3\pi} g^4 - \frac{11}{3\pi} g^2 v_3 + \frac{2}{3\pi} v_3^2$$

RG in 3-body sector

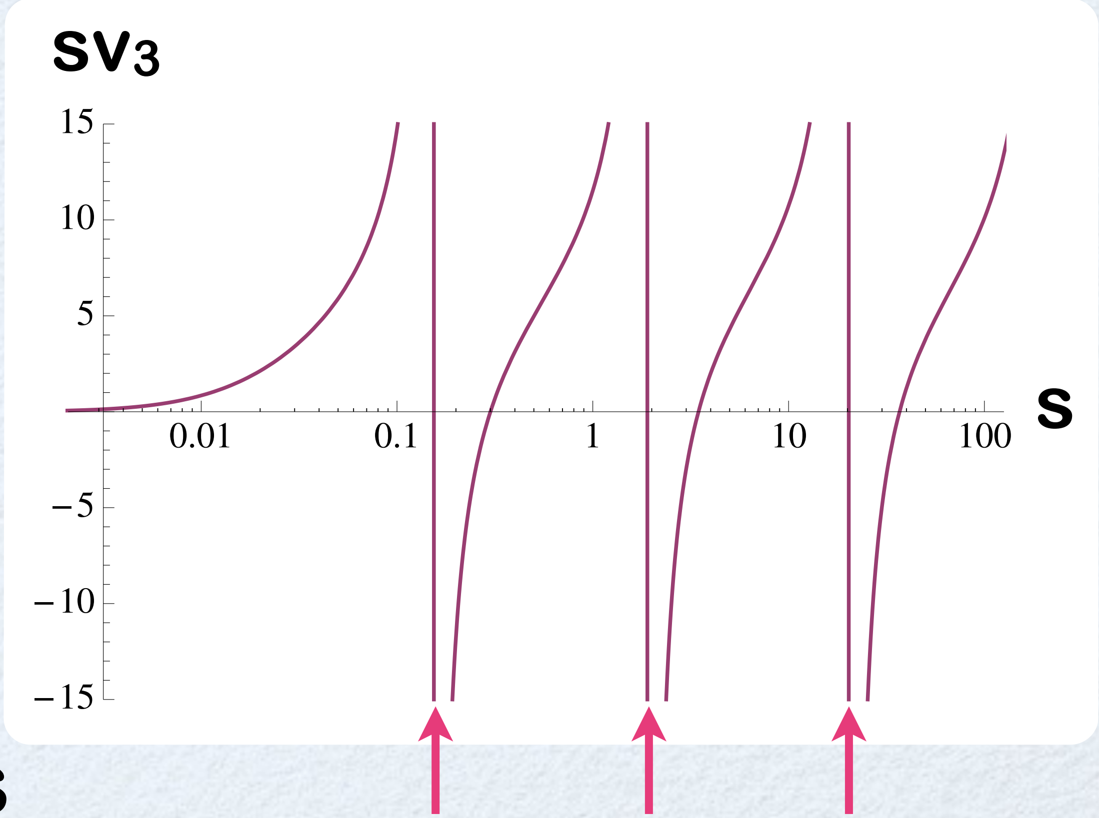
3-body problem ⇔ fermion + dimer scattering

$$\mathcal{L}_{3\text{-body}} = v_3 \sum_{a=\pm} \psi^\dagger \phi_a^\dagger \phi_a \psi + \cdots$$

$\cdots$: irrelevant

v_3 : marginal coupling @ low-energy limit s→∞

$$v_3(s) \to \frac{2\pi}{s}\left\{1 - \cot\left[\frac{4}{3}(\ln s - \theta)\right]\right\}$$



diverges at $\ln s = \frac{3\pi n}{4} + \theta$

$\ln s$: $\ln\ln \Lambda/\kappa$ θ : non-universal

=> characteristic energy scales

$$E_n \propto \frac{\Lambda^2}{m} e^{-2e^{3\pi n/4+\theta}}$$

Super Efimov effect !

Model confirmation

$$H = \int \frac{dk}{(2\pi)^2} \frac{k^2}{2m} \psi_k^\dagger \psi_k$$

$$- v_0 \sum_{a=\pm} \int \frac{dk\,dp\,dq}{(2\pi)^6} \underbrace{\psi^\dagger_{\frac{k}{2}+p} \chi_a(p) \psi^\dagger_{\frac{k}{2}-p}}_{} \times \underbrace{\psi_{\frac{k}{2}-q} \chi_{-a}(q) \psi_{\frac{k}{2}+q}}_{}$$

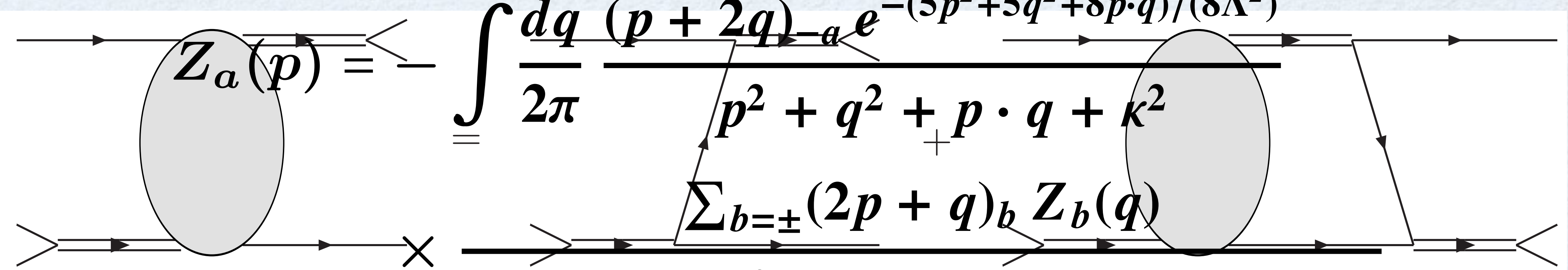
Spinless fermions with a separable potential

resonance (a→∞)

$$\chi_\pm(p) = (p_x \pm i p_y)\, e^{-p^2/(2\Lambda^2)}$$

3-body binding energies $\lambda_n = \ln\ln(mE_n/\Lambda^2)^{-1/2}$

⇒ solve STM equation numerically



$$Z_a(p) = -\int \frac{dq}{2\pi} \frac{(p+2q)_{-a}\, e^{-(5p^2+5q^2+8p\cdot q)/(8\Lambda^2)}}{p^2+q^2+p\cdot q+\kappa^2} \times \frac{\sum_{b=\pm}(2p+q)_b\, Z_b(q)}{(\frac{3}{4}q^2+\kappa^2)\, e^{(\frac{3}{4}q^2+\kappa^2)/\Lambda^2}\, \mathrm{E}_1[(\frac{3}{4}q^2+\kappa^2)/\Lambda^2]}$$

Model confirmation

$$H = \int \frac{dk}{(2\pi)^2}\frac{k^2}{2m}\psi_k^\dagger \psi_k - v_0 \sum_{a=\pm}\int \frac{dk\,dp\,dq}{(2\pi)^6}\,\psi^\dagger_{\frac{k}{2}+p}\chi_a(p)\psi^\dagger_{\frac{k}{2}-p} \times \psi_{\frac{k}{2}-q}\chi_{-a}(q)\psi_{\frac{k}{2}+q}$$

Spinless fermions with a separable potential

resonance (a→∞) ← v_0

$$\chi_\pm(p) = (p_x \pm i p_y)\, e^{-p^2/(2\Lambda^2)}$$

3-body binding energies $\lambda_n = \ln\ln(mE_n/\Lambda^2)^{-1/2}$

n	λ_n	$\lambda_n - \lambda_{n-1}$
0	0.5632	—
1	2.770	2.207
2	5.078	2.308
3	7.430	2.352
4	9.785	2.355
5	12.141	2.356
∞	—	2.35619 ← $3\pi/4$

Super Efimov effect !

⇒ **doubly exponential scaling** $mE_n/\Lambda^2 \propto e^{-2e^{3\pi n/4+\theta}}$

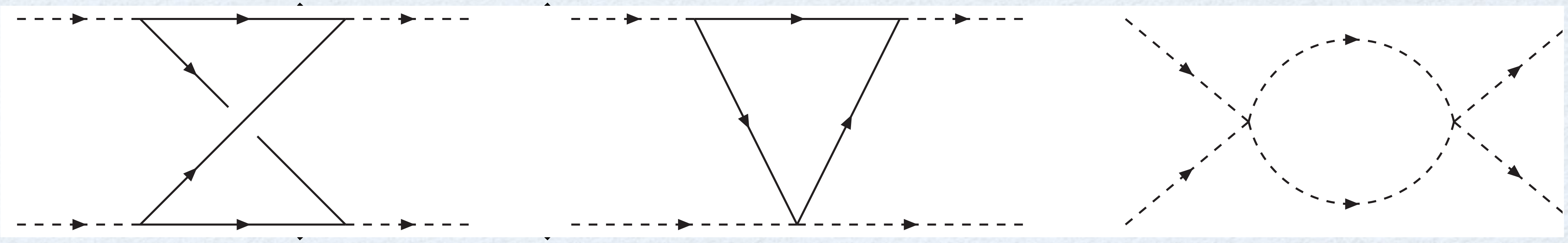
RG in 4-body sector

4-body problem ⇔ dimer+dimer scattering

$$\mathcal{L}_{4\text{-body}} = \sum_{a=\pm}\left[v_4\phi_a^\dagger\phi_{-a}^\dagger\phi_{-a}\phi_a + v_4'\phi_a^\dagger\phi_a^\dagger\phi_a\phi_a\right] + \cdots$$

irrelevant

marginal couplings renormalized by



⇒ RG equations

$$\frac{dv_4}{ds} = -\frac{8}{\pi}g^4 + \frac{2}{\pi}g^2v_3 - \frac{2}{\pi}g^2v_4 + \frac{2}{\pi}v_4^2$$

$$\frac{dv_4'}{ds} = -\frac{4}{\pi}g^4 + \frac{2}{\pi}g^2v_3 - \frac{2}{\pi}g^2v_4' + \frac{2}{\pi}v_4'^2$$

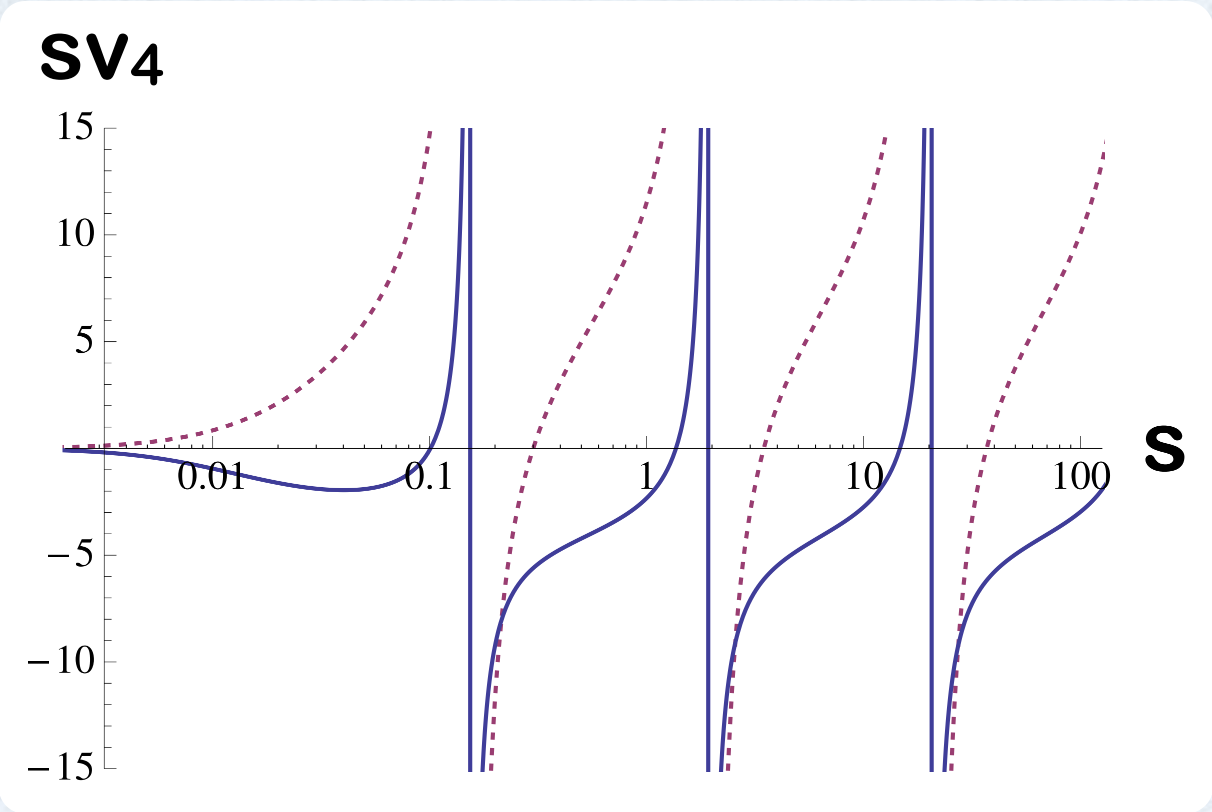
RG in 4-body sector

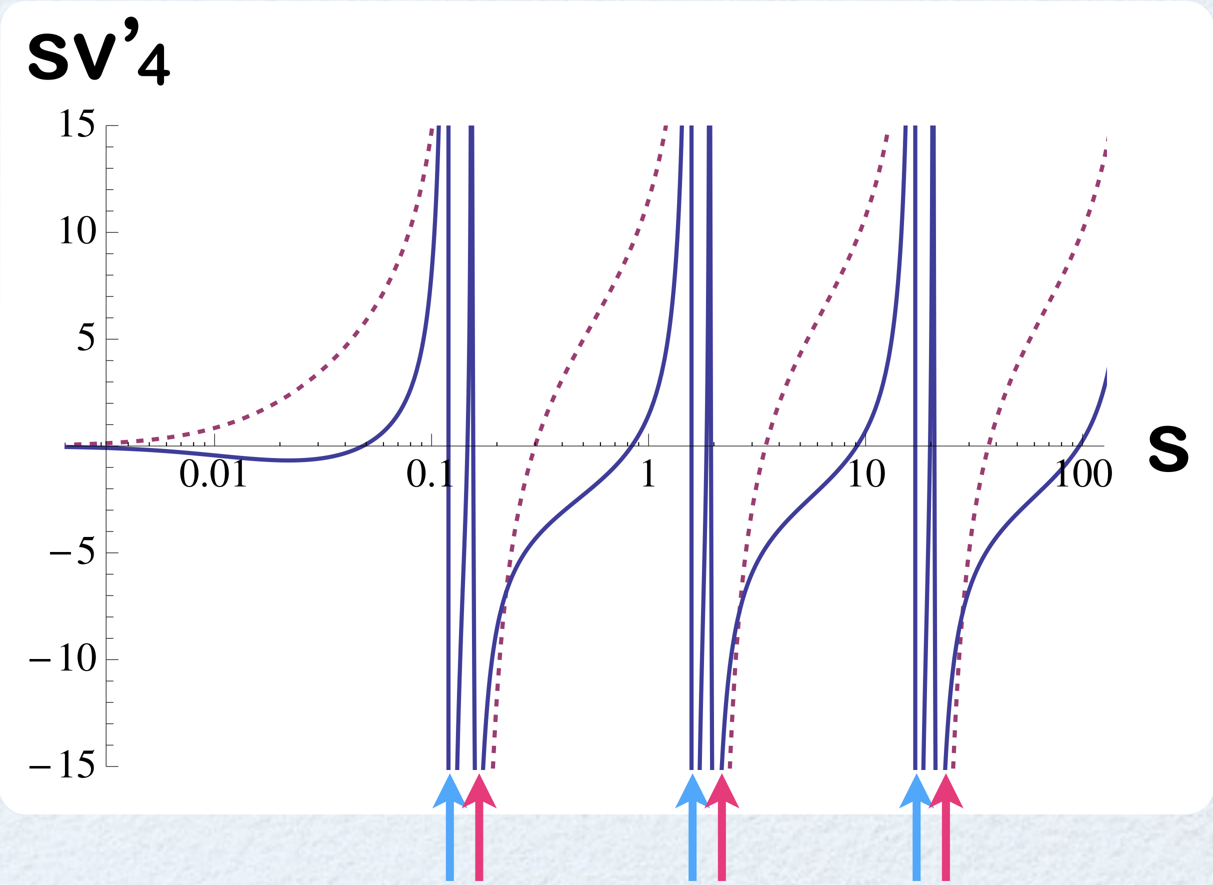
4-body problem ⇔ dimer+dimer scattering

$$\mathcal{L}_{4\text{-body}} = \sum_{a=\pm}\left[v_4 \phi_a^\dagger \phi_{-a}^\dagger \phi_{-a}\phi_a + v_4' \phi_a^\dagger \phi_a^\dagger \phi_a \phi_a\right] + \cdots$$

marginal couplings

irrelevant



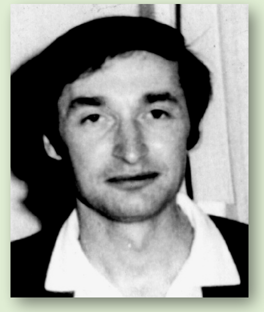


L=±2 tetramers attached to every trimer

with resonance energy $E_n \sim e^{-2e^{3\pi n/4+\theta-0.188}}$

Efimov vs super Efimov

Efimov effect



- 3 bosons
- 3 dimensions
- s-wave resonance

exponential scaling

$$E_n \sim e^{-2\pi n}$$

Super Efimov effect

New !

- 3 fermions
- 2 dimensions
- p-wave resonance

"doubly" exponential

$$E_n \sim e^{-2e^{3\pi n/4}}$$

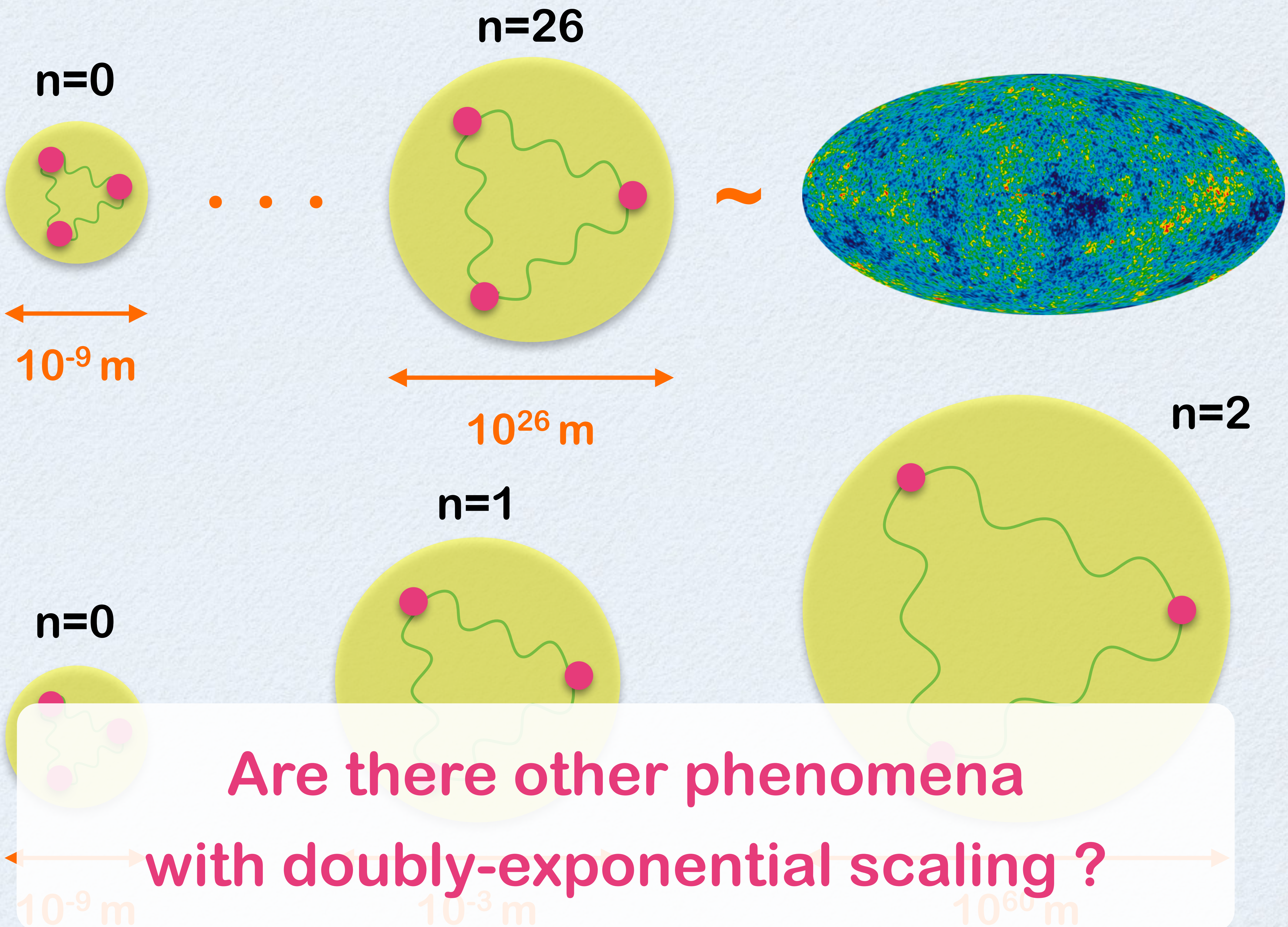
PRL 110, 235301 (2013) PHYSICAL REVIEW LETTERS week ending 7 JUNE 2013

Super Efimov Effect of Resonantly Interacting Fermions in Two Dimensions

Yusuke Nishida,[1] Sergej Moroz,[2] and Dam Thanh Son[3]

[1]*Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*
[2]*Department of Physics, University of Washington, Seattle, Washington 98195, USA*
[3]*Enrico Fermi Institute, University of Chicago, Chicago, Illinois 60637, USA*
(Received 18 January 2013; published 4 June 2013)

Efimov vs super Efimov



Are there other phenomena
with doubly-exponential scaling ?

Efimov vs super Efimov

WIKIPEDIA
The Free Encyclopedia

- Main page
- Contents
- Featured content
- Current events
- Random article
- Donate to Wikipedia
- Wikimedia Shop

Interaction
Help
Abou
Com
Rece
Cont

Tools
Print/
Langu
Бела
Бела

Article | Talk

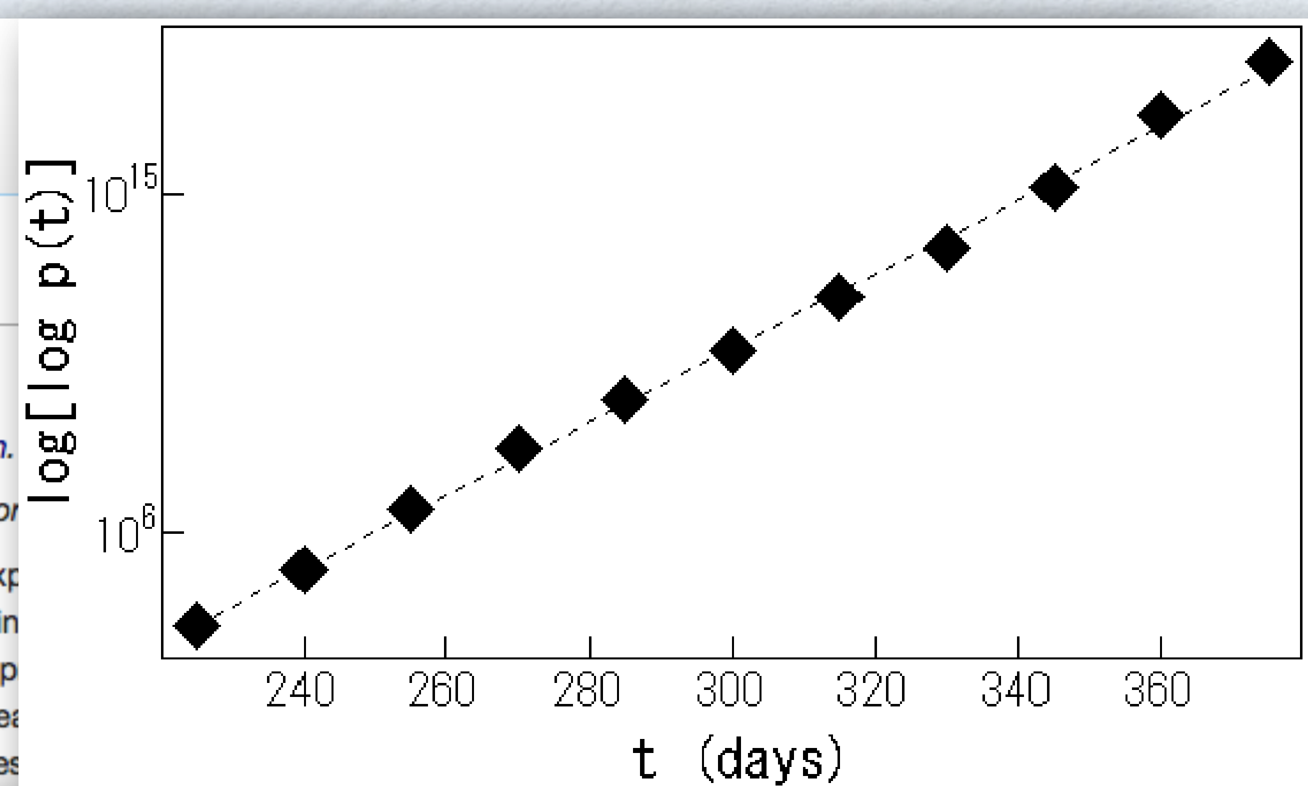
Hyperinflation

From Wikipedia, the free encyclopedia

For lungs filling with excessive air, see Hyperaeration.

Certain figures in this article use scientific notation for

In economics, **hyperinflation** occurs when a country exp
accelerating rates of monetary and price inflation, causin
holdings of money. Under such conditions, the general p
increases rapidly as the official currency quickly loses rea
value of economic items generally stay the same with res



Cornell University Library

We gratefully acknowledge support from the Simons Foundation and member institutions

arXiv.org > cond-mat > arXiv:cond-mat/0112441

Search or Article-id (Help | Advanced search) All papers Go!

Condensed Matter > Statistical Mechanics

The mechanism of double exponential growth in hyper-inflation

Takayuki Mizuno, Misako Takayasu, Hideki Takayasu

(Submitted on 24 Dec 2001)

Ana... order to explain such behavior we introduce the general coarse-graining technique in physics, the Monte Carlo renormalization group method, to the price dynamics. Starting from a microscopic stochastic equation ... is shown to be responsible for the double-exponential behavior.

Download:
- PDF
- PostScript
- Other formats

Current browse context:
cond-mat

References & Citations
- NASA ADS

Bookmark (what is this?)

Are there other "physics" phenomena with doubly-exponential scaling ?

Extension:
Mass imbalance mixtures

Efimov vs super Efimov

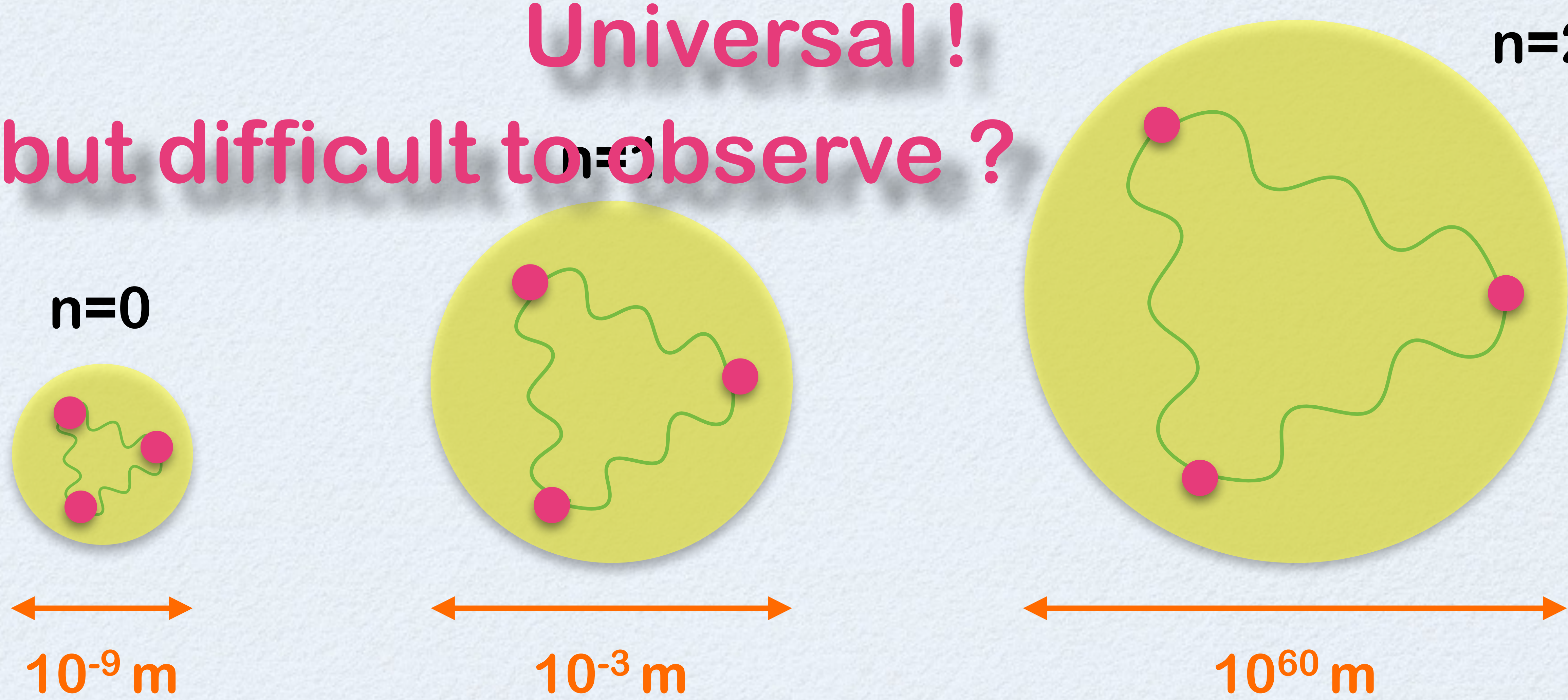
Super Efimov effect

- 3 fermions
- 2 dimensions
- p-wave resonance

Infinite bound states with doubly exponential scaling $E_n \sim e^{-2e^{3\pi n/4}}$

Universal !

but difficult to observe ?



Efimov vs super Efimov

Efimov effect

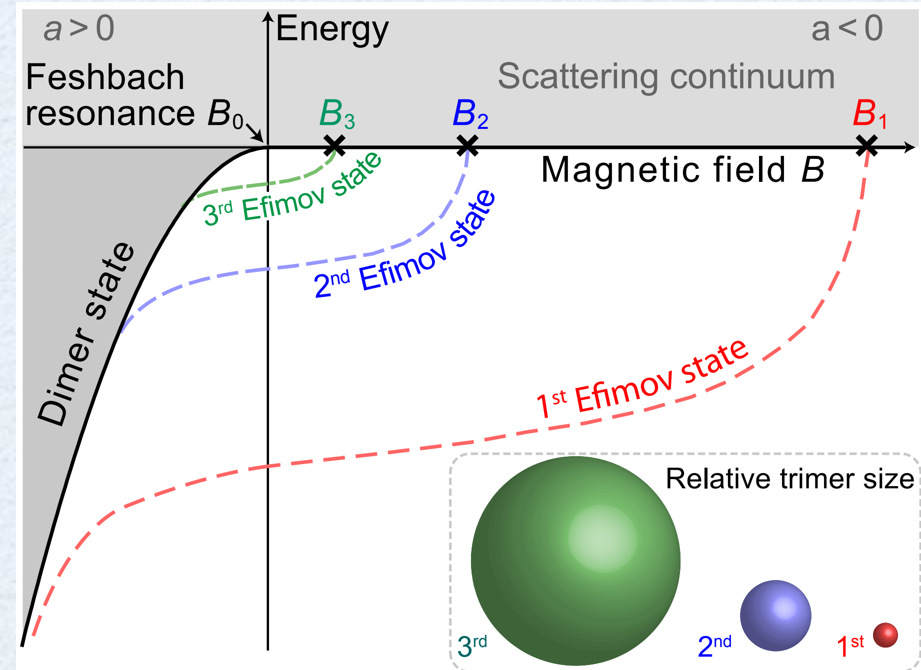
- 3 identical bosons
- 3 dimensions
- s-wave resonance

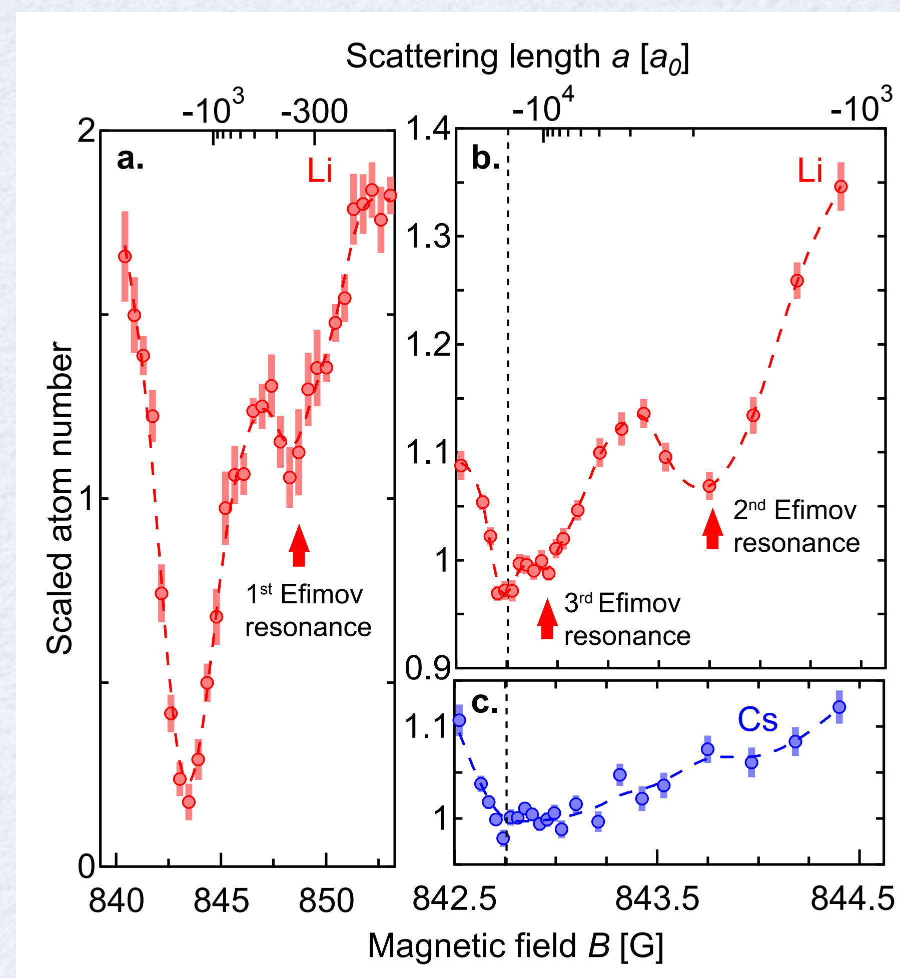
exponential scaling

$$\frac{E_{n+1}}{E_n} \to e^{-2\pi} \approx (22.7)^{-2}$$

$$(4.88)^{-2}$$

for ^{6}Li-^{133}Cs mixture





S.-K. Tung et al, arXiv:1402.5943; R. Pires et al, arXiv:1403.7246

Efimov vs super Efimov

Efimov effect

- 3 identical bosons
- 3 dimensions
- s-wave resonance

exponential scaling

$$\frac{E_{n+1}}{E_n} \to e^{-2\pi} \approx (22.7)^{-2}$$

$$(4.88)^{-2}$$

for ^{6}Li-^{133}Cs mixture

Super Efimov effect

- 3 identical fermions
- 2 dimensions
- p-wave resonance

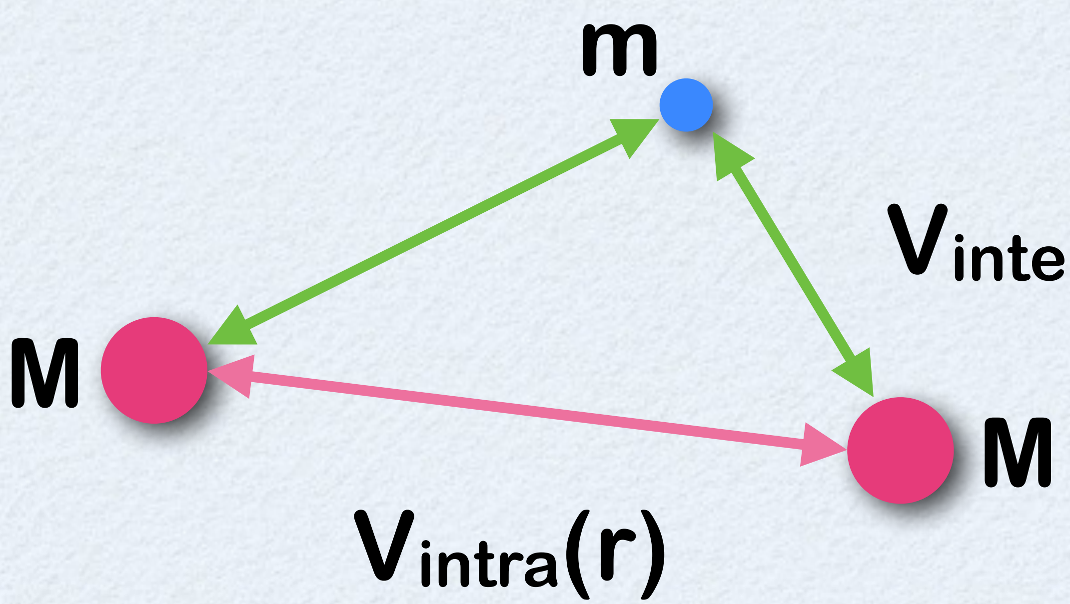
"doubly" exponential

$$\frac{\ln E_{n+1}}{\ln E_n} \to e^{3\pi/4} \approx 10.55$$

???

for ^{6}Li-^{133}Cs mixture

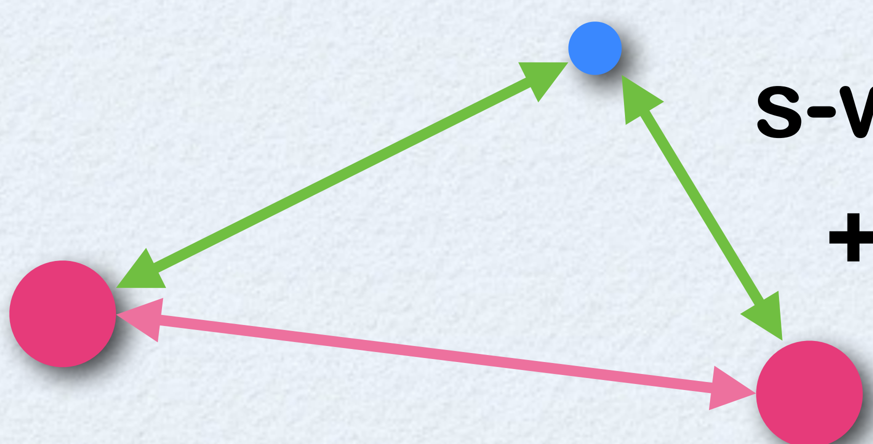
Mass imbalance mixtures



$V_{inter}(r)$ @ p-wave resonance

Low-energy limit

(higher partial waves die out)



s-wave interaction ~ 1/log(k)

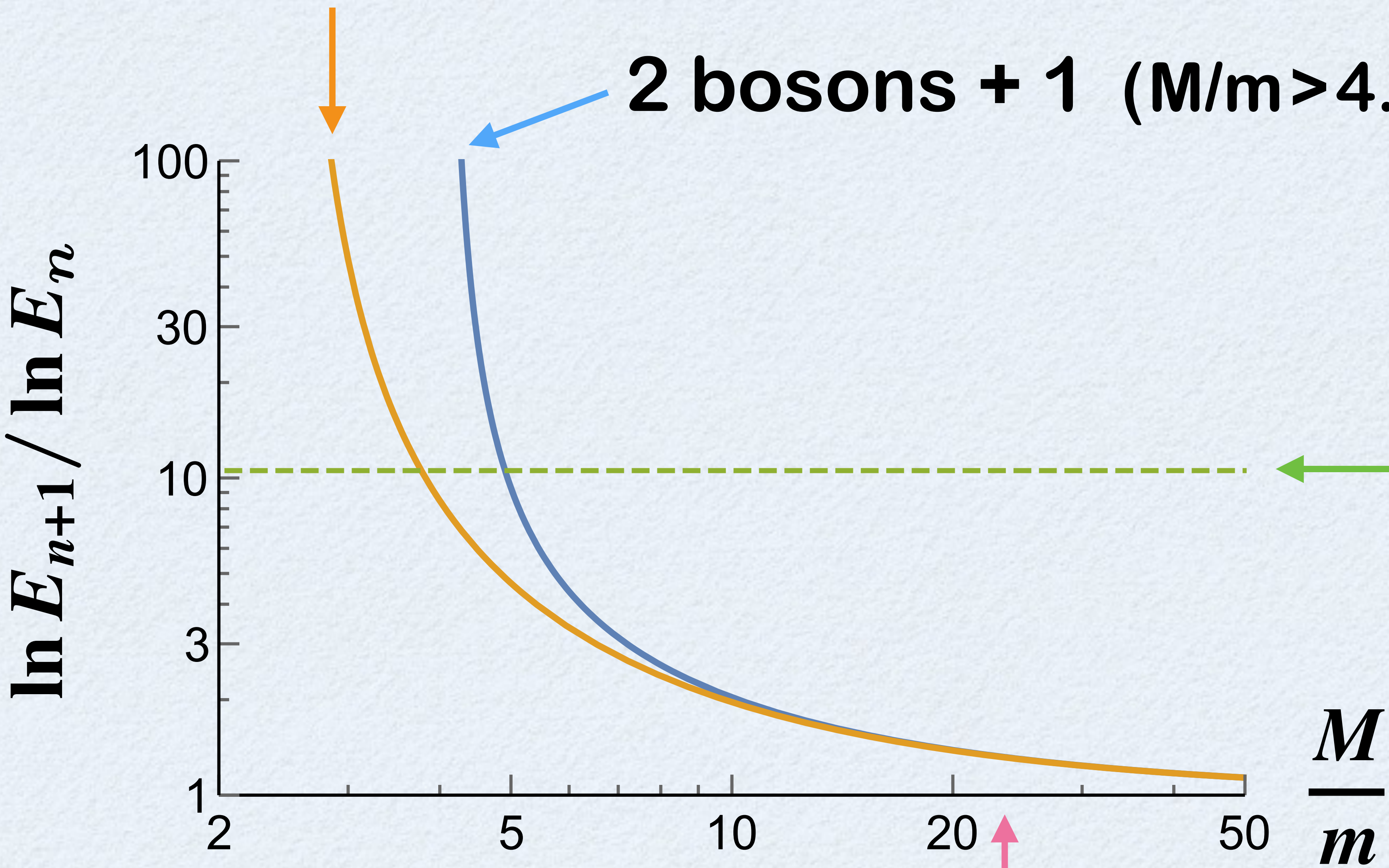
+ p-wave resonance ~ 1/log(k)

s-wave interaction (only for bosons)

Mass imbalance mixtures

2 fermions + 1 (M/m>2.41)

2 bosons + 1 (M/m>4.03)



$e^{3\pi/4} \approx 10.55$

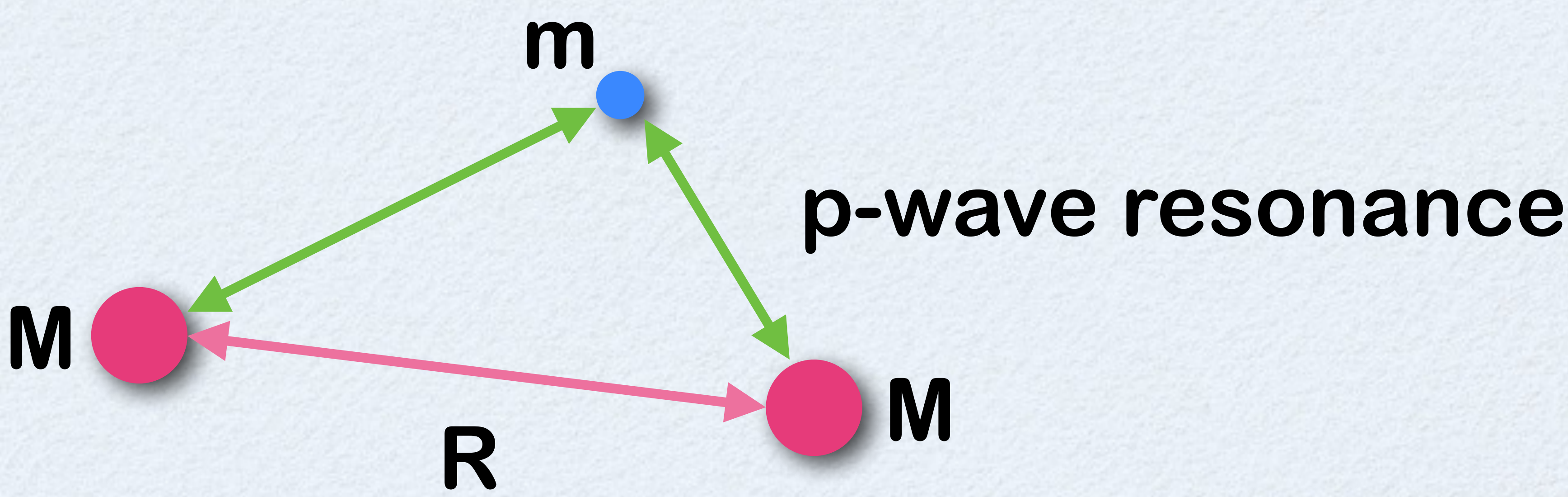
for 3 identical fermions

$$\frac{\ln E_{n+1}}{\ln E_n} \approx 1.34 \text{ !}$$ for ^{6}Li-^{133}Cs mixture

- p-wave resonance observed but 2D confinement necessary

M. Repp et al, Phys. Rev. A 87, 010701(R) (2013)

Born-Oppenheimer approx.



Effective potential induced by light particle

$$V_{\text{eff}}(R) \to -\frac{1}{mR^2 \ln(R/r_0)}\bigg|_{R \gg r_0}$$

$$\Rightarrow \quad E_n \sim e^{-\frac{m\pi^2}{2M} n^2}$$

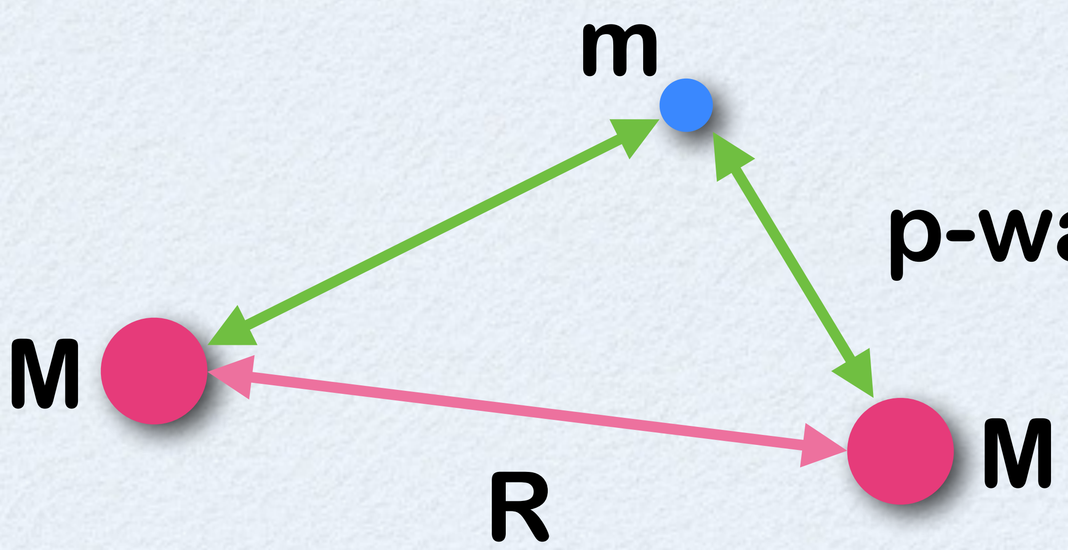
C. Gao & Z. Yu, arXiv:1401.0965

M. A. Efremov & W. P. Schleich, arXiv:1407.3352

inconsistent with our prediction for large M/m

$$E_n \sim e^{-2e^{(2m/M)\pi n}}$$

Born-Oppenheimer approx.



p-wave resonance

Characteristic time scales

$$T_{\mathrm{heavy}} \sim MR^2 \quad \gg \quad T_{\mathrm{light}} \sim mR^2 \ln(R/r_0)$$

adiabatic condition holds only for

$$R/r_0 \ll e^{M/m} \quad \Rightarrow \quad E/E_0 \gg e^{-2M/m}$$

Crossover from Born-Oppenheimer $E_n \sim e^{-\frac{m\pi^2}{2M}n^2}$

to super Efimov $E_n \sim e^{-2e^{(2m/M)\pi n}}$ at $E \sim e^{-2M/m}$???

Summary

Super Efimov effect

- 3 fermions
- 2 dimensions
- p-wave resonance

⇒ Infinite bound states with doubly exponential scaling $E_n \sim e^{-2e^{3\pi n/4}}$

1. New few-body universality
2. RG analysis & Model confirmation
3. First doubly exponential scaling (?)
4. Easier to observe with mass imbalance
5. Born-Oppenheimer approximation fails