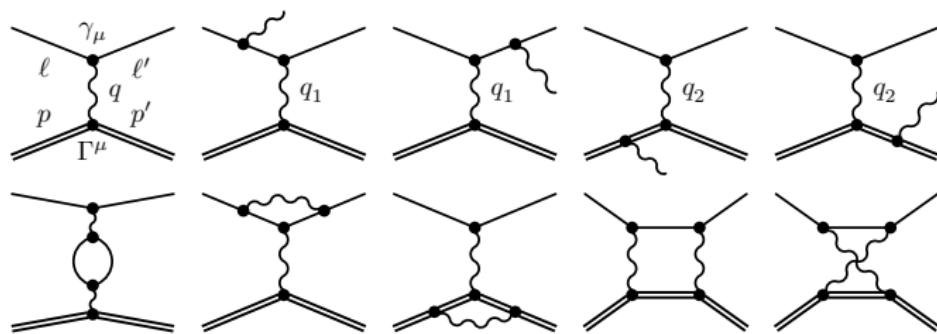


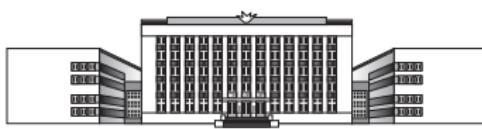
ESEPP: an event generator for elastic scattering of electrons and positrons on protons

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ESEPP event generator: motivation

ESEPP (which stands for **E**lastic **S**cattering of **E**lectrons and **P**ositrons on **P**rotons) is a new multi-purpose event generator developed for Monte Carlo simulation of unpolarized elastic scattering of charged leptons (e^- , e^+ , μ^- , μ^+) on protons. The generator takes into account the lowest-order QED radiative corrections to the Rosenbluth cross section including the first-order bremsstrahlung process beyond the soft-photon and ultrarelativistic approximations.

The generator can be useful for several current and planned experiments:

- Two-photon exchange experiments (VEPP–3, OLYMPUS, CLAS)
Main processes: $e^- p \rightarrow e^- p$, $e^+ p \rightarrow e^+ p$
- The proton charge radius experiment PRad at JLab
Main process: $e^- p \rightarrow e^- p$ with $-q^2$ from 10^{-4} to 10^{-2} GeV 2
- The muon-proton scattering experiment MUSE at PSI
Main processes: $\mu^\pm p \rightarrow \mu^\pm p$, $e^\pm p \rightarrow e^\pm p$ with $p_{\text{beam}} = 115, 153$, and 210 MeV
- Search for a heavy photon at VEPP–3 ($e^+ e^- \rightarrow \gamma A'$) arXiv:1207.5089
Background process: $e^+ p \rightarrow e^+ p \gamma$
- The DarkLight experiment at JLab ($e^- p \rightarrow e^- p A'$)
Background process: $e^- p \rightarrow e^- p \gamma$

ESEPP event generator: advantages and disadvantages

The main advantages of ESEPP:

- ✓ Four types of incident particles are possible: e^- , e^+ , μ^- , μ^+ ;
- ✓ All the kinematic parameters of the final particles are known \Rightarrow ESEPP is a general-purpose generator;
- ✓ We use an accurate calculation for the first-order bremsstrahlung instead of the usual soft-photon approximation;
- ✓ Not only the lepton bremsstrahlung is considered, but also the proton bremsstrahlung and the interference term;
- ✓ We do not use the ultrarelativistic approximation $m \ll Q^2$ (except for the lepton vertex correction); it should be important for experiments with muons (PSI) and for measurements with extremely small values of Q^2 (JLab).

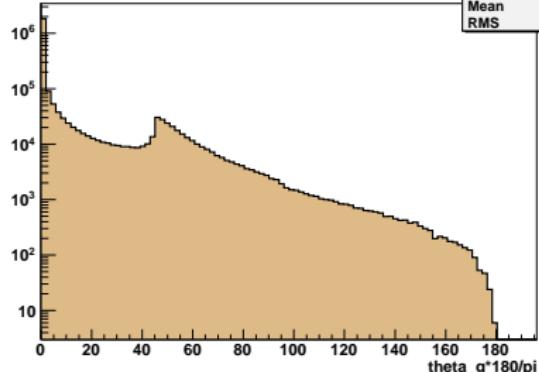
The main disadvantage:

- ✓ Only the first-order bremsstrahlung is taken into account.

The source code of the generator is freely available (under the GNU GPL license) on the page <http://www.inp.nsk.su/~gramolin/esepp/>.

Some examples of events generated

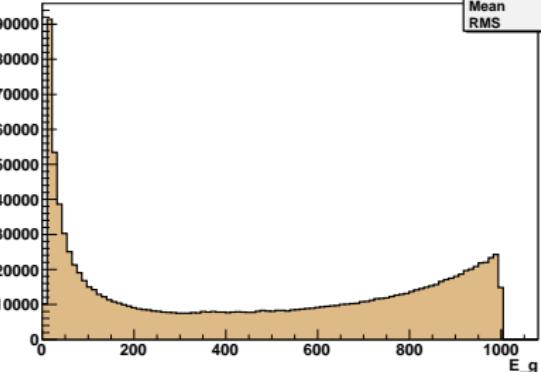
theta_g*180/pi



htemp

Entries	2518354
Mean	9.733
RMS	21.95

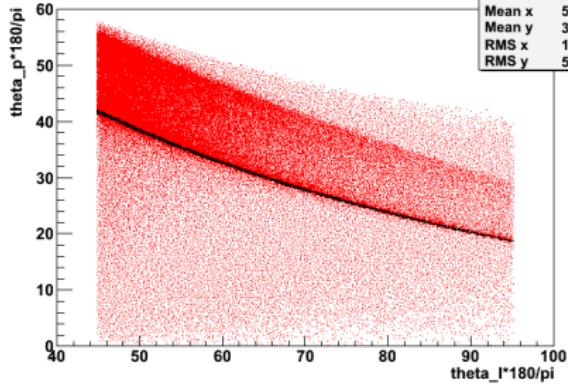
E_g {E_g > 0}



htemp

Entries	1259826
Mean	478
RMS	349.3

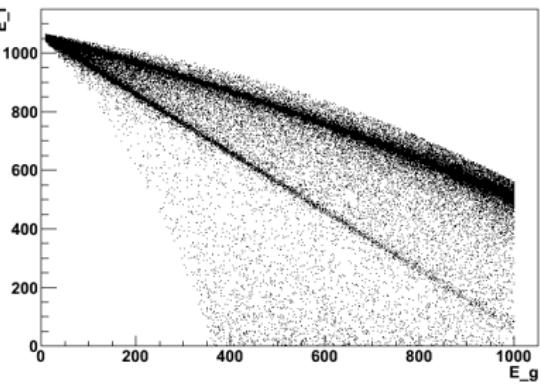
theta_p*180/pi



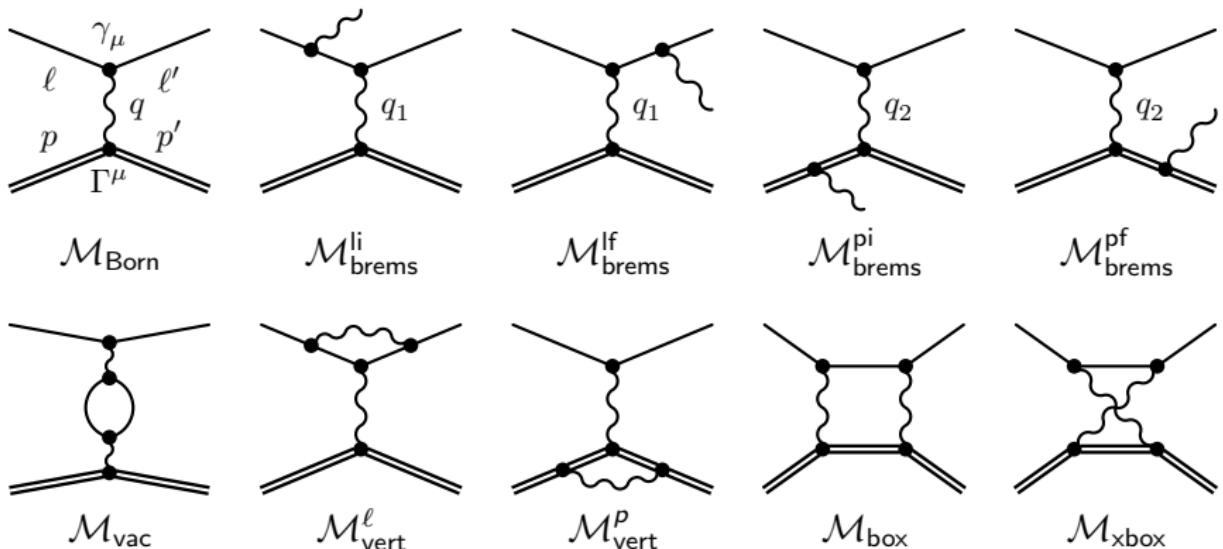
h2

Entries	1258528
Mean x	55.76
Mean y	35.49
RMS x	10.66
RMS y	5.417

E_I:E_g {theta_l*180/pi < 46}



The first-order QED radiative corrections



$$\begin{aligned} \sigma(\ell^\pm p) &\propto |\mathcal{M}_{\text{Born}}|^2 + 2 \operatorname{Re} [\mathcal{M}_{\text{Born}}^\dagger (\mathcal{M}_{\text{vac}} + \mathcal{M}_{\text{vert}}^\ell + \mathcal{M}_{\text{vert}}^p)] \\ &\mp 2 \operatorname{Re} [\mathcal{M}_{\text{Born}}^\dagger (\mathcal{M}_{\text{box}} + \mathcal{M}_{\text{xbox}})] + |\mathcal{M}_{\text{brems}}^{\text{li}} + \mathcal{M}_{\text{brems}}^{\text{lf}}|^2 + |\mathcal{M}_{\text{brems}}^{\text{pi}} + \mathcal{M}_{\text{brems}}^{\text{pf}}|^2 \\ &\mp 2 \operatorname{Re} [(\mathcal{M}_{\text{brems}}^{\text{li}} + \mathcal{M}_{\text{brems}}^{\text{lf}})^\dagger (\mathcal{M}_{\text{brems}}^{\text{pi}} + \mathcal{M}_{\text{brems}}^{\text{pf}})] + \mathcal{O}(\alpha^4) \end{aligned}$$

Elastic scattering of charged leptons on protons

- In the one-photon exchange approximation and assuming $-q^2 \gg m^2$:

$$\boxed{\frac{d\sigma_{\text{Born}}}{d\Omega_\ell} = \frac{1}{\varepsilon(1+\tau)} [\varepsilon G_E^2(q^2) + \tau G_M^2(q^2)] \frac{d\sigma_{\text{Mott}}}{d\Omega_\ell}},$$

where $\tau = -q^2/4M^2$, $\varepsilon = [1 + 2(1+\tau)\tan^2(\theta_\ell/2)]^{-1}$ ($0 < \varepsilon < 1$), and the Mott cross section is

$$\frac{d\sigma_{\text{Mott}}}{d\Omega_\ell} = \frac{\alpha^2}{4E_\ell^2} \frac{\cos^2(\theta_\ell/2)}{\sin^4(\theta_\ell/2)} \frac{E'_\ell}{E_\ell}.$$

- In the general case (for any values of $-q^2$):

$$\tilde{\varepsilon} = \left[1 - \left(2\tau - \frac{m^2}{M^2} \right) \frac{1+\tau}{\tau} \frac{q^2/(4E_\ell E'_\ell)}{1+q^2/(4E_\ell E'_\ell)} \right]^{-1} = \left[1 - (1+\tau) \frac{2q^2 + 4m^2}{4E_\ell E'_\ell + q^2} \right]^{-1} \\ (0 < \tilde{\varepsilon} < \infty),$$

$$\frac{d\sigma_{\text{Mott}}}{d\Omega_\ell} = \frac{\alpha^2}{4E_\ell^2} \frac{1+q^2/(4E_\ell E'_\ell)}{q^4/(4E_\ell E'_\ell)^2} \frac{1}{d} \frac{M(E'_\ell{}^2 - m^2)}{ME_\ell E'_\ell + m^2(E'_\ell - E_\ell - M)},$$

$$q^2 = 2M(E'_\ell - E_\ell), \quad d = \frac{E'_\ell}{E_\ell} \sqrt{\frac{E_\ell^2 - m^2}{E'_\ell{}^2 - m^2}}.$$

Different parametrizations for the proton form factors

- The dipole formula:

$$G_E(q^2) = \left(1 - \frac{q^2}{\Lambda^2}\right)^{-2}, \quad G_M(q^2) = \mu G_E(q^2), \quad \Lambda^2 = 0.71 \text{ GeV}^2.$$

- Kelly parametrization (PRC **70** (2004) 068202):

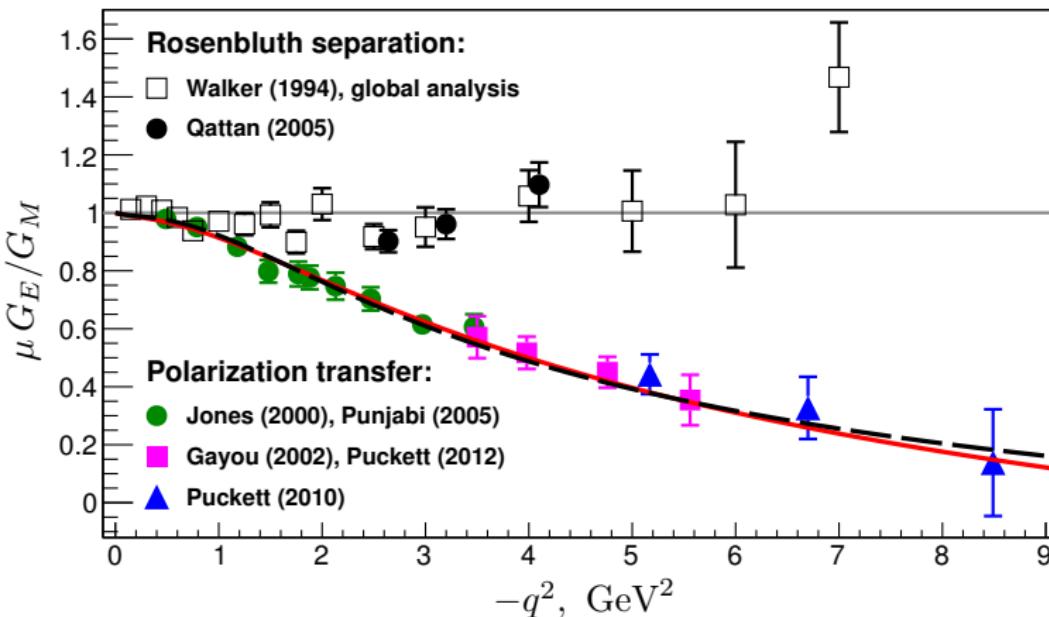
$$G(q^2) = \frac{1 + a_1\tau}{1 + b_1\tau + b_2\tau^2 + b_3\tau^3}.$$

Parametri-zation	G_E				G_M/μ			
	a_1	b_1	b_2	b_3	a_1	b_1	b_2	b_3
Dipole	0	9.92	24.6	0	0	9.92	24.6	0
Kelly	-0.24	10.98	12.82	21.97	0.12	10.97	18.86	6.55
Puckett	-0.299	11.11	14.11	15.7	0.081	11.15	18.45	5.31

- Dirac F_1 and Pauli F_2 form factors are linear combinations of G_E and G_M :

$$F_1(q^2) = \frac{G_E(q^2) + \tau G_M(q^2)}{1 + \tau}, \quad F_2(q^2) = \frac{G_M(q^2) - G_E(q^2)}{1 + \tau}.$$

Different parametrizations for the proton form factors



The curves correspond to the parametrizations of the proton form factors by Kelly (black dashed line) and Puckett (red solid line).

- J. J. Kelly, Phys. Rev. C **70** (2004) 068202
- A. J. R. Puckett, arXiv:1008.0855

Kinematics of the process $\ell^\pm p \rightarrow \ell^\pm p \gamma$

- Four-momenta of all particles in the laboratory frame:

$$\ell = (E_\ell, \ell) = (E_\ell, 0, 0, |\ell|),$$

$$p = (M, \mathbf{p}) = (M, 0, 0, 0),$$

$$\ell' = (E'_\ell, \ell') = (E'_\ell, |\ell'| \sin \theta_\ell, 0, |\ell'| \cos \theta_\ell),$$

$$p' = (E_p, \mathbf{p}') = (E_p, |\mathbf{p}'| \sin \theta_p \cos \phi_p, |\mathbf{p}'| \sin \theta_p \sin \phi_p, |\mathbf{p}'| \cos \theta_p),$$

$$k = (E_\gamma, \mathbf{k}) = (E_\gamma, E_\gamma \sin \theta_\gamma \cos \phi_\gamma, E_\gamma \sin \theta_\gamma \sin \phi_\gamma, E_\gamma \cos \theta_\gamma).$$

- Our 4 basic kinematic variables: θ_ℓ , E_γ , θ_γ and ϕ_γ (the energy E_ℓ of the incident lepton is assumed to be known). Other kinematic variables can be calculated from them.
- Energy E'_ℓ of the scattered lepton can be found from the equation

$$\sqrt{E'^2_\ell - m^2} A = E'_\ell B + C,$$

where the coefficients A , B , and C are known:

$$A = \sqrt{E_\ell^2 - m^2} \cos \theta_\ell - E_\gamma \cos \psi, \quad B = E_\ell + M - E_\gamma,$$

$$C = E_\gamma \left(E_\ell + M - \sqrt{E_\ell^2 - m^2} \cos \theta_\gamma \right) - M E_\ell - m^2.$$

$$\cos \psi = \cos \theta_\ell \cos \theta_\gamma + \sin \theta_\ell \sin \theta_\gamma \cos \phi_\gamma.$$

- E'_ℓ is known $\Rightarrow \ell$, p , ℓ' , and k are known $\Rightarrow p' = \ell + p - \ell' - k$.

$\ell^\pm p \rightarrow \ell^\pm p \gamma$: the soft-photon approximation

- The differential cross section ($z = -1$ for e^-/μ^- and $z = +1$ for e^+/μ^+):

$$\frac{d\sigma_{\text{brems}}}{dE_\gamma d\Omega_\gamma d\Omega_\ell} = -\frac{\alpha E_\gamma}{4\pi^2} \left[z \frac{\ell}{k \cdot \ell} - z \frac{\ell'}{k \cdot \ell'} + \frac{p}{k \cdot p} - \frac{p'}{k \cdot p'} \right]^2 \frac{d\sigma_{\text{Born}}}{d\Omega_\ell}.$$

- The cross section integrated over all photon directions and energies $E_\gamma < E_\gamma^{\text{cut}}$:

$$\begin{aligned} \frac{d\sigma_{\text{brems}}}{d\Omega_\ell} \Big|_{E_\gamma < E_\gamma^{\text{cut}}} &= \frac{-\alpha}{4\pi^2} \frac{d\sigma_{\text{Born}}}{d\Omega_\ell} \int_{E_\gamma < E_\gamma^{\text{cut}}} \frac{d^3 k}{E_\gamma} \left[z \frac{\ell}{k \cdot \ell} - z \frac{\ell'}{k \cdot \ell'} + \frac{p}{k \cdot p} - \frac{p'}{k \cdot p'} \right]^2 \\ &= -2\alpha \frac{d\sigma_{\text{Born}}}{d\Omega_\ell} \sum_{i,j} \Theta(p_i) \Theta(p_j) B(p_i, p_j, E_\gamma^{\text{cut}}), \end{aligned}$$

where

$$\begin{aligned} B(p_i, p_j, E_\gamma^{\text{cut}}) &= \frac{1}{8\pi^2} \int_{E_\gamma < E_\gamma^{\text{cut}}} \frac{d^3 k}{\sqrt{|k|^2 + \lambda^2}} \frac{p_i \cdot p_j}{(k \cdot p_i)(k \cdot p_j)} \\ &= \frac{p_i \cdot p_j}{4\pi} \int_0^1 \frac{dx}{p_x^2} \left(\ln \frac{4(E_\gamma^{\text{cut}})^2}{p_x^2} + \frac{p_x^0}{|\mathbf{p}_x|} \ln \frac{p_x^0 - |\mathbf{p}_x|}{p_x^0 + |\mathbf{p}_x|} + \ln \frac{p_x^2}{\lambda^2} \right), \end{aligned}$$

$$p_x = (p_x^0, \mathbf{p}_x) = x p_i + (1-x) p_j,$$

$\Theta(\ell) = z$, $\Theta(\ell') = -z$, $\Theta(p) = 1$, $\Theta(p') = -1$, and λ is a fictitious mass of the photon.

$\ell^\pm p \rightarrow \ell^\pm p \gamma$: an accurate QED calculation

The differential cross section of the process $\ell^\pm p \rightarrow \ell^\pm p \gamma$ is expressed in terms of the square of the amplitude $|\mathcal{M}_{\text{brems}}|^2$ as follows:

$$\frac{d\sigma_{\text{brems}}}{dE_\gamma d\Omega_\gamma d\Omega_\ell} = \frac{1}{(2\pi)^5} \frac{1}{32I} \times \frac{E_\gamma (E'_\ell{}^2 - m^2)}{\left| E'_\ell \left(E_\gamma \cos \psi - \sqrt{E_\ell^2 - m^2} \cos \theta_\ell \right) + (E_\ell + M - E_\gamma) \sqrt{E'_\ell{}^2 - m^2} \right|} |\mathcal{M}_{\text{brems}}|^2,$$

where

$$I = \sqrt{(\ell \cdot p)^2 - m^2 M^2},$$

$$|\mathcal{M}_{\text{brems}}|^2 = \mathcal{M}_{\text{brems}}^\ell + \mathcal{M}_{\text{brems}}^p - z \mathcal{M}_{\text{brems}}^{\ell p}.$$

Expressions for the terms $\mathcal{M}_{\text{brems}}^\ell$, $\mathcal{M}_{\text{brems}}^p$ and $\mathcal{M}_{\text{brems}}^{\ell p}$ can be easily obtained in the framework of QED without using the soft-photon approximation.

$\ell^\pm p \rightarrow \ell^\pm p \gamma$: the lepton bremsstrahlung term

$$\mathcal{M}_{\text{brems}}^\ell = \frac{e^6}{q_1^4} (\mathcal{L}_{1\mu\nu} + \mathcal{L}_{2\mu\nu}) \mathcal{P}^{\mu\nu},$$

where

$$\begin{aligned}\mathcal{L}_{1\mu\nu} &= \frac{1}{2} \text{tr} \left[(\ell' + m) \gamma^\alpha \frac{\ell' + k + m}{2(k \cdot \ell')} \gamma_\mu (\ell + m) \gamma_\alpha \frac{\ell - k + m}{2(k \cdot \ell)} \gamma_\nu \right] \\ &\quad - \frac{1}{2} \text{tr} \left[(\ell' + m) \gamma^\alpha \frac{\ell' + k + m}{2(k \cdot \ell')} \gamma_\mu (\ell + m) \gamma_\nu \frac{\ell' + k + m}{2(k \cdot \ell')} \gamma_\alpha \right], \\ \mathcal{P}^{\mu\nu} &= \frac{1}{2} \text{tr} \left[(\not{p} + M) \left\{ (F_1(q_1^2) + F_2(q_1^2)) \gamma^\nu - \frac{F_2(q_1^2)}{2M} P^\nu \right\} \right. \\ &\quad \left. (\not{p}' + M) \left\{ (F_1(q_1^2) + F_2(q_1^2)) \gamma^\mu - \frac{F_2(q_1^2)}{2M} P^\mu \right\} \right],\end{aligned}$$

and an expression for the tensor $\mathcal{L}_{2\mu\nu}$ is obtained from the expression for $\mathcal{L}_{1\mu\nu}$ after changing $\ell \leftrightarrow -\ell'$.

Squares of the four-momenta transferred to the proton:

$$q_1^2 = (p' - p)^2 = 2M(E'_\ell - E_\ell + E_\gamma),$$

$$q_2^2 = (\ell - \ell')^2 = 2\sqrt{(E_\ell^2 - m^2)(E'^2_\ell - m^2)} \cos\theta_\ell - 2E_\ell E'_\ell + 2m^2.$$

$\ell^\pm p \rightarrow \ell^\pm p \gamma$: the proton bremsstrahlung term–1

$$\mathcal{M}_{\text{brems}}^p = \frac{e^6}{q_2^4} \mathcal{L}_{\mu\nu} (\mathcal{P}_1^{\mu\nu} + \mathcal{P}_2^{\mu\nu}),$$

where

$$\mathcal{L}_{\mu\nu} = \frac{1}{2} \text{tr} \left[(\ell + m) \gamma_\nu (\ell' + m) \gamma_\mu \right],$$

$$\begin{aligned} \mathcal{P}_1^{\mu\nu} &= \frac{1}{2} \text{tr} \left[(\not{p}' + M) \left\{ (F_1(0) + F_2(0)) \gamma^\alpha \right. \right. \\ &\quad \left. \left. - \frac{F_2(0)}{2M} [(2p' + k)^\alpha - \gamma^\alpha (\not{p}' + \not{k} - M)] \right\} \frac{\not{p}' + \not{k} + M}{2(k \cdot p')} \right. \\ &\quad \left. \left\{ (F_1(q_2^2) + F_2(q_2^2)) \gamma^\mu - \frac{F_2(q_2^2)}{2M} [P_+^\mu - (\not{p}' + \not{k} - M) \gamma^\mu] \right\} (\not{p} + M) \right. \\ &\quad \left. \left\{ (F_1(0) + F_2(0)) \gamma^\alpha - \frac{F_2(0)}{2M} [(2p - k)^\alpha - \gamma^\alpha (\not{p} - \not{k} - M)] \right\} \frac{\not{p} - \not{k} + M}{2(k \cdot p)} \right. \\ &\quad \left. \left\{ (F_1(q_2^2) + F_2(q_2^2)) \gamma^\nu - \frac{F_2(q_2^2)}{2M} [P_-^\nu - (\not{p} - \not{k} - M) \gamma^\nu] \right\} \right] \end{aligned}$$

$\ell^\pm p \rightarrow \ell^\pm p \gamma$: the proton bremsstrahlung term–2

$$\begin{aligned}
& -\frac{1}{2} \text{tr} \left[(\not{p}' + M) \left\{ (F_1(0) + F_2(0)) \gamma^\alpha \right. \right. \\
& - \frac{F_2(0)}{2M} \left[(2p' + k)^\alpha - \gamma^\alpha (\not{p}' + \not{k} - M) \right] \left. \right\} \frac{\not{p}' + \not{k} + M}{2(k \cdot p')} \\
& \left\{ (F_1(q_2^2) + F_2(q_2^2)) \gamma^\mu - \frac{F_2(q_2^2)}{2M} [P_+^\mu - (\not{p}' + \not{k} - M) \gamma^\mu] \right\} (\not{p} + M) \\
& \left\{ (F_1(q_2^2) + F_2(q_2^2)) \gamma^\nu - \frac{F_2(q_2^2)}{2M} [P_+^\nu - \gamma^\nu (\not{p}' + \not{k} - M)] \right\} \frac{\not{p}' + \not{k} + M}{2(k \cdot p')} \\
& \left. \left\{ (F_1(0) + F_2(0)) \gamma^\alpha - \frac{F_2(0)}{2M} [(2p' + k)^\alpha - (\not{p}' + \not{k} - M) \gamma^\alpha] \right\} \right],
\end{aligned}$$

and an expression for the tensor $\mathcal{P}_2^{\mu\nu}$ is obtained from the expression for $\mathcal{P}_1^{\mu\nu}$ after changing $p \leftrightarrow -p'$.

Some notations used:

$$P = p + p', \quad P_+ = p + p' + k, \quad P_- = p + p' - k.$$

$\ell^\pm p \rightarrow \ell^\pm p \gamma$: the interference term

$$\mathcal{M}_{\text{brems}}^{\ell p} = \frac{e^6}{q_1^2 q_2^2} \left(\frac{1}{k \cdot \ell'} \mathcal{L}_{1\mu\nu}^\alpha + \frac{1}{k \cdot \ell} \mathcal{L}_{2\mu\nu}^\alpha \right) (\mathcal{P}_{1\alpha}^{\mu\nu} - \mathcal{P}_{2\alpha}^{\mu\nu}),$$

where

$$\begin{aligned} \mathcal{L}_{1\mu\nu}^\alpha &= \frac{1}{2} \text{tr} \left[(\ell' + m) \gamma^\alpha (\ell' + k + m) \gamma_\mu (\ell + m) \gamma_\nu \right], \\ \mathcal{P}_{1\alpha}^{\mu\nu} &= \frac{1}{2} \text{tr} \left[(\not{p}' + M) \left\{ (F_1(q_1^2) + F_2(q_1^2)) \gamma^\mu - \frac{F_2(q_1^2)}{2M} P^\mu \right\} (\not{p} + M) \right. \\ &\quad \left\{ (F_1(q_2^2) + F_2(q_2^2)) \gamma^\nu - \frac{F_2(q_2^2)}{2M} [P_+^\nu - \gamma^\nu (\not{p}' + k - M)] \right\} \frac{\not{p}' + k + M}{2(k \cdot p')} \\ &\quad \left. \left\{ (F_1(0) + F_2(0)) \gamma_\alpha - \frac{F_2(0)}{2M} [(2p' + k)_\alpha - (\not{p}' + k - M) \gamma_\alpha] \right\} \right], \end{aligned}$$

and expressions for the lepton $\mathcal{L}_{2\mu\nu}^\alpha$ and proton $\mathcal{P}_{2\alpha}^{\mu\nu}$ tensors are obtained from the expressions for $\mathcal{L}_{1\mu\nu}^\alpha$ and for $\mathcal{P}_{1\alpha}^{\mu\nu}$ after changing $\ell \leftrightarrow -\ell'$ and $p \leftrightarrow -p'$ respectively.

- Full FeynCalc calculation can be found at <http://gramolin.com/esepp/>

Virtual-photon corrections: approach of Mo & Tsai

$$\begin{aligned}\mathcal{M}_{\text{vac}}^e &= \frac{\alpha}{\pi} \left[-\frac{5}{9} + \frac{1}{3} \ln \frac{-q^2}{m^2} \right] \mathcal{M}_{\text{Born}}, \\ \mathcal{M}_{\text{vert}}^\ell &= -\frac{\alpha}{2\pi} \left[K(\ell, \ell') - K(\ell, \ell) - \frac{3}{2} \ln \frac{-q^2}{m^2} + 2 \right] \mathcal{M}_{\text{Born}}, \\ \mathcal{M}_{\text{vert}}^p &= -\frac{\alpha}{2\pi} [K(p, p') - K(p, p)] \mathcal{M}_{\text{Born}}, \\ \mathcal{M}_{\text{box}} &= z \frac{\alpha}{2\pi} [K(\ell, -p) + K(\ell', -p')] \mathcal{M}_{\text{Born}}, \\ \mathcal{M}_{\text{xbox}} &= -z \frac{\alpha}{2\pi} [K(\ell', p) + K(\ell, p')] \mathcal{M}_{\text{Born}},\end{aligned}$$

where

$$K(p_i, p_j) = (p_i \cdot p_j) \int_0^1 \frac{dx}{p_x^2} \ln \frac{p_x^2}{\lambda^2}.$$

The terms $K(\ell, -p)$ and $K(\ell', -p')$ are complex, but only their real parts contribute to the cross section. For this reason, Mo and Tsai resorted to the following simplification of these terms:

$$\text{Re } K(\ell, -p) \approx K(\ell, p), \quad \text{Re } K(\ell', -p') \approx K(\ell', p'),$$

Cancellation of IR divergences and full set of formulas

$$\frac{d\sigma_{\text{elast}}}{d\Omega_\ell} + \frac{d\sigma_{\text{brems}}}{d\Omega_\ell} \Big|_{E_\gamma < E_\gamma^{\text{cut}}} = (1 + \delta_{\text{virt}} + \delta_{\text{brems}}) \frac{d\sigma_{\text{Born}}}{d\Omega_\ell},$$

where

$$\delta_{\text{virt}} = \delta_{\text{vac}}^e + \delta_{\text{vert}},$$

$$\delta_{\text{brems}} = \delta_{\text{brems}}^{\ell\ell} + \delta_{\text{brems}}^{pp} - z \delta_{\text{brems}}^{\ell p},$$

$$\delta_{\text{vac}}^e = \frac{2\alpha}{\pi} \left(-\frac{5}{9} + \frac{1}{3} \ln \frac{-q^2}{m_e^2} \right),$$

$$\delta_{\text{vert}} = \frac{\alpha}{\pi} \left(\frac{3}{2} \ln \frac{-q^2}{m^2} - 2 \right),$$

$$\delta_{\text{brems}}^{\ell\ell} = -2\alpha \left[\tilde{B}(\ell, \ell, E_\gamma^{\text{cut}}) - 2\tilde{B}(\ell, \ell', E_\gamma^{\text{cut}}) + \tilde{B}(\ell', \ell', E_\gamma^{\text{cut}}) \right],$$

$$\delta_{\text{brems}}^{pp} = -2\alpha \left[\tilde{B}(p, p, E_\gamma^{\text{cut}}) - 2\tilde{B}(p, p', E_\gamma^{\text{cut}}) + \tilde{B}(p', p', E_\gamma^{\text{cut}}) \right],$$

$$\delta_{\text{brems}}^{\ell p} = 4\alpha \left[\tilde{B}(\ell, p, E_\gamma^{\text{cut}}) - \tilde{B}(\ell, p', E_\gamma^{\text{cut}}) - \tilde{B}(\ell', p, E_\gamma^{\text{cut}}) + \tilde{B}(\ell', p', E_\gamma^{\text{cut}}) \right],$$

$$\tilde{B}(p_i, p_j, E_\gamma^{\text{cut}}) = \frac{p_i \cdot p_j}{4\pi} \int_0^1 \frac{dx}{p_x^2} \left(\ln \frac{4(E_\gamma^{\text{cut}})^2}{p_x^2} + \frac{p_x^0}{|\mathbf{p}_x|} \ln \frac{p_x^0 - |\mathbf{p}_x|}{p_x^0 + |\mathbf{p}_x|} \right).$$

Vacuum polarization: further improvements

- Leptonic contribution:

$$\delta_{\text{vac}}^{e, \mu, \tau} = \frac{2\alpha}{\pi} \left\{ -\frac{5}{9} - \frac{4}{3} \frac{m^2}{q^2} + \left(\frac{1}{3} + \frac{2}{3} \frac{m^2}{q^2} \right) \sqrt{1 - \frac{4m^2}{q^2}} \times \ln \left[\frac{-q^2}{4m^2} \left(1 + \sqrt{1 - \frac{4m^2}{q^2}} \right)^2 \right] \right\},$$

where m is the mass of the electron, muon or tau lepton correspondingly.

- In addition to the leptonic contribution, we also need to consider the hadronic contribution to the vacuum polarization. The total amplitude \mathcal{M}_{vac} (which includes both the leptonic and hadronic parts) is expressed through the so-called photon polarization operator $\mathcal{P}(q^2)$ as

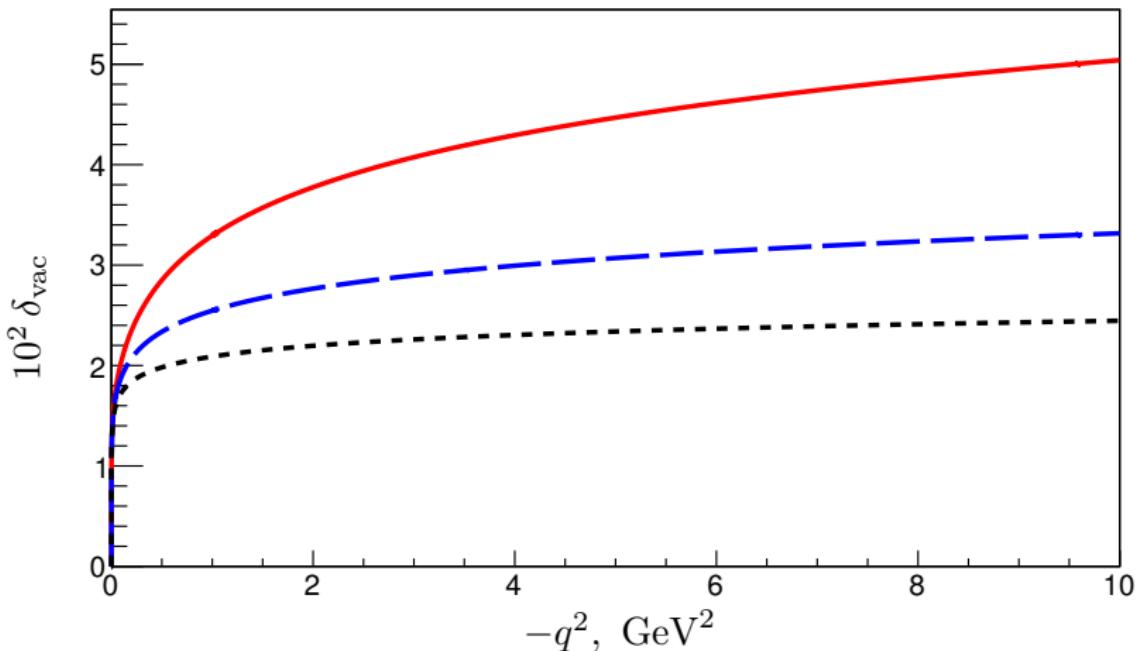
$$\mathcal{M}_{\text{vac}} = [\text{Re } \mathcal{P}(q^2)] \mathcal{M}_{\text{Born}}, \quad (1)$$

which implies

$$\delta_{\text{vac}} = 2 \text{Re } \mathcal{P}(q^2). \quad (2)$$

Numerical values of $\mathcal{P}(q^2)$ are extracted from the experimental data on the cross sections of the annihilation processes $e^+ e^- \rightarrow \text{hadrons}$. For our event generator we use the results of a global analysis by F. V. Ignatov (see <http://cmd.inp.nsk.su/~ignatov/vpl/>).

Different contributions to vacuum polarization



Different contributions to the δ_{vac} correction as functions of q^2 : contribution δ_{vac}^e from electron-positron loops only — **black short-dashed curve**; full leptonic contribution $\delta_{\text{vac}}^e + \delta_{\text{vac}}^\mu + \delta_{\text{vac}}^\tau$ — **blue long-dashed curve**; full vacuum polarization correction δ_{vac} — **red solid curve**.

Lepton vertex correction and the TPE correction

- Mo and Tsai neglected several terms when deriving the formula for the lepton vertex correction. More precisely, this correction is given by the expression

$$\delta'_{\text{vert}} = \frac{\alpha}{\pi} \left(\frac{3}{2} \ln \frac{-q^2}{m^2} - \frac{1}{2} \ln^2 \frac{-q^2}{m^2} + \frac{\pi^2}{6} - 2 \right), \quad (3)$$

which is still based on the approximation $-q^2 \gg m^2$.

- We can refine the expressions for the amplitudes \mathcal{M}_{box} and $\mathcal{M}_{\text{xbox}}$ describing the infrared-divergent parts of the TPE contribution. Maximon and Tjon obtained the following expressions using a less drastic approximation than that employed by Mo and Tsai:

$$\mathcal{M}_{\text{box}}^{\text{MTj}} = z \frac{\alpha}{\pi} \frac{E_\ell}{|\ell|} \ln \left(\frac{E_\ell + |\ell|}{m} \right) \ln \left(\frac{-q^2}{\lambda^2} \right) \mathcal{M}_{\text{Born}},$$

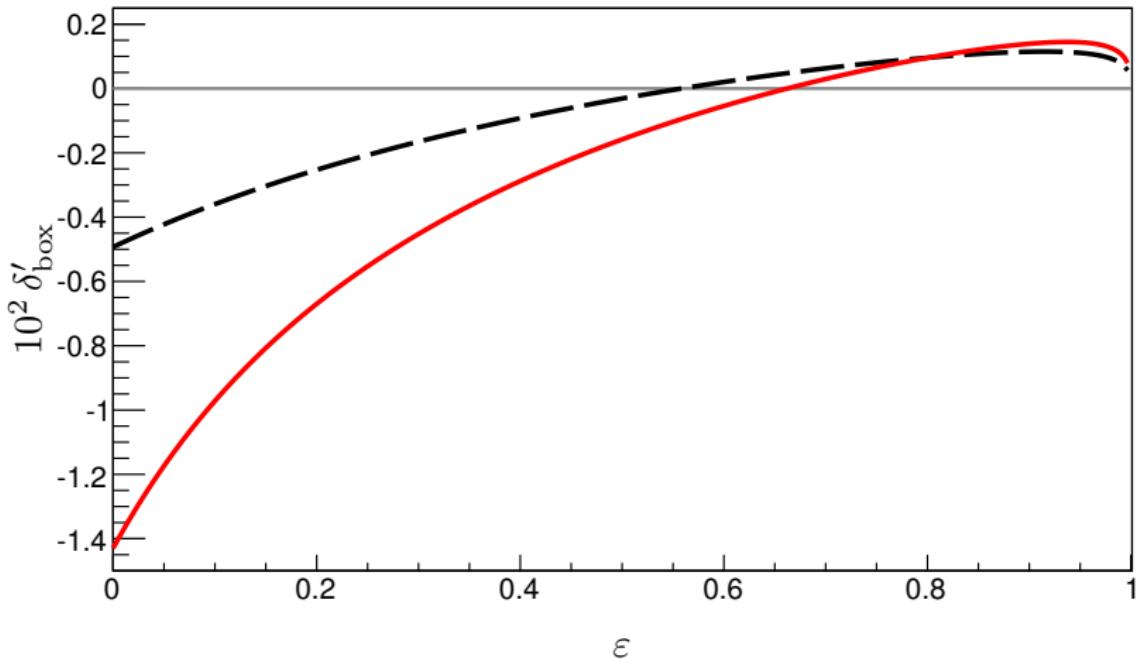
$$\mathcal{M}_{\text{xbox}}^{\text{MTj}} = -z \frac{\alpha}{\pi} \frac{E'_\ell}{|\ell'|} \ln \left(\frac{E'_\ell + |\ell'|}{m} \right) \ln \left(\frac{-q^2}{\lambda^2} \right) \mathcal{M}_{\text{Born}}.$$

The amplitudes and by Maximon and Tjon, as well as the amplitudes and by Mo and Tsai, are infrared divergent, but the difference $(\mathcal{M}_{\text{box}}^{\text{MTj}} + \mathcal{M}_{\text{xbox}}^{\text{MTj}} - \mathcal{M}_{\text{box}} - \mathcal{M}_{\text{xbox}})$ is finite and gives the following addition to δ_{virt} :

$$\delta'_{\text{box}} = -\frac{\alpha}{\pi} \left[\ln \frac{E_\ell}{E'_\ell} \ln \left(\frac{q^4}{4M^2 E_\ell E'_\ell} \right) + 2\Phi \left(1 - \frac{M}{2E_\ell} \right) - 2\Phi \left(1 - \frac{M}{2E'_\ell} \right) \right],$$

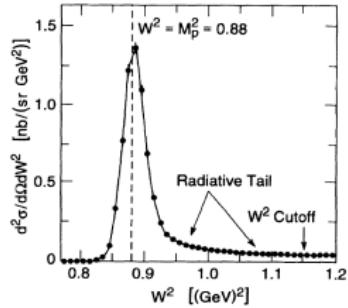
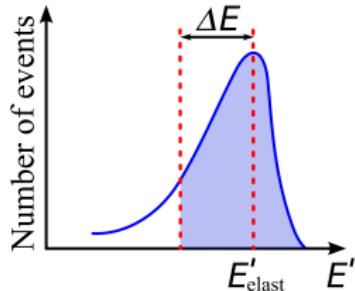
where the function Φ (Spence's function or dilogarithm) is $\Phi(x) = - \int_0^x \frac{\ln |1-x|}{x} dx$.

Dependence of δ'_{box} on ε



Dependence of δ'_{box} on ε for the two fixed values of the four-momentum transfer squared: $-q^2 = 1 \text{ GeV}^2$ (black dashed line) and $-q^2 = 5 \text{ GeV}^2$ (red solid line).

Radiative corrections in single arm experiments



To select elastic events the following condition is commonly used:

$$E' > E'_{\text{elast}}(E, \theta_\ell) - \Delta E, \text{ where } E'_{\text{elast}}(E, \theta_\ell) = \frac{ME}{M + E(1 - \cos \theta_\ell)}.$$

Or another condition:

$$W^2 < W_{\text{cut}}^2, \text{ where } W^2 = M^2 + 2M(E - E') - 4EE' \sin^2 \frac{\theta_\ell}{2},$$

and W^2 is the missing mass squared. Typically used value is $W_{\text{cut}}^2 = 1.1 \div 1.15 \text{ GeV}^2$ (and $W^2 = M^2 = 0.88 \text{ GeV}^2$ in the case of purely elastic scattering). It is easy to express ΔE through W_{cut}^2 :

$$\Delta E = \frac{W_{\text{cut}}^2 - M^2}{2M + 4E \sin^2(\theta_\ell/2)}.$$

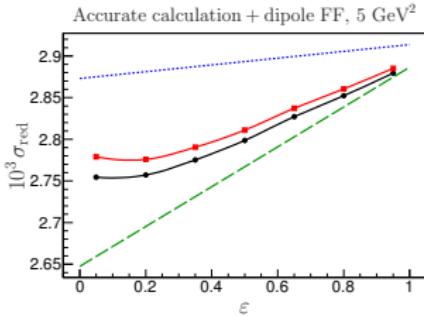
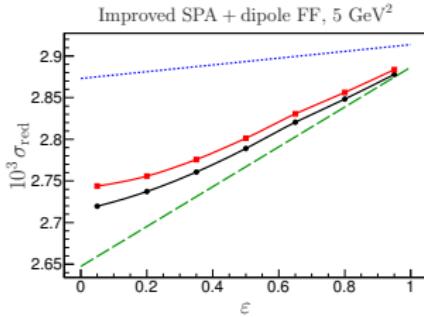
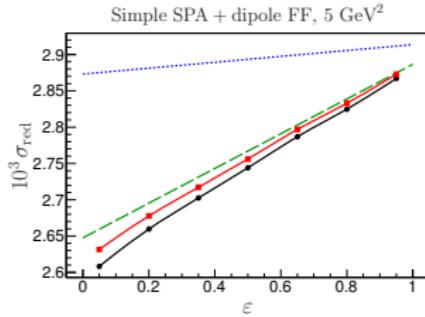
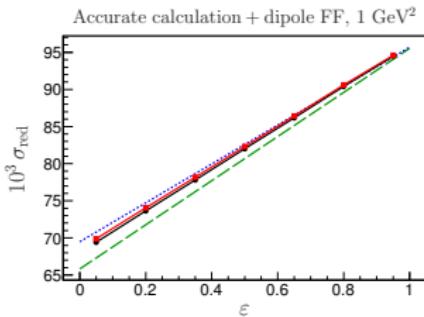
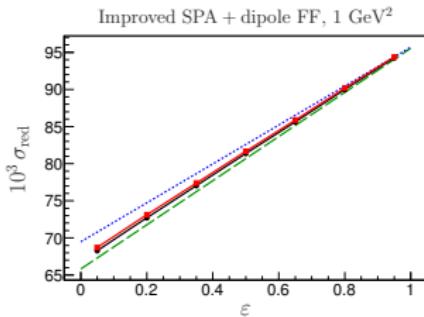
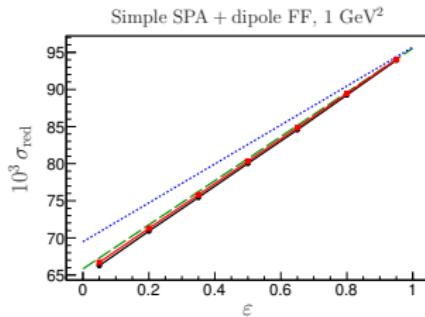
Application to Rosenbluth measurements: kinematics

$$W_{\text{cut}}^2 = 1.1 \text{ GeV}^2, \quad \Delta\theta_\ell = \pm 0.05^\circ$$

E_ℓ , GeV	θ_ℓ , deg.	$-q^2$, GeV 2	ε	$d\sigma_{\text{Mott}}/d\Omega_\ell$, nb/sr	ΔE , MeV	δ'_{box}	δ_{MTs}	δ_{MTj} $-\delta'_{\text{box}}$
0.862	139.6	1	0.05	0.410	44.7	-0.0042	-0.052	-0.058
0.960	102.6		0.20	2.639	52.1	-0.0025	-0.062	-0.067
1.083	80.76		0.35	7.394	59.4	-0.0013	-0.072	-0.077
1.248	63.93		0.50	17.47	67.1	-0.0003	-0.084	-0.087
1.497	49.21		0.65	40.97	75.4	0.0004	-0.098	-0.100
1.966	34.66		0.80	113.2	85.3	0.0010	-0.117	-0.119
3.805	16.29		0.95	748.9	100.7	0.0011	-0.163	-0.164
3.161	126.4	5	0.05	0.026	18.4	-0.0117	-0.163	-0.171
3.462	84.55		0.20	0.266	27.0	-0.0067	-0.175	-0.181
3.839	63.55		0.35	1.011	35.8	-0.0036	-0.186	-0.191
4.345	48.89		0.50	3.001	45.3	-0.0016	-0.198	-0.201
5.108	36.89		0.65	8.537	56.0	-0.0001	-0.212	-0.215
6.550	25.61		0.80	28.24	69.4	0.0010	-0.232	-0.234
12.19	11.91		0.95	232.8	91.5	0.0014	-0.279	-0.280

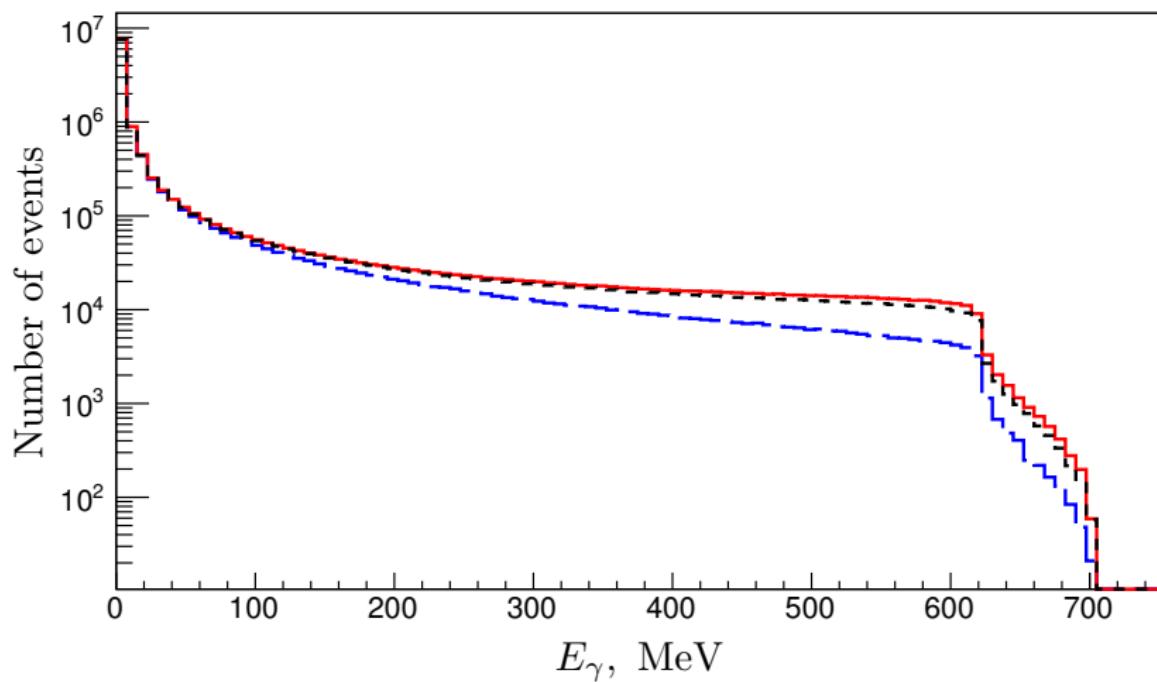
Kinematic parameters and the corresponding values of the standard radiative corrections, which we use in our analysis of the Rosenbluth measurements.

Application to Rosenbluth measurements: results



Bremsstrahlung spectra for the selected events

$$-q^2 = 5 \text{ GeV}^2, \quad \varepsilon = 0.05, \quad W_{\text{cut}}^2 = 1.1 \text{ GeV}^2, \quad \Delta E = 18.4 \text{ MeV}$$



Bremsstrahlung spectra for the selected events in the three considered models:
simple SPA (**long-dashed blue line**), improved SPA (**short-dashed black line**),
and accurate QED calculation (**solid red line**).

Conclusion and acknowledgements

- The new event generator for elastic scattering of charged leptons on protons was developed. It allows to take into account bremsstrahlung more accurately. A detailed description of the generator will be available very soon (November 2013).
- Comprehensive simulation of the detector (for example, using the Geant4 toolkit) with a realistic event generator is the best way to do radiative correction in modern experiments.
- Not only the two-photon exchange, but also bremsstrahlung can affect the results of Rosenbluth measurements.

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Thank you for your attention!

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