# Introduction to Radiative Corrections

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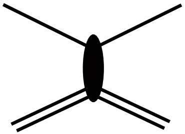
# Talk plan

The plan of this talk is the following:

- Classical electrodynamics introduction to radiative corrections
  - Ultraviolet catastrophe
  - Infrared catastrophe
- Quantum electrodynamics definition
- Illustrative process: electron–proton scattering
- Virtual Corrections
  - Infrared divergency
  - Ultraviolet divergency
- Ultraviolet divergency treatment
- Infrared divergency treatment
- Collinear singularity
- Structure Functions method
- Application of the radiative corrections to a selected process
- Conclusion

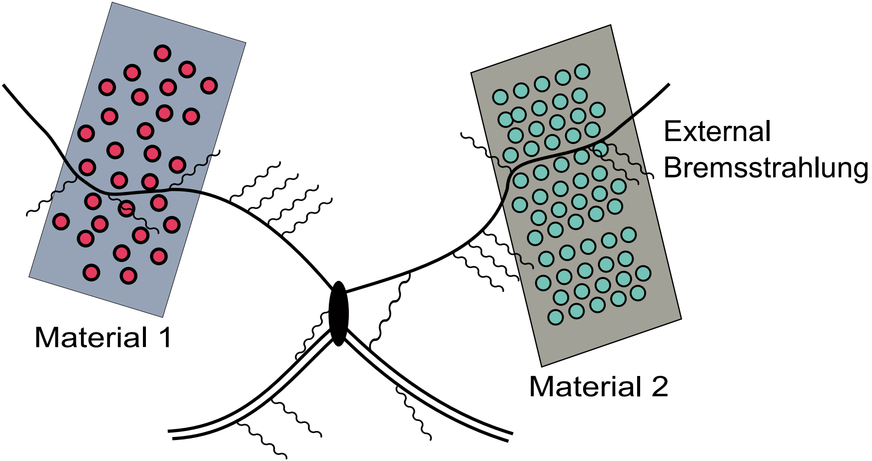
# Types of Radiative Corrections

In experiment one usually wants to study the process of hard scattering with simple interaction.



# Types of Radiative Corrections

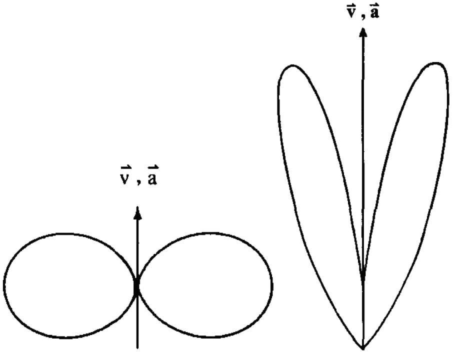
But reality is slightly more complicated...

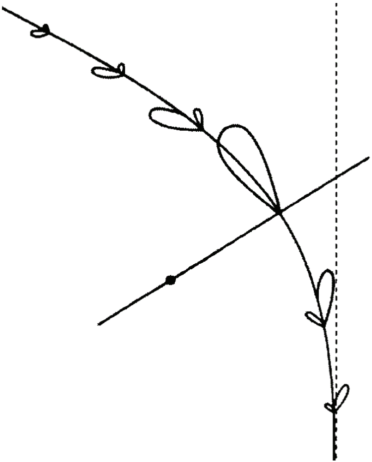


# Classic Electrodynamics

It is know from the classical electrodynamics that any electrically charged matter being accelerated emits electromagnetic waves.

$$\frac{dP}{d\Omega} = \frac{e^2 \dot{v}^2}{4\pi c^3} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}.$$

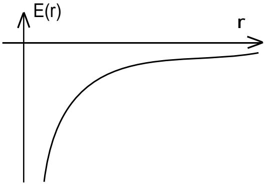


Illustrations are taken from "The Elementary Process of Bremsstrahlung" by Eberhard Haug and Werner Nakel

# Classic Electrodynamics: Self-Interaction (UV)

The problem of infinities first arose in the classical electrodynamics of point particles in the 19th century.



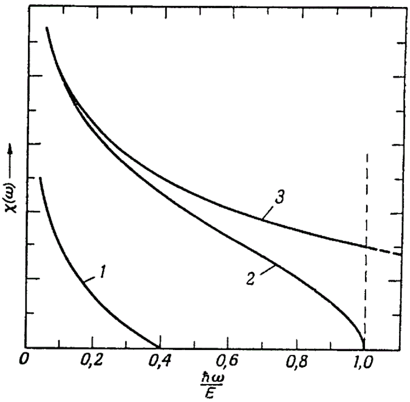

Once electromagnetic field possess energy–momentum it was tempting to associate this energy as the source of the electron mass. Assuming that the electron is a charged spherical shell of radius $r_e$ one can estimate the energy (mass) in its field as

$$mc^2 = \frac{1}{2}\int E^2(r)\, d\mathbf{r} \sim \int_{r_e}^{\infty} \left(\frac{e}{r^2}\right)^2 r^2 dr = \frac{e^2}{r_e},$$

which becomes infinite in the limit as $r_e$ approaches zero. This implies that the point particle would have infinite inertia, making it unable to be accelerated. Thus we can define the "radius of electron" $r_e$ , which determines the spatial region size where classical approach becomes unapplicable:

$$r_e = \frac{e^2}{4\pi\varepsilon_0\, mc^2} = \alpha\frac{\hbar}{mc} \approx 2.8 \times 10^{-15}\ \text{m}.$$

# Classic Electrodynamics: Small Energy Emission (IR)




1. Classical spectrum (1).
2. Quasi-classical spectrum.
3. Bethe–Heitler quantum result.

The spectrum of bremsstrahlung in scattering of fast but not relativistic particle with charge $q_1$ and mass $M$ on charge $q_2$ at rest within classical electrodynamics:

$$\chi(\omega)\, d\omega = \\ = \frac{16}{3}\frac{q_2^2}{c}\left(\frac{q_1^2}{Mc^2}\right)^2\left(\frac{c}{v}\right)^2 \ln\left(\frac{aMv^3}{q_1 q_2 \omega}\right) d\omega.$$

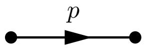
$$\hbar\omega\, \sigma_{brems}(\omega)\, d(\hbar\omega) = \chi(\omega)\, d\omega = \frac{16}{3}\frac{q_2^2}{c}\left(\frac{q_1^2}{Mc^2}\right)^2\left(\frac{c}{v}\right)^2 \ln\left(\frac{aMv^3}{q_1 q_2 \omega}\right) d\omega. \quad (1)$$

# Quantum Electrodynamics: Definition

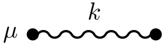
The Quantum Electrodynamics (QED) is the gauge $U(1)$ theory with the following lagrangian:

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma_\mu\psi A^\mu, \tag{2}$$

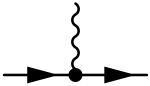
where $e = \sqrt{4\pi\alpha}$ is the lepton charge modulus and $\alpha \approx 1/137$ is the fine-structure constant. Electromagnetic field tensors $F_{\mu\nu}$ has the form: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This gives us **Feynman Rules**:




$$S(p) = i\frac{p\!\!\!/ + m}{p^2 - m^2 + i\varepsilon}$$

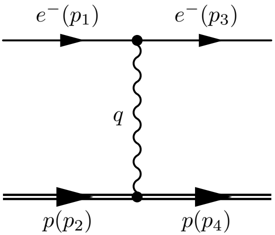



$$D_{\mu\nu}(k) = -i\frac{g_{\mu\nu}}{k^2 - \lambda^2 + i\varepsilon}$$



$$-ie\,\bar{\psi}\gamma_\mu\psi\,A^\mu$$

# Example: Electron–Proton Scattering




Here $q = p_1 - p_3$, and

$$\tau = \frac{-q^2}{4M_p^2}.$$

$$\mathcal{M}_B = \frac{e^2}{q^2}\left[\bar{u}\left(p_3\right)\gamma^\mu u\left(p_1\right)\right]\left[\bar{u}\left(p_2\right)\Gamma_\mu\left(q\right)u\left(p_4\right)\right],$$

where $\Gamma_\mu\left(q\right)$ is the electromagnetic vertex which is parameterized in terms of proton form factors $F_{1,2}\left(q^2\right)$:

$$\Gamma_\mu\left(q\right) = F_1\left(q^2\right)\gamma_\mu + \frac{1}{4M_p}\left(\gamma_\mu \not{q} - \not{q}\gamma_\mu\right)F_2\left(q^2\right),$$

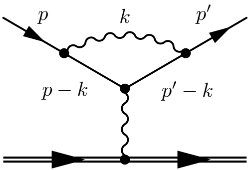
which are normalized as follows:

$$F_1\left(0\right) = 1, \qquad F_2\left(0\right) = \mu_p,$$

and are related with the electric (Dirac) and magnetic (Sachs) form factors as:

$$G_E = F_1 - \tau F_2, \qquad G_M = F_1 + F_2. \qquad (3)$$

# Lepton Vertex Corrections




Let us consider vertex correction first. It leads to the modification of the vertex:

$$-ie\gamma_\mu \to -ie\Lambda_\mu\left(p, p'\right), \tag{4}$$

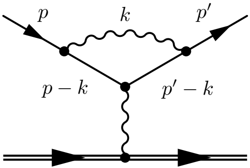
where $\Lambda_\mu\left(p, p'\right)$ has the form:

$$\Lambda_\mu\left(p, p'\right) = \left(-ie\right)^2 i \int \frac{d^4 k}{\left(2\pi\right)^4} \frac{\gamma_\nu\left(\not{p}' - \not{k} + m\right)\gamma_\mu\left(\not{p} - \not{k} + m\right)\gamma_\nu}{\left((p'-k)^2 - m^2\right)\left((p-k)^2 - m^2\right)\left(k^2 - \lambda^2\right)}, \tag{5}$$

where $\lambda$ is the photon fictitious mass which is necessary for infra-red divergence regularization. One can see that after some algebra the numerator of the integral can be transformed into the following expression:

$$\gamma_\nu\left(\not{p}' - \not{k} + m\right)\gamma_\mu\left(\not{p} - \not{k} + m\right)\gamma_\nu = 4\left(pp'\right)\gamma_\mu - 2\not{p}\not{k}\gamma_\mu - 2\gamma_\mu\not{k}\not{p}' - 2\not{k}\gamma_\mu\not{k}.$$

# Lepton Vertex Corrections



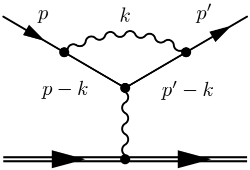

The first term is ultra-violet finite but has infra-red divergence (when $k \to 0$):

$$I = \int \frac{d^4k}{\left(k^2 - \lambda^2\right)\left(k^2 - 2\left(kp\right)\right)\left(k^2 - 2\left(kp'\right)\right)} =$$

$$= -i\pi^2 \int\limits_0^1 dy \int\limits_0^1 \frac{xdx}{x^2 p_y^2 + \lambda^2 \left(1 - x\right)} = -\frac{i\pi^2}{2} \int\limits_0^1 \frac{dy}{p_y^2} \ln\left(\frac{p_y^2}{\lambda^2}\right) \sim \ln\left(\frac{\lambda}{m}\right),$$

where $p_y = yp + \left(1 - y\right)p'$. But we will discuss this later and now consider ultra-violet divergent part.

# Lepton Vertex Corrections: Ultra-Violet Divergencies




The third term is ultra-violet divergent:

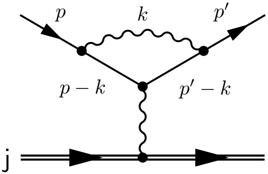
$$I^{\mu\nu} = \int \frac{d^4k\ k^\mu k^\nu}{(k^2-\lambda^2)\left(k^2-2\,(kp)\right)\left(k^2-2\,(kp')\right)} =$$

and first it should be regularized. We will do this by injecting the regularization multiplier with some auxiliary parameter $\Lambda$:

$$= \int \frac{d^4k\ k^\mu k^\nu}{(k^2-\lambda^2)\left(k^2-2\,(kp)\right)\left(k^2-2\,(kp')\right)} \frac{-\Lambda^2}{k^2-\Lambda^2} =$$

This factor will be eliminated at the end of calculation going to limit $\Lambda \to \infty$. But now we can calculate the integrals:

$$= -\frac{i\pi^2}{2}\left[\int_0^1 dy \frac{p_y^\mu p_y^\nu}{p_y^2} - g^{\mu\nu}\left(\ln\left(\frac{\Lambda}{m}\right) - 1 - \frac{1}{2}\int_0^1 dy \ln\left(\frac{p_y^2}{m^2}\right)\right)\right]. \qquad (6)$$




Here the momentum transferred is $q = p - p'$.

Thus we have for divergent part of the vertex $\Lambda_\mu(p, p')$:

$$\Lambda_\mu^{UV}(p, p') = \frac{\alpha}{4\pi} \int\limits_0^1 dy \frac{\not{p}_y \gamma^\mu \not{p}_y}{p_y^2} -$$

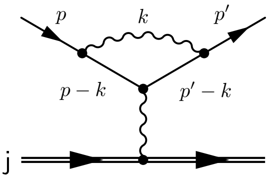
$$- \frac{\alpha}{4\pi} \gamma_\mu \int\limits_0^1 dy \ln\left(\frac{p_y^2}{m^2}\right) + \frac{\alpha}{8\pi} \gamma_\mu \left(\ln\left(\frac{\Lambda}{m}\right) - 1\right).$$

So, in order to get finite physical result we should apply some regularization scheme. Here we can use the requirement of electric charge normalization, i.e. that at $q^2 \to 0$ the vertex corrections tends to zero:

$$\Lambda_\mu(p, p')\big|_{q \to 0} = \Lambda_\mu(p, p) \to 0, \qquad (7)$$

so in our calculations we can do the following substitution which ensures this requirement:

$$\Lambda_\mu(p, p') \quad \to \quad \tilde{\Lambda}_\mu(p, p') = \Lambda_\mu(p, p') - \Lambda_\mu(p, p). \qquad (8)$$



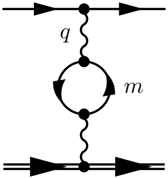

$q = p - p'$

Thus we get that the renormalized vertex can be written in the form of vertex with form factors:

$$\tilde{\Lambda}_\mu(p,p') = \Lambda_\mu(p,p') - \Lambda_\mu(p,p) =$$
$$= G_1(q^2)\gamma_\mu + \frac{1}{4m}(\gamma_\mu \not{q} - \not{q}\gamma_\mu) G_2(q^2) =$$
$$= -\frac{\alpha}{2\pi}(K(p,p') - K(p,p))\gamma_\mu +$$

$$+\frac{\alpha}{2\pi}\left[-3 + \left(3 - \frac{q^2}{m^2}\right)\int_0^1 dy \frac{m^2}{p_y^2} - \frac{1}{2}\int_0^1 dy \ln\left(\frac{p_y^2}{m^2}\right)\right]\gamma_\mu -$$
$$-\frac{\alpha}{2\pi}\left(\int_0^1 dy \frac{m^2}{p_y^2}\right)\frac{1}{4m}(\gamma_\mu \not{q} - \not{q}\gamma_\mu), \qquad K(p,p') = (pp')\int_0^1 dy \ln\left(\frac{p_y^2}{\lambda^2}\right).$$

# Ultra-Violet Divergencies: Renormalization




We can consider the vacuum excitations in photon propagator in the similar way. This diagram leads to the following modification of the photon propagator:

$$-i\frac{g_{\mu\nu}}{q^2} \to -i\frac{g_{\mu\nu}}{q^2}\left(1+\Pi\left(q^2,m^2\right)\right) \to -i\frac{g_{\mu\nu}}{q^2-\Pi\left(q^2,m^2\right)},$$

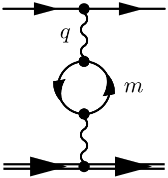
where $\Pi\left(q^2,m^2\right)$ is the vacuum excitation operator:

$$\Pi\left(q^2,m^2\right) = \frac{2\alpha}{\pi}\int\limits_0^1 dz\, z\left(1-z\right)\ln\left(1-\frac{q^2 z\left(1-z\right)}{m^2-i\varepsilon}\right) =$$

$$= \frac{\alpha}{3\pi}\left\{\left(1-\frac{2}{x^2}\right)\sqrt{1+4/x^2}\ln\left(\frac{\sqrt{1+4/x^2}+1}{\sqrt{1+4/x^2}-1}\right)+\frac{4}{x^2}-\frac{5}{3}\right\},$$

where $x^2 = -q^2/m^2$. The expression corresponding to the Feynman diagram diverges quadratically in high–frequency limit, but we already regularized the expression above with the use of condition that photon should not obtain any mass as a result of radiative corrections: $\Pi\left(q^2=0,m^2\right)=0$.

# Ultra-Violet Divergencies: Renormalization




There also known forms in asymptotic regions:

$$\Pi\left(q^2, m^2\right)\Big|_{-q^2 \ll m^2} = \frac{\alpha}{16\pi}\left(\frac{-q^2}{m^2}\right),$$

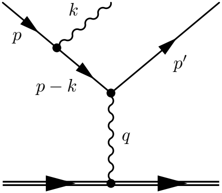
$$\Pi\left(q^2, m^2\right)\Big|_{-q^2 \gg m^2} = \frac{\alpha}{\pi}\left(\frac{1}{3}\ln\left(\frac{-q^2}{m^2}\right) - \frac{5}{9}\right).$$

Thus we can take into account the contributions from vacuum excitations (VE) like:

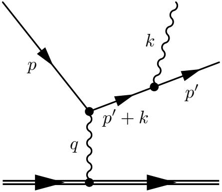
$$\mathcal{M}_{VE} = \mathcal{M}_B \sum_{i=e,\mu,\tau} \Pi\left(q^2, m_i^2\right). \tag{9}$$

But usually one prefers iterate this procedure and sum up all these contributions in the form:

$$\mathcal{M}_B + \mathcal{M}_{VE} = \frac{\mathcal{M}_B}{1 - \sum\limits_{i=e,\mu,\tau} \Pi\left(q^2, m_i^2\right)}. \tag{10}$$




Let us now consider the emission of the real photon from lepton line. In general this process worths for separate consideration ("hard" photon emission), but we will concentrate on the case when emitted photon is "soft", i.e. its energy is small $\omega \to 0$. This approximation allows to use the semi–classical approach (the so called "current approximation") when the amplitude of the bremsstrahlung factorizes as follows:

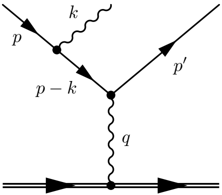


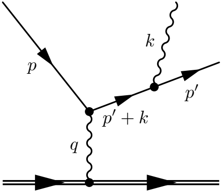

$$\mathcal{M}_{Brems.} = e \left( \frac{(ep')}{(kp')} - \frac{(ep)}{(kp)} \right) \mathcal{M}_B, \tag{11}$$

where $e_\mu$ is the emitted photon polarization vector. And this factor in braces gives the following multiplier to the cross section, which has the sense of the probability of emission of the photon with energy $\omega$ along the direction $\mathbf{n} = \mathbf{k}/\omega$:

$$P(k)\, d\mathbf{k} = \frac{\alpha}{(2\pi)^2} \left( \frac{(ep)}{E - (\mathbf{pn})} - \frac{(ep')}{E' - (\mathbf{p'n})} \right)^2 \frac{d\omega}{\omega}\, d\Omega_k. \tag{12}$$

# Real Photon Emission: Infrared Divergencies




It is seen that this probability of photon emission

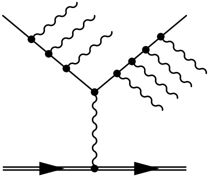
$$P\left(k\right)d\mathbf{k}=\frac{\alpha}{\left(2\pi\right)^{2}}\left(\frac{(ep)}{E-(\mathbf{pn})}-\frac{(ep')}{E'-(\mathbf{p'n})}\right)^{2}\frac{d\omega}{\omega}\,d\Omega_{k},$$

shows the typical infrared behavior $d\omega/\omega$ and strongly peaked angular dependence about the directions of $\mathbf{p}$ and $\mathbf{p}'$. The probability for photon emission into the energy range $d\omega$ is

$$\omega^{2}d\omega\int P\left(k\right)d\Omega_{k}=\frac{2\alpha}{\pi}\left[\ln\frac{2\left(E^{2}-(\mathbf{pp'})\right)}{m^{2}}-1\right]\frac{d\omega}{\omega}.$$

Thus the probability of photon emission increases logarithmically with the energy. And the integral probability of photon emission diverges:

$$\int P\left(k\right)d\mathbf{k}=\int\limits_{\omega_{min}}^{\omega_{max}}\omega^{2}d\omega\int P\left(k\right)d\Omega_{k}=\frac{2\alpha}{\pi}\left[\cdots\right]\ln\frac{\omega_{max}}{\omega_{min}}.$$



The essential idea for the understanding of the infrared divergence problem was first brought out by Bloch and Nordsieck in their famous paper [F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937)]. In brief, their idea is:

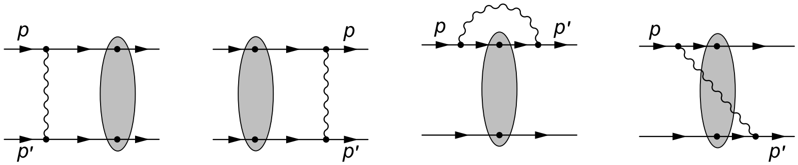
- Individual photons can be emitted with arbitrarily small energies $\omega < \omega_{min}$ and thus there is a possibility that some photons will escape detection.
- The probability that only a finite number of photons will escape detection is precisely zero.
- The observed cross section is very close to the cross section where all radiative corrections are ignored.

This is the well-known **cancellation between the real and virtual infrared divergences**.

# Infrared Divergencies: Yennie–Frautschi–Suura

More detailed and systematic analysis of infrared divergencies was done in the paper [D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. **13**, 379 (1961)] where virtual and real photon emission was analyzed in a series of the process and some general procedure was presented. Let us follow it in a sketch.

# Infrared Divergencies: Yennie–Frautschi–Suura – Virtual Photons




First they make a statement that infrared contributions goes from the virtual photon diagrams where photon is emitted and absorbed from external legs. And the amplitude of one additional virtual photon emission $\mathcal{M}^{(1)}$ has the form:

$$\mathcal{M}^{(1)} = \alpha B \, \mathcal{M}^{(0)} + \tilde{\mathcal{M}}^{(1)}, \tag{13}$$

where $\mathcal{M}^{(0)}$ is the amplitude of the hard sub-process with no extra photon emission and factor $B$ has the common form:

$$B = \frac{i}{(2\pi)^3} \int \frac{d^4k}{k^2 - \lambda^2} \left( \frac{2p'_\mu - k_\mu}{2\,(p'k) - k^2} - \frac{2p_\mu - k_\mu}{2\,(pk) - k^2} \right)^2 . \tag{14}$$

# Infrared Divergencies: Yennie–Frautschi–Suura – Virtual Photons

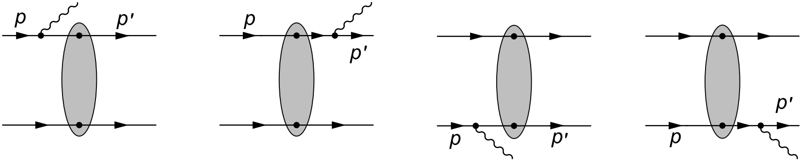
Considering the processes with more then one additional photon in the same approach one obtains the following result:

$$
\begin{aligned}
\mathcal{M}^{(0)} &= \mathcal{M}^{(0)}, \\
\mathcal{M}^{(1)} &= \alpha B\, \mathcal{M}^{(0)} + \tilde{\mathcal{M}}^{(1)}, \\
\mathcal{M}^{(2)} &= \frac{(\alpha B)^2}{2} \mathcal{M}^{(0)} + \alpha B\, \tilde{\mathcal{M}}^{(1)} + \tilde{\mathcal{M}}^{(2)}, \\
&\dots \\
\mathcal{M}^{(n)} &= \sum_{r=0}^{n} \frac{(\alpha B)^r}{r!} \tilde{\mathcal{M}}^{(n-r)},
\end{aligned}
$$

and summing the contributions with any number of virtual photons emitted one gets the exponentiated form:

$$
\mathcal{M} = \sum_{n=0}^{\infty} \mathcal{M}^{(n)} = \exp(\alpha B) \sum_{n=0}^{\infty} \tilde{\mathcal{M}}^{(n)}. \tag{15}
$$

# Infrared Divergencies: Yennie–Frautschi–Suura




Now we consider the cross section with the emission of the additional $n$ real photons with their total energy equal to $\epsilon$:

$$\frac{d\sigma_n}{d\epsilon} = \exp\left(2\alpha B\right)\frac{1}{n!}\int\prod_{m=1}^{n}\frac{d\mathbf{k_m}}{\left(\mathbf{k_m}^2+\lambda^2\right)^{1/2}}\delta\left(\epsilon-\sum_{i=1}^{n}\omega_n\right)\tilde{\rho}_n\left(p,p',k_1,\cdots,k_n\right).$$

And if we take into account the contributions from the emission of all the possible number of real photons we get:

$$\frac{d\sigma}{d\epsilon} = \sum_{n=0}^{\infty}\frac{d\sigma_n}{d\epsilon} = \exp\left(2\alpha B\right)\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}dy\, e^{iy\epsilon}\exp\left[2\alpha\tilde{B}+D\right]\times$$

$$\times\left\{\tilde{\beta}_0+\sum_{n=0}^{\infty}\frac{1}{n!}\int\prod_{m=1}^{n}\frac{d\mathbf{k_m}}{\omega_m}e^{-iy\omega_m}\tilde{\beta}_n\left(p,p',k_1,\cdots,k_n\right)\right\}.$$

# Infrared Divergencies: Yennie–Frautschi–Suura

Thus the contributions with singular infrared behavior factorizes as:

$$\frac{d\sigma}{d\epsilon} = \exp\left(2\alpha\left(B + \tilde{B}\right)\right)\frac{d\hat{\sigma}}{d\epsilon}, \tag{16}$$

where $d\hat{\sigma}$ is the "cross section" which is independent of the soft photon limit (i.e. it is finite as $\lambda \to 0$) and

$$B = \frac{i}{(2\pi)^3}\int\frac{d^4k}{k^2 - \lambda^2}\left(\frac{2p'_\mu - k_\mu}{2(p'k) - k^2} - \frac{2p_\mu - k_\mu}{2(pk) - k^2}\right)^2,$$

$$\tilde{B} = \frac{-1}{8\pi^2}\int' \frac{d\mathbf{k}}{(\mathbf{k}^2 + \lambda^2)^{1/2}}\left(\frac{p'_\mu}{(p'k)} - \frac{p_\mu}{(pk)}\right)^2.$$

# Infrared Divergencies: Yennie–Frautschi–Suura

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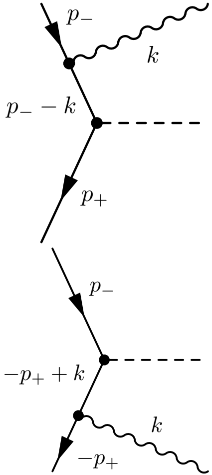
where $d\hat{\sigma}$ is the "cross section" which is independent of the soft photon limit (i.e. it is finite as $\lambda \to 0$) and

$$B = -\frac{1}{2\pi}\left[\ln\frac{2\,(pp')}{m^2}\left(\ln\frac{m^2}{\lambda^2}+\frac{1}{2}\ln\frac{2\,(pp')}{m^2}-\frac{1}{2}\right)-\ln\frac{m^2}{\lambda^2}\right],$$
$$\tilde{B} = \frac{1}{2\pi}\left[\ln\frac{2\,(pp')}{m^2}\left(\ln\frac{m^2}{\lambda^2}+\frac{1}{2}\ln\frac{2\,(pp')}{m^2}-\ln\frac{EE'}{\epsilon^2}\right)-\ln\frac{m^2}{\lambda^2}+\ln\frac{EE'}{\epsilon^2}\right],$$

and thus the exponent argument is infrared finite:

$$2\alpha\left(B+\tilde{B}\right) = -\frac{\alpha}{\pi}\left(\ln\frac{2\,(pp')}{m^2}-1\right)\ln\frac{EE'}{\epsilon^2}+\frac{\alpha}{2\pi}\ln\frac{2\,(pp')}{m^2}.$$

# Collinear Singularity

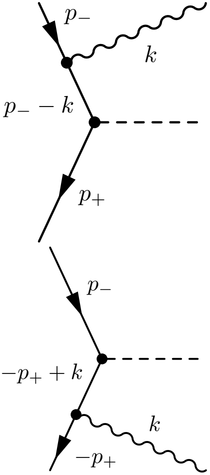



Typical example of collinear singularity can be demonstrated in electron–positron annihilation. The singularity appears from the propagator of electron (positron) after emitting the photon. The **denominators of these propagators** has the form:

$$(p_- - k)^2 - m^2 = -2\,(p_- k) = \\ = -2kp\left(\frac{m^2}{p\,(E+p)} + (1-\cos\theta)\right),$$

$$(p_+ - k)^2 - m^2 = -2\,(p_+ k) = \\ = -2kp\left(\frac{m^2}{p\,(E+p)} + (1+\cos\theta)\right),$$

where $p_\pm = (E, 0, 0, \mp p)$ are the momenta of initial $e^\pm$ beams and $\theta$ is the angle between $\mathbf{p}_-$ and $\mathbf{k}$.




For example we consider the following integral over photon angle $\theta$. One can change the variable of integration which absorbs the singularity and obtain:

$$\int_{-1}^{1} d\cos\theta \frac{f(\cos\theta)}{4(p_-k)(p_+k)} =$$

$$= \frac{1}{4(kp)^2} \int_{-1}^{1} dz \frac{f(z)}{\left[\frac{m^2}{p(E+p)} + (1-z)\right]\left[\frac{m^2}{p(E+p)} + (1+z)\right]} =$$

$$= \frac{1}{4(kp)^2} \frac{1}{a} L \int_{0}^{1} dx\, f(x)\,,$$

where $z \equiv \cos\theta$ and (using $s = (p_+ + p_-)^2 = 4E^2$)

$$a = 1 + \frac{m^2}{p(E+p)}, \qquad L = \ln\left(1 + \frac{2p(E+p)}{m^2}\right) \sim \ln\frac{s}{m^2}.$$

# Collinear Singularity: Kinoshita–Lee–Nauenberg theorem

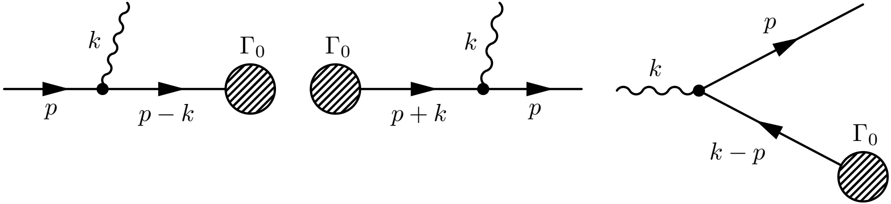
Indeed this kind of singularities often appears in the processes where large scale difference is present. For instance, here, in $e^+e^-$ annihilation, we have dangerous terms like:

$$|\mathcal{M}|^2 \sim \frac{1}{(p_\pm k)} \text{ and } \frac{m^2}{(p_\pm k)^2} \quad \text{which leads to} \quad \int |\mathcal{M}|^2 \, d\Gamma \sim \ln \frac{s}{m^2}. \tag{18}$$

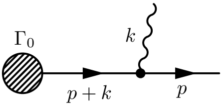
Since quantity $|\mathcal{M}|^2$ usually enters some finite observable, like cross sections or decay widths, **there should exist the finite limit at vanishing mass** $m \to 0$. This means that in total cross section or decay width the terms singular in mass disappears. This statement was proven in general way long time ago [T. Kinoshita, J. Math. Phys. **3**, 650 (1962); T. D. Lee and M. Nauenberg, Phys. Rev. **133**, B1549 (1964)].

# Collinear Factorization: Quasi-Real Electron Method

Let us consider **Quasi-Real Electron Method** (see V. N. Baier, V. S. Fadin and V. A. Khoze, Nucl. Phys. B **65**, 381 (1973)) which will allow us to investigate the useful consequences of collinear singularity. Thus we consider the following processes




when the electron with large energy $E \gg m$ emits photon with energy $\omega$ along the direction of it's momenta (assuming that $(E-\omega) \gg m$). In this kinematics the denominator of electron propagator becomes small and if all the others momenta transferred in the process is large then the cross section factorizes and is equal to the product of the cross section without emission of the photon and the probability of photon emission.




The amplitude of the process is given by the expression:

$$\mathcal{M} = e\bar{u}(p)\,\not{e}^{\lambda}\frac{\not{p}+\not{k}+m}{2(pk)}\Gamma_0(p+k,\ldots), \qquad (19)$$

where $e^{\lambda}$ is the vector of polarization of emitted photon. These spinors satisfy the following relations of completeness:

$$\sum_{s=\pm 1} u_s(p+k)\,\bar{u}_s(p+k) = E_{p+k}\gamma_0 - (\mathbf{p}+\mathbf{k})\gamma + m,$$

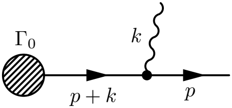
$$\sum_{s=\pm 1} v_s(p+k)\,\bar{v}_s(p+k) = E_{p+k}\gamma_0 - (\mathbf{p}+\mathbf{k})\gamma - m,$$

where $E_{p+k} = \sqrt{(\mathbf{p}+\mathbf{k})^2 + m^2}$. Using these relations we can rewrite the lepton propagator in (19) in the following way:

$$\frac{\not{p}+\not{k}+m}{2(pk)} = \frac{1}{2E_{p+k}}\sum_{s=\pm 1}\left[\frac{u_s(p+k)\,\bar{u}_s(p+k)}{E_p+\omega-E_{p+k}} + \frac{v_s(-p+k)\,\bar{v}_s(-p+k)}{E_p+\omega+E_{p+k}}\right],$$

where $E_p = \sqrt{\mathbf{p}^2+m^2}$.

# Collinear Factorization: Quasi-Real Electron Method




If angle $\theta$ between $\mathbf{p}$ and $\mathbf{k}$ is small (we assume that $|\mathbf{p}| \gg m$, $|\mathbf{p}-\mathbf{k}| \gg m$), then denominator of first term is small:

$$E_p + \omega - E_{p+k} \approx \frac{\omega E_p}{2\left(E_p+\omega\right)}\left(\frac{m^2}{E_p^2}+\theta^2\right),$$

$$E_p + \omega + E_{p+k} \approx 2\left(E_p+\omega\right),$$

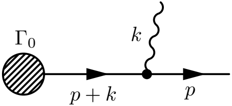
Thus amplitude of the process factorizes:

$$\mathcal{M} = \frac{e}{2E_{p+k}} \frac{\bar{u}\left(p\right) \not{e}^{\lambda} u\left(p+k\right)}{E_p+\omega-E_{p+k}} \mathcal{M}_0\left(p+k,...\right), \tag{20}$$

where $\mathcal{M}_0$ is the amplitude of the process without photon emission in shifter kinematics:

$$\mathcal{M}_0\left(p+k,...\right) = \bar{u}\left(p+k\right)\Gamma_0\left(p+k,...\right). \tag{21}$$

# Collinear Factorization: Quasi-Real Electron Method

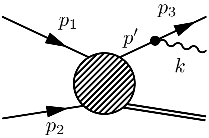



The square of modulus of the amplitude $\mathcal{M}$ from (20) has the form:

$$|\mathcal{M}|^2 = e^2 |\mathcal{M}_0|^2 \left( \frac{E_p^2 + (E_p+\omega)^2}{\omega (E_p+\omega)(kp)} - \frac{m^2}{(kp)^2} \right), \qquad (22)$$

where in braces we keep only singular on $(kp)$ terms. This approximation is valid if other momenta transferred are large ($t_0 \gg (kp)$). Thus defining the probability of emission of hard photon with energy $\omega$ by the electron with momentum $\mathbf{p}$ as:

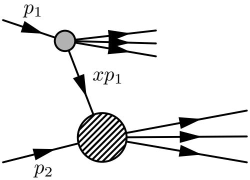
$$dW_{\mathbf{p}}(\mathbf{k}) = \frac{\alpha}{4\pi} \left( \frac{E_p^2 + (E_p+\omega)^2}{E_p \omega (kp)} - \frac{m^2}{(kp)^2} \frac{E_p+\omega}{E_p} \right) \frac{d\mathbf{k}}{\omega}, \qquad (23)$$




we can write the cross section of the process with the emission of the photon in the factorized form:

$$d\sigma = \left. \frac{d\sigma_0}{d\mathbf{p}'} \right|_{p'=p_3+k} dW_{\mathbf{p_3}+\mathbf{k}}(\mathbf{k})\, d\mathbf{p_3}. \qquad (24)$$

# Structure Functions Approach



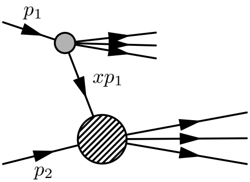

Iterating the formulae of Quasi Real Electron Method and taking into account the virtual corrections we obtain the **Structure Functions Approach**:

$$d\sigma(s) = \int\limits_0^1 dx\, D\left(x, Q^2\right) d\hat{\sigma}\left(sx\right). \qquad (25)$$

where $D(x, Q^2)$ is the **Structure Function** of the lepton, which means that probability to find the lepton with momentum $xp$ inside the lepton with momentum $p$, i.e. $x$ is the fraction of the initial momentum carried by the final lepton, and $Q^2$ is the scale of hard subprocess.

This approach (based on the partonic picture of QCD hard processes of DGLAP equations) was elaborated in a series of the papers of L.N. Lipatov, V.S. Fadin and E.A. Kuraev (see for example E. A. Kuraev and V. S. Fadin, Sov. J. Nucl. Phys. **41**, 466 (1985)).

# Structure Functions Approach




The structure functions take into account the corrections with large logarithms of type $\ln\left(Q^2/m^2\right)$ and satisfy the Lipatov's evolution equations (which is similar to DGLAP equations):

$$\frac{dD\left(x,Q^2\right)}{d\ln Q^2} = \frac{\alpha}{2\pi}\int\limits_x^1 \frac{dz}{z} P^{(1)}\left(\frac{x}{z}\right) D\left(z,Q^2\right), \qquad (26)$$

where $P(x)$ is the evolution equation kernel, which describes the elementary act of photon emission and the explicit form of this kernel is defined from Quasi Real Electron Method:

$$P^{(1)}(x) = \left(\frac{1+x^2}{1-x}\right)_+ = \lim_{\Delta\to 0}\left[\frac{1+x^2}{1-x}\theta\left(1-x-\Delta\right) + \left(2\ln\left(\Delta\right)+\frac{3}{2}\right)\delta\left(1-x\right)\right].$$

# Structure Functions Approach

Solving the evolution equations one gets the following result for the Structure Function $D\left(x, Q^2\right)$:

$$D\left(x, Q^2\right) = \delta\left(1 - x\right) + \frac{\alpha}{2\pi}\left(\ln\left(\frac{Q^2}{m^2}\right) - 1\right) P^{(1)}\left(x\right) + \\ + \frac{1}{2!}\left(\frac{\alpha}{2\pi}\right)^2\left(\ln\left(\frac{Q^2}{m^2}\right) - 1\right)^2 P^{(2)}\left(x\right) + \cdots \tag{27}$$

where $P^{(n)}\left(x\right)$ are the kernels of evolution equations of $n$-th order:

$$P^{(1)}\left(x\right) = \left(\frac{1+x^2}{1-x}\right)_+ = \lim_{\Delta\to 0}\left[\frac{1+x^2}{1-x}\theta\left(1 - x - \Delta\right) + \left(2\ln\left(\Delta\right) + \frac{3}{2}\right)\delta\left(1-x\right)\right],$$

$$P^{(n)}\left(x\right) = \int\limits_x^1 \frac{dy}{y} P^{(1)}\left(y\right) P^{(n-1)}\left(\frac{x}{y}\right).$$

# Structure Functions Approach: Kinoshita–Lee–Nauenberg theorem

It is worth to be mentioned that Structure Function Approach satisfies the requirements of Kinoshita–Lee–Nauenberg theorem. This is easy to see if one notices the following property of Structure Function $D\left(x, Q^2\right)$:

$$\int\limits_0^1 dx\, D\left(x, Q^2\right) = 1, \qquad \int\limits_0^1 dx\, xD\left(x, Q^2\right) = 1, \tag{28}$$

which follows from the corresponding property of evolution kernel:

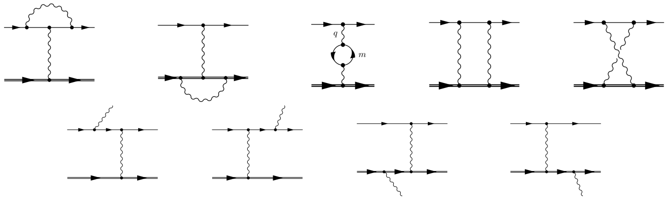
$$\int\limits_0^1 dx\, P^{(1)}\left(x\right) = 0. \tag{29}$$

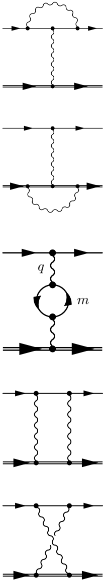
Thus in total cross section or decay width which is written within the Structure Function Approach all the dangerous terms with mass singularities like $\ln\left(Q^2/m^2\right)$ are canceled out.

We consider the electron–proton scattering as an example of application of above–mentioned approaches to estimate the radiative corrections. In fact it was done in a series of well-known papers (here is the most famous ones: L. W. Mo and Y.-S. Tsai, Rev. Mod. Phys. **41**, 205 (1969); L. C. Maximon and J. A. Tjon, Phys. Rev. **C62**, 054320 (2000)). These calculations were rather cumbersome and complicated. We will not discuss them in details. But we present only the result of calculation from the latter one. The cross section of elastic electron–proton scattering with radiative corrections in first order of perturbation theory included is:

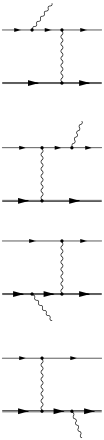
$$d\sigma = d\sigma_0 \left(1 + \delta\right), \tag{30}$$

where $\delta$ is the contribution of total radiative correction which includes

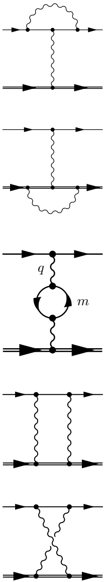





$$\delta = \frac{\alpha}{\pi}\left\{\frac{13}{6}\ln\left(\frac{-q^2}{m^2}\right) - \frac{28}{9} - \frac{1}{2}\ln^2\eta + \mathsf{Li}_2\left(\cos^2\left(\theta/2\right)\right) - \right.$$
$$\left. -\frac{\pi^2}{6} - \left[\ln\left(\frac{-q^2}{m^2}\right) - 1\right]\ln\left(\frac{4E_1E_3}{\left(2\eta\epsilon\right)^2}\right)\right\} +$$
$$+\frac{2\alpha Z}{\pi}\left\{-\ln\eta\,\ln\left(\frac{-q^2x}{\left(2\eta\epsilon\right)^2}\right) + \right.$$
$$\left. +\mathsf{Li}_2\left(1-\frac{\eta}{x}\right) - \mathsf{Li}_2\left(1-\frac{1}{\eta x}\right)\right\} +$$
$$+\frac{\alpha Z^2}{\pi}\left\{\frac{E_4}{|\mathbf{p_4}|}\left(-\frac{1}{2}\ln^2 x - \ln x\ln\left(\frac{\rho^2}{M_p^2}\right) + \ln x\right) + \right.$$
$$+\left(\frac{E_4}{|\mathbf{p_4}|}\ln x - 1\right)\ln\left(\frac{M_p^2}{\left(2\eta\epsilon\right)^2}\right) + 1 +$$
$$\left. +\frac{E_4}{|\mathbf{p_4}|}\left(-\mathsf{Li}_2\left(1-\frac{1}{x^2}\right) + 2\,\mathsf{Li}_2\left(-\frac{1}{x}\right) + \frac{\pi^2}{6}\right)\right\}.$$



# $e-p$ Scattering with RC: Structure Function Approach




Within Structure Function Approach all the large logarithms in orders of perturbation theory is summed up with the structure function $D(x)$ as (see YMB, E. A. Kuraev, and E. Tomasi-Gustafsson, PRC **75**, 015207 (2007)):

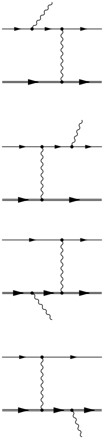
$$\frac{d\sigma}{d\Omega} = \int\limits_{z_0}^{1} dx\, D(x) \frac{\Phi_0(x)}{\left|1-\Pi\left(Q_x^2\right)\right|^2} \left(1 + \frac{\alpha}{\pi} K_0\right),$$

where $\Phi_0(x)$ is the Born cross section calculated in the kinematics where initial electron has the momentum $xp_1$ instead of $p_1$. The quantity $K_0$ takes into account term which is not enhanced with large logarithms:

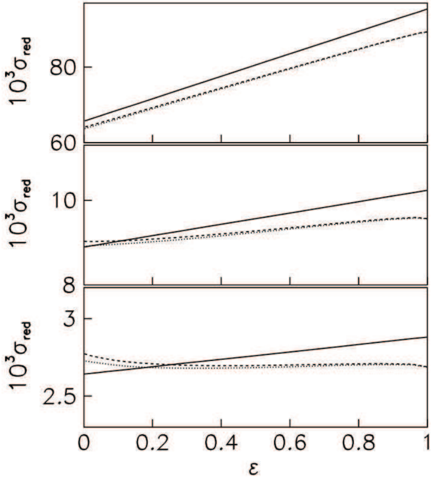
$$K_0 = K_e + K_p + K_{int}, \qquad z_0 = \frac{E_3 - \epsilon}{E_3 + \epsilon(\rho - 1)},$$

$$K_e = \mathrm{Li}_2\left(\cos^2(\theta/2)\right) - \frac{1}{2}\ln^2\eta - \frac{\pi^2}{6} - \frac{1}{2},$$

$$K_p = (\ln x - \beta)\ln\frac{M_p^2}{4\epsilon^2} - \frac{1}{2}\ln^2 x - \ln x \ln\left(\frac{\rho^2}{M_p^2}\right) + \cdots .$$



# $e - p$ Scattering with RC: Structure Function Approach



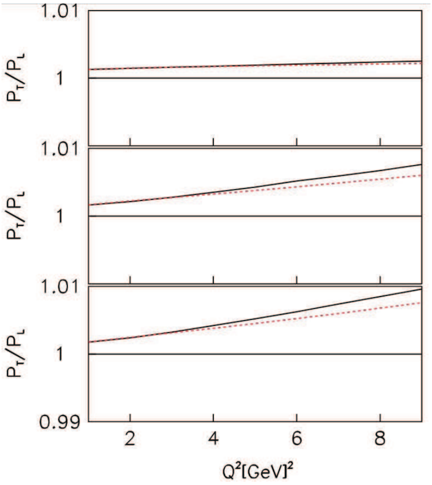

In Rosenbluth method of form factor measurement one usually plot and fit the so called "reduced cross section" $\sigma_{red}$ as a function of $\varepsilon$ which has linear form in Born approximation:

$$\sigma_{red} = \frac{\varepsilon\rho\left(1+\tau\right)}{\sigma_M}\Phi_0 = \\ = \varepsilon\, G_E^2\left(Q^2\right) + \tau\, G_M^2\left(Q^2\right),$$

where $\tau = Q^2/\left(4\mathcal{M}_p^2\right)$ and $Q^2 = -q^2$. The quantity $\varepsilon$ is the polarization degree of virtual photon:

$$\varepsilon^{-1} = 1 + 2\left(1+\tau\right)\tan^2\left(\theta/2\right).$$

$$\frac{d\sigma}{d\Omega} = \int\limits_{z_0}^{1} dx\, D\left(x\right)\frac{\Phi_0\left(x\right)}{\left|1-\Pi\left(Q_x^2\right)\right|^2}\left(1+\frac{\alpha}{\pi}K_0\right).$$




In Polarization Transfer method of form factor measurement one measures the ratio of two polarized cross sections:

$$\Phi_T = C_T\left(Q^2,\varepsilon\right) G_E\left(Q^2\right) G_M\left(Q^2\right),$$
$$\Phi_L = C_L\left(Q^2,\varepsilon\right) G_M^2\left(Q^2\right),$$

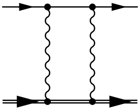
where $C_{T,L}\left(Q^2,\varepsilon\right)$ is some kinematical coefficients. Thus the ratio of this cross sections gives the ratio of form factors:

$$R\left(Q^2\right) = \frac{\mu_p G_E\left(Q^2\right)}{G_M\left(Q^2\right)} \sim \frac{\Phi_T}{\Phi_L}.$$

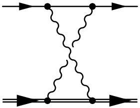
A. I. Akhiezer and M. .P. Rekalo, Sov. Phys. Dokl. **13**, 572 (1968)

$$\frac{d\sigma_{T,L}}{d\Omega} = \int_{z_0}^{1} dx\, D\left(x\right) \frac{\Phi_{T,L}\left(x\right)}{\left|1-\Pi\left(Q_x^2\right)\right|^2}\left(1+\frac{\alpha}{\pi}K_{T,L}\right).$$

# $2\gamma$ Contribution



Two photon contribution is of great importance at the moment since there is an argument that this contribution can be the reason of inconsistency in the measurement of electromagnetic form factors of the proton.



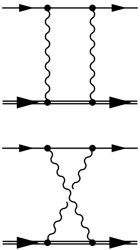
The well known estimation was done in Y.-S. Tsai, Phys.Rev. **122**, 1898 (1961) using the soft–photon approximation which gave the following result:

$$\mathcal{M}_{2\gamma} = \frac{\alpha Z}{2\pi}\left[K\left(p_2, p_3\right) + K\left(p_4, p_1\right) - K\left(p_2, -p_1\right) - K\left(p_4, -p_3\right)\right]\mathcal{M}_B,$$

where $K\left(p_i, p_j\right)$ is the function already used above:

$$K\left(p_i, p_j\right) = \frac{2\left(p_i p_j\right)}{-i\pi^2}\int \frac{d^4k}{\left(k^2 - \lambda^2\right)\left(k^2 - 2\left(kp_i\right)\right)\left(k^2 - 2\left(kp_j\right)\right)}.$$

# $2\gamma$ Contribution



Another estimation was done in paper L. C. Maximon and J. A. Tjon, Phys. Rev. **C62**, 054320 (2000) where more accurate calculation of the integrals were performed (but still in soft–photon approximation). The result was:

$$\mathcal{M}_{2\gamma} = \frac{\alpha Z}{\pi} \left[ \frac{E_3}{|\mathbf{p_3}|} \ln\left(\frac{E_3 + |\mathbf{p_3}|}{m}\right) - \frac{E_1}{|\mathbf{p_1}|} \ln\left(\frac{E_1 + |\mathbf{p_1}|}{m}\right) \right] \ln\left(\frac{-q^2}{\lambda^2}\right) \mathcal{M}_B.$$

But both these estimations lacks for contributions from the hard photon exchange kinematical region. This contribution is finite but **cannot be evaluated in model–independent way at the moment**.

# Conclusions

1. Radiative corrections are important ingredient of any precise measurement.
2. The method of Structure Functions is simple and convenient method for estimation of radiative corrections in leading logarithmical approximation.
3. Two–photon contribution and hard photon emission in electron–proton scattering cannot be evaluated in a model independent way as they require adequate description of excited proton states.
4. Modern experimental setups (large acceptance detectors and spectrometers, coincidence measurements, high resolution, etc) need precise modeling of radiative corrections contributions, leading to Monte–Carlo generators with radiative corrections implemented.