Karlsruhe Institute of Technology

VBFNLO

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Institute for Theoretical Physics

Introduction

VBFNLO

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Vector-Boson-Fusion at Next-to-Leading Order

Introduction

VBFNLO

Physics
Vector-Boson-~~Fusion~~ at Next-to-Leading Order

Introduction

VBFNLO

F
~~Ph~~ysics
Vector-Boson-~~Fusion~~ at Next-to-Leading Order

- Fully flexible parton-level Monte Carlo for processes with electroweak bosons
  - accurate predictions needed for LHC (both signal and background)
  - MC efficient solution for high number of final-state particles (decays of electroweak bosons included)
- general cuts and distributions of final-state particles
- various choices for renormalization and factorization scales
- any pdf set available from LHAPDF (or hard-wired CTEQ6L1, CT10, MRST2004qed, MSTW2008)
- event files in Les Houches Accord (LHA) or HepMC format (LO only)

Process overview

List of implemented processes

(New in VBFNLO 2.7.0)

- vector-boson fusion production at NLO QCD of
  - Higgs (+NLO EW, NLO SUSY)
  - Higgs plus third hard jet
  - Higgs plus photon
  - Higgs pair

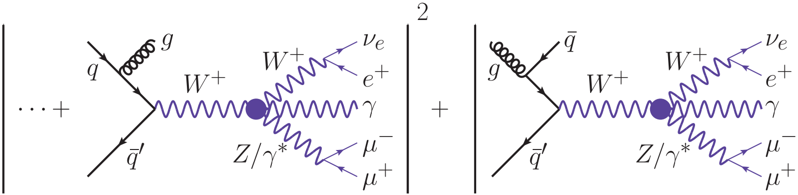
  } (including Higgs decays)
  - vector boson (W, Z, γ)
  - two vector bosons (W^+W^-, $W^\pm W^\pm$, WZ, ZZ; $W\gamma$)
- diboson production
  - diboson (WW, WZ, ZZ, $W\gamma$, $Z\gamma$, $\gamma\gamma$) (NLO QCD)
  - diboson via gluon fusion (WW, ZZ, $Z\gamma$, $\gamma\gamma$) (part of NNLO QCD contribution to diboson)
  - diboson (WZ, $W\gamma$) plus hard jet (NLO QCD)
  - diboson ($W^\pm W^\pm$, WZ, $W\gamma$) plus two hard jets (NLO QCD)
- triboson production (NLO QCD)
  - triboson (all combinations of W, Z, γ)
  - triboson ($W\gamma\gamma$) plus hard jet
- Higgs plus vector boson (NLO QCD) (including Higgs decays)
  - Higgs plus vector boson (WH)
  - Higgs plus vector boson plus hard jet (WH)
- Higgs plus two jets via gluon fusion (one-loop LO) (including Higgs decays)
- new physics models
  - anomalous Higgs couplings
  - anomalous triple and quartic gauge couplings
  - Higgsless and spin-2 models
  - Two-Higgs model

Intermediate state Higgs boson in all processes included where applicable

Implementation Details

[Hagiwara, Zeppenfeld]

- Helicity amplitude method
- Same building blocks for different Feynman graphs
  ⇒ Compute only once per phase-space point and reuse (”leptonic tensors”)
  → Significantly faster than generated code (up to factor 10)



- Catani-Seymour dipole subtraction scheme

$$\sigma_{\mathrm{NLO}} = = \underbrace{\int_{m+1} [\mathrm{d}\sigma^R|_{\epsilon=0} - \mathrm{d}\sigma^A|_{\epsilon=0}]}_{\text{real emission}} + \underbrace{\int_m [\mathrm{d}\sigma^V + \int_1 \mathrm{d}\sigma^A]_{\epsilon=0}}_{\text{virtual contributions}} + \underbrace{\int_m \mathrm{d}\sigma^C}_{\text{finite collinear term}}$$

- Photon isolation à la Frixione
  Processes with real photons in final state can have configurations with photon collinear to final-state quark → QED divergence

  Simple (e.g. *R*) separation cut between photon and jet not infrared safe
  → Frixione photon isolation

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$$\sum_i E_{T_i} \Theta(\delta - R_{i\gamma}) \leq p_{T_\gamma} \frac{1-\cos\delta}{1-\cos\delta_0} \qquad (\text{for all} \quad \delta \leq \delta_0 = 0.7)$$

  $\Rightarrow$ Efficiently suppresses fragmentation contribution

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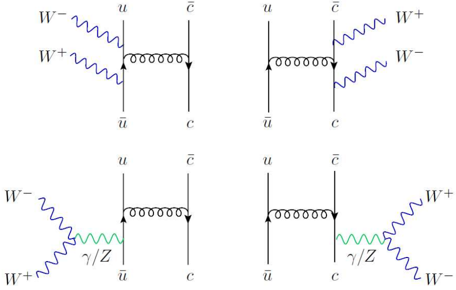
QCD-Diboson production

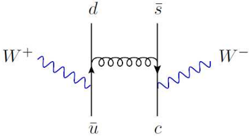
$W^+ W^+ jj$ and $W^+ W^- jj$ known at NLO QCD for some time

[Melia, Melnikov, Röntsch, Zanderighi; Greiner, Heinrich, Mastrolia, Ossola, Reiter, Tramontano]

W^+W^-jj



W^+W^-jj & W^+W^+jj
(latter after changing quark flavors appropriately)



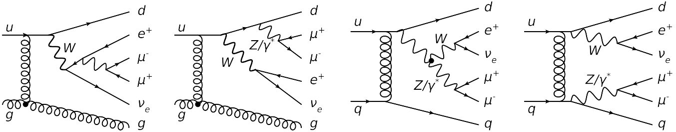
+ diagrams where quark line without attached vector bosons is replaced by gluons

WZjj production

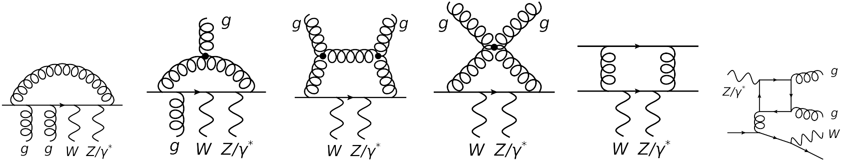
[Campanario, Kerner, Le, Zeppenfeld; new in VBFNLO 2.7.0]

QCD-induced *WZjj* production ($\mathcal{O}(\alpha^4 \alpha_s^2)$)

- Born: 90 subprocesses



- virtual corrections: up to rank-5 hexagons



- real emission: 146 subprocesses

Speed comparison

W^+Zjj @ LO

runtime for 0.1% accuracy:

- VBFNLO: 8 min
- Sherpa: 5 h
- MadGraph 5: 1 month
  (based on extrapolation:
  0.7% in 14 hours)

W^+Zjj @ NLO

runtime for 1% accuracy:

- VBFNLO: 2-3 h

Origin of better performance

- get rid of numerical instabilities
- combine and reuse similar contributions
- good phase-space generator

W^+W^+jj @ NLO

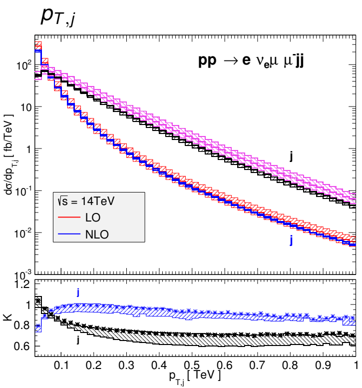
VBFNLO:

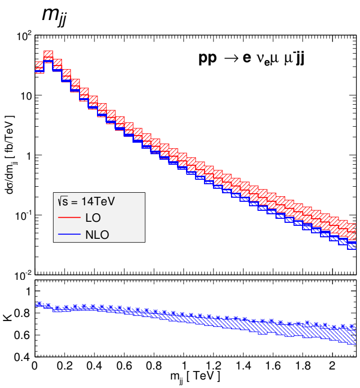
- 30 min for 1% accuracy

POWHEG-BOX:

- 200 jobs, 3 days each
- best result:
  8% accuracy
- median:
  60% accuracy
- worst result:
  off by factor 10^6
  (with error as large)
- combined result:
  1.1% (weighted mean)

Distributions



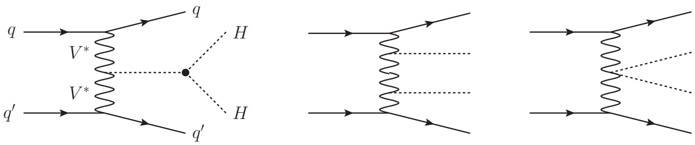


→ Differential K factor varying in distributions

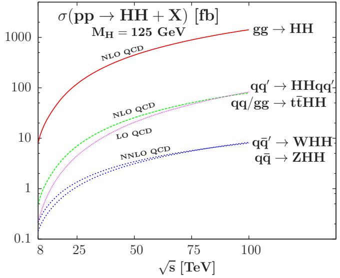
VBF Higgs Pair Production

Higgs pair production via VBF at NLO QCD

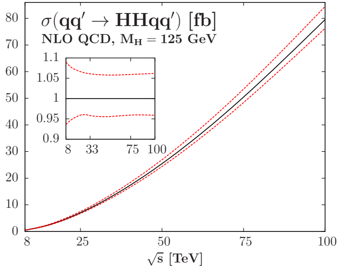
[Figy; Baglio, Djouadi, Gröber, Mühlleitner, Quevillon, Spira]



Total cross section for HH production at the LHC



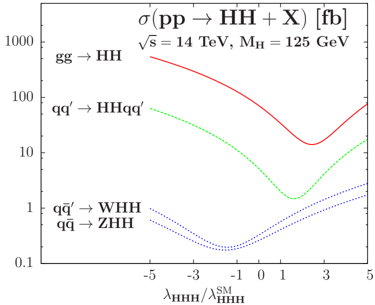
Total uncertainty of VBF-HH c.s. (PDF+α_s+scale $\mu = \mu_{F,R} = [\frac{1}{2}; 2]Q^*$)

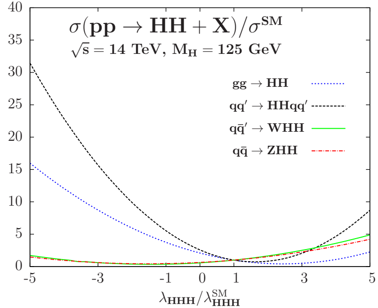


VBF Higgs Pair Production

[Baglio, Djouadi, Gröber, Mühlleitner, Quevillon, Spira]

Sensitivity of Higgs pair production processes to trilinear Higgs self-coupling λ_{HHH}





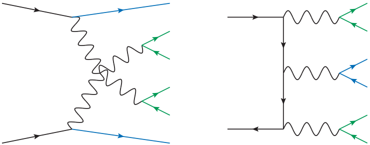
VBF most sensitive channel

Semileptonic Decays

[New in VBFNLO 2.7.0: Feigl]

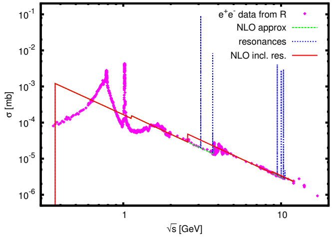
- Vector bosons have significant branching fractions into $q\bar{q}$ final states
  $BR(W \to q\bar{q}) \simeq BR(Z \to q\bar{q}) \simeq 70\%$
- $\Rightarrow$ Extend decay modes and let one vector boson decay hadronically
- $\Rightarrow$ semi-leptonic final states

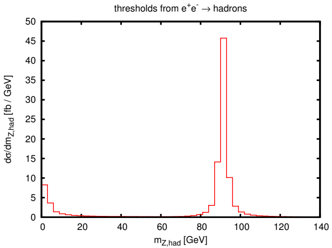
- Vector-boson-fusion processes only contain t-channel exchange
- s-channel contribution can be viewed as
  triboson production with one vector boson decaying hadronically
- interference between s- and t-channel small
  $\Rightarrow$ can be treated as separate process
- $\leftrightarrow$ s-channel contribution small in phenomenologically interesting phase-space regions ($M_{jj,\text{cut}} \gg M_V$)



Semileptonic Decays

- Semileptonic decays implemented in the following channels:
  - $pp \to VVjj$ via VBF with $VV \in W^+W^-,\ W^\pm W^\pm,\ W^\pm Z,\ ZZ$
  - $pp \to Hjj \to VVjj$ via VBF with $VV \in W^+W^-, ZZ$
  - $pp \to VV$ with $VV \in W^+W^-,\ W^\pm Z,\ ZZ$
  - $pp \to VVV$ with $VVV \in WWW,\ WWZ,\ WZZ,\ ZZZ,\ WW\gamma,\ WZ\gamma,\ ZZ\gamma$
- Factor $K = 1 + \frac{\alpha_s}{\pi}$ for $V \to q\bar{q}$ decay can be added as NLO approximation for decay (assuming resonant V production dominates)
- Non-resonant contributions included
- Virtual photon decays $\gamma^* \to q\bar{q}$
  - can form single jet in real emission part
  - quarks massless $\Rightarrow$ divergence $\leftrightarrow$ pion mass as regulator in reality
  - $\Rightarrow$ technical cut $\to$ cross section depends on value
  - $\Rightarrow$ estimate size from $e^+e^- \to$ hadrons without modelling resonances explicitly





Anomalous quartic gauge couplings

Vector-boson scattering ideal process to test anomalous quartic gauge couplings

[New in VBFNLO 2.7.0: Feigl, Schlimpert]

Dimension-8 operators in Lagrangian
(Φ Higgs doublet, $W^{\mu\nu}/B^{\mu\nu}$: SU(2)/U(1) field strength tensors):

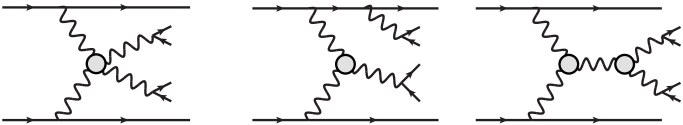
$$\mathcal{L}_{M,2} \propto [B^{\mu\nu} B_{\mu\nu}] \times \left[\left(D^\beta \Phi \right)^\dagger D_\beta \Phi \right]$$

$$\mathcal{L}_{T,1} \propto \left[W^{\alpha\nu} W_{\mu\beta} \right] \times \left[W^{\mu\beta} W_{\alpha\nu} \right]$$

$$\ldots$$

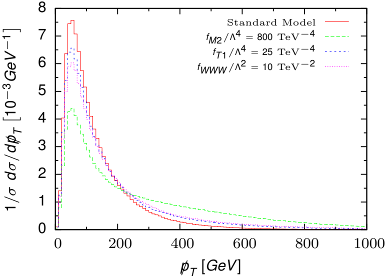
(at least) four gauge fields in each term $\rightarrow$ modify quartic gauge couplings

triple gauge couplings contribute as well

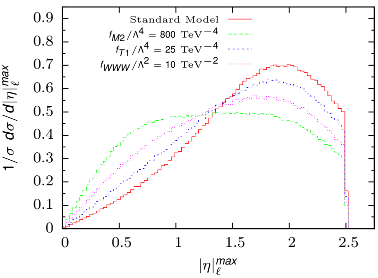


Anomalous quartic gauge couplings

Normalized $\not{p}_t$ distribution



Normalized $|\eta|_\ell^{max}$ distribution



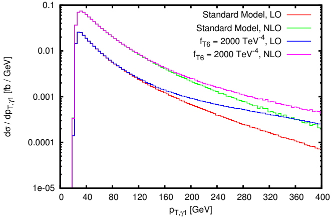
- Anomalous couplings enhance predominantly high-energy region
- $\Delta\sigma \sim \mathcal{O}(1-4\%)$ for total cross section,
  $\Delta\sigma \sim \mathcal{O}(20-100\%)$ in high-energy region, $m_{WW}^T > 800$ GeV
- Visible changes in distributions, different for individual couplings
- $\rightarrow$ distinguish between different couplings

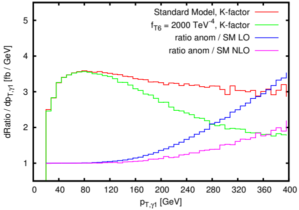
NLO QCD vs. anomalous couplings

Example: $W\gamma\gamma$ with VBFNLO, $\frac{f_{T,6}}{\Lambda^4} = 2000\ \text{TeV}^{-4}$, $\Lambda_{FF} = 1606$ GeV, $n_{FF} = 4$

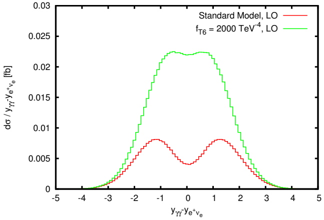
[SM: Bozzi, Campanario, MR, Zeppenfeld; Snowmass White Paper on aQGC]

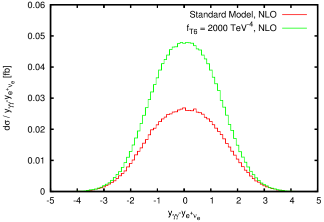
Transverse momentum distribution of the hardest photon





Rapidity difference photon pair – W for $p_{T,\gamma_1} > 200$ GeV





aQGC break radiation zero already at tree-level $\Rightarrow$ different NLO K-factors

Form factor tool

Contribution of higher-dimensional operators can violate unitarity above certain energy scale → unphysical

- Determine energy scale of unitarity violation → Partial-wave analysis
  - Consider amplitudes for on-shell $VV \to VV$ scattering ($V \in W, Z, \gamma$)
  - Decompose into series of partial waves with coefficients a_i, $i = 0, 1, 2, \ldots$
  - → Condition for unitarity conservation: $|\mathrm{Re}(a_i)| < \frac{1}{2}$
  - Strongest bound typically from $i = 0$ → check only this contribution

  ⇒ maximal energy scale Λ_{max}
- Ensure unitarity at higher energies by applying form factor
  - Unitarity preserved by new-physics contributions entering at or before Λ_{max} → acts as cut-off
  - effective implementation in low-energy theory ⇒ form factor
  - explicit form model-dependent → choice arbitrary
  - VBFNLO: dipole form factor

$$\mathcal{F}(s) = \frac{1}{\left(1 + \frac{s}{\Lambda_{\mathrm{FF}}^2}\right)^n}$$

Λ_{FF}^2, n: free parameters

- Determine maximal Λ_{FF} from given anomalous couplings, n and maximum energy considered

→ implemented in form factor tool available from VBFNLO web site
`http://www.itp.kit.edu/~vbfnloweb/wiki/doku.php?id=download:formfactor`

Example output

```
[...]
 Reading in anomalous couplings parameter:
   SQRT_S                = 14000.
   FFEXP                 = 2.0000
   FS0                   = 0.10000E-09
   FS1                   = 0.10000E-09
[...]
 Checking tree-level unitarity violation with on-shell W+W- -> W+W- scattering
 using the largest helicity combination of the zeroth partial wave...
[...]
 Checking tree-level unitarity violation with on-shell VV->VV scattering
 including all Q=0 channels involving W and Z bosons using the largest
 helicity combination of the zeroth partial wave...
[...]
 Results for each channel, taking only the helicity combination with the largest
 contribution to the zeroth partial wave into account:

 FFscale_WWWW =     688. GeV    ( without FF: |Re(pwave_0)| > 0.5 at   0.8 TeV )
[...]
 No tree-level unitarity violation in W+W- -> AA scattering found.
[...]
 Results for each channel, taking contributions from all helicity combinations to
 the zeroth partial wave into account by diagonalizing the T-matrix:

 FFscale_WWWW_diag =     688. GeV    ( without FF: |Re(pwave_0)| > 0.5 at   0.8 TeV )
[...]
 FFscale_VVVV_Q_0 =     622. GeV    ( without FF: |Re(pwave_0)| > 0.5 at   0.7 TeV )
[...]
```

Two-Higgs Model

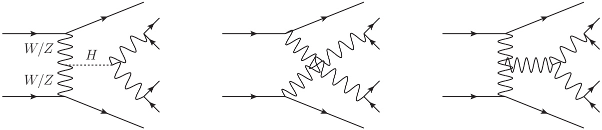
[New in VBFNLO 2.7.0: MR]

Two Higgs model for VBF processes

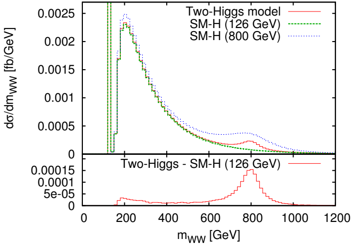
Search for heavy Higgs bosons:

- significant signal-background interference due to large Higgs width
- What defines "background"?

  $\leftrightarrow$ continuum-only diagrams violate unitarity $\leftrightarrow$ 126 GeV Higgs



$\rightarrow$ Model with two Higgs resonances



Example:

- h_0: M_{h_0}=126 GeV,
  $g^2_{h_0 VV}/g^2_{HVV,SM} = 0.7$
- H_0: M_{H_0}=800 GeV,
  $g^2_{H_0 VV}/g^2_{HVV,SM} = 0.3$

$\rightarrow$ Consistent definition possible

Reweighting events (REPOLO)

[F. Schissler, available on request]

Generating events at detector-level time-consuming (shower, detector simulation, . . .)

$\rightarrow$ Reuse SM Higgs events and reweight for different BSM scenarios

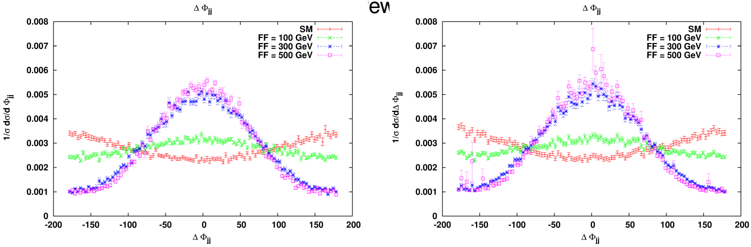
$\rightarrow$ REPOLO (REweighting POwheg events at Leading Order)

uses VBFNLO framework to multiply each event by a factor $\frac{|\mathcal{M}_{\text{BSM}}|^2}{|\mathcal{M}_{\text{SM}}|^2}$

Limitation:

event with high reweighting factor ($|\mathcal{M}_{\text{SM}}|^2 \ll |\mathcal{M}_{\text{BSM}}|^2$) can destroy distributions
$\rightarrow$ only SM-like distributions can be safely reweighted

Example: VBF-$H \rightarrow \gamma\gamma$, SM $\rightarrow$ anomalous Higgs couplings ($+HW_+^{\mu\nu}W_{\mu\nu}^-$, $HZ^{\mu\nu}Z_{\mu\nu}$)



$\Rightarrow$ distributions correctly reproduced, larger errors in SM-suppressed regions

Conclusions

- New processes in VBFNLO 2.7.0
  - QCD-$WZjj$, $W\gamma jj$, $W^{\pm}W^{\pm}jj$ at NLO QCD
    fast implementation in VBFNLO
  - Higgs pair production via VBF
- New features in VBFNLO 2.7.0
  - Semi-leptonic decays for $VVjj$, VV, VVV
  - Anomalous quartic gauge couplings
  - Form factor tool
    determine energy bound for unitarity violation in anomalous gauge couplings
  - Two-Higgs model for diboson-VBF processes

VBFNLO is a flexible parton-level Monte Carlo for processes with electro-weak bosons

Code available at `http://www.itp.kit.edu/vbfnlo`

VBFNLO is collaborative effort:

K. Arnold, J. Baglio, J. Bellm, G. Bozzi, M. Brieg, F. Campanario, C. Englert, B. Feigl, J. Frank, T. Figy, F. Geyer, N. Greiner, C. Hackstein, V. Hankele, B. Jäger, N. Kaiser, M. Kerner, G. Klämke, M. Kubocz, L.D. Ninh, C. Oleari, S. Palmer, S. Plätzer, S. Prestel, MR, R. Roth, H. Rzehak, F. Schissler, O. Schlimpert, M. Spannowsky, M. Worek, D. Zeppenfeld

Contact: `vbfnlo@itp.kit.edu`

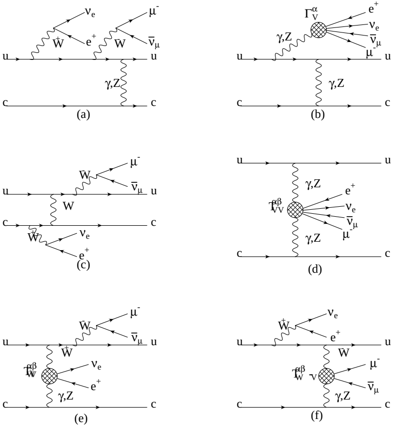
Diboson-VBF production

[Bozzi, Jäger, Oleari, Zeppenfeld; hep-ph/0603177, hep-ph/0604200, hep-ph/0701105]
[Denner, Hosekova, Kallweit (W^+W^+)]

- Part of the NLO wishlist
  [Les Houches 2005]
- background to Higgs searches
- access to anomalous triple and quartic gauge couplings

Implementation:

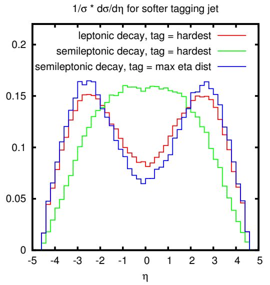
- modular structure
  $\rightarrow$ reuse building blocks
- leptonic decays included
- only t- and u-channel diagrams (s-channel implemented separately as triboson process)
- no interference effects from identical leptons



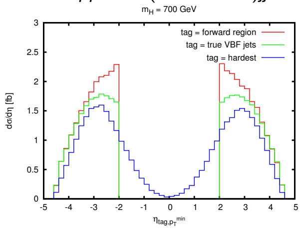
Tagging jet definition

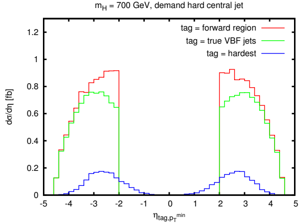
- two widely separated jets characteristic VBF signature
- leptonic decays: two hardest jets
- semileptonic decays: jet can come from vector boson decay $\Rightarrow$ not a good choice
- alternatives:
  - two jets with largest distance in rapidity
    $\rightarrow$ works for signal, but bad background rejection
  - separate phase space explicitly:
    - tagging jets: $|\eta_{\text{tag}}| > \eta_c$, $\eta_1 \times \eta_2 < 0$
    - vector boson decay products: $|\eta_{\text{decay}}| < \eta_c$
    - require high-p_T jet in central region

$pp \rightarrow W^+W^- jj$ via VBF, inclusive cuts



$pp \rightarrow H(\rightarrow W^+W^-)jj$





Scale variation

$pp \to e^+ \nu_e \mu^+ \mu^- jj$, $\sqrt{S} = 14$ TeV

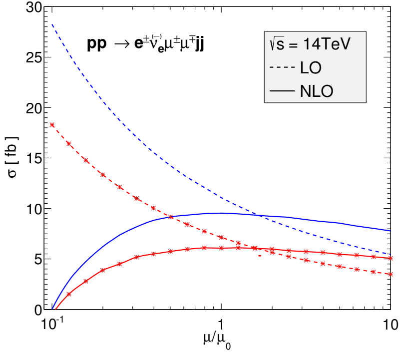
- cuts

$$\begin{aligned} p_{T,j} &> 20\text{ GeV} & |\eta_j| &< 4.5 \\ p_{T,\ell} &> 20\text{ GeV} & |\eta_\ell| &< 2.5 \\ m_{\ell^+\ell^-} &> 15\text{ GeV} & \not{p}_T &> 30\text{ GeV} \\ R_{jj} &> 0.4 & R_{\ell\ell} &> 0.4 \\ R_{j\ell} &> 0.4 \end{aligned}$$

- PDF
  MSTW 2008 with $n_F = 4$
- scale

$$\mu = \mu_F = \mu_R = \frac{1}{2}\left(\sum_{\text{jet}} p_{T,\text{jet}} + E_{T,W} + E_{T,Z}\right)$$

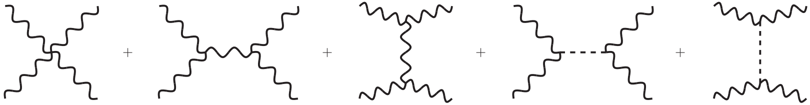
with $E_{T,V} = \sqrt{p_{T,V}^2 + m_V^2}$



→ scale dependence strongly reduced

Continuum-Higgs interference

Four-vector-boson leptonic tensors contain Higgs resonance



necessary for correct high-energy behaviour (otherwise unitarity violation at $\sim$ 1 TeV)

Notation: $\mathcal{M}_H \sim \frac{s}{v^2}$ Signal amplitude for s-, t- and u-channel exchange of H

$\mathcal{M}_B \sim \frac{-s}{v^2}$ continuum electroweak background amplitude

- Light Higgs bosons:
  - sharp resonance at m_H
  - basically no signal-background interference
- Heavy Higgs bosons:
  - width becomes large (Γ_H^{OS} =123 (304, 647) GeV at $m_H = 600$ (800, 1000) GeV)
  - significant signal-background interference
  - What defines background?

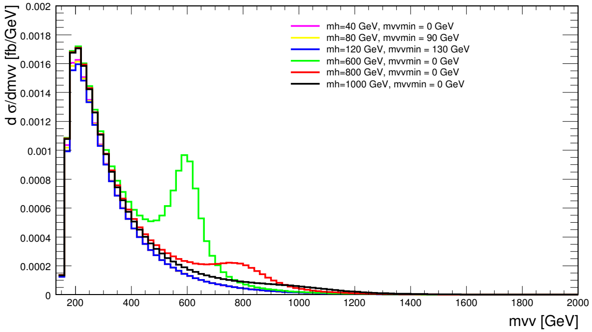
Continuum-Higgs interference

What defines background?

$B = \int d\Phi |\mathcal{M}_B|^2$ or $S = \int d\Phi \left[|\mathcal{M}_H|^2 + 2\mathrm{Re}\mathcal{M}_H\mathcal{M}_B^*\right]$ violate unitarity at large s

$\Rightarrow$ Compare to SM light Higgs scenario

VBF-WW cross section for different m_H ($m_{VV,\mathrm{min}}$ to remove Higgs peak)

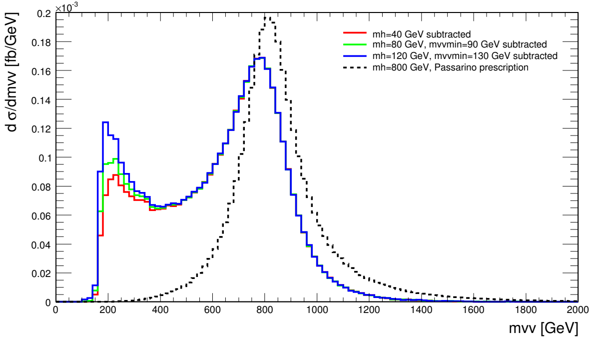


$\rightarrow$ light-Higgs curves indistinguishable at large m_{VV}

Continuum-Higgs interference

- Define $S = \int d\Phi |\mathcal{M}_B + \mathcal{M}_H(m_H)|^2 - B$ with $B = \int d\Phi |\mathcal{M}_B + \mathcal{M}_h(m_h)|^2$
- Integrate over suitable mass range $[m_H - \Gamma_1, m_H + \Gamma_2]$
- $\Rightarrow$ S and B well defined and do not violate unitarity

$m_H = 800$ GeV, Passarino prescription vs. subtraction at different m_h



- Some light-Higgs mass dependence in threshold region around $m_{VV} = 200$ GeV
$\Rightarrow$ eliminate by cuts
- True resonance shape is not reproduced by modified Breit-Wigner distribution

Matching with Parton Shower

[work in progress: Arnold, Plätzer, MR et al.]

Karlsruhe Institute of Technology

VBFNLO interfaced to Herwig++

requires some process-dependent modifications
⇒ currently only VBF-Higgs production, further processes will follow

→ Matching NLO calculations with parton shower

NLO calculation

- normalization correct to NLO
- additional jet at high-p_T accurately described
- theoretical uncertainty reduced
- low-p_T jet emission badly modelled
- parton level description

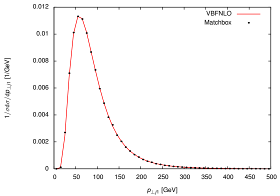
LO + parton shower

- LO normalization only
- further high-p_T jets badly described
- Sudakov suppression at small p_T
- events at hadron level possible

⇒ Herwig++ package Matchbox [Gieseke, Plätzer]

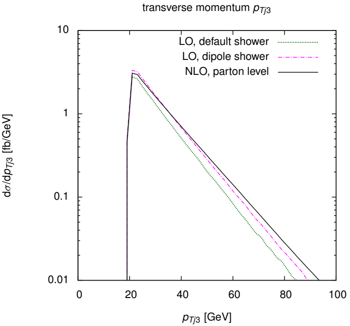
Parton shower based on Catani-Seymour dipoles

Matching methods: MC@NLO and POWHEG

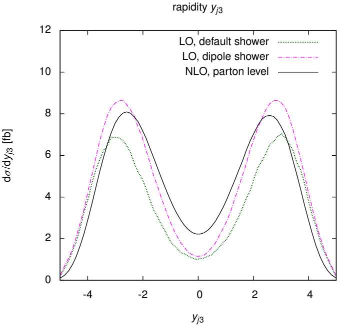


NLO validation plot

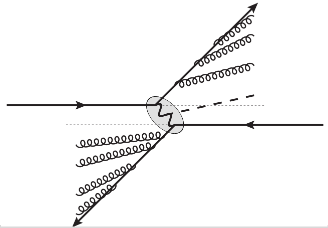
Shower effects at LO



Transverse momentum of third hardest jet

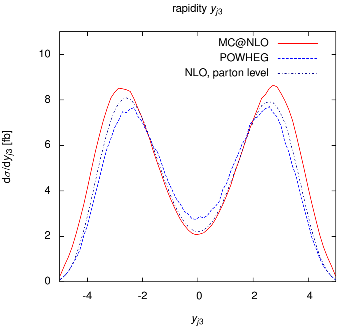


Rapidity of third hardest jet

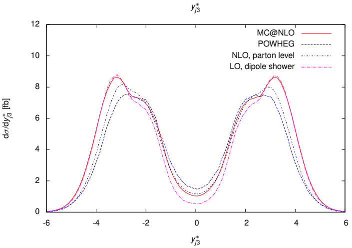


- additional radiation by shower created mainly between jets and beam axis (color connections)
- dipole shower "interpolates" between NLO behavior in central region and shower behavior at small angles

NLO matching



Rapidity of third hardest jet



$$y^*_{j3} = y_{j3} - \frac{1}{2}\left(y_{j1} + y_{j2}\right)$$