Topological phases and surface states:

Realizing Majorana fermions in superconducting hybrid systems

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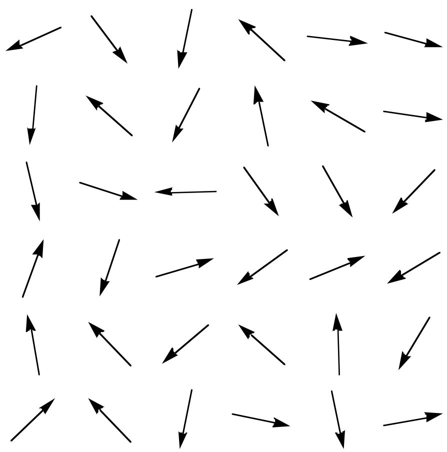
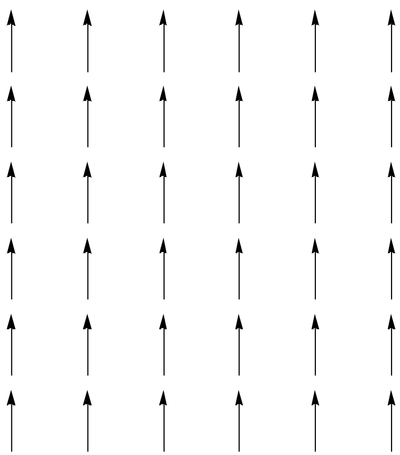
phases of condensed matter

disordered state
paramagnet



$T > T_c$

ordered state
ferromagnet



$T < T_c$

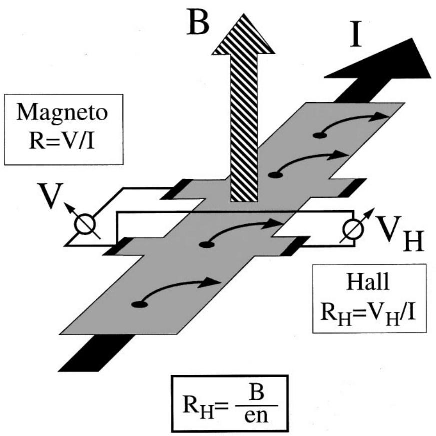
order parameter: *magnetization*

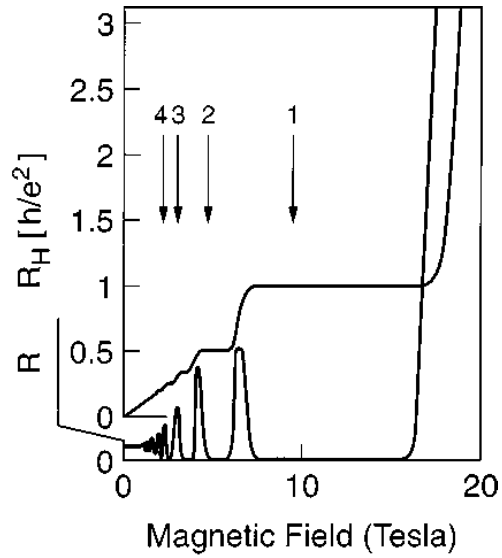
spontaneously broken symmetry: *time-reversal symmetry*

many examples: superconductivity, superfluidity, ferroelectricity, liquid crystals…

Landau theory of phase transitions

The integer quantum Hall effect: a topological phase

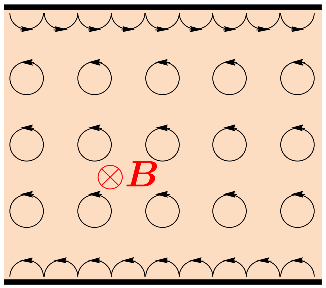




Quantized Hall conductivity:

$$R_H = \frac{h}{e^2}\frac{1}{n} \qquad n = 1, 2, 3\ldots$$

(filling factor)



- insulating bulk (Landau levels)
- n topologically protected metallic edge states (skipping orbits)
- n cannot be changed without closing an energy gap

Since ~ 2005, many examples of such topological phases

Outline

1. **Topological insulators and edge states**
    - topological band theory
    - bulk-edge correspondence
    - example: quantum spin Hall insulator
2. **Topological superconductors and Majorana fermions**
    - Bogoliubov quasiparticles and Majorana fermions
    - models of topological superconductors
    - fractional Josephson effect

1. topological insulators and edge states

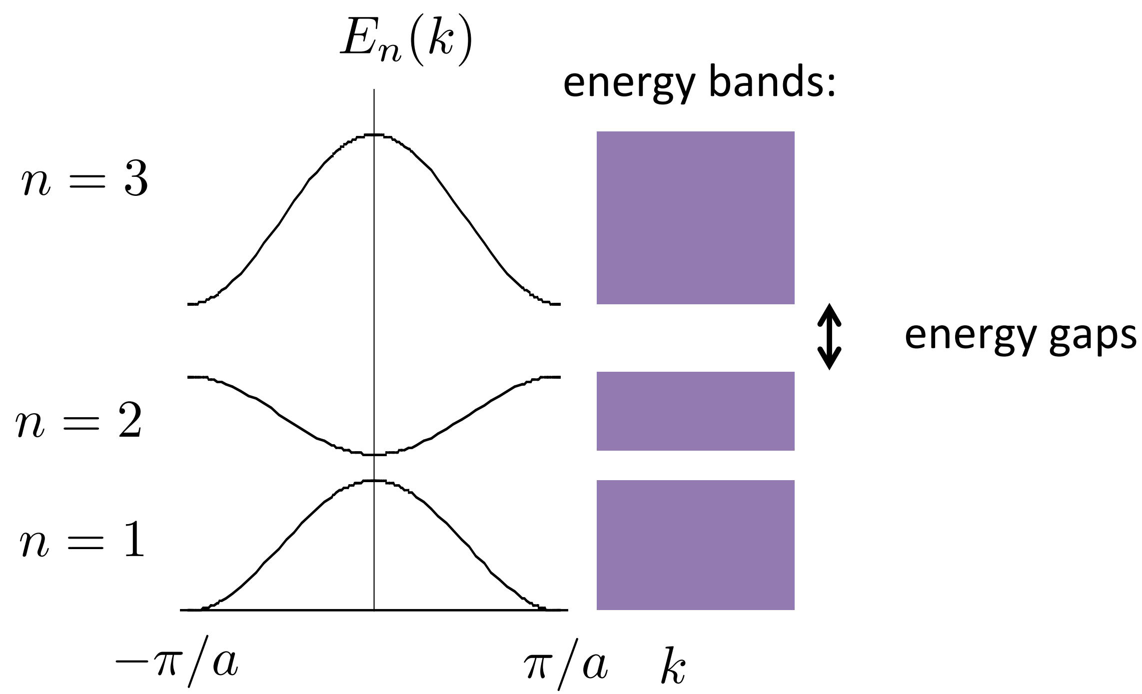
Bloch band theory

Schrödinger equation for electrons in a periodic potential $U(x+a) = U(x)$

$$\left[\frac{p^2}{2m} + U(x)\right]\psi(x) = E\psi(x)$$

Bloch theorem: $\psi_{n,k}(x) = e^{ikx}u_{n,k}(x)$ with $u_{n,k}(x+a) = u_{n,k}(x)$

quasi-momentum: $-\dfrac{\pi}{a} < k \leq \dfrac{\pi}{a}$ (Brillouin zone)

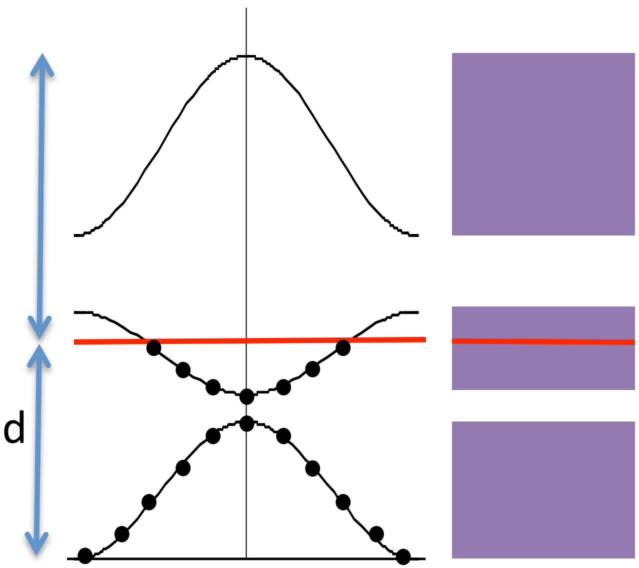


Metals vs. insulators

electrons are fermions ⟶ Pauli principle

metals

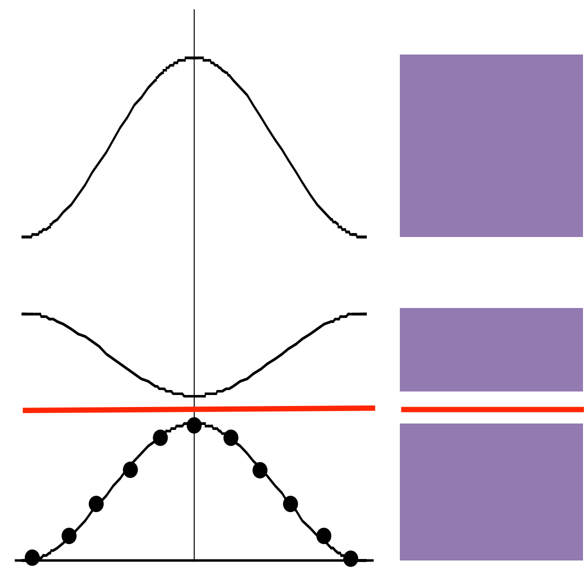
empty states



occupied states

Fermi level

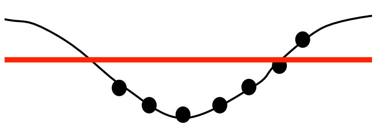
insulators



conduction band

valence band

response to an electric field:



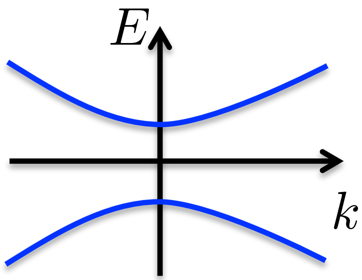
flow of electric current

electrons cannot move

Not all insulators are similar!

Topology of the Fermi surface

two-band model in 2D
Dirac spectrum with a mass term



Dirac cone

$$H = v(k_x\sigma_x + k_y\sigma_y) + m\sigma_z$$

$$E_\pm(\boldsymbol{k}) = \pm\sqrt{v^2\boldsymbol{k}^2 + m^2}$$

$$H = \boldsymbol{h}(\boldsymbol{k}).\boldsymbol{\sigma}$$

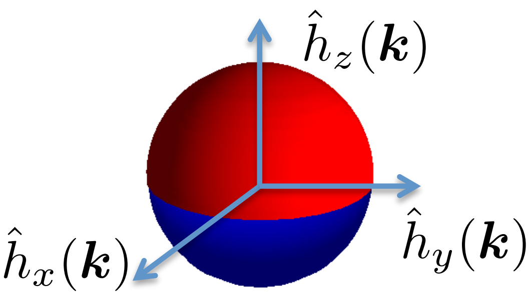
$$\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

$$\hat{\boldsymbol{h}}(\boldsymbol{k}) = \frac{\boldsymbol{h}(\boldsymbol{k})}{|\boldsymbol{h}(\boldsymbol{k})|}$$

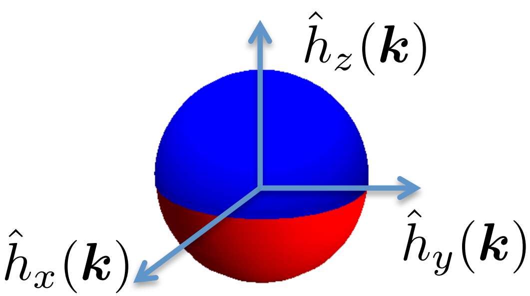
Berry's curvature:

$$n = \frac{1}{4\pi}\int d^2k\,\hat{\boldsymbol{h}}(\boldsymbol{k}).\left[\frac{\partial\hat{\boldsymbol{h}}(\boldsymbol{k})}{\partial k_x}\times\frac{\partial\hat{\boldsymbol{h}}(\boldsymbol{k})}{\partial k_y}\right]$$

= solid angle swept by $\hat{\boldsymbol{h}}(\boldsymbol{k})$



$m > 0$
$n = 1/2$



$m < 0$
$n = -1/2$

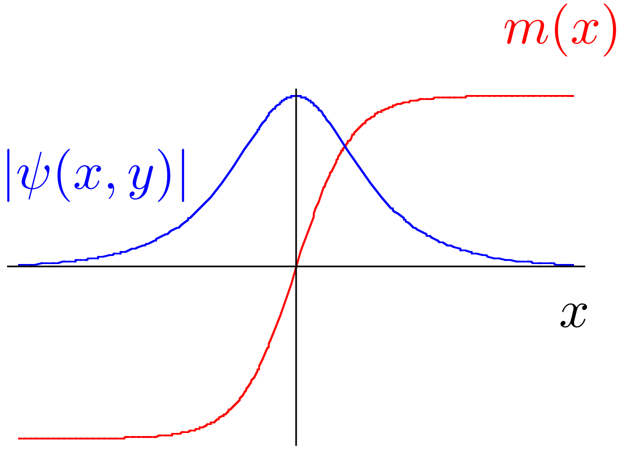
Fermion doubling theorem: even number of Dirac Cones $\rightarrow$ $(\Sigma\, n)$ is integer

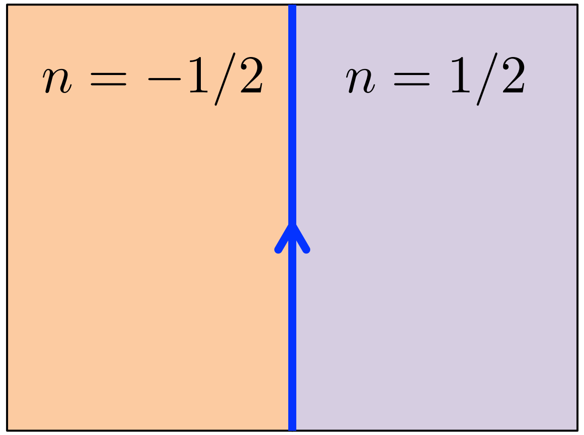
Bulk-boundary correspondence

$$H = -i\hbar v \left(\sigma_x \partial_x + \sigma_y \partial_y\right) + m(x)\sigma_z$$

$$\psi(x,y) \propto \begin{pmatrix} 1 \\ i \end{pmatrix} e^{ik_y y} e^{\frac{1}{\hbar v}\int^x m(x)dx}$$

$$E_0(k_y) = \hbar v k_y$$





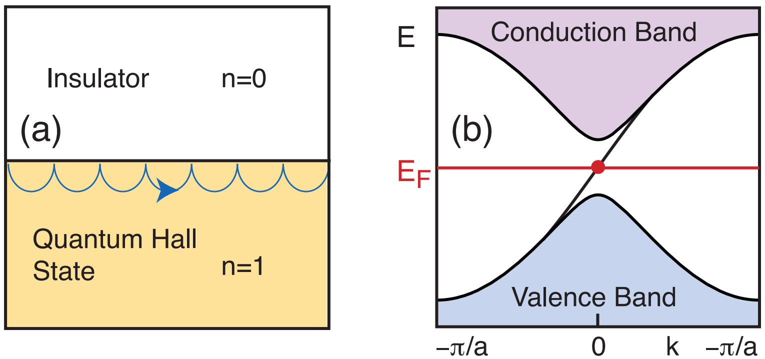
$$\Delta n = 1$$

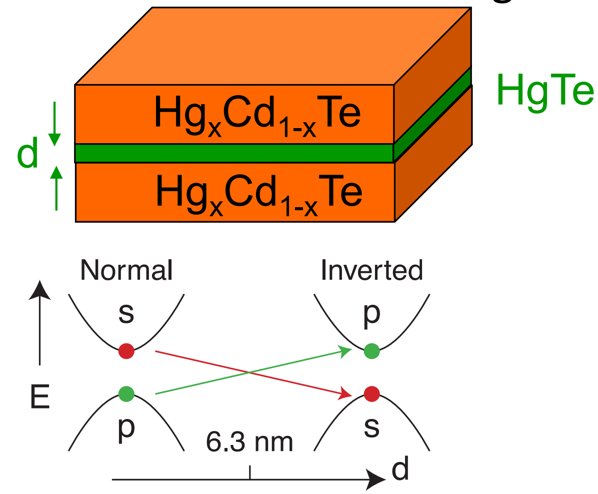
1 chiral metallic edge state
robust to (smooth) potential disorder

Quantum spin Hall insulators

König et al, 2007

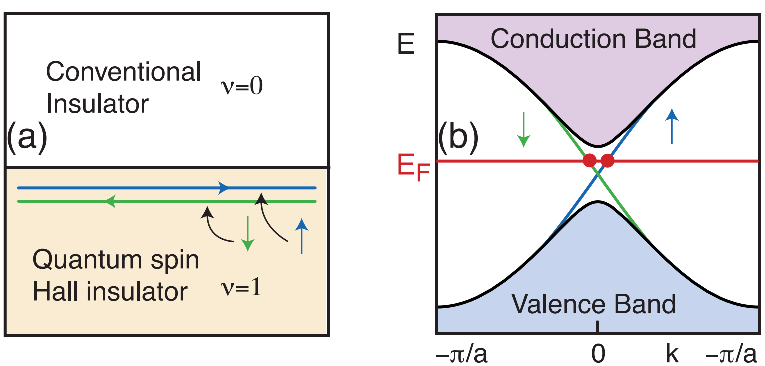
Quantum Hall insulator
Time-reversal symmetry broken

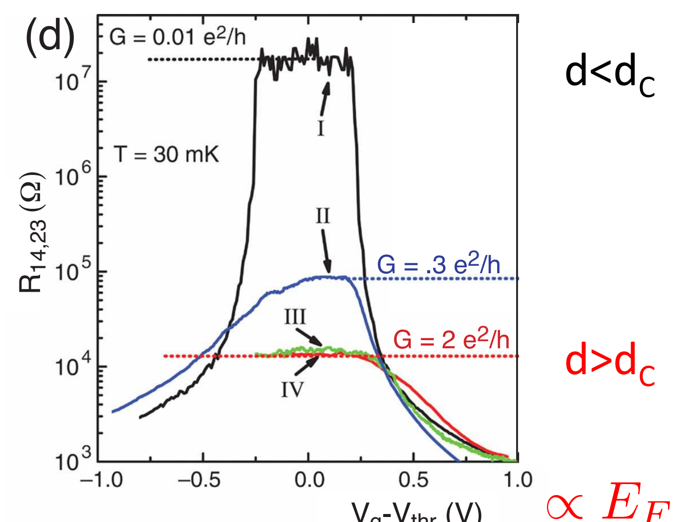




energy gap $E_c - E_v$

Quantum spin Hall insulator
Time-reversal symmetry preserved





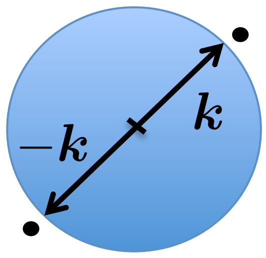
2. Topological superconductors and Majorana fermions

Superconductivity

Bardeen-Cooper-Schrieffer theory (1957)

attractive interaction between electrons in metals

➢ Cooper instability: binding of two electrons



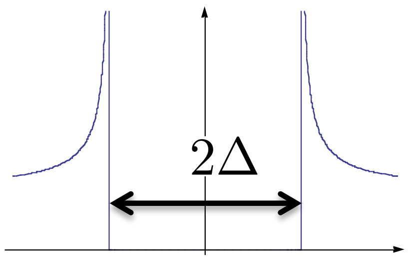
Fermi sea

➢ Cooper pairs condense into a superfluid ground-state

zero-resistance state

➢ energy gap in the excitation spectrum

density of states



energy

specific heat

$$C \sim e^{-\Delta/T} \qquad T \ll \Delta$$

Bogoliubov-de Gennes theory

BCS Hamiltonian (for spinless electrons)

$$\mathcal{H} = \sum_k \left(\xi_k c_k^\dagger c_k + \underbrace{\Delta_k c_k^\dagger c_{-k}^\dagger + \Delta_k^* c_{-k} c_k}_{\textit{pair potential}} \right) \qquad \xi_k = \epsilon_k - \mu$$

BdG formulation

$$\mathcal{H} = \frac{1}{2} \sum_k \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} \begin{pmatrix} \xi_k & \Delta_k \\ \Delta_k^* & -\xi_k \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} + \mathrm{cst}$$

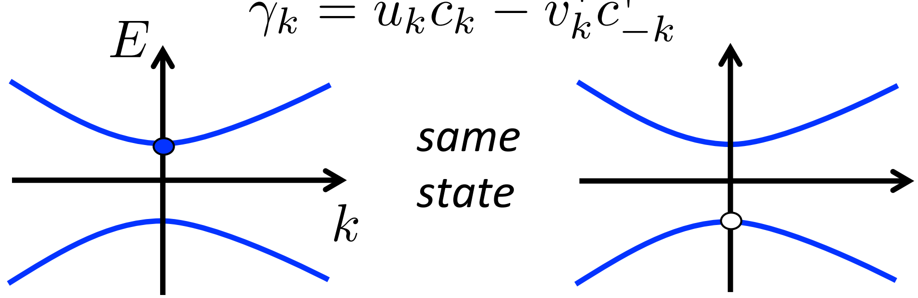
particle-hole spinor

the energy spectrum is gapped and particle-hole symmetric:

$$E_k = \pm\sqrt{\xi_k^2 + |\Delta_k|^2}$$

elementary excitations = superposition of electron creation and annihilation

$$\gamma_k = u_k c_k - v_k^* c_{-k}^\dagger$$



$$\gamma(-E) = \gamma^\dagger(E)$$

Majorana fermions

particle-hole symmetry in superconductors

$$\gamma(-E) = \gamma^\dagger(E)$$

$$E = 0 : \quad \gamma(0) = \gamma^\dagger(0)$$

$\left\{ \begin{array}{l} \text{no charge} \\ \text{no spin} \end{array} \right.$

(Dirac) fermions:

$$\{c_i, c_j^\dagger\} = \delta_{ij} \qquad \{c_i, c_j\} = 0$$

Majorana fermions

$$\Gamma_i = \Gamma_i^\dagger$$

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}$$

- conjectured by Majorana (1937)
- no known elementary particle (maybe neutrino?)

1 fermion = 2 Majoranas

$$c = \frac{1}{2}(\Gamma_1 + i\Gamma_2) \qquad c^\dagger = \frac{1}{2}(\Gamma_1 - i\Gamma_2)$$

Kitaev 1D toy model

N sites:

Kitaev, 2001

$$\mathcal{H} = \sum_n \left[-(t c_n^\dagger c_{n+1} + \text{h.c.}) - \mu c_n^\dagger c_n + (\Delta c_n^\dagger c_{n+1}^\dagger + \text{h.c.}) \right]$$

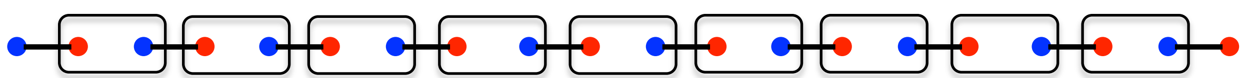
hopping *pairing* (p-wave)

$$c_n = (\Gamma_{L,n} + i\Gamma_{R,n})/2$$
$$c_n^\dagger = (\Gamma_{L,n} - i\Gamma_{R,n})/2$$

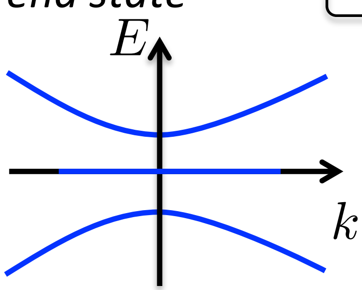
specific choice: $\mu = 0$, $t = \Delta$

$$\mathcal{H} = -it \sum_{n=1}^{N-1} \Gamma_{R,n}\Gamma_{L,n+1}$$

Majorana end state



Majorana end state

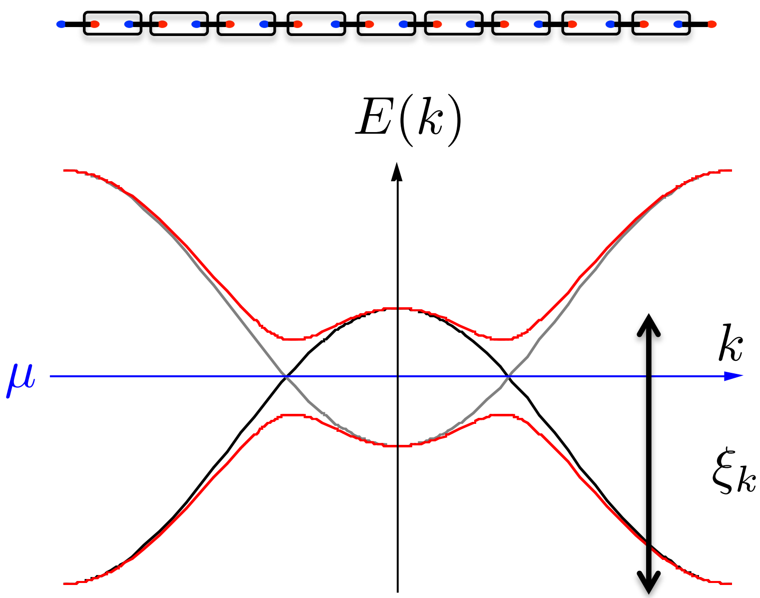


zero-energy non-local fermion: $$d^\dagger d = \frac{1}{2}(1 + i\Gamma_{L,1}\Gamma_{R,N})$$

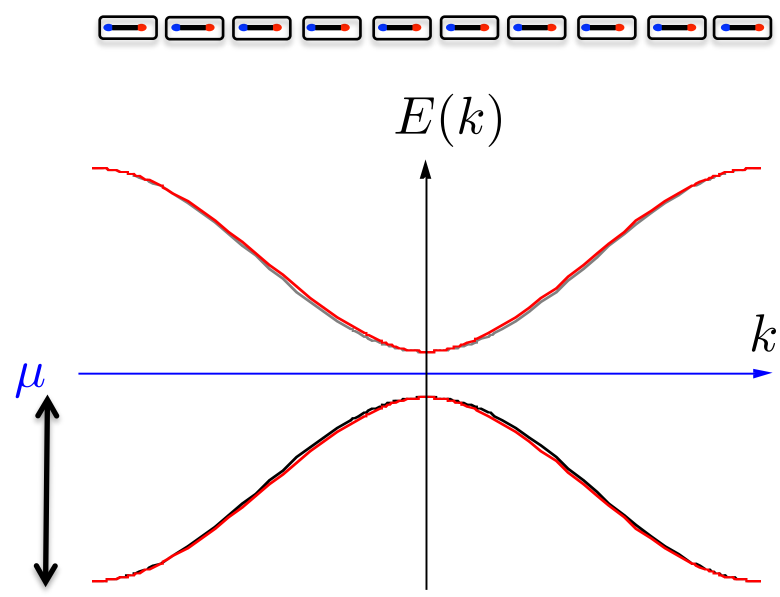
Kitaev model and topology

$|\mu| < 2t$

$|\mu| > 2t$



$$\xi_k = 4t\cos k$$



n=1 (topological)

n=0 (trivial)

$\mathbb{Z}_2$ topological index = # crossing of the Fermi surface (k>0)

= # pairs of Majorana end states

$p+ip$ two-dimensional superconductivity

Read and Green, 2000

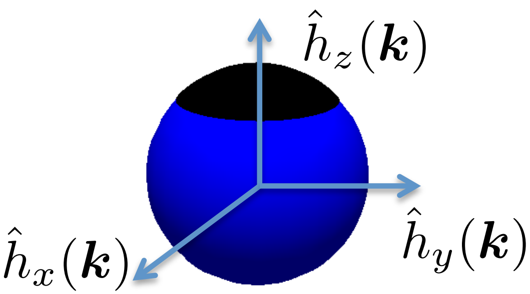
$$\mathcal{H} = \int d^2r \left\{ \psi^\dagger(r) \left(-\frac{\partial_x^2 + \partial_y^2}{2m} - \mu \right) \psi(r) - i\Delta \left[\psi(r) \underbrace{(\partial_x + i\partial_y)}_{p+ip} \psi(r) + \text{h.c.} \right] \right\}$$

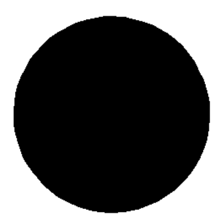
$$\mathcal{H} = \boldsymbol{h}(\boldsymbol{k}).\boldsymbol{\tau} \qquad \boldsymbol{\tau} = (\tau_x, \tau_y, \tau_z) \qquad \begin{cases} h_x(\boldsymbol{k}) = \Delta k_x \\ h_y(\boldsymbol{k}) = \Delta k_y \\ h_z(\boldsymbol{k}) = \dfrac{\boldsymbol{k}^2}{2m} - \mu \end{cases}$$

$\mathbb{Z}$ topological index:

$$\hat{\boldsymbol{h}}(\boldsymbol{k}) = \frac{\boldsymbol{h}(\boldsymbol{k})}{|\boldsymbol{h}(\boldsymbol{k})|} \qquad n = \frac{1}{4\pi} \int d^2k \, \hat{\boldsymbol{h}}(\boldsymbol{k}). \left[\frac{\partial \hat{\boldsymbol{h}}(\boldsymbol{k})}{\partial k_x} \times \frac{\partial \hat{\boldsymbol{h}}(\boldsymbol{k})}{\partial k_y} \right]$$

$\mu < 0$ $\quad n = 0$ (trivial) $\qquad \mu > 0$ $\quad n = 1$ (topological)





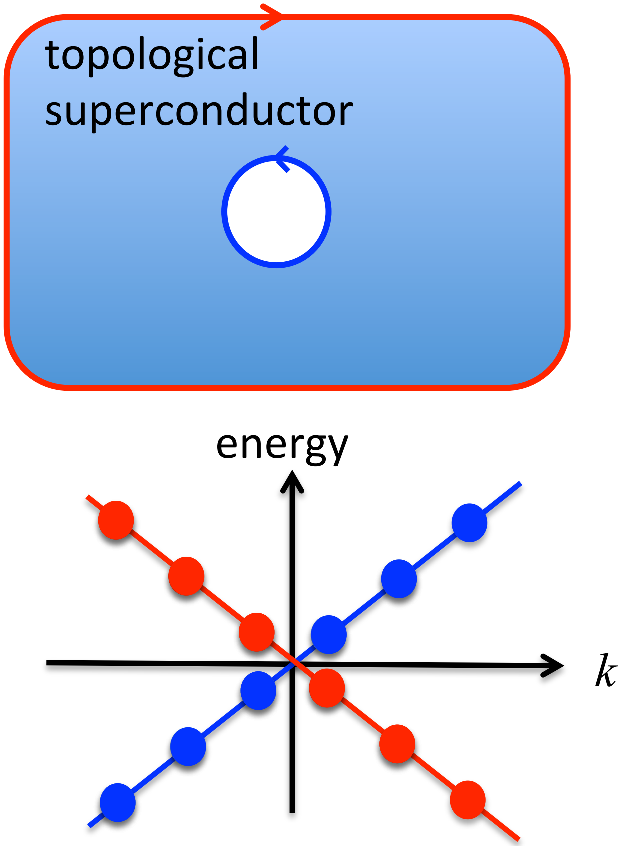
$n = 2\nu + 1$ with

$\Delta(\boldsymbol{k}) \propto (k_x + ik_y)^{2\nu+1}$

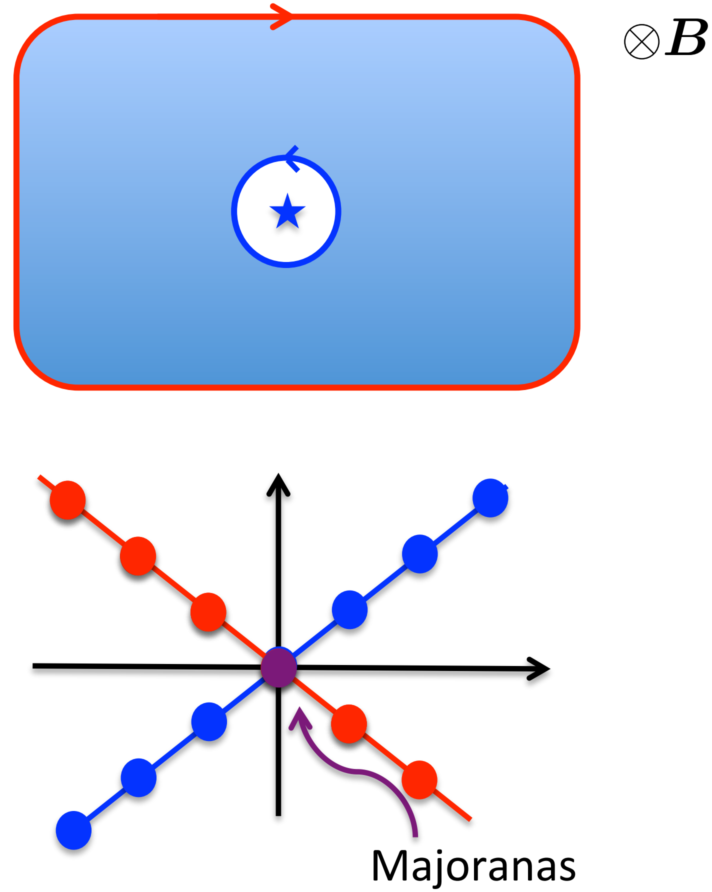
Majorana modes and vortex core states

Read and Green, 2000

hole



quantized flux
through a vortex $\Phi = \dfrac{hc}{2e}$



non-abelian statistics

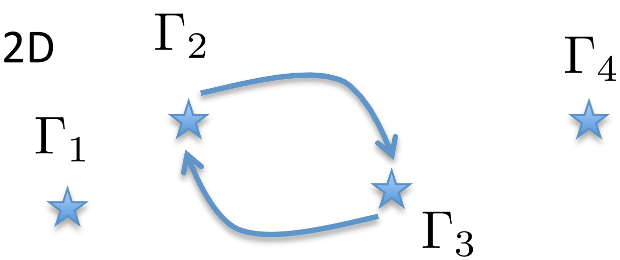
Ivanov, 2003

2N Majoranas $\Gamma_1, \Gamma_2, \ldots, \Gamma_{2p}, \Gamma_{2p+1}, \ldots$

N fermions $\ldots, d_p = \frac{1}{2}(\Gamma_{2p} + i\Gamma_{2p+1}), \ldots$

2^N degeneracy of the ground state $|n_1, \ldots, n_N\rangle$ $n_p = d_p^\dagger d_p$

clockwise exchange of two (vortex core) Majoranas in 2D



$$U_{23}\Gamma_2 U_{23}^\dagger = +\Gamma_3$$

$$U_{23}\Gamma_3 U_{23}^\dagger = -\Gamma_2$$

braiding of exchanges → rotations within the ground-state manifold

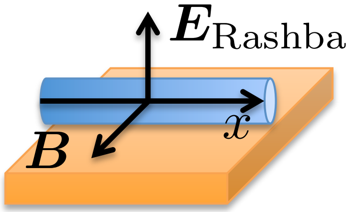
prospects for adiabatic quantum computing

proximity-induced topological superconductivity

semiconducting nanowire

Lutchyn et al., 2010;
Oreg et al, 2010

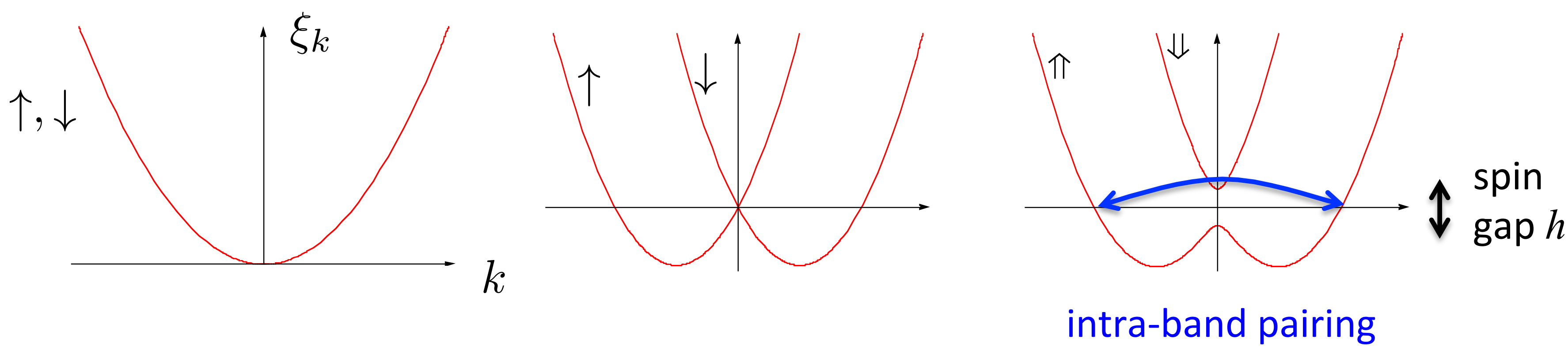
$$\mathcal{H} = \int dx \left[\psi^\dagger(x)\left(\frac{p_x^2}{2m} - \mu + \alpha\sigma_z p_x + h\sigma_y\right)\psi(x) + \Delta\left(\psi_\uparrow\psi_\downarrow + \text{h.c.}\right)\right]$$



Rashba spin-orbit coupling

transverse Zeeman field

effective pairing induced by proximity with a conventional superconductor

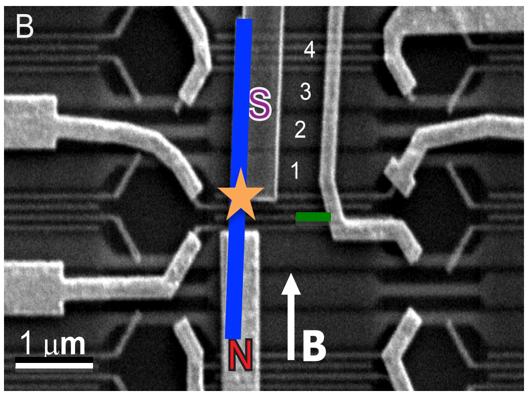


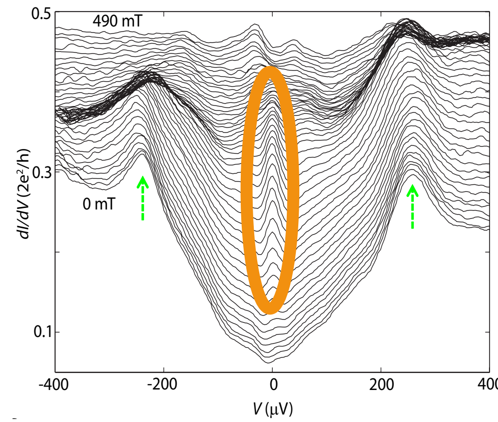
topological if $\sqrt{\Delta^2 + \mu^2} < h$

Zero-bias anomaly

Mourik et al., 2013

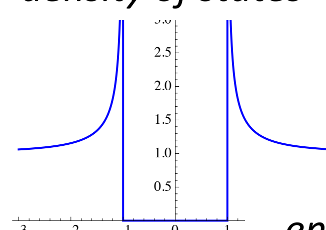
Au/InSb/NbTi





the tunneling current probes
the local density of state

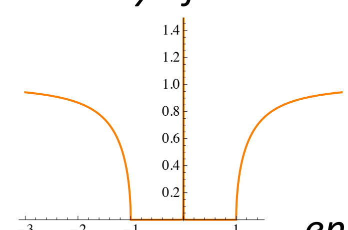
density of states



energy

trivial

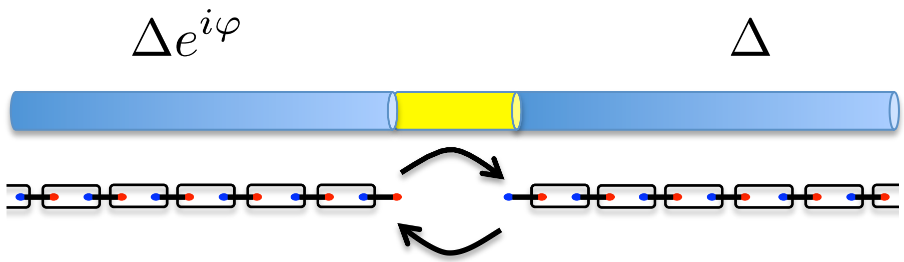
density of states



energy

topological

topological Josephson junctions



$$d = (\Gamma_R + i\Gamma_L)/2$$
$$d^\dagger = (\Gamma_R - i\Gamma_L)/2$$

$$\mathcal{H}_J = i2t' \cos\left(\frac{\varphi}{2}\right) \Gamma_L \Gamma_R = t' \cos\left(\frac{\varphi}{2}\right)\left(d^\dagger d - \frac{1}{2}\right)$$

$$d^\dagger d = 0 \,\mathrm{or}\, 1$$

$$E_J(\varphi) = \mp \frac{t'}{2} \cos\left(\frac{\varphi}{2}\right)$$

equilibrium Josephson current

$$I(\varphi) = \frac{2e}{\hbar}\frac{\partial E_J}{\partial \varphi} = \pm \frac{et'}{2\hbar} \sin\left(\frac{\varphi}{2}\right)$$

conventional:

$$I \propto \sin\varphi$$

2π-periodic

4π-periodic current-phase relation
for a given occupation of the Majorana bound state

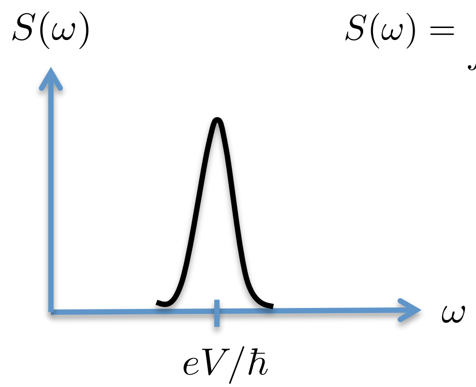
fractional ac Josephson effect

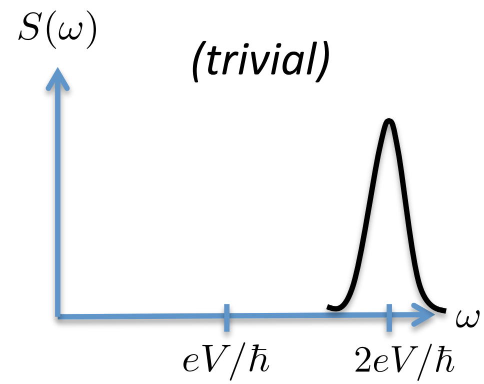
Kitaev, 2001
Kwon et al, 2004
Fu and Kane, 2009

$$I \propto \sin\left(\frac{\varphi}{2}\right) \qquad \dot{\varphi} = \frac{2e}{\hbar}V(t)$$

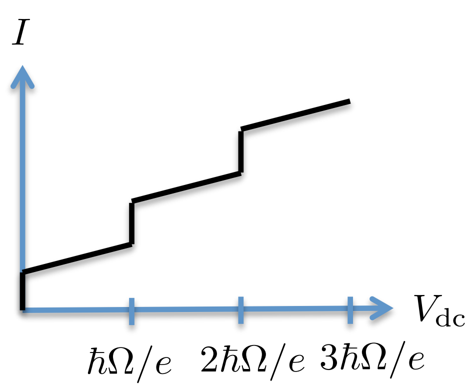
➢ Josephson radiation at half the Josephson frequency $\omega = eV_{\rm dc}/\hbar$

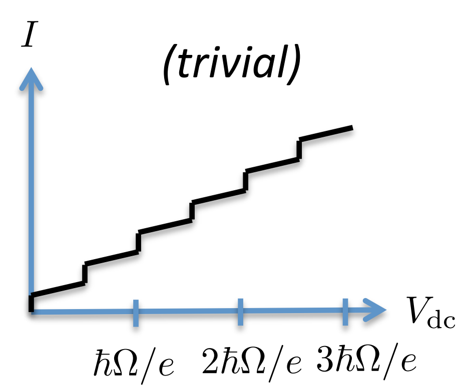
$$S(\omega) = \int d\tau e^{i\omega\tau}\overline{I(t+\tau)I(t)}$$





➢ absence of odd Shapiro steps in the presence of microwaves $V(t) = V_{\rm dc} + V_{\rm ac}\cos(\Omega t)$





Conclusion

➢ Surprises in the (non-interacting) band theory of solids

➢ topological insulating phases:
- insulating bulk characterized by a topological index
- robust metallic edge states
- quantized response function

➢ emergent Majorana fermions in hybrid junctions with conventional superconductors

General references

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- D.M. Badiane, MH, J.S. Meyer, PRL **107**, 177002 (2011)
- MH, J.S. Meyer, L.I. Glazman, arXiv:1303.4909