Theories of the glass transition

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- Description of glass forming liquids
- Theories of glass formation
- Ageing and nonequilibrium dynamics
- The jamming transition: « glass transition » at zero temperature.
The **liquid-glass transition** is observed in many polymers and other liquids that can be supercooled far below the melting point of the crystalline phase.

It is not a transition between thermodynamic ground states. Glass is a quenched disorder state, and its entropy, density, and so on, depend on the thermal history.

Therefore, the **glass transition is primarily a dynamic phenomenon**: on cooling a liquid, internal degrees of freedom successively fall out of equilibrium.

Some theoretical methods predict an underlying phase transition in the hypothetical limit of infinitely long relaxation times.

*No direct experimental evidence supports the existence of these transitions.*
Definition of a glass?

\[ \tau_{\text{micro}} \ll \tau_{\text{exp}} \ll \tau_{\text{relax}} \]

Time scale separation between microscopic, experimental, relaxation; the system is out of equilibrium on the experimental time scale.

(cf. S.K. Ma, Statistical Physics)
- **hard** glasses: (SiO2, bulk metallic glasses) Large shear modulus (Gpa); spectroscopy, X-ray and neutron scattering, calorimetry

- **soft** glasses: Colloids, foams, granular systems. Small elastic modulus (Mpa). Rheology, light scattering

-other examples: vortex glasses, spin glasses
Glass transition defined by typical viscosity $\eta$ (or relaxation time $\tau$) of $10^{13}$ Poise. Arbitrary but convenient

Arrhenius plot: log(time) or log(viscosity) versus $1/T$.  

Similar behaviour for relaxation times obtained using different methods (dielectric relaxation, NMR). $\alpha$ relaxation time $\tau_\alpha$
Angell’s classification: « strong » and « fragile » glasses

$\eta(T)$ defines $T_g$: $\eta(T_g) = 10^{13}$ Poise (arbitrary definition)

N.B.: $10^{13}$P corresponds to alpha relaxation time of 100 seconds

• $\log(\eta)$ vs $T_g/T$ : quantifies deviations from Arrhenius behaviour

Angell-plot (Uhlmann)

Vogel-Tamman-Fulcher

$$\eta(T) = \eta_0 \exp\left(\frac{B}{T - T_0}\right)$$

Bässler

$$\eta(T) = \eta_0 \exp\left(\frac{A}{T^2}\right)$$
Strong curvature: ‘fragile’ glass. Organic (OTP: Orthoterphenyl), ionic (CKN: CaK NO3); Energy barrier increases as T decreases, cooperative aspects.

Weak curvature: ‘strong’ glass. Covalent bonding, SiO2, ZnCl2, BeF2
What about the microscopic structure? Not much happens...

\[ S(q) = N^{-1} \left\langle \sum_k \sum_j \exp(i \mathbf{q} \cdot (\mathbf{r}_k - \mathbf{r}_j)) \right\rangle \]

Hard to see a « critical phenomenon » - or perhaps not the right order parameter.
Time dependent correlations: microscopic dynamics

\[ \phi_{AB}(t,t') = \langle A(t) B(t') \rangle \]

\[ \phi_{AB}(t,t') = \langle A(t) B(t') \rangle = \langle A(t-t') B(0) \rangle = \phi_{AB}(t-t') \]
Transformation to frequency domain

\( \phi''(\omega) \): imaginary part of FT(\( \phi(t) \))

\( \chi''(\omega) \): imaginary part of the susceptibility

\( \chi''(\omega) = \frac{\omega}{k_B T} \phi''(\omega) \)
Example: dielectric relaxation

Lunkenheimer et al. (2001)
Summary
• strong slowing down (temperature, pressure,...)
• transition to a nonergodic ‘phase’
• increase of the apparent free energy barriers with decreasing $T$
• no long range order
• no obvious length scale
• complex time dependent relaxation
• stretched alpha relaxation (Non Debye)
• thermodynamic “anomalies” associated with glass “transition”

⇒ at low temperature : ageing
Philip W. Anderson, a Nobel Prize-winning physicist at Princeton, wrote in 1995: "The deepest and most interesting unsolved problem in solid state theory is probably the theory of the nature of glass and the glass transition."

He added, "This could be the next breakthrough in the coming decade."

Thirteen years later, scientists still disagree, with some vehemence, about the nature of glass.

David A. Weitz, a physics professor at Harvard, joked, "There are more theories of the glass transition than there are theorists who propose them."

Dr. Weitz performs experiments using tiny particles suspended in liquids to mimic the behavior of glass, and he ducks out of the theoretical battles. "It just can get so controversial and so many loud arguments, and I don't want to get involved with that myself."
Theories of glass formation

• Adams-Gibbs approach and random first order theory
• Mode-coupling theories: dynamical phase transition
• P-spin mean field models – a unified view.
• Kinetically frustrated models
• Frustrated domain growth

Energy (free energy ?) landscape

Configuration entropy = log(number of minima with energy \( u \))

Vibrational free energy of the minimum at energy \( u \)

\[
 f_v(\beta, u) = \frac{k_B}{V} \ln \Omega(u) 
\]

\[
 Z = \int du \exp \left( -\beta V(u + f_v(\beta, u) - T s_c(u)) \right) 
\]
Assume $f_0(\beta, u)$ independent of $u$

Minima occupied at temperature $k_B T = 1/\beta$ verify

$$\frac{\partial s_c}{\partial u} = -\beta,$$

Problem if $s_c(u)$ vanishes at $u_{\text{min}}$ with a finite slope

$$s_c(u) = (u - u_{\text{min}})/T_0$$

Equilibrium impossible at temperatures below $T_0$

« Entropic crisis »
Calculating $s_c(u)$ is the difficult part but has been done using various approximations (Mézard-Parisi, Kirpatrick-Wolynes, replica approach); these schemes confirm the existence of an ‘entropic crisis’ at a finite temperature.

In the replica scheme, the order parameter is the correlation between two replica of the system coupled by a vanishingly small potential (symmetry breaking field).

Below $T_0$, all replica in the same state, configurational entropy vanishes.

(See Mézard and Parisi, J. Chem. Phys. 1999)
Connection to dynamics: Adams Gibbs 1965
(See Bouchaud Biroli cond-mat/0406317 for modern variants)

- $N_d$ independent ‘cooperative rearranging regions’ (CRR).

- Number of atoms within one region $= N/N_d = z$

- Configuration entropy $s_c = N_d k/V$

- Time scale for a re-arrangement $= \tau_0 \exp( +z\delta/k_BT)$

$$\tau(T) = \tau_0 \exp\left(\frac{A}{T s_c(T)}\right)$$

$A = N\delta/V$

Correlation well verified in simulation and experiment
Mode coupling approaches

Simplified mode coupling (Gezti 1984)

\[ \eta = \left( \frac{V}{k_B T} \right) \int_0^t < \sigma_{xz}(t) \sigma_{xz}(0) > dt \]

\[ \sigma_{xz}(t) \sim \sum_k A(k) \rho(k, t) \]

\[ < \rho(k, t) \rho(k, 0) > \sim \exp(-k^2 D t) \]
Finally:

Combine with Stokes Einstein

\[ \eta = \eta_0 + \frac{B(T)}{D} \]

\[ D = \frac{k_B T}{6\pi \eta \sigma} \]

\[ \eta(T) = \frac{\eta_0}{1 - 6\pi B(T)/\sigma} \]

« viscosity feedback » ; divergence at a finite value of the control parameter T (or pressure or density). Dynamical phase transition.
Eventually, closed equations for correlation functions:

\[
\phi(q, t) = \langle \rho_q(t) \rho_{-q}(0) \rangle / NS(q)
\]

\[
\ddot{\phi}(q, t) + \Omega_0(q) \phi(q, t) + \int_0^t M(q, t-s) \phi(q, s) ds = 0
\]

\[
\Omega_0(q) = q^2 k_B T / m S(q)
\]

\[
M_{MC}^{MC}(q, t) = \int d^3 q' V(q, q') \phi(q', t) \phi(q-q', t)
\]

\(V(q, q')\) dependent on liquid structure factor \(S(q)\), hence on temperature and density
Simplified model, $S(q) = \lambda \delta(q-q_0)$:

$$\ddot{\phi}(t) + \Omega_0^2 \phi(t) + \lambda_2 \Omega_0^2 \int_0^t \phi(t-s)^2 \dot{\phi}(s) ds = 0$$

Laplace transform

$$\phi(z) = 1/(z + \Omega_0^2/(z + M(z)))$$

Look for nonergodic behaviour

$$\phi(z) \sim f/z$$

$$f/(1-f) = M(f)/\Omega_0^2 = \lambda_2 f^2$$

non ergodic solution if $\lambda_2 > \lambda_{2c} = 4$
Nonlinear differential equation with memory term. Sophisticated analysis possible close to the « critical point » $\lambda_2 = \lambda_{2c} = 4$ (Götze).

\[
\frac{\phi(z)}{1 - z\phi(z)} = \frac{z}{\Omega_0^2} + M(z) \quad \phi(z) = f_c z^{-1} + G(z)
\]

\[
z^{-1}\left(\frac{f_c}{1 - f_c} - \lambda_2 f_c^2 - \lambda_2 LT(G(t)^2) + zG(z)^2(1 - f_c)^{-3} = G(z)(2\lambda_2 f_c - (1 - f_c)^{-2} + \frac{z}{\Omega_0^2} + O(z^2 G(z)^3))
\]

for $\lambda_2 = \lambda_{2c} = 4$ we have

\[
\lambda LT(G(t)^2)(z) - zG(z)^2 = \text{constant} \times \left(\frac{z}{\Omega_0^2} + O(z^2 G(z)^3)\right)
\]

With $\lambda = 1/2$
Solution:

\[ G(t) = A/(\Omega_0 t)^a, \quad G(z) \sim z^{a-1} \]

\[ \lambda \Gamma(1 - 2a) = \Gamma(1 - a)^2 \]

\[ \phi(t) = f_c + A/(\Omega_0 t)^a \]

Correlation function approaches a plateau with a power law, nonuniversal exponent \( a \).

Multiple time scale analysis, different time scales depend on the distance from critical point \( \varepsilon = \lambda_2 - \lambda_{2c} \).
Generic properties of solutions

• Numerical solution, hard spheres

- Temperature (density) $T_c$ at which the relaxation time diverges
- Close to $T_c$ relaxation time diverges as $\tau \propto (T-T_c)^{-\gamma}$
Generic properties – approach to the plateau (β relaxation) –
time scale \( \tau_\varepsilon \sim \varepsilon^{-1/2a} \)

- Near the plateau
  \[ \pm B/z + zG(z)^2 - \lambda \text{L}[G(t)^2](z) = 0 \]
  B and \( \lambda \) can be computed from the full equations.

- Near the plateau, factorization property

\[
\phi(q, t) = f(q) + \varepsilon^{1/2} \varepsilon(q) G(t/\tau_\varepsilon)
\]

- Short times \( G(x) \propto x^{-a} \)
- Long times \( G(x) \propto x^b \) (von Schweidler)
- \( a, b, \lambda \) verify
- \[ \Gamma(1-a)^2/\Gamma(1-2a)=\Gamma(1+b)^2/\Gamma(1+2b) = \lambda \]
- exponent \( \gamma \) of \( \tau \propto (T-T_c)^{-\gamma} \) verifies \( \gamma = 1/2a + 1/2b \)
Generic properties – terminal relaxation ($\alpha$) – Time scale

\[ \frac{t}{T_e} \sim \varepsilon^{-1/2a - 1/2b} \]

- Shape of the curve independent of $T$ during $\alpha$ relaxation for $T > T_c$; “time-temperature superposition principle”
- Good approximation Kohlrausch-Williams-Watts (streched exponential)

\[ \phi(t) = A \exp(- (t/\tau)^\beta) \]

N.B. not an exact solution!

- For $T < T_c$, $\phi$ (plateau value) increases as $(T_c - T)^{1/2}$
Comparison with experiments/simulation:

• $T_c$ does not exist in practice

• Numerical predictions for $T_c$ are above $T_g$

• Good description for the first stages of slowing down (typically relaxation times up to $10^{-8}$ s), Probably better for colloids

• Predictive theory: « reentrant » glass transition for attractive colloidal systems
Reentrant glass transition in attractive colloids

**Experiment**

![Graph showing the reentrant glass transition](image1)

‘Unsticking’ a colloidal glass, and sticking it again

W C K Poon¹, K N Pham, S U Egelhaaf and P N Pusey


**Theory**

![Graph showing the theoretical model](image2)

![Inset showing the temperature dependence](image3)

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‘p-spin’ models in mean field – A unified approach?

\( \vec{\phi} \) vector with \( N \) components \((\phi_1, ... \phi_N)\) on a sphere

\[
\frac{\partial \phi_i}{\partial t} + \mu \phi_i = -\frac{\partial V}{\partial \phi_i} + \xi_i(t)
\]

\[
V(\vec{\phi}) = g \sum_{i<j<k} J_{ijk} \phi_i \phi_j \phi_k
\]

\( p=3 \)

\( J_{ijk} \) random variable with variance

\[
[J_{ijk}^2]_J = 1/N^{p-1} - 1/N^2
\]

\[
\phi_i(t) = (G_0 \otimes \xi_i)(t) - J_{ijk} G_0 \otimes (G_0 \otimes \xi_j \bullet G_0 \otimes \xi_k)(t) + \ldots
\]

« mean field » limit (infinite \( N \)) ‘Mode coupling approximation’ becomes exact
Model has

dynamical transition at $T_d$; described by nonlinear equations of MCT in liquids. Appearance of many minima in free energy landscape, separated by infinite barriers (mean field).

- Static transition of Adams-Gibbs type (entropy vanishes with finite slope) at $T_K$

Both aspects come from the mean field limit, but probably something remains in finite dimensions... (Mézard –Parisi)
Kinetically frustrated spin models

Introduced by Fredrickson and Andersen in 1984. Many similar models, for a review see

General idea: at low T weak concentration of mobile regions. To allow structural relaxation these mobile regions must explore the volume, and this takes time
Fredrickson-Andersen: spins $n_i=0,1$ on lattice. No interactions, trivial Hamiltonian

$n_i = 1 \text{ mobile } n_i = 0 \text{ frozen}$

Rule of the game: evolution $1 \rightarrow 0$ or $0 \rightarrow 1$ possible only if spin $i$ has at least one mobile neighbor. Transition rates verify detailed balance

1. $0 \rightarrow 1$ rate $w_i(0 \rightarrow 1) = (1 - \Pi_{j,i}(1 - n_j)) \exp(-1/T)$
2. $1 \rightarrow 0$ rate $w_i(0 \rightarrow 1) = (1 - \Pi_{j,i}(1 - n_j))$

At equilibrium concentration of mobile zones:

$c(T) = \langle n_i \rangle = (1 + \exp(1/T))^{-1}$
At low $T$, relaxation proceeds through diffusion of isolated mobile zones.

Diffusion coefficient of a zone: $D = \exp(-1/T)$

Distance between mobile zones: $1/c$ (in $d=1$)

Relaxation time: $D \times (1/c)^2$

$$\tau(T) = \exp(3/T)$$
East: spins $n_i=0,1$ No interactions, trivial Hamiltonian

$$H = \sum_{i=1}^{N} n_i$$

$n_i=1$ mobile; $n_i=0$ frozen

Rule of the game: evolution $1 \rightarrow 0$ ou $0 \rightarrow 1$ possible only if spin $i$ a has its left neighbour mobile. Transition rates verify detailed balance

$$0 \rightarrow 1 \text{ rate } w_i(0 \rightarrow 1) = n_{i-1} \exp(-1/T)$$
$$1 \rightarrow 0 \text{ rate } w_i(0 \rightarrow 1) = n_{i-1}$$

At equilibrium mobile zones concentration:

$$c(T) = \langle n_i \rangle = (1 + \exp(1/T))^{-1}$$
\[ \tau(T) = \exp(1/T^2 \ln(2)) \]

Fragile behaviour

100000..0001 domain of length \( d=2^k \)

\( n(k) = \) number of spins to relax the one on the right

\( n(k) = n(k-1) + 1 = k+1 \)

Time \( t(d) = \exp(n(k)/T) = \exp(\ln(d)/T \ln(2)) \)

Using \( d=1/c = \exp(1/T) \) one gets the result

Results can be modified by adjusting the rules of the game…
Models explain dynamical heterogeneities in terms of space/time correlations

Static configurations are the same in all three cases!
- Highlights the importance of **dynamical heterogeneities**
- **Phase transition** in the space of trajectories, with a critical point at $T=0$ (Garrahan, van Wijland)

**Dynamical heterogeneities**

Displacement of particles over a time interval $50\tau_\alpha$

Can also be characterized in experiments (confocal microscopy, hole burning)
Notion of cooperativity: \( X_i(t) = d_i(t) - \langle d_i(t) \rangle \) where \( d_i(t) \) is the displacement of particle \( i \) in the interval \([0,t]\). The idea (Heuer and Doliwa, PRE 2000) is to compare the fluctuations at the one particle level and over the whole system:

\[
N_x^{coop} = \frac{\text{Var}[\sum X_i]}{\sum \text{Var}[X_i]} = 1 + \frac{\sum_{i \neq j} \langle X_i X_j \rangle}{\sum \langle X_i^2 \rangle}
\]

Cooperativity goes through a maximum at \( t^* \).
Approach can be generalized to any observable mobility field

\[ c(r; t, 0) = \sum_{i=1}^{N} c_i(t, 0) \delta[r - r_i(0)], \]

\[ G_4(r; t) = \langle c(r; t, 0) c(0; t, 0) \rangle - \langle c(r; t, 0) \rangle^2; \]

\[ \chi_4(t) = \int d^d r G_4(r; t). \]

« 4-point susceptibility »

Relates to the correlation volume at time \( t \), and is also the variance of the global correlation:

\[ \chi_4(t) = N \left[ (C(t, 0))^2 - \langle C(t, 0) \rangle^2 \right]. \]

where

\[ C(t, 0) = \int d^d r c(r; t, 0). \]
Variation on the theme:

- domains of size $L(T)$, increases as $T$ decreases
- $E(L) \sim L^\alpha$
- Distribution of $L$, hence distribution of $E$
- Time $\tau \sim \exp(E(L)/T)$
Hypothesis: system approaches spinodal instability, but domain growth limited by frustration

Possible realizations:
- Icosahedral order preferred locally, frustrated in 3d.
- Magnetic system with ferromagnetic interactions at long range, antiferro or dipolar at long range

\[ \xi = a |T - T^*|^{-\nu} \]

- Frustration limits domain growth to a size \( R \)

\[ MKR^5 = k_B T (M^2/\xi^2)R^3 \]

- \( M \) order parameter
- \( K \) characterizes the strength of frustration
- \( R^5 \): for long range elastic interactions \( 1/R \)
Finally, \( R \sim 1/\xi \sqrt{K} \)

Each domain has a rearrangement energy

\( E(R) = \gamma R^2 \) with \( \gamma = k_B T/\xi^2 \) interfacial tension

and \( \tau = \exp(E(R)/k_B T) \) so that finally

\[
\ln(\tau/\tau_0) = K^{-1/2} (T^\ast - T)^{4\nu}
\]

Good fits for viscosities with \( \nu = 2/3 \)

Refinements with size distribution, translation-rotation decoupling, etc… See
A small problem…


\[ \mathcal{H}[\phi(r)] = \int d^2r \left[ \phi(r) \left( r + k_0^{-2}(\nabla^2 + k_0^2)^2 \right) \phi(r) + \frac{\lambda}{4!} \phi^4(r) \right] \]

Formation of a lamellar phase and associated critical dynamics (bloc copolymers, magnetic layers), rather different from glassy dynamics
Some conclusions…

• « THE » theory does not exist

• Many different and complementary approaches. Models can capture some aspects of real systems, but also miss many – risk of studying model artefacts.

• NB: not mentioned: free volume (recent version by D. Long and F. Lequeux), trap models, lattice gas models....
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Inside the glassy state: nonequilibrium relaxation (ageing)

http://en.wikipedia.org/wiki/Phase_transition

The **liquid-glass transition** is observed in many polymers and other liquids that can be supercooled far below the melting point of the crystalline phase.

**It is not a transition between thermodynamic ground states.** Glass is a quenched disorder state, and its entropy, density, and so on, depend on the thermal history.
Mechanical response (compliance = apply a stress suddenly, measure strain response) of a glassy polymer

« Time-aging » superposition:

\[ J(t, t_w) = J(t/t_w^\mu) \]

Response properties depend on the « age » \( t_w \) (time spent in the glassy state)
Stress relaxation after step strain, in a dense colloidal suspension. (C. Derec, thesis Paris 2001)
Correlation functions $C(t_w + t, \ t_w)$ in a Lennard-Jones at $T=0.3$

$C = f(t/ t_w)$

V. Viasnoff, thesis Paris 2002

Correlation function (dynamic light scattering) in a dense colloidal suspension.

Relaxation time proportional to $t_w$
A model to rationalize results: the trap model (J-P. Bouchaud)

Escape time from trap $\alpha$

$$\tau_\alpha = \exp(+|E_\alpha|/k_B T)$$

The distribution of trap depths is given by

$$\rho(E) = \exp(-|E|/k_B T_c)$$

Distribution of trapping times $P(\tau) \sim \left(\frac{\tau_0}{\tau}\right)^{(1+T/T_c)}$

$\rightarrow <\tau>$ is infinite $T < T_c$
Central object: distribution in energies at time $t$, $P(E, t)$

\[
\frac{\partial P(E, t)}{\partial t} = -P(E, t)e^{-E/\mathcal{T}} + \Gamma(t)\rho(E),
\]

\[
\Gamma(t) = \int_0^\infty dE' P(E', t) \exp(-E'/k_BT)
\]

Correlation defined by the fraction of particles that do not change energy between $t$ and $t'$:

\[
C(t, t') = \int_0^\infty d\epsilon P(\epsilon, t') \exp\left[-|\Gamma_0\rho\exp\left(-\frac{\epsilon}{T}\right)(t-t')|\right]
\]
\( N(t_w) \) energies visited after \( t_w \), \( E_{\text{max}} \) the maximum value, \( \tau_{\text{max}} \) the associated trapping time.

\[
N(t_w) \int_{\tau_{\text{max}}}^{\infty} d\tau \psi(\tau) = 1
\]

\[
t_w = \sum_{j=1, N(t_w)} \tau_j = N(t_w) \int_{0}^{E_{\text{max}}} d\tau \tau \psi(\tau)
\]

Where \( \psi(\tau) \) is the trapping time distribution

\[
\tau_{\text{max}} \sim t_w \text{ pour } T < T_c
\]
Other descriptions of ageing:

• domain growth (coarsening)
• Mean field p-spin (Cugliandolo-Kurchan)

Interesting notions emerge such as effective temperature associated with the fluctuation dissipation ratio.
The jamming transition: « glass transition » at zero temperature.
J point reached by progressively increasing the density of a hard sphere (granular) packing until overlap removal becomes impossible. Packing at point J can support external stress at zero T.
The J point corresponds to an isostatic solid

Minimum number of contacts needed for mechanical stability

Match unknowns (# interparticle normal forces) to equations

Frictionless spheres in D dimensions:

Number of unknowns per particle = \( \frac{Z}{2} \)
Number of equations per frictionless sphere = \( D \)

\[ \Rightarrow Z_c = 2D \]

Maxwell criterion for rigidity: global condition.

Friction changes \( Z_c \)

Nonaffine deformation dominates close to point J (see recent review by M. Van Hecke, “Jamming of Soft Particles: Geometry, Mechanics, Scaling and Isostaticity”.)
Isostatic solids (Maxwell)

Number of constraints matches number of degrees of freedom

(from Z. Zeravic Ph.D thesis, Leiden (2010))
See also Mathieu Wyart PhD,, « On the rigidity of amorphous solids » http://arxiv.org/abs/cond-mat/0512155 + Annals of Physics
Isostatic solids have an anomalous density of vibrational states at small frequencies
Normal solids: Debye, \( g(\omega) \sim \omega^{(d-1)} \)
Isostatic packing: excess density of states, \( g(\omega) \sim \omega^0 \)

Construct low-\( \omega \) modes from soft modes
(Matthieu Wyart, Tom Witten, Sid Nagel)

\( n \) modes with \( \omega = 0 \)

\( N(L) \sim L^{D-1} \) floppy modes (from cutting boundary)
\( \omega \sim 1/L \) (variational calculation, Matthieu Wyart)

\( \omega \sim 1/L \) (variational calculation, Matthieu Wyart)
$N(L) \sim L^{d-1}$ floppy modes (from cutting boundaries)

These modes fall within a frequency range $\omega \sim 1/L$ (Wyart)

$g(\omega) \sim N / (\omega L^d) \sim L^0$
(Liu Silbert, Nagel, O’Hern, Wyart, Witten)
See also recent experiments:
Density of states of colloidal glasses and supercooled liquids
Isostatic length

If \( z > z_c \) one has \( \delta z = z - z_c \) excess contact per particle

The argument holds for systems smaller than the isostatic length

\[ \ell^* \sim \frac{1}{\delta z} \]

Above jamming: a system smaller than \( \ell^* \) becomes unstable if contacts are removed at its boundaries

Below jamming: \( z < z_c \), a system smaller than \( \ell^* = 1/|\delta z| \) becomes solid if its boundaries are blocked.