# THE COST OF GAUGE COUPLING UNIFICATION IN THE $SU(5)$ MODEL AT THREE LOOPS

L. MIHAILA
*Institut für Theoretische Teilchenphysik, Karlsruhe Institute of Technology (KIT), D-76128 Karlsruhe, Germany*

The non-supersymmetric $SU(5)$ model can accommodate heavy neutrinos and gauge coupling unification when augmented with an adjoint fermionic multiplet $24_F$. Among the most important phenomenological implications of the model is the prediction of light fermions and scalars, charged under the $SU(2)_L$ gauge group, in the reach of the Large Hadron Collider (LHC). In this talk, we report on the recent calculation [1] of the correlation function between the mass scale of the new electroweak multiplets and the gauge coupling unification scale at three loop accuracy.

## 1 Introduction

Nowadays, it is well established that the minimal non-supersymmetric $SU(5)$ model in its original form [2] is phenomenologically ruled out. In principle, this is due to the lack of gauge coupling unification and the massless neutrinos. These unsatisfactory aspects of the original model can be simultaneously eliminated, if one adds an additional fermionic multiplet in the adjoint representation $24_F$ [3,4]. To understand the role of the $24_F$ multiplet in the model, let us recall its decomposition w.r.t. the Standard Model (SM) gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$.

$$24_F = \underbrace{(1,1,0)_F}_{S_F} \oplus \underbrace{(1,3,0)_F}_{T_F} \oplus \underbrace{(8,1,0)_F}_{O_F} \oplus \underbrace{(3,2,-\tfrac{5}{6})_F}_{X_F} \oplus \underbrace{(\overline{3},2,+\tfrac{5}{6})_F}_{\overline{X}_F}\,, \tag{1}$$

where $S_F$, $T_F$, $O_F$ ($X_F$) are Majorana (Dirac) degrees of freedom. A special role in the model is played by the electroweak singlet and triplet states $S_F$ and $T_F$. They are involved in the Yukawa interactions that after the SU(5) gauge symmetry breaking will generate masses for neutrinos through a hybrid type-I+III seesaw mechanism [5,6,7,8,9]. The electroweak singlet $S_F$ resembles a sterile neutrino, whereas the electroweak triplet is sometimes referred to as a heavy lepton.

As can be read from the above decomposition, the fermionic states $T_F, O_F, X_F$ are charged under the groups $SU(2)_L$, $SU(3)_C$ and $SU(3)_c \times SU(2)_L \times U(1)_Y$, respectively. Thus, they will give also contributions to the gauge coupling evolution with the energy scale. In particular, the electroweak triplets $T_F$ can delay the meeting of the couplings $\alpha_1$ and $\alpha_2$ from the energy scale of about $10^{13}$ GeV in the minimal $SU(5)$ model to values in agreement with the bounds enforced by the non-observation of proton decay [10] of about $10^{15.5}$ GeV. The underlaying condition is that $T_F$ states are rather light, in the TeV range, in order to have the maximal impact on the evolution of $\alpha_2$. In contrast, the states $X_F$, that are charged both under the $SU(2)_L$ and $U(1)_Y$ have always the opposite effects, due to their contributions to the beta functions of the coupling constants $\alpha_1$ and $\alpha_2$. Thus, in order to reach a high enough unification scale for the coupling $\alpha_1$ and $\alpha_2$, one needs in addition a very heavy mass scale for $X_F$ states. However, this mass scale

can be at most of the order of $M_G^2/\Lambda$, where $M_G$ denotes the unification scale and $\Lambda$ is the cutoff scale of the $SU(5)$ model. The latter have to be chosen in such a way that the low-energy value of the ratio $m_b/m_\tau$ (where $m_b$ stands for the bottom quark mass and $m_\tau$ for the tau lepton mass) is correctly reproduced in the model and to maximize the perturbativity domain. It was shown [3,4] that a value of about $\Lambda = 100\, M_G$ is a reasonable choice.
Furthermore, for a complete unification it is also necessary that the strong coupling constant $\alpha_3$ meets the electroweak couplings $\alpha_1$ and $\alpha_2$ at the right energy scale. Obviously, the states that have a direct impact on the energy evolution of $\alpha_3$ are the colour octet fermions $O_F$. As we show in the next section, a proper unification requires that the states $O_F$ live at intermediate mass-scale of about $10^8$ GeV.

The crucial parameter for phenomenology is actually the effective mass of the electroweak triplet states. This mass scale is defined as an average between the mass scale of the electroweak triplet fermions $T_F$ and the similar components of the scalar multiplet that lives in the 24-dimensional representation[a]. Both types of triplets, if light enough, can give interesting signature at the LHC. The fermionic components lead to lepton number violation effects in same sign di-lepton events [3,4]. The bosonic triplet instead can easily modify the decay properties of the Higgs boson (see e.g. [11]), that will be measured with increasing precision at the LHC.

Let us also mention that, the Higgs sector is the one of the genuine SU(5) model

$$5_H = \underbrace{(3,1,-\tfrac{1}{3})_H}_{\mathcal{T}} \oplus \underbrace{(1,2,+\tfrac{1}{2})_H}_{h} \quad \text{and} \tag{2}$$

$$24_H = \underbrace{(1,1,0)_H}_{S_H} \oplus \underbrace{(1,3,0)_H}_{T_H} \oplus \underbrace{(8,1,0)_H}_{O_H} \oplus \underbrace{(3,2,-\tfrac{5}{6})_H}_{X_H} \oplus \underbrace{(\overline{3},2,+\tfrac{5}{6})_H}_{\overline{X}_H}, \tag{3}$$

where $S_H$, $T_H$ and $O_H$ ($\mathcal{T}$, $h$ and $X_H$) are real (complex) scalars. In our notation, $h$ stands for the SM Higgs doublet. The mass spectrum of the model is derived as usual from the minimization conditions of the scalar potential. In this respect, it is a nontrivial fact that the tree-level calculation of the spectrum allows the mass pattern required by unification

$$m_{T_F} \approx m_{T_H} \ll m_{O_F} \approx m_{O_H} \ll M_G\,. \tag{4}$$

Nevertheless, the required mass hierarchy strengthen the fine-tuning issue typical for non-supersymmetric GUTs.

## 2 Framework

The effective mass scale for the electroweak triplets can be determined from the constraint of gauge coupling unification. More precisely, it depends only on the unification scale of the electroweak couplings $\alpha_1$ and $\alpha_2$ at one- and two-loop oder in perturbation theory. The dependence on the strong coupling constant occurs starting from three loops. To study the energy evolution of the electroweak couplings for the case of a largely split mass spectrum as required in Eq. (4), it is convenient to apply the method of effective field theories (EFT)s. It consists in integrating out the heavy degrees of freedom that cannot influence the physics at the low-energy scale.
In physical renormalizations schemes like the momentum subtraction scheme or the on-shell scheme, the effects due to heavy particle thresholds are encountered in the renormalization constants of the parameters. However, for the analysis of the gauge coupling unification that requires the running of the couplings over many orders of magnitude, higher order radiative corrections to the RGEs are essential. But, their calculation beyond one-loop order in mass dependent renormalization schemes is quite involved. A much more suited scheme for this purpose is the minimal subtraction scheme ($\overline{\text{MS}}$), for which the gauge coupling beta functions are mass

[a] We denote the latter states by $T_H$ to distinguish their scalar origin.

independent and their computation is substantially simplified. Nevertheless, in this scheme the Appelquist-Carazzone [12] theorem does not hold anymore and the threshold effects have to be taken into account explicitly. The latter are parametrized through the decoupling ( matching ) coefficients. They can be computed perturbatively using the physical constraint that the Green's functions involving light particles have to be equal in the original and the effective theory. For the computation presented here, we adopt this second method and apply it up to the third order in perturbation theory.

The computation of the renormalization constants up to the three-loop order in the $\overline{\rm MS}$ scheme can be reduced to the evaluation of only massless propagator diagrams. For the present calculation, we use a well-tested chain of programs: the Feynman rules of the model are obtained with the help of the program `FeynRules` [13] and translated into `QGRAF` [14] syntax. `QGRAF` generates further all contributing Feynman diagrams. The output is passed via `q2e` [15,16], which transforms Feynman diagrams into Feynman amplitudes, to `exp` [15,16] that generates `FORM` [17] code. The latter is processed by `MINCER` [18] that computes analytically massless propagator diagrams up to three loops and outputs the $\epsilon$ expansion of the result. The three-loop expressions for the beta functions of the gauge couplings in the low-energy theories can be found in Ref. [1].

For the computation of the matching coefficients of the gauge couplings, when two different theories are matched together, one has to consider Green's functions involving light particles and a vertex that contains the gauge couplings $\alpha_i$. Since the matching coefficients are universal quantities, they must be independent of the momentum transfer of the specific process taken under consideration. For convenience of the calculation, one chooses vanishing external momenta. Thus, in dimensional regularization only diagrams containing at least one heavy particle inside the loops contribute have to be taken into account. As a consequence, the resulting Feynman amplitudes can be mapped to massive tadpole topologies that are handled with the help of the program MATAD [19]. Explicit two-loop results for the matching coefficients of the gauge couplings in the $SU(5) + 24_F$ model can be found in Ref. [1].

## 3 Numerical Results

In this section we study the numerical impact of the three-loop corrections on the evolution of the gauge couplings and on the correlation function between the electroweak triplet mass and the GUT scale. In practice, we integrate numerically the $n$-loop beta functions of the gauge couplings taking into account also the $(n-1)$-loop running of the top-Yukawa coupling and the $(n-2)$-loop running of the Higgs boson self-coupling together with the $(n-1)$-loop matching conditions for the gauge couplings. In the present analysis $n = 1, 2, 3$. We can safely neglect the contribution of the bottom and tau Yukawa couplings. We also neglect in this analysis the effects due to the new scalar self-interactions of the scalar triplet $T_H$. As input parameters for the running analysis we take [20]

$$\alpha_1^{\overline{\rm MS}}(M_Z) = 0.0169225 \pm 0.0000039\,, \tag{5}$$
$$\alpha_2^{\overline{\rm MS}}(M_Z) = 0.033735 \pm 0.000020\,, \tag{6}$$
$$\alpha_3^{\overline{\rm MS}}(M_Z) = 0.1173 \pm 0.00069\,, \tag{7}$$
$$\alpha_t^{\overline{\rm MS}}(M_Z) = 0.07514\,, \tag{8}$$

given in the full SM, i.e. with the top quark threshold effects taken into account. The Higgs self-coupling is determined assuming a Higgs boson with mass 125 GeV. Thus, we obtain

$$\alpha_{\lambda_h} \approx 0.010\,. \tag{9}$$

For illustration we show in Fig. 1 an unification pattern for the inverse of the gauge couplings, taking into account three-loop order RGEs and two-loop order threshold corrections. For the

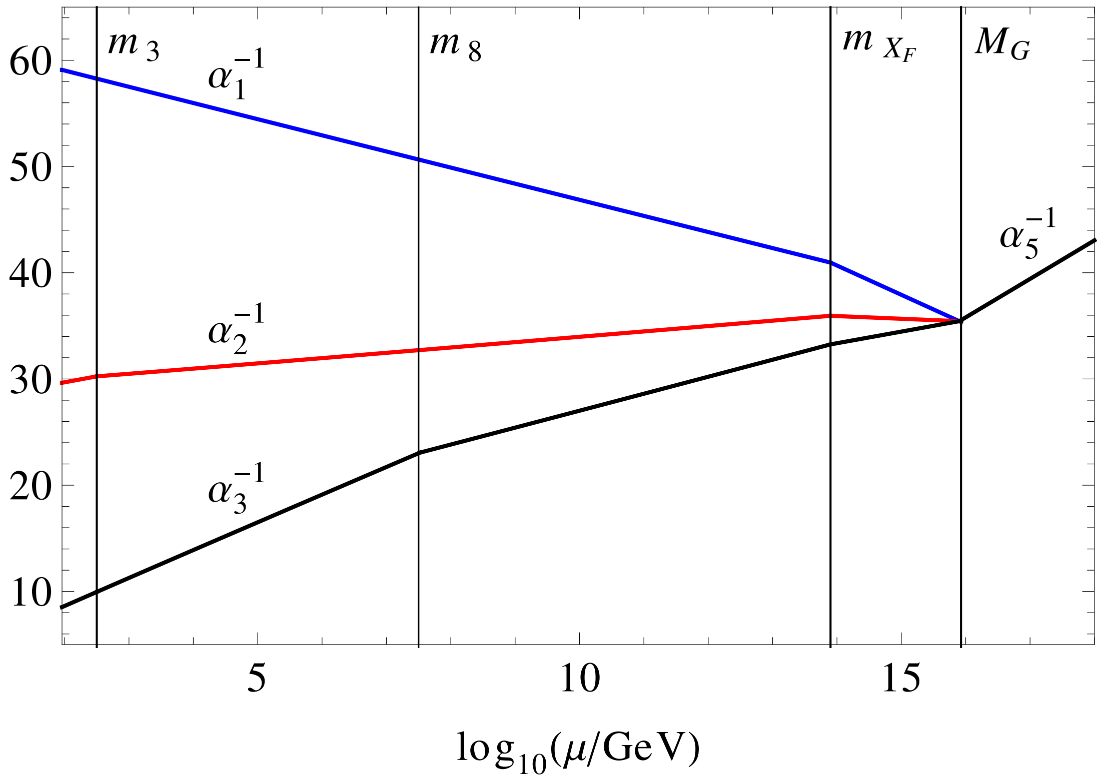


Figure 1: Sample three-loop unification pattern for $m_{T_F} = m_{T_H} = 10^{2.5}\,\mathrm{GeV}$, $m_{O_F} = m_{O_H} = 10^{7.5}\,\mathrm{GeV}$, $m_{X_F} = M_G/100$ and $m_{\mathcal{T}} = m_{X_V} = M_G$. The lines with different slopes from top to bottom correspond to $\alpha_1^{-1}$ (blue), $\alpha_2^{-1}$ (red) and $\alpha_3^{-1}$ (black). The dashed vertical lines denote the masses of the intermediate-scale thresholds.

intermediate mass scales we choose: $m_{T_F} = m_{T_H} = 10^{2.5}\,\mathrm{GeV}$, $m_{O_F} = m_{O_H} = 10^{7.5}\,\mathrm{GeV}$ and $m_{X_F} = M_G/100$. Here, $M_G$ is defined as the scale where the electroweak couplings $\alpha_1$ and $\alpha_2$ meet. The colour triplet Higgs $\mathcal{T}$ and the $SU(5)$ gauge bosons $X_V$ also play a role for the unification. However, their effects are sub-leading as compared to those generated by the electroweak triplets $T_{F,H}$ and colour octets $O_{F,H}$, because they are predicted to live at very high energies between the unification and the Planck scales. For convenience, we fix their masses at the unification scale $m_{\mathcal{T}} = m_{X_V} = M_G$. The vertical lines mark the scales where the electroweak triplet $T_{H,F}$ and color octet states $O_{H,F}$ are decoupled. At lower order in perturbation theory it is advisable to choose the decoupling scale at the effective mass scales defined as $m_3 = (m_{T_F}^4 m_{T_H})^{1/5}$ and $m_8 = (m_{O_F}^4 m_{O_H})^{1/5}$. The dependence of the physical observables like the value of the unification scale or of the unified gauge coupling become more and more insensitive on this unphysical parameter, once higher order corrections are taken into account.
In order to quantify the impact of the newly computed corrections let us mention that for such a sample unification pattern the relative difference between the two- and three-loop values of $\alpha_1$, $\alpha_2$ and $\alpha_3$ evaluated at $M_G$ amounts to 0.015%, 0.061% and 0.08% respectively. This has to be compared with the relative experimental uncertainties: $\Delta\alpha_1/\alpha_1 = 0.023\%$, $\Delta\alpha_2/\alpha_2 = 0.059\%$ and $\Delta\alpha_3/\alpha_3 = 0.59\%$. Hence, for $\alpha_1$ and $\alpha_2$ the three-loop corrections are of the same order of magnitude as the experimental uncertainties.

The effective triplet mass scale is an important parametr by itself. More precisely, its upper bound represents the worse case scenario for the posibility to observe such states at the LHC. The maximal value of the $m_3$ parameter is obtained when the masses of the super heavy particles $m_{\mathcal{T}}$ and $m_{X_F}$ are set to their maximally allowed values. Apart from these parameters, $m_3$ depends only on the gauge coupling unification scale. At the one-loop order, this dependence is known completely analytically. Starting from two-loop order, one has to solve a system of coupled differential equations. Its solution is shown in Fig. 2 as a function of the unification scale. From the low left corner to the high right one the corelation function between the maximal value $m_3^{\rm max}$ and the unification scale $M_G$ is shown at one-, two-and three-loop order accuracy. As can be read from the figure the predictions for $m_3^{\rm max}$ at one- and two-loop orders differ by several TeV. In turn, this translates into a variation of the unification scale by about an oder of magnitude. To be able to use the electroweak triplet mass scale as a validity check for the $SU(5)+24_F$ model, we need more precise theoretical predictions, at least in the range of experimental precision. This requirement is nicely fulfilled at the three-loop order in perturbation theory, for which the theoretical uncertainties are reduced by about a factor ten.

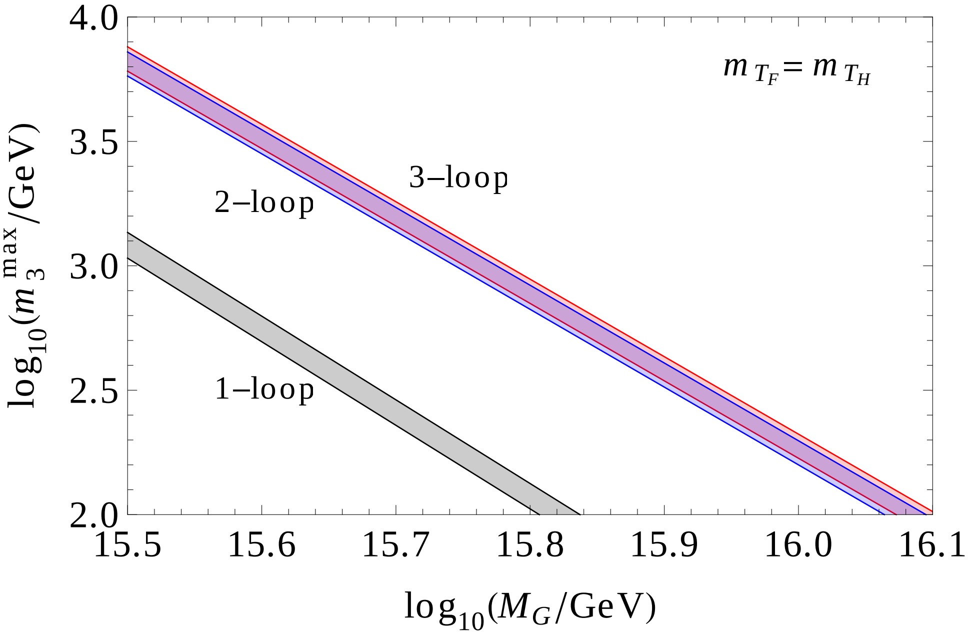


Figure 2: $m_3^{\rm max} - M_G$ correlation between. The black, blue and red bands (from bottom-left to top-right) correspond respectively to the one-, two- and three-loop running analysis. The error bands are obtained by varying the low-energy couplings $\alpha_1(M_Z)$ and $\alpha_2(M_Z)$ into their $1\sigma$ values (cf. Eqs. (**??**)–(**??**)).

The upper bound on the electroweak triplet mass scale is actually the parameter relevant for the phenomenology. It can be used as a validity test of the model in the sense that if the electroweak triplet states escape detection at the LHC, i.e. they live beyond the TeV scale, than the predicted unification scale of the model should be below $10^{5.5}$ GeV. This in turn renders the proton lifetime to be in the reach of the future generation of megaton-scale experiments. Thus, non-observation of the electroweak triplet states in the TeV range as well as of proton instability are sufficient to refute the model.

In this talk, we present the recent computation of three-loop order corrections to the predicted mass scale for the electroweak tripltes. These higher order corrections are necessary in order to reduce the theoretical uncertainties on a level compatible with those induced by the experimenatal accuracy on the determination of the electroweak couplings at low energies. Moreover, from a theoretical point of view, the three-loop order corrections are indispensable in order to establish the convergence of the perturbative series. This can be understood from the fact that the relative difference between the one- and two-loop order corrections amounts to more than 100%. In contrast, the three-loop corrections lay on top of the two-loop ones (see Fig. 2) and reduce the relative errors at around 25%, in the range of the experimental precision.

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