Higher Spin 6-Vertex Model and Macdonald Polynomials

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LAPTh, CNRS, Annecy

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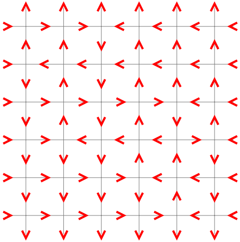
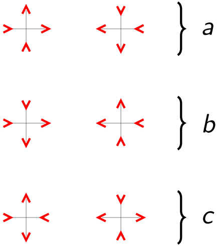
Joint work with Ferenc Balogh

Outline

Definition of the 6-Vertex model



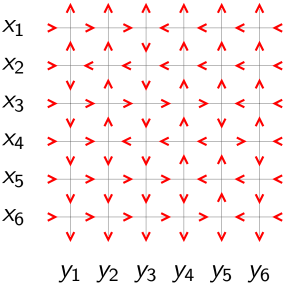


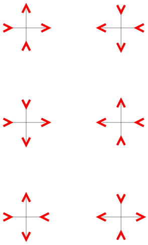
Definition (Partition function)

The partition function is defined as usual:

$$\mathcal{Z}_n \propto \sum_{\text{configurations}} \prod_{i,j} \omega_{i,j}$$

Definition of the 6-Vertex model





$\} \; a = qx - q^{-1}y$

$\} \; b = x - y$

$\} \; c = (q - q^{-1})\sqrt{xy}$

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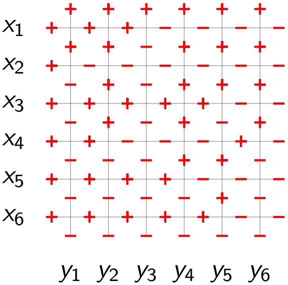

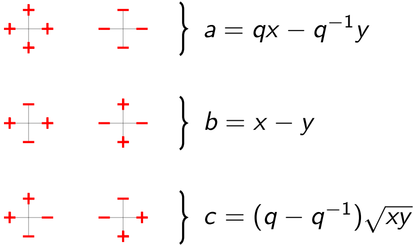

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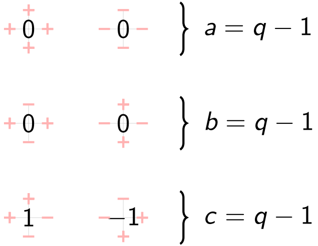
Definition of the 6-Vertex model

1	0	0	1	0	0	0
1	1	0	0	0	0	0
1	0	0	0	0	1	0
1	0	1	0	0	−1	1
1	0	0	0	1	0	0
1	0	0	0	0	1	0
	q	q	q	q	q	q



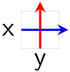
Definition (Partition function)

We can use it to count Alternating Sign Matrices (at $q^3 = 1$):

$$\mathcal{Z}_n(\mathbf{1}, \mathbf{q}) = \sharp\{\text{ASM of size } n \times n\} = 1,\ 2,\ 7,\ 42,\ 429,\ \ldots$$

The $\check{R}$ matrix

We can build a configuration using boxes:



$$= \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix} =: \check{R}(x, y)$$

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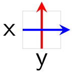

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Then

$$\check{R}(x,y) : V_1 \otimes V_2 \longrightarrow V_2 \otimes V_1$$

where V_i is the spin $\frac{1}{2}$ representation of $U_q(sl_2)$.

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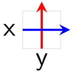

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Example

$$\check{R}(x,y)\,|{+}{-}\rangle = c(x,y)\,|{+}{-}\rangle + b(x,y)\,|{-}{+}\rangle$$

Yang–Baxter equation

Identity equation:

$$\check{R}(x,y)\check{R}(y,x) = (qx - q^{-1}y)(qy - q^{-1}x)Id.$$

Yang–Baxter equation:

$$\check{R}_{2,3}(y_2,y_3)\check{R}_{1,3}(y_1,y_3)\check{R}_{1,2}(y_1,y_2) = \check{R}_{1,2}(y_1,y_2)\check{R}_{1,3}(y_1,y_3)\check{R}_{2,3}(y_2,y_3)$$

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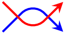

$$= (qx - q^{-1}y)(qy - q^{-1}x)$$

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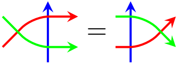

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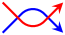

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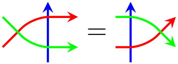

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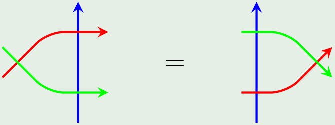

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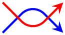 $= (qx - q^{-1}y)(qy - q^{-1}x)$ 

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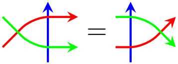

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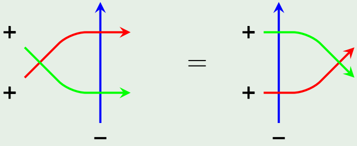

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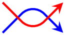 $= (qx - q^{-1}y)(qy - q^{-1}x)$ 

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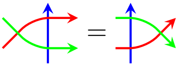

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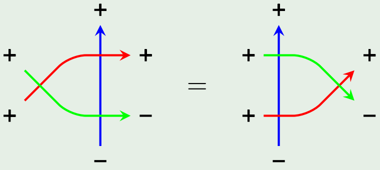

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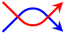 $= (qx - q^{-1}y)(qy - q^{-1}x)$ 

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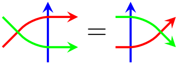

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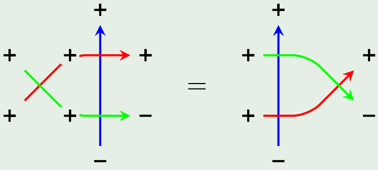

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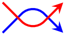 $= (qx - q^{-1}y)(qy - q^{-1}x)$ 

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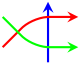
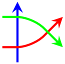
 $=$ 

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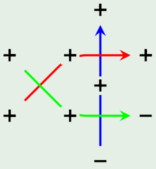
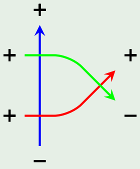


$=$



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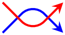 $= (qx - q^{-1}y)(qy - q^{-1}x)$ 

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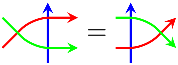

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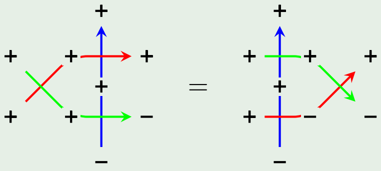

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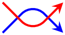 $= (qx - q^{-1}y)(qy - q^{-1}x)$ 

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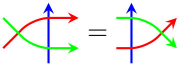

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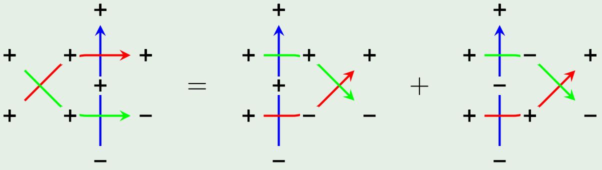

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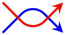 $= (qx - q^{-1}y)(qy - q^{-1}x)$ 

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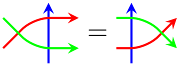

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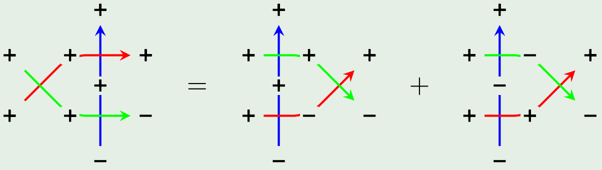
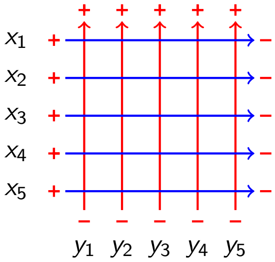


$$a(x,y)a(y,z)c(x,z) = c(y,z)a(x,z)c(x,y) + b(y,z)c(x,z)b(x,y)$$

The partition function



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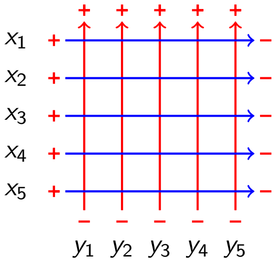
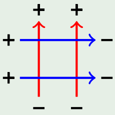


Example ($n = 2$)

$\mathcal{Z}_2 =$



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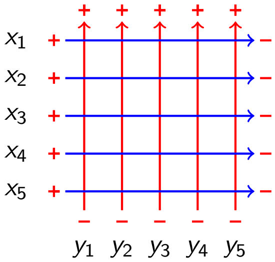

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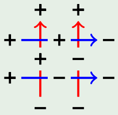
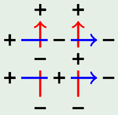
$\mathcal{Z}_2 =$  $+$ 

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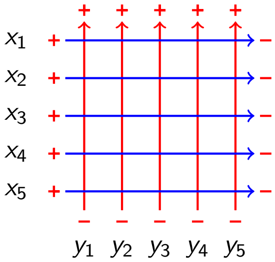

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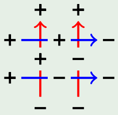
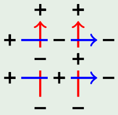
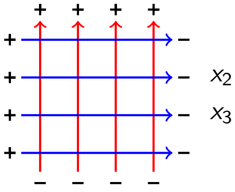
$\mathcal{Z}_2 =$  $+$ 

$$c(x_2, y_1)a(x_2, y_2)a(x_1, y_1)c(x_1, y_2) + b(x_2, y_1)c(x_2, y_2)c(x_1, y_1)b(x_1, y_2)$$

Properties of the partition function

Symmetry in **x** and **y**. Proved using Yang–Baxter equation:



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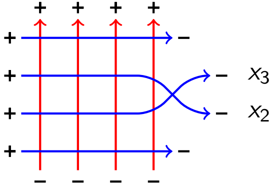

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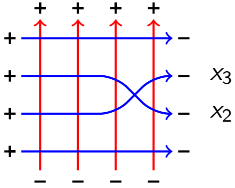

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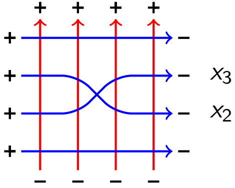

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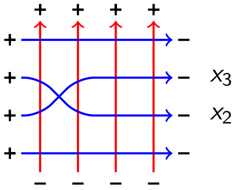

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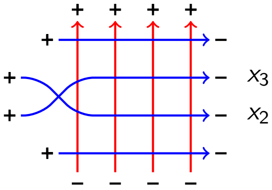

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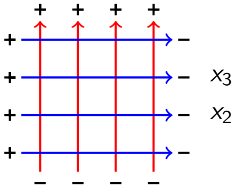

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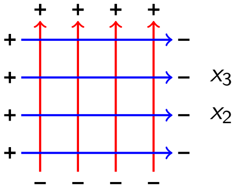
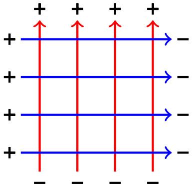


Korepin's recursion relation. At $y_1 = q^2 x_1$, $a(x_1, y_1) = 0$, then:



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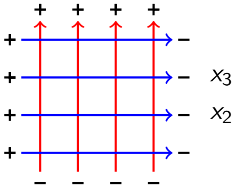

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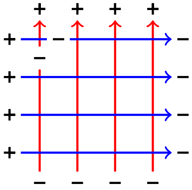

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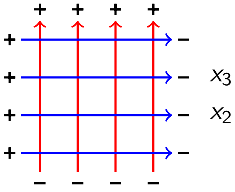

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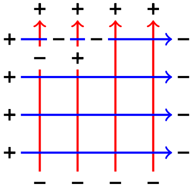

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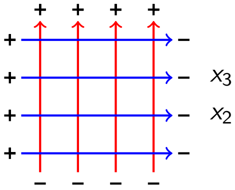

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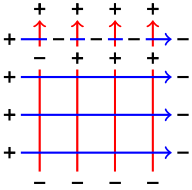

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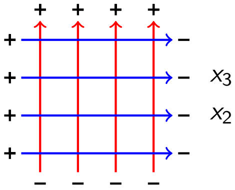

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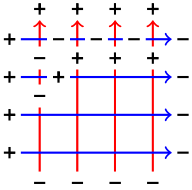

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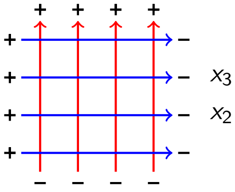

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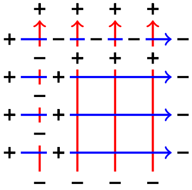

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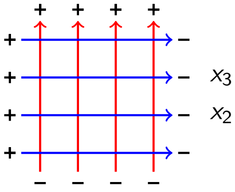

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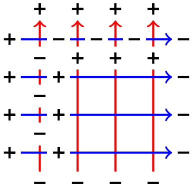

$$\mathcal{Z}_n\big|_{y_1=q^2x_1} \propto \prod_{i\neq 1}(y_i - x_1)(x_i - y_1)\sqrt{x_1 y_1}\,\mathcal{Z}_{n-1}$$

Properties of the partition function

A weight $c(x_i, y_j)$ appears an odd number of times at each row and column. Normalize

$$\frac{\mathcal{Z}_n(\mathbf{x}, \mathbf{y})}{\prod_i \sqrt{x_i y_i}} \rightarrow \mathcal{Z}_n(\mathbf{x}, \mathbf{y})$$

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- *Satisfies the Korepin's recursion relation;*
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- *Has total degree $n(n-1)$;*
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- *Is symmetric in $\mathbf{x}$ and $\mathbf{y}$.*

Explicit formula for the partition function

Using Korepin's recursion relation, Izergin showed that:

$$\mathcal{Z}_n(\mathbf{x},\mathbf{y}) = (\text{Pre-factor}) \det \left| \frac{1}{(x_i - qy_j)(qx_i - y_j)} \right|_{i,j}$$

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A Schur polynomial

Set $q = e^{\frac{2i\pi}{3}}$, the so-called combinatorial point.

$$Y_n = \{n-1, n-1, n-2, n-2, \ldots, 1, 1, 0, 0\}$$

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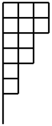
$Y_4 =$ 

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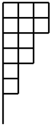

It turns out that:

$$\mathcal{Z}_n(\mathbf{x}, \mathbf{y}) = S_{Y_n}(\mathbf{x}, \mathbf{y})$$

Definition (Schur polynomial)

Let $\lambda = \{\lambda_1, \ldots, \lambda_N\}$, then the Schur polynomial is given by:

$$S_\lambda(\mathbf{z}) = \frac{\det \left| z_i^{N+\lambda_j-j} \right|_{i,j}}{\det \left| z_i^{N-j} \right|_{i,j}}$$

Higher Spin Generalization

We can build a higher spin version, of the model:

$$\check{R}^{(\ell)}(x,y) : V_1^{(\ell)} \otimes V_2^{(\ell)} \longrightarrow V_2^{(\ell)} \otimes V_1^{(\ell)}$$

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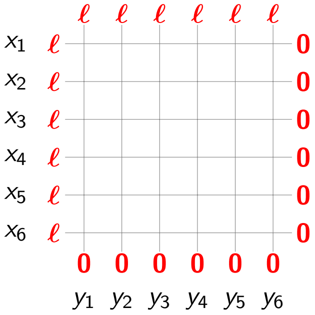

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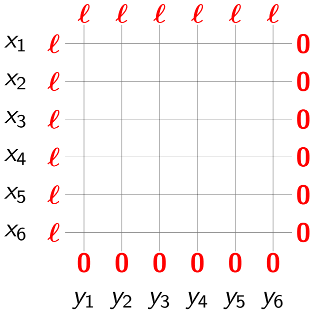

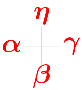

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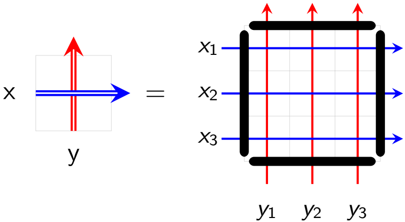

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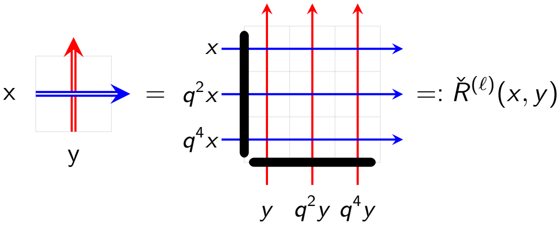

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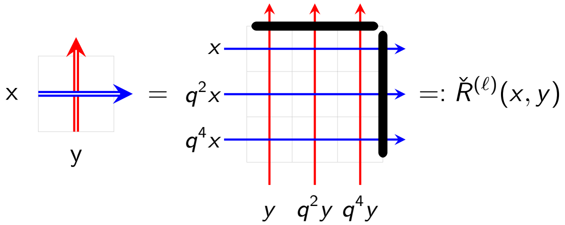

Yang-Baxter equation

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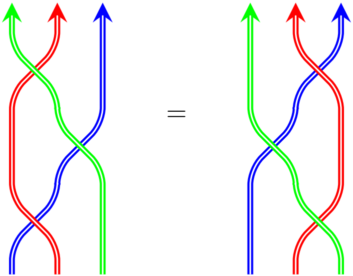

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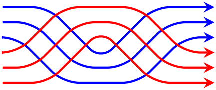

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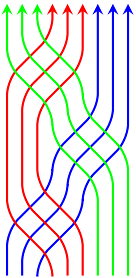

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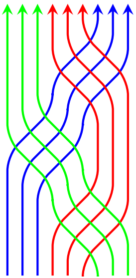

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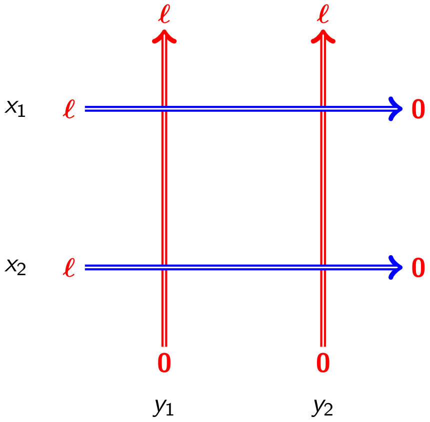

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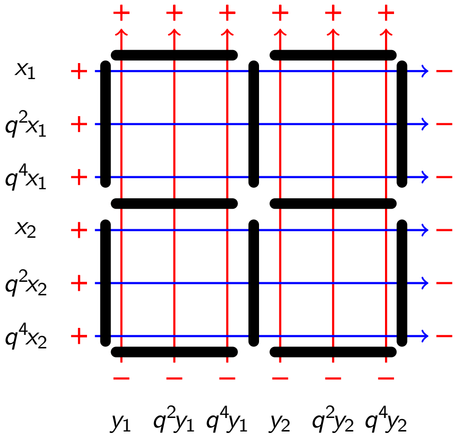

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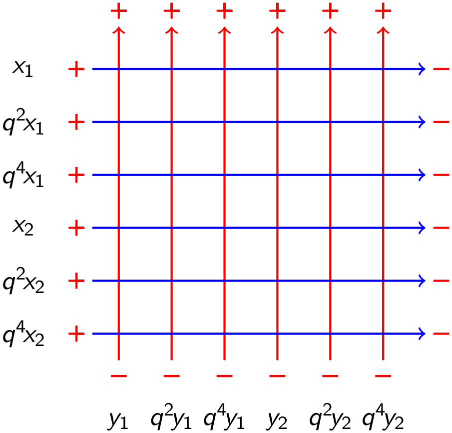

An Explicit Formula for the Partition Function

Let

$$\bar{\mathbf{x}} = \{x_1, q^2 x_1, \dots, q^{2\ell-2} x_1, \dots, x_n, q^2 x_n, \dots, q^{2\ell-2} x_n\}$$
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Wheel Condition

When $q^{2\ell+1} = 1$, the partition function satisfies the wheel condition:

Definition (Wheel condition)

A polynomial $P(\mathbf{z})$ staisfies the wheel condition if:

$$P(\mathbf{z}) = 0 \qquad \text{if } z_k = q^{1+2s_2} z_j = q^{2+2s_1+2s_2} z_i$$

for all $s_1, s_2 \in \mathbb{N}_0$ such that $s_1 + s_2 \leq \ell - 1$ and for any choice $i < j < k$.

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We call this point $q^{2\ell+1} = 1$, the combinatorial point.

Wheel condition - an example

Example ($\ell = 2$)

Impose $q^5 = 1$. It means that the polynomial vanishes whenever:

$$z_k = qz_j = q^2 z_i \qquad z_k = q^3 z_j = q^4 z_i \qquad z_k = qz_j = q^4 z_i$$

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$$P(z_1, \ldots, z_{2n}) = \prod_{1 \leq i < j \leq n} (qz_i - z_j)(q^3 z_i - z_j) \prod_{n < i < j \leq 2n} (qz_i - z_j)(q^3 z_i - z_j)$$

Symmetric polynomials - different basis

Let $\lambda = \{\lambda_1, \ldots, \lambda_N\}$, such that $\lambda_i \geq \lambda_{i+1}$.

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$$m_\lambda = \mathcal{S}\left(z_1^{\lambda_1} z_2^{\lambda_1} \ldots z_N^{\lambda_N}\right)$$

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Macdonald polynomials

Define an inner product by:

$$\langle p_\lambda(\mathbf{z}), p_\mu(\mathbf{z}) \rangle_{\tilde{q},t} = z_\mu \delta_{\lambda\mu} \prod_i \frac{1-\tilde{q}^{\mu_i}}{1-t^{\mu_i}}$$

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The Macdonald polynomials are defined by the Gram–Schmidt process:

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They depend in two parameters $\tilde{q}$ and t.

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Theorem (Jimbo, Miwa, Feigin and Mukhin)

The symmetric polynomials obeying to the wheel condition are spanned by the Macdonald polynomials P_λ (with $\tilde{q} = q^2$ and $t = q$), such that:

$$\lambda_i - \lambda_{i+2} \geq \ell$$

The Partition Function as a Macdonald Polynomial

Fix the number of boxes to $\ell n(n-1)$, then we have no choice.

$$Y_{n,\ell} = \{\ell(n-1), \ell(n-1), \ell(n-2), \ell(n-2), \ldots, \ell, \ell, 0, 0\}$$

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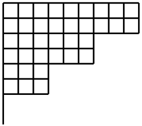
$Y_{4,3} =$ 

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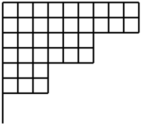
$Y_{4,3} =$ 

Theorem (Main result)

At the combinatorial point, the partition function is the Macdonald polynomial:

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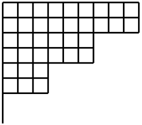
$Y_{4,3} =$ 

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Corollary (Symmetry)

The partition function is a fully symmetric homogeneous polynomial.

Open questions

- The 6–Vertex model is in bijection with Alternating Sign Matrices.
  - Can this generalization lead us to nice combinatorics?
  - What about the homogeneous limit?
- Wheel conditions and determinants.
  - Can we obtain different wheel conditions, with similar methods?
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