

Higher Spin 6-Vertex Model and Macdonald Polynomials

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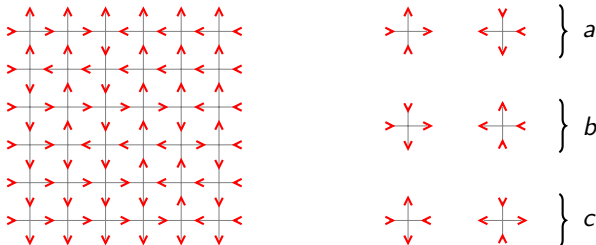
October 4, 2012

Joint work with Ferenc Balogh

Outline

- 1 6-Vertex Model
 - Definition
 - Integrability
 - Combinatorial point
- 2 Higher Spin Generalization
 - Definition and Integrability
 - Partition Function
- 3 Macdonald Polynomial
 - Wheel Condition
 - Macdonald polynomials
 - Main Result

Definition of the 6-Vertex model

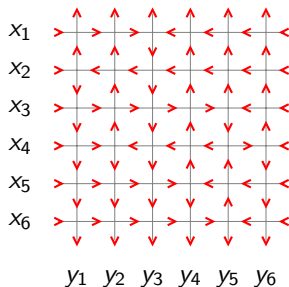


Definition (Partition function)

The partition function is defined as usual:

$$\mathcal{Z}_n \propto \sum_{\text{configurations}} \prod_{i,j} \omega_{i,j}$$

Definition of the 6-Vertex model



$$\left\{ \begin{array}{cc} \begin{array}{c} \uparrow \\ \times \\ \downarrow \end{array} & \begin{array}{c} \leftarrow \\ \times \\ \rightarrow \end{array} \end{array} \right\} a = qx - q^{-1}y$$

$$\left\{ \begin{array}{cc} \begin{array}{c} \uparrow \\ \times \\ \downarrow \end{array} & \begin{array}{c} \leftarrow \\ \times \\ \rightarrow \end{array} \end{array} \right\} b = x - y$$

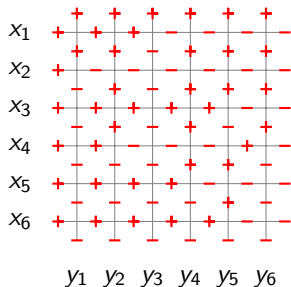
$$\left\{ \begin{array}{cc} \begin{array}{c} \uparrow \\ \times \\ \downarrow \end{array} & \begin{array}{c} \leftarrow \\ \times \\ \rightarrow \end{array} \end{array} \right\} c = (q - q^{-1})\sqrt{xy}$$

Definition (Partition function)

The partition function is defined as usual:

$$\mathcal{Z}_n(\mathbf{x}, \mathbf{y}) \propto \sum_{\text{configurations}} \prod_{i,j} \omega_{i,j}(x_i, y_j)$$

Definition of the 6-Vertex model



$$\left. \begin{array}{cc} \begin{array}{c} + \\ | \\ + \end{array} & \begin{array}{c} - \\ | \\ - \end{array} \end{array} \right\} a = qx - q^{-1}y$$

$$\left. \begin{array}{cc} \begin{array}{c} + \\ | \\ - \end{array} & \begin{array}{c} - \\ | \\ + \end{array} \end{array} \right\} b = x - y$$

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$$\mathcal{Z}_n(\mathbf{x}, \mathbf{y}) \propto \sum_{\text{configurations}} \prod_{i,j} \omega_{i,j}(x_i, y_j)$$

Definition of the 6-Vertex model

$$\begin{array}{cccccc}
 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 1 & 0 & 1 & 0 & 0 & -1 & 1 \\
 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 q & q & q & q & q & q & q
 \end{array}
 \quad
 \left.
 \begin{array}{cc}
 \begin{array}{c} + \\ +0+ \\ + \end{array} & \begin{array}{c} - \\ -0- \\ - \end{array}
 \end{array}
 \right\} a = q - 1$$

$$\left.
 \begin{array}{cc}
 \begin{array}{c} - \\ +0+ \\ - \end{array} & \begin{array}{c} + \\ -0- \\ + \end{array}
 \end{array}
 \right\} b = q - 1$$

$$\left.
 \begin{array}{cc}
 \begin{array}{c} + \\ +1- \\ - \end{array} & \begin{array}{c} - \\ -1+ \\ + \end{array}
 \end{array}
 \right\} c = q - 1$$

Definition (Partition function)

We can use it to count Alternating Sign Matrices (at $q^3 = 1$):

$$\mathcal{Z}_n(\mathbf{1}, \mathbf{q}) = \#\{\text{ASM of size } n \times n\} = 1, 2, 7, 42, 429, \dots$$

The $\check{R}$ matrix

We can build a configuration using boxes:

$$x \begin{array}{|c|} \hline \begin{array}{c} \uparrow \\ \hline \rightarrow \\ \hline \end{array} \\ \hline y \end{array} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix} =: \check{R}(x, y)$$

The $\check{R}$ matrix

We can build a configuration using boxes:

$$x \begin{array}{|c|} \hline \begin{array}{c} \uparrow \\ \square \\ \downarrow \end{array} \\ \hline \end{array} \begin{array}{|c|} \hline \begin{array}{c} \leftarrow \\ \square \\ \rightarrow \end{array} \\ \hline \end{array} y = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix} =: \check{R}(x, y)$$

Then

$$\check{R}(x, y) : V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$$

where V_i is the spin $\frac{1}{2}$ representation of $U_q(sl_2)$.

The $\check{R}$ matrix

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$$x \begin{array}{c} \uparrow \\ \boxed{\begin{array}{c} \text{ } \\ \text{ } \end{array}} \\ \leftarrow \text{y} \end{array} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix} =: \check{R}(x, y)$$

Then

$$\check{R}(x, y) : V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$$

where V_i is the spin $\frac{1}{2}$ representation of $U_q(sl_2)$.

Example

$$\check{R}(x, y) |+-\rangle = c(x, y) |+-\rangle + b(x, y) |-+\rangle$$

Yang–Baxter equation

Identity equation:

$$\check{R}(x, y)\check{R}(y, x) = (qx - q^{-1}y)(qy - q^{-1}x)Id.$$

Yang–Baxter equation:

$$\check{R}_{2,3}(y_2, y_3)\check{R}_{1,3}(y_1, y_3)\check{R}_{1,2}(y_1, y_2) = \check{R}_{1,2}(y_1, y_2)\check{R}_{1,3}(y_1, y_3)\check{R}_{2,3}(y_2, y_3)$$

Yang–Baxter equation

Identity equation:

$$\begin{array}{c} \text{Diagram: A crossing of a red line and a blue line, with a blue arc connecting the two lines on the left side.} \end{array} = (qx - q^{-1}y)(qy - q^{-1}x) \begin{array}{c} \text{Diagram: Two parallel horizontal lines, the top one red and the bottom one blue, both with arrows pointing to the right.} \end{array}$$

Yang–Baxter equation:

$$\begin{array}{c} \text{Diagram: A crossing of a red line and a green line, with a blue arc connecting the two lines on the left side.} \end{array} = \begin{array}{c} \text{Diagram: A crossing of a red line and a green line, with a blue arc connecting the two lines on the right side.} \end{array}$$

Yang–Baxter equation

Identity equation:

$$\begin{array}{c} \text{Diagram: A blue line and a red line crossing. The blue line has an arrow pointing up-right, and the red line has an arrow pointing down-left.} \end{array} = (qx - q^{-1}y)(qy - q^{-1}x) \begin{array}{c} \text{Diagram: Two parallel horizontal lines. The top line is red with an arrow pointing right. The bottom line is blue with an arrow pointing right.} \end{array}$$

Yang–Baxter equation:

$$\begin{array}{c} \text{Diagram: A red line and a green line crossing. The red line has an arrow pointing up-right, and the green line has an arrow pointing down-left.} \end{array} = \begin{array}{c} \text{Diagram: A red line and a green line crossing. The red line has an arrow pointing up-left, and the green line has an arrow pointing down-right.} \end{array}$$

Example

$$\begin{array}{c} \text{Diagram: A red line and a green line crossing. The red line has an arrow pointing up-right, and the green line has an arrow pointing down-left.} \end{array} = \begin{array}{c} \text{Diagram: A red line and a green line crossing. The red line has an arrow pointing up-left, and the green line has an arrow pointing down-right.} \end{array}$$

Yang–Baxter equation

Identity equation:

$$\text{Diagram} = (qx - q^{-1}y)(qy - q^{-1}x) \text{Diagram}$$

The diagram on the left shows a crossing of a red line and a blue line. The diagram on the right shows two parallel horizontal lines, one red and one blue, both with arrows pointing to the right.

Yang–Baxter equation:

$$\text{Diagram} = \text{Diagram}$$

The diagram on the left shows a crossing of a red line and a green line. The diagram on the right shows a crossing of a green line and a red line. Both diagrams have a blue vertical line passing through the crossing.

Example

$$\text{Diagram} = \text{Diagram}$$

The diagram on the left shows a crossing of a red line and a green line. The diagram on the right shows a crossing of a green line and a red line. Both diagrams have a blue vertical line passing through the crossing. The left diagram has a '+' sign above the crossing and a '-' sign below it. The right diagram has a '+' sign above the crossing and a '-' sign below it.

Identity equation:

$$\text{Diagram of two crossing lines (red and blue)} = (qx - q^{-1}y)(qy - q^{-1}x) \begin{matrix} \text{red arrow} \\ \text{blue arrow} \end{matrix}$$

Yang–Baxter equation:

Yang–Baxter equation

Identity equation:

$$\begin{array}{c} \text{Diagram: a crossing of a red line and a blue line} \end{array} = (qx - q^{-1}y)(qy - q^{-1}x) \begin{array}{c} \text{Diagram: two parallel horizontal lines, top red, bottom blue} \end{array}$$

Yang–Baxter equation:

$$\begin{array}{c} \text{Diagram: a crossing of a red line and a green line} \end{array} = \begin{array}{c} \text{Diagram: a crossing of a green line and a red line} \end{array}$$

Example

$$\begin{array}{c} \text{Diagram: a crossing of a red line and a green line, with a blue line passing through it} \end{array} = \begin{array}{c} \text{Diagram: a crossing of a green line and a red line, with a blue line passing through it} \end{array}$$

Yang–Baxter equation

Identity equation:

$$\begin{array}{c} \text{red} \swarrow \quad \searrow \text{blue} \\ \text{blue} \swarrow \quad \searrow \text{red} \end{array} = (qx - q^{-1}y)(qy - q^{-1}x) \begin{array}{c} \text{red} \rightarrow \\ \text{blue} \rightarrow \end{array}$$

Yang–Baxter equation:

$$\begin{array}{c} \text{red} \swarrow \quad \searrow \text{blue} \\ \text{blue} \swarrow \quad \searrow \text{red} \end{array} = \begin{array}{c} \text{blue} \swarrow \quad \searrow \text{red} \\ \text{red} \swarrow \quad \searrow \text{blue} \end{array}$$

Example

$$\begin{array}{c} + \\ + \quad \text{red} \rightarrow \quad + \\ + \quad \text{blue} \rightarrow \quad - \\ - \end{array} = \begin{array}{c} + \\ + \quad \text{blue} \rightarrow \quad + \\ + \quad \text{red} \rightarrow \quad - \\ - \end{array}$$

Yang–Baxter equation

Identity equation:

$$\begin{array}{c} \text{Red line over blue line} \\ \text{Blue line over red line} \end{array} = (qx - q^{-1}y)(qy - q^{-1}x) \begin{array}{c} \text{Red line over blue line} \\ \text{Blue line over red line} \end{array}$$

Yang–Baxter equation:

$$\begin{array}{c} \text{Red line over green line} \\ \text{Green line over blue line} \end{array} = \begin{array}{c} \text{Green line over blue line} \\ \text{Blue line over red line} \end{array}$$

Example

$$\begin{array}{c} + \\ + \text{---} + \text{---} + \\ | \text{---} + \text{---} + \\ | \text{---} + \text{---} - \\ - \end{array} = \begin{array}{c} + \\ + \text{---} + \text{---} + \\ | \text{---} + \text{---} - \\ | \text{---} - \text{---} - \\ - \end{array}$$

Yang–Baxter equation

Identity equation:

$$\begin{array}{c} \text{Red line over blue line} \\ \text{Blue line under red line} \end{array} = (qx - q^{-1}y)(qy - q^{-1}x) \begin{array}{c} \text{Red line over blue line} \\ \text{Blue line under red line} \end{array}$$

Yang–Baxter equation:

$$\begin{array}{c} \text{Red line over green line} \\ \text{Green line under blue line} \end{array} = \begin{array}{c} \text{Green line over red line} \\ \text{Red line under blue line} \end{array}$$

Example

$$\begin{array}{c} + \\ + \quad + \quad + \\ + \quad + \quad + \\ - \end{array} \begin{array}{c} + \\ + \quad + \quad + \\ + \quad + \quad + \\ - \end{array} = \begin{array}{c} + \\ + \quad + \quad + \\ + \quad + \quad + \\ - \end{array} \begin{array}{c} + \\ + \quad + \quad + \\ + \quad + \quad + \\ - \end{array} + \begin{array}{c} + \\ + \quad + \quad + \\ + \quad + \quad + \\ - \end{array} \begin{array}{c} + \\ + \quad + \quad + \\ + \quad + \quad + \\ - \end{array}$$

Yang–Baxter equation

Identity equation:

$$\begin{array}{c} \text{Diagram: a blue line and a red line crossing, with arrows indicating a braid-like interaction.} \end{array} = (qx - q^{-1}y)(qy - q^{-1}x) \begin{array}{c} \text{Diagram: two parallel horizontal lines, one red on top and one blue on bottom, both with arrows pointing right.} \end{array}$$

Yang–Baxter equation:

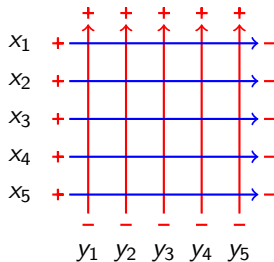
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Example

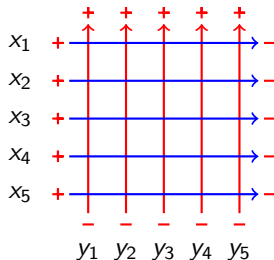
$$\begin{array}{c} \text{Diagram 1: A 3x3 grid of vertices. The top row has three '+' signs. The bottom row has a '+' on the left, a '-' in the middle, and a '-' on the right. The middle row has a '+' on the left, a '+' in the middle, and a '-' on the right. A red line enters from the top-left and exits from the bottom-right. A green line enters from the top-right and exits from the bottom-left. A blue line enters from the top-middle and exits from the bottom-middle.} \end{array} = \begin{array}{c} \text{Diagram 2: Similar to Diagram 1, but with different line colors and orientations.} \end{array} + \begin{array}{c} \text{Diagram 3: Similar to Diagram 1, but with different line colors and orientations.} \end{array}$$

$$a(x, y)a(y, z)c(x, z) = c(y, z)a(x, z)c(x, y) + b(y, z)c(x, z)b(x, y)$$

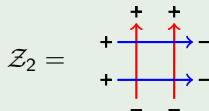
The partition function



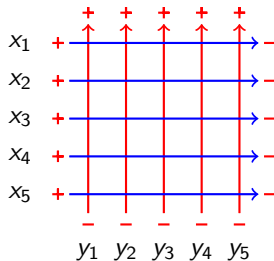
The partition function



Example ($n = 2$)



The partition function

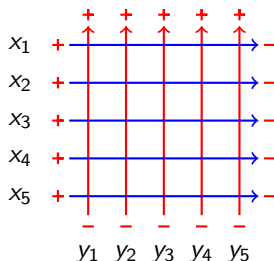


Example ($n = 2$)

$$\mathcal{Z}_2 = \begin{array}{c} \begin{array}{cc} + & + \\ \uparrow & \uparrow \\ + & + \\ + & - \\ + & - \\ - & - \end{array} & + & \begin{array}{cc} + & + \\ \uparrow & \uparrow \\ + & - \\ - & + \\ - & + \\ - & - \end{array} \end{array}$$

The equation shows the partition function $\mathcal{Z}_2$ as the sum of two 2x2 grids of vertices. Each vertex in the grids has a red '+' sign on its top edge and a red '-' sign on its bottom edge. The horizontal edges are blue with arrows pointing right, and the vertical edges are red with arrows pointing up. The left edges are labeled x_1, x_2, x_3, x_4 and the right edges are labeled $-$. The bottom edges are labeled y_1, y_2, y_3, y_4 .

The partition function



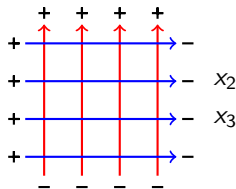
Example ($n = 2$)

$$\mathcal{Z}_2 = \begin{array}{c} \begin{array}{cc} + & + \\ \uparrow & \uparrow \\ + & + \\ \downarrow & \downarrow \\ + & + \\ - & - \end{array} & + & \begin{array}{cc} + & + \\ \uparrow & \uparrow \\ + & - \\ \downarrow & \downarrow \\ + & + \\ - & - \end{array} \end{array}$$

$$c(x_2, y_1)a(x_2, y_2)a(x_1, y_1)c(x_1, y_2) + b(x_2, y_1)c(x_2, y_2)c(x_1, y_1)b(x_1, y_2)$$

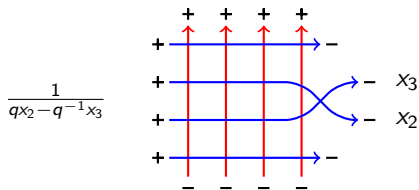
Properties of the partition function

Symmetry in x and y . Proved using Yang–Baxter equation:



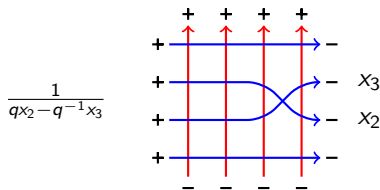
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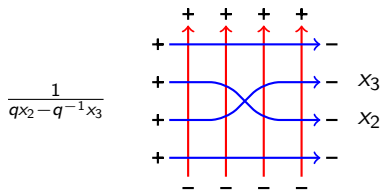
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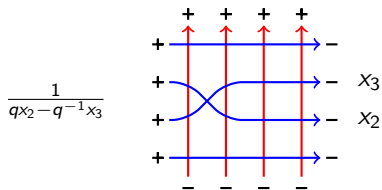
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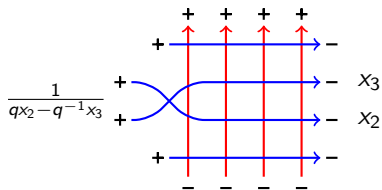
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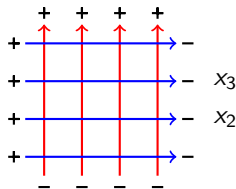
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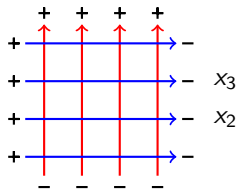
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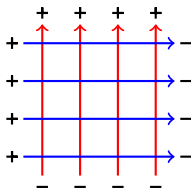


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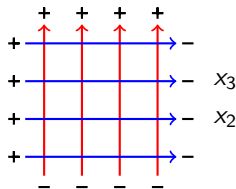


Korepin's recursion relation. At $y_1 = q^2 x_1$, $a(x_1, y_1) = 0$, then:

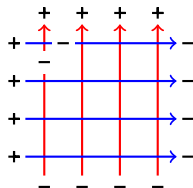


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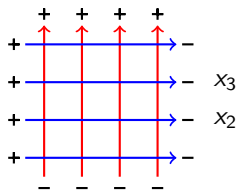


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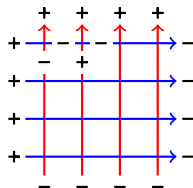


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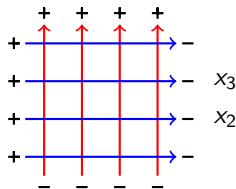


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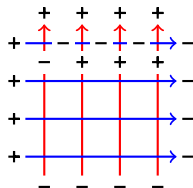


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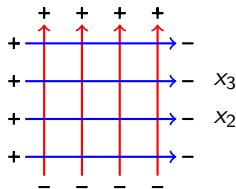


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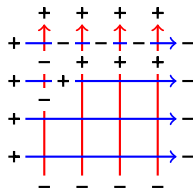


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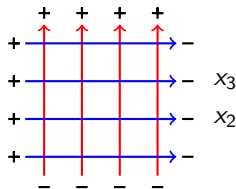


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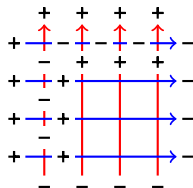


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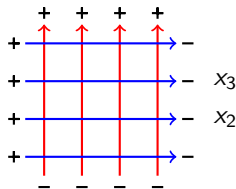


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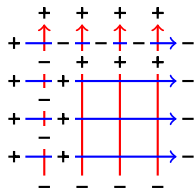


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Korepin's recursion relation. At $y_1 = q^2 x_1$, $a(x_1, y_1) = 0$, then:



$$\mathcal{Z}_n|_{y_1=q^2 x_1} \propto \prod_{i \neq 1} (y_i - x_1)(x_i - y_1) \sqrt{x_1 y_1} \mathcal{Z}_{n-1}$$

Properties of the partition function

A weight $c(x_i, y_j)$ appears an odd number of times at each row and column. Normalize

$$\frac{\mathcal{Z}_n(\mathbf{x}, \mathbf{y})}{\prod_i \sqrt{x_i y_i}} \rightarrow \mathcal{Z}_n(\mathbf{x}, \mathbf{y})$$

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Explicit formula for the partition function

Using Korepin's recursion relation, Izergin showed that:

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A Schur polynomial

Set $q = e^{\frac{2i\pi}{3}}$, the so-called combinatorial point.

$$Y_n = \{n-1, n-1, n-2, n-2, \dots, 1, 1, 0, 0\}$$

100

Higher Spin Generalization

We can build a higher spin version, of the model:

$$\check{R}^{(\ell)}(x, y) : V_1^{(\ell)} \otimes V_2^{(\ell)} \rightarrow V_2^{(\ell)} \otimes V_1^{(\ell)}$$

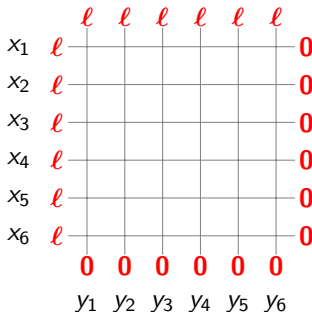
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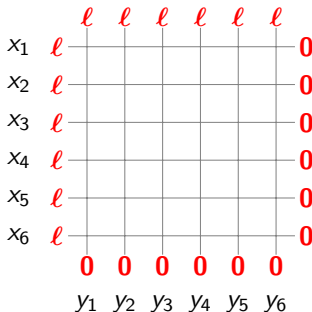


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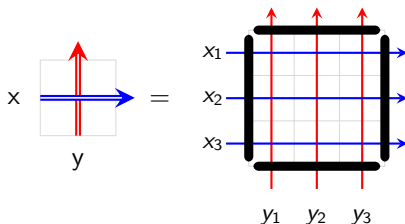
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$$\begin{array}{c}
 \text{Diagram illustrating the definition of the } \check{R}^{(\ell)} \text{ matrix for } \ell=3. \\
 \text{Left side: A single vertex with a horizontal blue arrow labeled } x \text{ and a vertical red arrow labeled } y. \\
 \text{Right side: A } 3 \times 3 \text{ grid of vertices. The vertical line is labeled } x, q^2 x, q^4 x \text{ from top to bottom. The horizontal line is labeled } y, q^2 y, q^4 y \text{ from left to right.} \\
 \text{The grid is labeled } =: \check{R}^{(\ell)}(x, y) \text{ on the right.}
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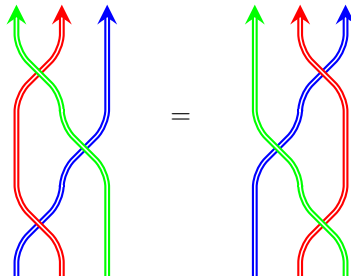
$$\begin{array}{c}
 \begin{array}{|c|} \hline \text{6-vertex model} \\ \hline \end{array} \\
 = \\
 \begin{array}{|c|c|c|} \hline \text{3x3 grid of 6-vertex models} \\ \hline \end{array} \\
 =: \check{R}^{(\ell)}(x, y)
 \end{array}$$

Yang-Baxter equation

Identity equation:



Yang-Baxter equation:

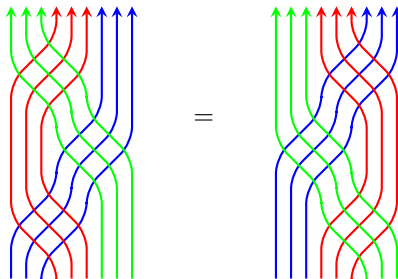


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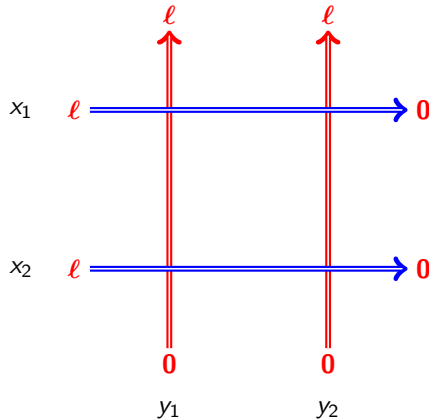


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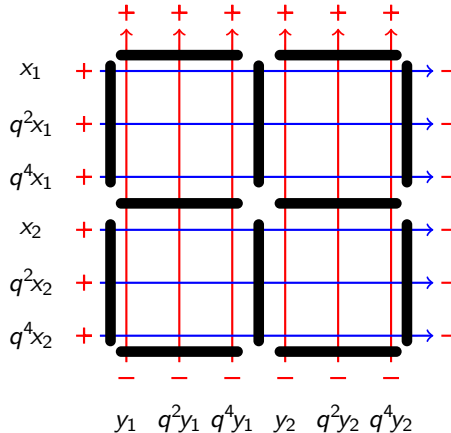
The Partition Function

We construct the partition function in the same way:



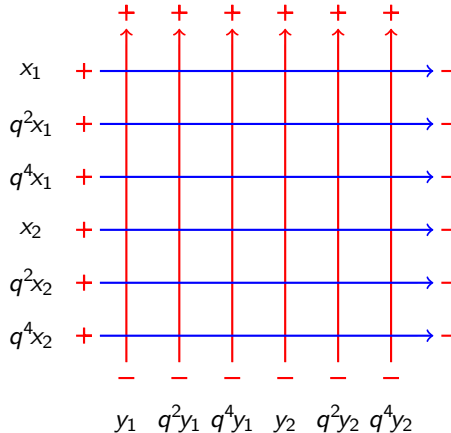
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An Explicit Formula for the Partition Function

Let

$$\bar{\mathbf{x}} = \{x_1, q^2 x_1, \dots, q^{2\ell-2} x_1, \dots, x_n, q^2 x_n, \dots, q^{2\ell-2} x_n\}$$

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Caradoc, Foda and Kitanine showed that:

$$\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y}) = (\text{Pre-factor}) \det \left| \frac{1}{(\bar{x}_i - q\bar{y}_j)(q\bar{x}_i - \bar{y}_j)} \right|_{i,j}^{\ell n \times \ell n}$$

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Wheel Condition

When $q^{2\ell+1} = 1$, the partition function satisfies the wheel condition:

Definition (Wheel condition)

A polynomial $P(\mathbf{z})$ satisfies the wheel condition if:

$$P(\mathbf{z}) = 0 \quad \text{if } z_k = q^{1+2s_2} z_j = q^{2+2s_1+2s_2} z_i$$

for all $s_1, s_2 \in \mathbb{N}_0$ such that $s_1 + s_2 \leq \ell - 1$ and for any choice $i < j < k$.

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We call this point $q^{2\ell+1} = 1$, the combinatorial point.

Wheel condition - an example

Example ($\ell = 2$)

Impose $q^5 = 1$. It means that the polynomial vanishes whenever:

$$z_k = qz_j = q^2 z_i \qquad z_k = q^3 z_j = q^4 z_i \qquad z_k = qz_j = q^4 z_i$$

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The following polynomial satisfies the wheel condition:

$$P(z_1, \dots, z_{2n}) = \prod_{1 \leq i < j \leq n} (qz_i - z_j)(q^3 z_i - z_j) \prod_{n < i < j \leq 2n} (qz_i - z_j)(q^3 z_i - z_j)$$

Symmetric polynomials - different basis

Let $\lambda = \{\lambda_1, \dots, \lambda_N\}$, such that $\lambda_i \geq \lambda_{i+1}$.

Definition (Monomials)

$$m_\lambda = \mathcal{S} \left(z_1^{\lambda_1} z_2^{\lambda_1} \dots z_N^{\lambda_N} \right)$$

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$$p_{\{1,1,0\}} = (z_1 + z_2 + z_3)^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1 z_2 + z_1 z_3 + z_2 z_3)$$

Macdonald polynomials

Define an inner product by:

$$\langle p_\lambda(\mathbf{z}), p_\mu(\mathbf{z}) \rangle_{\tilde{q}, t} = z_\mu \delta_{\lambda\mu} \prod_i \frac{1 - \tilde{q}^{\mu_i}}{1 - t^{\mu_i}}$$

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Definition (Macdonald polynomials)

The Macdonald polynomials are defined by the Gram–Schmidt process:

$$\begin{aligned} 0 &= \langle P_\lambda(\mathbf{z}; \tilde{q}, t), P_\mu(\mathbf{z}; \tilde{q}, t) \rangle_{\tilde{q}, t} && \text{if } \lambda \neq \mu \\ P_\lambda(\mathbf{z}; \tilde{q}, t) &= m_\lambda(\mathbf{z}) + \sum_{\mu \prec \lambda} c_{\lambda, \mu}(\tilde{q}, t) m_\mu(\mathbf{z}) \end{aligned}$$

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Theorem (Jimbo, Miwa, Feigin and Mukhin)

The symmetric polynomials obeying to the wheel condition are spanned by the Macdonald polynomials P_λ (with $\tilde{q} = q^2$ and $t = q$), such that:

$$\lambda_i - \lambda_{i+2} \geq \ell$$

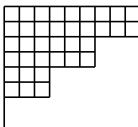
The Partition Function as a Macdonald Polynomial

Fix the number of boxes to $\ell n(n-1)$, then we have no choice.

$$Y_{n,\ell} = \{\ell(n-1), \ell(n-1), \ell(n-2), \ell(n-2), \dots, \ell, \ell, 0, 0\}$$

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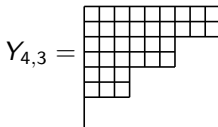


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$$\mathcal{Z}_{n,\ell}(\mathbf{x}, \mathbf{y}) \propto P_{Y_{n,\ell}}(\mathbf{x}, \mathbf{y}; q^2, q)$$

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Theorem (Main result)

At the combinatorial point, the partition function is the Macdonald polynomial:

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Corollary (Symmetry)

The partition function is a fully symmetric homogeneous polynomial.

Open questions

- The 6-Vertex model is in bijection with Alternating Sign Matrices.
 - Can this generalization lead us to nice combinatorics?
 - What about the homogeneous limit?
- Wheel conditions and determinants.
 - Can we obtain different wheel conditions, with similar methods?
 - And the same wheel condition but higher degree?
- Can we prove directly that the partition function is fully symmetric?

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