Asymptotic behaviour of correlation functions.

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Outline

Generalities about lattice models

- Linear operator $\mathcal{H}$ on Hilbert space $\mathscr{H} = \mathscr{V}_1 \otimes \cdots \otimes \mathscr{V}_L$.
- Spaces $\mathscr{V}_\ell$ can be finite or infinite dimensional. Often isomophic $\mathscr{V}_\ell \simeq \mathscr{V}_0$.
- Basis of operators $\mathcal{O}^{(\alpha)}$ on $\mathscr{V}_0 \rightsquigarrow$ operators $\mathcal{O}_\ell^{(\alpha)} = \underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{\ell-1 \mathrm{times}} \otimes \mathcal{O}^{(\alpha)} \otimes \underbrace{\mathrm{id} \cdots \otimes \mathrm{id}}_{N-\ell-1}$.

  Often $\mathcal{H}$ has nearest neighbor coupling structure

$$\mathcal{H} = \sum_{j=1}^{L} f(\mathcal{O}_j^{(\alpha)}, \mathcal{O}_{j+1}^{(\beta)}) + \mathrm{bdry\ \ terms}$$

- **Example** The periodic XXZ spin-1/2 chain:

$$\mathcal{H}_{XXZ} = J \sum_{n=1}^{L} \left\{ \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \cos(\zeta) \sigma_n^z \sigma_{n+1}^z + h \sigma_n^z \right\} \quad , \quad \sigma_{n+L} \equiv \sigma_n$$

  - Local spaces $\mathscr{V}_\ell \simeq \mathscr{V}_0 \simeq \mathbb{C}^2$ and local operators

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad h \equiv \text{magnetic field.}$$

What one would like to know?

i) Find the Eigenstates and Eigenvectors of $\mathcal{H}|\,\Psi_{\beta}\,\rangle \;=\; E_{\beta}|\,\Psi_{\beta}\,\rangle$;

ii) Compute in closed form and characterize the correlation functions

$$\langle\,\Psi_{\gamma}\,|\mathcal{O}_{1}^{(\alpha_1)}\dots\mathcal{O}_{m}^{(\alpha_m)}|\,\Psi_{\beta}\,\rangle \quad ;$$

- Characterize intrinsic & response properties of a system.
- Appear in perturbative expansions: $\mathcal{H} \;\hookrightarrow \mathcal{H}+\mathcal{H}_{\mathrm{pert}}$.

iii) Characterize the behaviour at finite temperatures

$$\langle\mathcal{O}_{m}^{(\alpha_m)}\mathcal{O}_{1}^{(\alpha_1)}\rangle_{T} \;\equiv\; \mathrm{tr}\Big[\mathrm{e}^{-\frac{\mathcal{H}}{T}}\mathcal{O}_{m}^{(\alpha_m)}\mathcal{O}_{1}^{(\alpha_1)}\Big]/\mathrm{tr}\Big[\mathrm{e}^{-\frac{\mathcal{H}}{T}}\Big]$$

⊛ Program *i*) – *iii*) especially interesting when $L \to +\infty$.

Low-lying excitations in 1D quantum Hamiltonians

★ **'84** Cardy Central charge ⇝ finite-size corrections to ground state energy ;

$$E_{G.S.} = L\varepsilon - c\frac{\pi v_F}{6L} + \mathrm{O}\left(\frac{1}{L^2}\right) \qquad \text{and} \qquad E_{\text{ex}} - E_{G.S.} = \frac{2\pi v_F}{L}\delta$$

★ Bethe Ansatz ⇝ spectrum given by solutions to algebraic equations

$$F^L(\lambda_j) = \prod_{a=1}^{N} S(\lambda_j, \lambda_k) \qquad \text{and} \qquad E(\{\lambda_j\}) = \sum_{j=1}^{N} \varepsilon_0(\lambda_j)$$

⇝ Extract the large N, L behavior.

★ Methods for computing finite-size corrections from Bethe Ansatz
'87-'95 (Batchelor, Destri, DeVega, Klumper, Pearce, Woynarowich, Wehner, Zittartz) ;

⊛ Proof of Cardy's predictions for the conformal structure of spectrum:

$$c = 1 \qquad \delta = \left(\frac{n_1}{2\mathcal{Z}}\right)^2 + (\mathcal{Z}n_2)^2 + n_3 \qquad \text{and} \quad \text{linear integral equations} \rightsquigarrow v_F\,,\ \mathcal{Z}$$

Typical long-distance behavior of correlators

- T>0 exponential decay at long-distance is expected:

$$\langle \mathcal{O}_m \mathcal{O}_1 \rangle_T = \langle \mathcal{O}_1 \rangle_T^2 + \mathcal{A} \exp\left(-m/\xi\right) + \dots$$

- T=0 Model becomes critical if gapless spectrum $\Longrightarrow$ algebraic decay

$$\langle \mathcal{O}_m \mathcal{O}_1 \rangle_{T=0} \equiv \frac{\langle \, G.S. \, | \mathcal{O}_m \mathcal{O}_1 | \, G.S. \, \rangle}{\langle G.S. \mid G.S. \rangle} \simeq \langle \mathcal{O}_1 \rangle_0^2 + \frac{C_1}{m^{\alpha_1}} + \frac{C_2}{m^{\alpha_2}} \cos\left(2m p_F\right) + \dots\dots$$

- Prediction of critical exponents α_i, correlation lengths ξ by approximate methods
  - Correspondence with a Conformal Field Theory (**'70 Polyakov** , **'84 Cardy**)
  - Correspondence with Luttinger liquid (**'75 Luther, Peschel** , **'81 Haldane**)

Predictions for the critical exponents

- Correlators in a two-dimensional CFT on a strip of width L

$$\left\langle \phi(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2)\right\rangle = C\left(\frac{\pi/L}{\sinh[\pi(z_1-z_2)/L]}\right)^{2\Delta_+}\left(\frac{\pi/L}{\sinh[\pi(\bar{z}_1-\bar{z}_2)/L]}\right)^{2\Delta_-} \qquad z_a = x_a + i v_F t_a \, .$$

- Excitation energy from form factor expansion

$$\left\langle \phi(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2)\right\rangle = \sum_{\Psi_{\mathrm{ex}}} |\langle\, 0\,|\phi(0,0)|\,\Psi_{\mathrm{ex}}\,\rangle|^2 \mathrm{e}^{-(t_1-t_2)(E_{\mathrm{ex}}-E_{G.S.})-i(x_1-x_2)(P_{\mathrm{ex}}-P_{G.S.})}$$

$$E_{\mathrm{ex}} - E_{G.S.} = \frac{2\pi}{L} v_F(\Delta_+ + \Delta_-) \qquad \text{and} \qquad P_{\mathrm{ex}} - P_{G.S.} = \frac{2\pi}{L}(\Delta_+ - \Delta_-)$$

- **'70** Polyakov Conformal invariance of correlators at large distances ;
- **'84** Cardy Central charge ⇝ finite-size corrections to ground state energy ;

Low-lying excitations ↔ conformal dimensions $\Delta_\pm$ ⇝ asymptotics

Asymptotic behavior of correlation functions

⊛ The non-linear Schrödinger model

$$H = \int_0^L \left\{ \partial_y \Psi^\dagger(y) \partial_y \Psi(y) + c \Psi^\dagger(y) \Psi^\dagger(y) \Psi(y) \Psi(y) - h \Psi^\dagger(y) \Psi(y) \right\} \mathrm{d}y$$

L: length of circle, $c > 0$ coupling constant (repulsive regime), $h > 0$ chemical potential.

- NLSM $\equiv$ quantum critical model at $T = 0K$
- low-lying excitations from large L analysis of Bethe Ansatz equations

♦ Density-density correlator $j(x) = \Psi^\dagger(x) \Psi(x)$:

$$\frac{\langle G.S. | j(x) j(0) | G.S. \rangle}{\langle G.S. | G.S. \rangle} = \langle j(x) j(0) \rangle \simeq \langle j(0) \rangle^2 + \frac{C_1}{x^2} + C_2 \frac{\cos(2x p_F)}{x^{2\mathcal{Z}^2}} + \dots$$

♦ Reduced density matrix

$$\langle \Psi(x) \Psi^\dagger(0) \rangle \simeq C_3 x^{-\frac{1}{2\mathcal{Z}^2}} + \dots$$

Turining the time on

♦ Predictions for the long-distance/long-time behavior at $T = 0K$ restricted to $x \gg v_F t$:

$$\left\langle j(x,t)\, j(0,0) \right\rangle \simeq \langle j(0,0)\rangle^2 + C_1' \frac{x^2 + v_F^2 t^2}{\left(x^2 - v_F^2 t^2\right)^2} + C_2' \frac{\cos(2x p_F)}{\left(x^2 - v_F^2 x^2\right)^{\mathcal{Z}^2}} + \ldots$$

⇒ *Consistency problem* with time-dependent asymptotics

$$\frac{x^2 + v_F^2 t^2}{\left(x^2 - v_F^2 t^2\right)^2}\left(1 + \mathrm{o}(1)\right) = \frac{1}{x^2}\left(1 + \mathrm{o}(1)\right) \quad \text{when } x \gg v_F t$$

- What happens when $x \sim v_F t$?

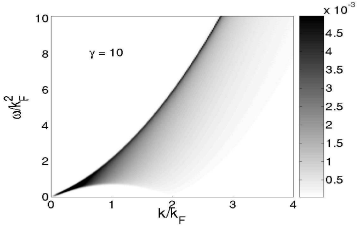
The edge exponents for Fourier transforms

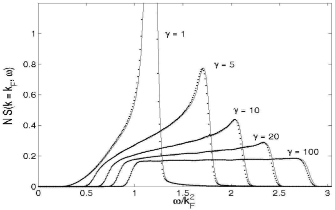
- Experiments measure Fourier transforms

$$S(k,\omega) = \int_{\mathbb{R}^2} \mathrm{e}^{i(\omega t - kx)} \langle j(x,t)\, j(0,0)\rangle \mathrm{d}x\mathrm{d}t$$

  - ⇝ DSF measured by Fourier sampling of time of flight images or Bragg spectroscopy.

★ **'06** (Caux, Calabrese) Density structure factor in NLSM





Predictions for the behavior near the edges

- ★ **'67** (Mahan), **'67** (Noziére, De Dominicus) Arguments for a power-law behavior near edges.
- ★ **'08** (Glazman, Imambekov) Non-linear Luttinger liquid $\rightsquigarrow$ predictions for edge exponents.

$$S(k,\omega) \simeq \mathscr{A}(k) \cdot \Xi(\omega - \varepsilon_h(k)) \cdot [\omega - \varepsilon_h(k)]^{\vartheta} \qquad \vartheta > 0$$

- ★ **'09** (Affleck, Pereira, White) X-ray edge-type model $\rightsquigarrow$ predictions for edge exponents.
- ★ **'10** (Caux, Glazman, Imambekov, Shashi) Predictions for $\mathscr{A}(k)$ (NLSM);
- Can these predictions be confirmed by an approach solely on the microscopic model?

The XXZ spin-1/2 chain

$$\mathcal{H}_{XXZ} = J \sum_{n=1}^{L} \left\{ \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \cos(\zeta) \sigma_n^z \sigma_{n+1}^z + h \sigma_n^z \right\} \quad , \quad \sigma_{n+L} \equiv \sigma_n$$

★ Coordinate Bethe Ansatz at $\cos(\zeta) = 1$ (**'31 Bethe**) :

Eigenvectors $|\lambda_1, \dots, \lambda_N\rangle = \sum_{\{n\}} c_{\{n\}}(\lambda_1, \dots, \lambda_N) | \{n\} \rangle$ & Eigenvalues $E = \sum_{a=1}^{N} \epsilon_0(\lambda_a)$

Parameterized by sols to Bethe equations

$$\left(\frac{\sinh(\lambda_j + i\zeta/2)}{\sinh(\lambda_j - i\zeta/2)} \right)^L = \prod_{\substack{a=1 \\ \neq j}}^{N} \frac{\sinh(\lambda_j - \lambda_k + i\zeta)}{\sinh(\lambda_j - \lambda_k - i\zeta)}$$

★ Further developments (**'58 Orbach , '66 Yang, Yang , '79 Faddeev, Sklyanin, Takhtadjan**)
Increase in rigor & simplification of expressions.

⊛ Eigenvectors *highly intricate* expression in basis where $\sigma_n^x, \sigma_n^y, \sigma_n^z$ are simple.

A correlator of interest

★ Computation of correlation functions $\langle \Psi_1 | \sigma_1 \sigma_m | \Psi_2 \rangle$ ⇝ highly complex problem.

Some simplifications

⊛ Correlation functions at $T = 0K$ ≡ expectation values in the ground state.

⊛ Study at first symmetric correlators

• Generating function $\mathcal{L}_m(\beta) \equiv \langle GS | \begin{pmatrix} 1 & 0 \\ 0 & e^{\beta} \end{pmatrix}_1 \cdots \begin{pmatrix} 1 & 0 \\ 0 & e^{\beta} \end{pmatrix}_m | GS \rangle$

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = \frac{\partial^2}{\partial \beta^2} \Big(\mathcal{L}_{m+1}(\beta) + \mathcal{L}_{m-1}(\beta) - 2\mathcal{L}_m(\beta) \Big)_{|\beta=0} - 4\langle \sigma_1^z \rangle + 1$$

⊛ Combinatorics strongly simplify at $\cos(\zeta) = 0$ (free fermion point)

⇝ 1^{st} results for free fermions ⟹ Toeplitz or Fredholm determinant representations

45 years of efforts:

Kaufman, Onsager, **Lieb, Mattis, Schulz**, **Lenard**, **McCoy, Wu**, **Korepin, Slavnov** . . .

Multiple integral representations at the free fermion point

♦ Fredholm determinant of pure sine kernel for $\mathcal{L}_m(\beta)$

$$\det\left[I+S_m\right] = \sum_{n\geq 0}\frac{1}{n!}\int_{-q}^{q}\det_n\begin{bmatrix} S_m(\lambda_1,\lambda_1)\dots S_m(\lambda_1,\lambda_n) \\ \dots \\ S_m(\lambda_n,\lambda_1)\dots S_m(\lambda_n,\lambda_n)\end{bmatrix}\cdot \mathrm{d}^n\lambda \quad \text{with} \quad S_m(\lambda,\mu) = \frac{e^{\beta}-1}{\pi}\,\frac{\sin\frac{m}{2}[p_0(\lambda)-p_0(\mu)]}{\sinh(\lambda-\mu)}$$

The number of integrals **varies** from 0 to $+\infty$ $\rightsquigarrow$ extract $m \to +\infty$ behavior.

- *Tour de force* direct analysis (**'79** **Tracy, Vaidya**);
- Sine kernel related to Painlevé V (**'80** **Jimbo, Miwa, Mori, Sato**);
- transverse Ising, imp. bosons (**'83-'86** **McCoy, Perk, Shrock, Tang**);
- Operator methods (**'94** **Widom** , **'94** **Budylin, Buslayev**) ;
- RHP setting for integrable integral operators (**'90** **Its, Izergin, Korepin, Slavnov**).

Beyond the free fermion point

★ Algebraic version of Bethe Ansatz (**'79 Faddeev, Takhtadjan, Sklyanin**)

- Algebraic construction of eigenstates $|\{\lambda_j\}\rangle = B(\lambda_1)\dots B(\lambda_N)|0\rangle$

- First series of multiple integrals at $T \neq 0$ and $h \neq 0$ (**'84 Izergin-Korepin**)

- $$\langle j(x,0)\, j(0,0)\rangle_T = \sum_{n=1}^{+\infty} \int_{-q}^{q} I_n^x(\lambda_1,\dots,\lambda_n)\cdot \mathrm{d}^n\lambda$$

$$I_n^x(\lambda_1,\dots,\lambda_n) = \text{partitions \& combinatorics \& non} - \text{linear integral equations}$$

⊛ Norms (**'81 Gaudin, McCoy, Wu** , **'82 Korepin**), Scalar products (**'89 Slavnov**),

- Dual fields based det. rep. (**'97 Kojima, Korepin, Slavnov**)

$$\langle \Psi(0,0)\Psi^\dagger(x,t)\rangle_T = \left(0\middle|\left(G(x,t)+\frac{\partial}{\partial\alpha}\right)_{|\alpha=0}\cdot\frac{\det[I+\widehat{V}_\alpha(x,t)]}{\det[I-K]}\middle|0\right)$$

Going beyond the free-fermion point: The vertex operator approach

- Multiple integrals representation matrix elements of reduced density matrix XXZ (T=0): ($\cos(\zeta) > 1$ **'92 Jimbo, Miki, Miwa, Nakayashiki** and $-1 < \cos(\zeta) < 1$ **'96 Jimbo, Miwa**)

$$\mathrm{tr}_{1,\dots,m}\left[\rho\,\sigma_1^z\,\sigma_m^z\right] = \langle\sigma_1^z\,\sigma_m^z\rangle \qquad \rho_{\epsilon_1\dots\epsilon_m}^{\epsilon_1',\dots,\epsilon_m'} = \int_{\mathscr{C}} \mathcal{G}(\lambda_1,\dots,\lambda_m)\,\mathrm{d}^m\lambda$$

- Small m separation of integrals ρ (**'03 Boos, Korepin, Smirnov**; **'06 Sato, Shiroishi, Takahashi**)

$$\langle\sigma_1^z\sigma_3^z\rangle = \frac{1}{3} - \frac{16}{3}\ln 2 + 3\zeta(3)$$

- Free fermionic structure & algebraic separation of integrals at generic m (**'04-'08 Boos, Jimbo, Miwa, Smirnov, Takeyama**)

Going beyond the free-fermion point: The Bethe Ansatz approach

- Solution of the inverse problem (**'99 Kitanine, Maillet, Terras**)
- Numerics: dynamical structure factors (XXZ, NLSM) $S(q,\omega) = \mathcal{F}\left[\langle j(x,t)\, j(0,0)\rangle_T\right](\omega, q)$
  (**'05 Caux, Hagemans, Maillet '06 Caux, Calabrese, Slavnov**)
- Series of mult. int. for 2 pt. functions (**'00-'05 Kitanine, Maillet, Slavnov, Terras**)

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = \sum_{n=1}^{+\infty} \int_{-q}^{q} \mathcal{F}_m^{(n)}(\mu_1, \dots, \mu_n)\, \mathrm{d}^n \mu$$

- Long-distance asymptotics $\Delta \neq 0$ from first principles (**'08 KKMST**)
- Long-distance & large-time asymptotics (**'11 K., Terras**) $\left\langle j(x,t) j(0,0) \right\rangle = \frac{1}{2} \partial_\beta^2 \partial_x^2 Q^{(\beta)}(x,t)$

$$Q^{(\beta)}(x,t) = Q^{(\beta)}_{\text{asym}}(x,t) + \underbrace{\sum_{n\geq 1} \sum_{\{\epsilon_t\}} \int_{\mathscr{C}_{\{\epsilon_t\}}} H_{n;\epsilon_t}(x,t;\{z_t\})\, \mathrm{d}^n z_t}_{\text{structure asymptotic series}}$$

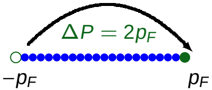
Long-distance asymptotics of densities at $T = 0K$

'11 Kitanine, K., Maillet, Slavnov, Terras

density-density correlation function of the NLS model at $T = 0K$:

$$\frac{\langle G.S. | j(0,0) j(x,0) | G.S. \rangle}{\langle G.S. | G.S. \rangle} = \langle j(0,0) \rangle^2 - \frac{\mathcal{Z}^2}{2\pi^2 x^2}\left(1 + \mathrm{o}(1)\right) + \sum_{\ell=1}^{+\infty} \frac{2\cos\left(2x\ell p_F\right)}{x^{2\ell^2 \mathcal{Z}^2}} \cdot |\mathcal{F}_\ell|^2 \left(1 + \mathrm{o}(1)\right)$$

$$|\mathcal{F}_\ell|^2 = \lim_{L \to +\infty} \left(\frac{L}{2\pi}\right)^{2\ell^2 \mathcal{Z}^2} \frac{\left| \langle G.S. | j(0,0) | \mathrm{umkp} \rangle \right|^2}{\| G.S. \|^2 \cdot \| \mathrm{umkp} \|^2}$$



- ★ ground state in positive chemical potential
- ★ one Umklapp excitation $\Delta E = 0$ $\Delta P = 2p_F$.

✔ Confirms C.F.T./Luttinger liquid-based predictions.

✔ Agrees with RHP approach (**'08** KKMST).

✔ Similar results for XXZ.

T=0K leading harmonics in long-time & distance asymptotics

to appear KKMST

Currents: $j(x,t) \equiv e^{iHt}\Psi^{\dagger}(x)\Psi(x)e^{-iHt}$ asymptotic regime $x \to +\infty$ and x/t fixed.

Overall structure of the asymptotic series (space-like regime):

$$\langle j(x,t)\, j(0,0)\rangle = \left(\frac{p_F}{\pi}\right)^2 - \frac{\mathcal{Z}^2}{2\pi^2}\frac{x^2+t^2v_F^2}{\left(x^2-t^2v_F^2\right)^2}\left(1+\mathrm{o}(1)\right)$$

$$+ \sum_{\substack{\ell_+;\ell_-\in\mathbb{Z} \\ \ell_++\ell_-\leq 0}}^{*} \frac{e^{ix\ell_+p_F}}{[-i(x-v_Ft)]^{\Delta^{(R)}_{\ell_+;\ell_-}}} \frac{e^{-ix\ell_-p_F}}{[i(x+v_Ft)]^{\Delta^{(L)}_{\ell_+;\ell_-}}}$$

$$\times\, e^{-i(\ell_++\ell_-)[xp(\lambda_0)-t\varepsilon(\lambda_0)]}\left(\frac{[p'(\lambda_0)]^2}{-i[xp''(\lambda_0)-t\varepsilon''(\lambda_0)]}\right)^{\frac{|\ell_++\ell_-|^2}{2}} \cdot \frac{(2\pi)^{\frac{|\ell_++\ell_-|}{2}}\left|\mathcal{F}^{(j)}_{\ell_+,\ell_-}\right|^2}{G\left(1+|\ell_++\ell_-|\right)}\left(1+\mathrm{o}(1)\right) .$$

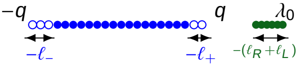
★ λ_0 **Saddle-point** of the oscillating phase: $p'(\lambda_0) - t\varepsilon'(\lambda_0)/x = 0$.

⇝ p dressed momentum & ε dressed energy.

Form factor interpretation of the amplitudes

$$\left|\mathcal{F}^{(j)}_{\ell_+,\ell_-}\right|^2 = \lim_{L\to+\infty}\left\{\left(\frac{L}{2\pi}\right)^{|\ell_+ + \ell_-|^2+\Delta^{(R)}_{\ell_+;\ell_-}+\Delta^{(L)}_{\ell_+;\ell_-}} \cdot \frac{\left|\left\langle G.S.\middle| j(0) \middle| \mathrm{Ex}(\ell_+;\ell_-)\right\rangle\right|^2}{\left\|G.S.\right\|^2\cdot\left\|\mathrm{Ex}(\ell_+;\ell_-)\right\|^2}\right\}$$

★ ℓ_+: # additional particles at q ℓ_-: # additional particles at $-q$ $|\ell_+ + \ell_-|$: # particles at λ_0



★ ground state in positive chemical potential

★ excitation $\begin{cases} \Delta E = |\ell_+ + \ell_-|\varepsilon(\lambda_0) \\ \Delta P = |\ell_+ + \ell_-|p(\lambda_0) - |\ell_+|p_F - |\ell_-|(-p_F) \end{cases}$

- Critical exponents $\Delta^{(R/L)}_{\ell_+;\ell_-}$ originate from excitations on Fermi boundaries.

$$\Delta^{(R)}_{\ell_+;\ell_-} = (\ell_+ + \ell_-)\phi(q,\lambda_0) - \ell_-\phi(q,-q) - \ell_+\phi(q,q) \qquad \left(I - \frac{K}{2\pi}\right)\cdot\phi(\lambda,\mu) = \theta(\lambda-\mu)$$

- Critical exponent $\dfrac{|\ell_+ + \ell_-|^2}{2}$ originates from gaussian saddle-point.

✔ Agrees with the first terms obtained through Natte series ('**11** K., Terras).

The power-law behavior of Fourier transforms (NLSM)

to appear KKMST (*to appear*)

(k,ω) configuartion close to the hole excitation line

$$\left(p_F - p(\lambda_0)\,,-\varepsilon(\lambda_0)\right) \qquad \text{with} \qquad \lambda_0 \in \,]-q\,;q[\ .$$

★ The *hole* treshold

$$S\Big(p_F - p(\lambda_0), -\varepsilon(\lambda_0) + \delta\omega\Big) \simeq \frac{\Xi\,(\delta\omega)\,[\delta\omega]^{\Delta_{1;0}^{(R)} + \Delta_{1;0}^{(L)} - 1}}{[v+v_F]^{\Delta_{1;0}^{(R)}}\,[v_F - v]^{\Delta_{1;0}^{(L)}}} \cdot \frac{(2\pi)^2\,\big|\mathcal{F}_{1,0}^{(j)}\big|^2}{\Gamma(\Delta_{1;0}^{(R)} + \Delta_{1;0}^{(L)})}\ .$$

★ v: velocity of the hole at λ_0 $\qquad$ v_F: velocity excitations on Fremi boundary.

$$\big|\mathcal{F}_{1,0}^{(j)}\big|^2 = \lim_{L\to+\infty}\left\{\left(\frac{L}{2\pi}\right)^{1+\Delta_{1;0}^{(R)} + \Delta_{1;0}^{(L)}} \frac{\Big|\big\langle G.S.\big|\,j(0)\,\big|\mathrm{Ex}\big\rangle\Big|^2}{\big\|G.S.\big\|^2\cdot\big\|\mathrm{Ex}\big\|^2}\right\}$$

★ ground state

★ excitation $\begin{cases} \Delta E &= -\varepsilon(\lambda_0) \\ \Delta P &= p_F - p(\lambda_0) \end{cases}$

(k,ω) configuartion close to the particle excitation line

$$\big(p(\lambda_0)-p_F,\varepsilon(\lambda_0)\big) \qquad \text{with} \qquad \lambda_0 \in \,]\,q\,;+\infty\,[\;.$$

★ The *particle* treshold

$$S\Big(p(\lambda_0)-p_F,\varepsilon(\lambda_0)+\delta\omega\Big) \simeq \frac{[\delta\omega]^{\Delta^{(R)}_{-1;0}+\Delta^{(L)}_{-1;0}-1}}{[v+v_F]^{\Delta^{(R)}_{-1;0}}[v_F-v]^{\Delta^{(L)}_{-1;0}}}\cdot\frac{(2\pi)^2\left|\mathcal{F}^{(j)}_{-1,0}\right|^2}{\Gamma\big(\Delta^{(R)}_{1;0}+\Delta^{(L)}_{1;0}\big)} \times \frac{\Xi(\delta\omega)\sin\big[\pi\Delta^{(L)}_{-1;0}\big]+\Xi(-\delta\omega)\sin\big[\pi\Delta^{(R)}_{-1;0}\big]}{\sin\pi\big[\Delta^{(R)}_{-1;0}+\Delta^{(L)}_{-1;0}\big]}$$

✔ Microscopic model approach ⇝ the non-linear Luttinger-based predictions.

The form factor approach

Form factor expansion for finite L of $\mathcal{O}(x,t) \equiv \mathrm{e}^{iHt}\mathcal{O}(x)\,\mathrm{e}^{-iHt}$

$$\begin{aligned}
\langle\, G.S.\,|\mathcal{O}(x,t)\,\mathcal{O}^{\dagger}(0,0)\,|\,G.S.\,\rangle &= \sum_{\{\mu\}_{\mathrm{ex}}} \langle\, G.S.\,|\mathrm{e}^{-ixP+itH}\mathcal{O}(0,0)\,\mathrm{e}^{ixP-itH}|\,\{\mu\}_{\mathrm{ex}}\,\rangle\langle\,\{\mu\}_{\mathrm{ex}}\,|\mathcal{O}^{\dagger}(0,0)\,|\,G.S.\,\rangle \\
&= \sum_{\{\mu\}_{\mathrm{ex}}} \mathrm{e}^{ix(P_{\mathrm{ex}}P_{\mathrm{G.S.}})-it(\mathcal{E}_{\mathrm{ex}}-\mathcal{E}_{G.S.})}\left|\langle\, G.S.\,|\mathcal{O}(0,0)\,|\,\{\mu\}_{\mathrm{ex}}\,\rangle\right|^{2}
\end{aligned}$$

presumed steps of the computation

- Characterize the excitations above the ground state;
- Asymptotic in *size* L formula for $\langle\, G.S.\,|\mathcal{O}(0,0)|\,\{\mu\}_{\mathrm{ex}}\,\rangle$;
- Localize sums at stationary-points: saddle-point, ends of Fermi zone ;
- Sum-up in the asymptotic regime.

Conslusion and perspectives

Review of the results

- ✔ Leading asymptotics of **any** harmonic in long-distance ;
- ✔ **All** harmonics in long-distance and large-time for pure particle-hole spectrum ;
- ✔ Reproduction of edge exponents with amplitudes from ABA ;

Next possible extensions

- ⊛ Include the effects of bound states (time dependent case) .