Symmetries of the ideal and the unitary Fermi gases

Elisa Meunier
based on joint work with X. Bekaert and S. Moroz
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Laboratoire de Mathématiques et Physique Théorique
Université François Rabelais de Tours

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Plan

Motivations

Unitary Fermi gas : between BCS (Bardeen Cooper Schrieffer) and BEC (Bose-Einstein condensate) regimes

In the large-N limit (N flavors of atoms) :

$$\text{unitary Fermi gas (interactions)} \underset{\text{Legendre}}{\longleftrightarrow} \text{ideal Fermi gas (free)}$$

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Symmetries

Galilei group (1)

The Galilei group acts on the d-dimensionnal spatial coordinates $\mathbf{x}$ and time t as

$$(t,\mathbf{x}) \to g(t,\mathbf{x}) = (t+\beta, \mathscr{R}\mathbf{x} + \mathbf{v}t + \mathbf{a}),$$

where

- $\beta \in \mathbb{R}$ and g_β is a time translation and its generator $\hat{P}_t$
- $\mathscr{R}$ is a rotation matrix and $\frac{d(d-1)}{2}$ spatial rotations generators $\hat{M}_{ij}$
- $\mathbf{v} \in \mathbb{R}^d$ and g_v are d Galilean boost and their generators $\hat{K}_i$
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Galilei group (2)

The Galilei group acts only on the coordinates.
The transformations are “geometrical“.
$\varphi(t, \mathbf{x}) \rightarrow \varphi'(t, \mathbf{x}) = \varphi(t', \mathbf{x}') = \varphi\left(g(t, \mathbf{x})\right)$.

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Symmetry : definitions

- symmetry of Schrödinger equation (linear) : $\widehat{S}\psi = 0$ with $\widehat{S} := \hat{P}_t - \hat{H}$:

$$\psi \to \psi' = \hat{A}\psi$$

- linear equation and if $\hat{A}_1$ and $\hat{A}_2$ are symmetries $\Rightarrow \hat{A}_1\hat{A}_2$ is symmetry also

- Relation of equivalence :

$$\hat{A}_1 \approx \hat{A}_2 \iff \hat{A}_1 = \hat{A}_2 + \hat{O}\widehat{S}\,.$$

with $\approx$ stands for equal on the mass-shell or proportionnal to the equations of motion
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The Schrödinger group of kinematical symmetries

The free Schrödinger equation (and chemical potential $\mu = 0$) is :

$$(2im\,\partial_t + \Delta)\psi(t, \mathbf{x}) = 0.$$

It is invariant under Galilei transformations if we allowed to modify wave function $\psi(t, \mathbf{x})$ (phase factor proportionnal to the mass).
These are kinematical symmetries :

$$\psi(t, \mathbf{x}) \to \gamma(t, \mathbf{x})\,\psi'(t, \mathbf{x}) = \gamma\big(g^{-1}(t', \mathbf{x}')\big)\,\psi\big(g(t, \mathbf{x})\big)$$

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Bargmann group

Projective representation (a phase) of Galilei group
= “genuine“ representation of Bargmann group

By enlarging the Galilei group through a central extension, known as the mass operator $\hat{M}$ (or the particle number operator) = Bargmann group

Generators of Bargmann group : $\hat{P}_t, \hat{M}_{ij}, \hat{K}_i, \hat{P}_i, \hat{M}$

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Bargmann group(2)

The generators of spatial translations and Galilean boosts don't commute : $[\hat{P}_i, \hat{K}_j] = -i\,\delta_{ij} m$.

These are the canonical commutation relations of the Heisenberg algebra $\mathfrak{h}_d$
where $\hat{K}_i$ play the role of the position operators $\hat{X}_i$
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Schrödinger group

The Schrödinger group :

- Bargmann group
- Scale transformations (their generator $\hat{D}$) :

$$(t, \mathbf{x}) \to q(t, \mathbf{x}) = \left(\frac{t}{\alpha^2}, \frac{\mathbf{x}}{\alpha}\right), \qquad \alpha \in \mathbb{R}.$$

- Expansion (non-relativistic analogue of the special conformal transformations) (its generator $\hat{C}$) : inversion $(t, \mathbf{x}) \to \Sigma(t, \mathbf{x}) = (-\frac{1}{t}, \frac{\mathbf{x}}{t})$ combines with time translation g_β

$$(t, \mathbf{x}) \to (\Sigma^{-1} g_\beta \Sigma)(t, \mathbf{x}) = \left(\frac{t}{1+\beta t}, \frac{\mathbf{x}}{1+\beta t}\right)$$

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Representations and Mathematical structure

Schrödinger algebra : $\mathfrak{sch}(d) = \mathfrak{h}_d \niplus \left(\mathfrak{o}(d) \oplus \mathfrak{sl}(2,\mathbb{R})\right)$

- $\mathfrak{h}_d$:

$$\hat{P}_i = -i\partial_i, \qquad \hat{K}_i = mx_i + it\partial_i, \qquad \hat{M} = m,$$

- $\mathfrak{o}(d)$:

$$\hat{M}_{ij} = -i(x_i\partial_j - x_j\partial_i),$$

- $\mathfrak{sl}(2,\mathbb{R})$:

$$\hat{P}_t = i\partial_t,$$

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Commutation relations

$\mathfrak{sch}(d) = \mathfrak{h}_d \ni \left(\mathfrak{o}(d) \oplus \mathfrak{sl}(2,\mathbb{R})\right)$
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$\mathfrak{sl}(2,\mathbb{R})$:

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Niederer Theorem

The maximal group of kinematical symmetries of free Schrödinger equation is the Schrödinger group.

geometrical (linear in the derivatives)
$\subset$ kinematical (one order differential operators : linear and constant in the derivatives)
$\subset$ higher (higher order differential operators)

Galilei $\subset$ Bargmann $\subset$ Schrödinger $\subset$ Weyl

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Symmetries : maximal algebra

Definition : The **maximal** symmetry algebra of the free Schrödinger equation is the algebra of **all** the inequivalent (not trivial) symmetries of the free Schrödinger equation.

Theorem : The maximal Lie algebra of symmetries for **the free Schrödinger equation** is generated algebraically by **the space translations and the Galilean boosts.**

$\Rightarrow$ Weyl algebra = envelopping Heisenberg algebra
$\mathfrak{A}(d) = \mathcal{U}(\mathfrak{h}_d) = Pol(\hat{K}, \hat{P})$

Theorem : (Eastwood, 2002) The maximal Lie algebra of symmetries for **d'Alembert equation** is generated algebraically by the **conformal Killing vectors (= algebra of Vasiliev higher-spin gravity).**

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Generators of degree two in $\hat{P}$ and $\hat{K}$

$\hat{\mathbf{X}}(t) \to \hat{\mathbf{K}}/m$ and $\hat{\mathbf{P}}(t) \to \hat{\mathbf{P}}$ (M. Valenzuela, 2009)

$$\hat{P}_t \approx \frac{\hat{P}^2}{2m},$$
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\end{aligned}$$

Bargmann framework

Symmetry algebra : from conformal to Schrödinger

The non-trivial commutation relations of the conformal algebra $\mathfrak{o}(d+2,2)$:

$$
\begin{aligned}
[\tilde{M}^{\mu\nu}, \tilde{M}^{\alpha\beta}] &= i(\eta^{\mu\alpha}\tilde{M}^{\nu\beta} + \eta^{\nu\beta}\tilde{M}^{\mu\alpha} - \eta^{\mu\beta}\tilde{M}^{\nu\alpha} - \eta^{\nu\alpha}\tilde{M}^{\mu\beta}),\\
[\tilde{M}^{\mu\nu}, \tilde{P}^{\alpha}] &= i(\eta^{\mu\alpha}\tilde{P}^{\nu} - \eta^{\nu\alpha}\tilde{P}^{\mu}),\\
[\tilde{D}, \tilde{P}^{\mu}] &= -i\tilde{P}^{\mu}, \quad [\tilde{D}, \tilde{K}^{\mu}] = i\tilde{K}^{\mu},\\
[\tilde{P}^{\mu}, \tilde{K}^{\nu}] &= -2i(\eta^{\mu\nu}\tilde{D} + \tilde{M}^{\mu\nu}),
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where Greek indices run from 0 to $d+1$.

The tilde signs are for relativistic generators
and hatted symbols for the non-relativistic operators.

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Symmetry algebra : from conformal to Schrödinger

The representations are :

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$$\tilde{K}_\mu = i\left(2x_\mu\left(x^\nu\partial_\nu + \frac{d}{2}\right) - x^2\partial_\mu\right), \quad \tilde{D} = i\left(x^\mu\partial_\mu + \frac{d}{2}\right)$$

The light-cone momentum $\tilde{P}^+ = (\tilde{P}^0 + \tilde{P}^{d+1})/\sqrt{2}$
$\longleftrightarrow$ the mass operator $\hat{M}$ in the non-relativistic theory.

All operators in the conformal algebra $(\mu = (-,+,i))$ that commute with $\tilde{P}^+$, form a subalgebra = Schrödinger algebra $\mathfrak{sch}(d)$:

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Symmetry algebra : from conformal to Schrödinger

The representations are :

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Embedding diagram

	Kinematical		*Higher*
Relativistic	$\mathfrak{o}(d+2,2)$	$\underset{\text{Eastwood}}{\subset}$	Vasiliev algebra (d+2,2)
	$\cup$		$\cup$
Non − relativistic	$\mathfrak{sch}(d)$	$\subset$	Weyl algebra (d)

Equations of motion : from K-G to Schrödinger

Massless Klein-Gordon (d'Alembert) equation in $d+2$-dimensional Minkowski spacetime

$$\Box\Psi(x) \equiv -\partial_0^2\Psi(x) + \sum_{i=1}^{d+1}\partial_i^2\Psi(x) = 0$$

- the light-cone coordinates : $x^{\pm} = \frac{x^0 \pm x^{d+1}}{\sqrt{2}}$
- the dimensionnal reduction along a light-like (or null) direction x^- (and time is $x^+ = t$) :

$$\Psi(x) = e^{-imx^-}\psi(x^+, \mathbf{x})$$

$\Rightarrow$ Free Schrödinger equation

$$(2im\,\partial_t + \Delta)\psi(t, \mathbf{x}) = 0$$

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Conclusion

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  - Symmetries : Weyl algebra is a maximal algebra of symmetries of free Schrödinger equation
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Thank you

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