Tasep, Asep et grandes matrices aléatoires.

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Plan

- I. Tasep: a review of definitions and some known results.
- II. Connection with queues, last passage percolation, and random matrices.
- III. Some results.
- IV. Extensions.

Tasep

The Totally Asymmetric Simple Exclusion Process (TASEP) is a non-reversible interacting particle system: configuration of particles $\eta_t \in \{0,1\}^{\mathbb{Z}}$, $t \geq 0$

$$\eta_t(i) = 1 : \text{there is a particle at site } i \text{ at time } t;$$

There is at most one particle at each site. Given η_o, the dynamics is defined as follows: Particles can jump to the neighboring right site only (Simple and Asymmetric) provided that the site is empty (Exclusion).

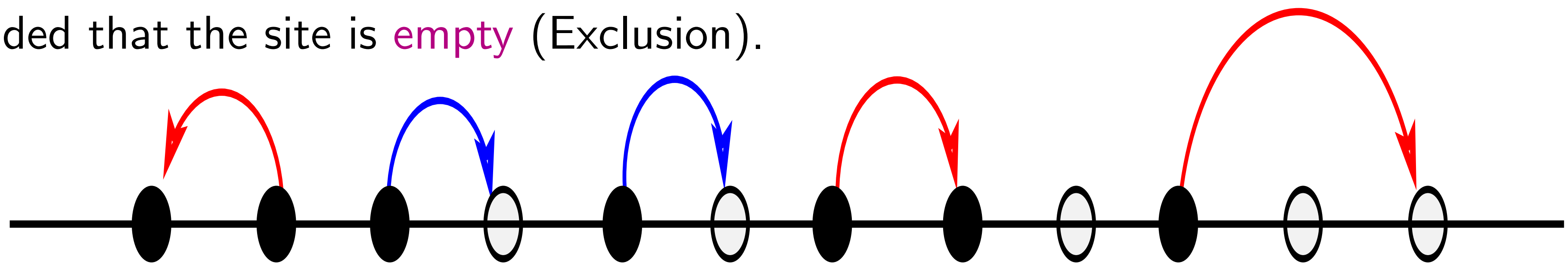


Figure 1: Allowed jumps

Jumps are independant and take place after an exponential waiting time with mean 1, which is counted from the time instant when the right neighbor site is empty.

Standard initial conditions

-step initial condition : $\eta_o(i) = 0$ if $i > 0$ and $\eta_o(i) = 1$ if $i \le 0$;
-flat initial condition : $\eta_o(i) = 0$ if i is odd and $\eta_o(i) = 1$ if i is even.
-Invariant measures:

- $\eta_o(i)$, $i \in \mathbb{Z}$ i.i.d. Bernoulli with a given density $\rho \in [0, 1]$ (translation invariant) known as equilibrium Tasep

- blocking measure (all sites occupied to the right of some site i)

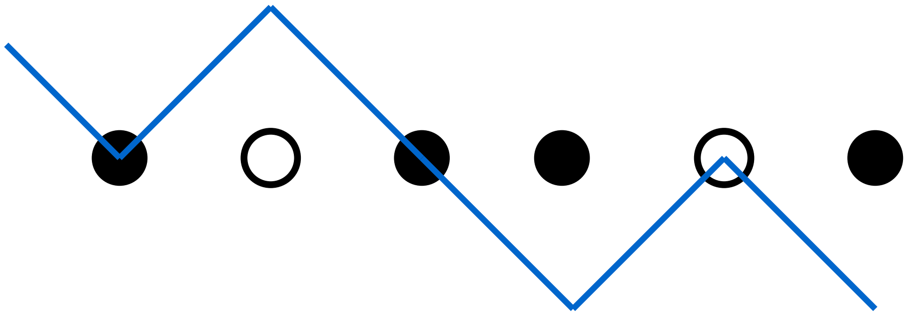
-two sided initial condition: Bernoulli independent random variables with density ρ_- (resp. ρ_+) on $\mathbb{Z}_-$ (resp. $\mathbb{Z}_+$).

What is the large time behavior ?

A lot of results using hydrodynamic approach (Ferrari-Fontes (94) e.g.)

Some quantities of interest I

The height function



$$h_t(j) = \begin{cases} 2N_t + \sum_{i=1}^{j}(1-2\eta_i(t)), & \text{for } j \geq 1, \\ 2N_t, & \text{for } j = 0, \\ 2N_t - \sum_{i=j+1}^{0}(1-2\eta_i(t)), & \text{for } j \leq -1, \end{cases} .$$

where N_t is the number of particles which jumped from site 0 to site 1 during the time-span $[0, t]$.

Assign label 0 to the particle sitting at the smallest positive integer site initially. Then use the ordering $\cdots < \mathbf{x}_2(0) < \mathbf{x}_1(0) < 0 \leq \mathbf{x}_0(0) < \mathbf{x}_{-1}(0) < \cdots$. Then $\mathbf{x}_k(t) > \mathbf{x}_{k+1}(t)$ for all $t \geq 0$.

$$\mathbb{P}(\cap_{k=1}^{m}\{h_{t_k}(x_k - y_k) \geq x_k + y_k\}) = \mathbb{P}(\cap_{k=1}^{m}\{\mathbf{x}_{y_k}(t_k) \geq x_k - y_k\}).$$

Random matrix limiting distribution

One fundamental result: for step initial condition.

Limiting shape (Rost (81)): $\bar{h}(v) := \lim_{t\to\infty} \frac{h_t(vt)}{t} = \begin{cases} \frac{1}{2}(v^2+1), & \text{if } |v| < 1, \\ |v|, & \text{if } |v| \geq 1. \end{cases}$

Theorem Johansson ('98) Let $v \in [0,1)$,

$$\lim_{t\to\infty} \mathbb{P}\left(h_t(vt) \geq \frac{1+v^2}{2}t - s\frac{(1-v^2)^{2/3}}{2^{1/3}}t^{1/3}\right) = F_{GUE}(s_k),$$

where $F_{GUE}(x)$ is the GUE Tracy-Widom distribution.

Reminder: let $H = H^*$ be a complex $N \times N$ Hermitian random matrix with i.i.d. $\mathcal{N}(0,1)$ entries above the diagonal. The suitably rescaled largest eigenvalue of $\frac{H}{\sqrt{N}}$ has Tracy-Widom F_{GUE} fluctuations as $N \to \infty$

Connections with queues, LPP and random matrices

Queues

Suppose that there are infinitely many servers with FIFO policy:

- the service time of customers at each server i.i.d. $\mathrm{Exp}(1)$.

- once a customer is served at the server i, she joins at the $(i+1)$th queue.

Consider a fixed time $t = 0$ (if not given) and (arbitrary) select one customer labeled 0: the queue where she is the 0th queue. We assign labels to the other customers so that the labels decrease for the customers ahead in the queues.
Let $Q_j(t)$ denote the label of the queue in which the jth customer is in at time t.

Tasep and queues

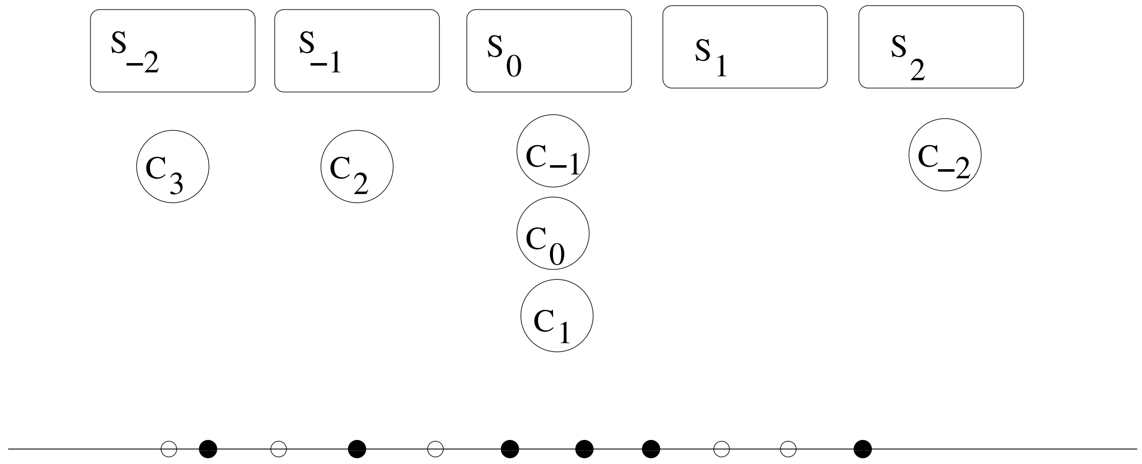


Figure 2: Black dots = customers; at every white dots one changes to the next counter.

Basic relationship between Tasep and Queues: step Tasep, stationnary queue=stationnary Tasep.

$$\mathbf{x}_j(t) = Q_j(t) - j.$$

Call also $E_j(i)$ be the time the jth customer exits the queue i, then we find that

$$\mathbb{P}(\cap_{k=1}^m \{E_{y_k}(x_k - 1) \leq t_k\}) = \mathbb{P}(\cap_{k=1}^m \{Q_{y_k}(t_k) \geq x_k\}) = \mathbb{P}(\cap_{k=1}^m \{\mathbf{x}_{y_k}(t_k) \geq x_k - y_k\}).$$

Last passage percolation

At each site $(i,j) \in \mathbb{N}^2$, a random variable w_{ij} is attached. The w_{ij}'s are independent (waiting time) not necessarily identically distributed.

An up-right path π from $(0,0)$ to $(x,y) \in \mathbb{N}^2$ is a sequence of points $(\pi_k \in \mathbb{Z}^2, k = 0, \ldots, x+y)$, with $\pi_0 = (0,0)$ and $\pi_{x+y} = (x,y)$, and satisfying $\pi_{k+1} - \pi_k \in \{(1,0),(0,1)\}$.

Set $L(\pi) = \sum_{(i,j)\in\pi} w_{i,j}$. Then, the last passage time is defined by

$$G(x,y) = \max_{\pi:(0,0)\to(x,y)} L(\pi).$$

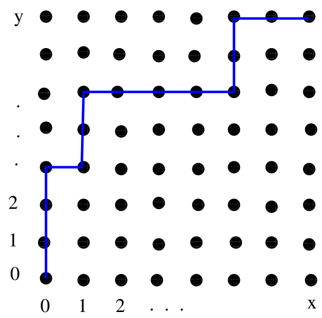
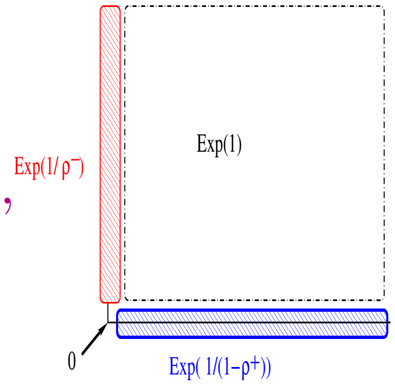


Figure 3: An upright path.

Tasep and LPP

Let $w_{i,j}$, $i,j \geq 0$, $i,j \in \mathbb{Z}$, be independent random variables with

$$
\begin{aligned}
&w_{0,0} = 0,\\
&w_{0,j} \sim \text{ exponential with mean } 1/\rho^-,\ j \geq 1,\\
&w_{i,0} \sim \text{ exponential with mean } (1-\rho^+)^{-1},\ i \geq 1,\\
&w_{i,j} \sim \text{ exponential with mean } 1, \quad i,j \geq 1.
\end{aligned}
$$



Associated last passage time : $G(x,y) = \max_{\pi:(0,0)\to(x,y)} L(\pi)$.

Two sided Tasep: initial configuration η_o is the Bernoulli $\rho^{\pm}$ product measure.

If $\underline{x_k, y_k \to \infty}$,

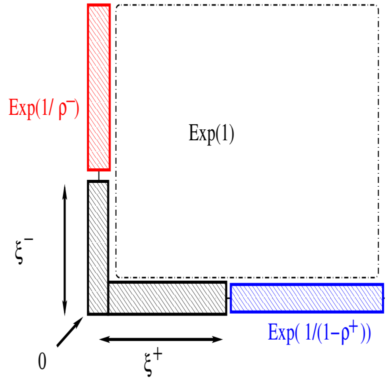
$$\lim \mathbb{P}(\cap_{k=1}^m \{\mathbf{x}_{y_k}(t_k) \geq x_k - y_k\}) = \lim \mathbb{P}(\cap_{k=1}^m \{G(x_k,y_k) \leq t_k\}).$$

Explanation: two sided boundary condition

Assume that η_o is the product measure of Bernoulli with parameter $\rho^{\pm}$ on $\mathbb{Z}^{\pm}$.

Theorem Praehofer-Spohn (2001)

Let ζ^+ (resp. ζ^-) be ind. geometric random variables with parameter $1-\rho^+$ (resp. ρ_-). The $\{w(i,j),(i,j)\in\mathbb{N}^2\}$ are independent as follows



Define $\widehat{G}(x,y)$ to be the last passage time to (x,y) in this LPP model. Then,

$$\mathbb{P}(\cap_{k=1}^m\{\mathbf{x}_{y_k}(t_k)\geq x_k-y_k\})=\mathbb{P}(\cap_{k=1}^m\{\widehat{G}(x_k,y_k)\leq t_k\}).$$

Think of $w(i,j)$ as the time needed for particle at $i-j-1$ to jump to $i-j$ (counted from the moment of time where $i-j$ is free)

LPP and random matrices

Let $(w_{i,j})$, $i, j \in \mathbb{N}$ independent exponentials r.v with parameter π_i, $i \in \mathbb{N}$.
Rate of exponentials depends on the row index only (or column only).

Let $X = (X_{ij})$ be a $(N+1) \times (p+1)$ matrix

$$X_{ij} \text{ i.i.d } \mathcal{N}(0,1) \text{ complex}$$

and

$$\Sigma = \text{diag}(\pi_0^{-1}, \pi_1^{-1}, \dots, \pi_N^{-1}).$$

Theorem: Johansson (2000) $G(N,p)$ and $\lambda_{max}(X\Sigma X^*)$ have the same distribution.

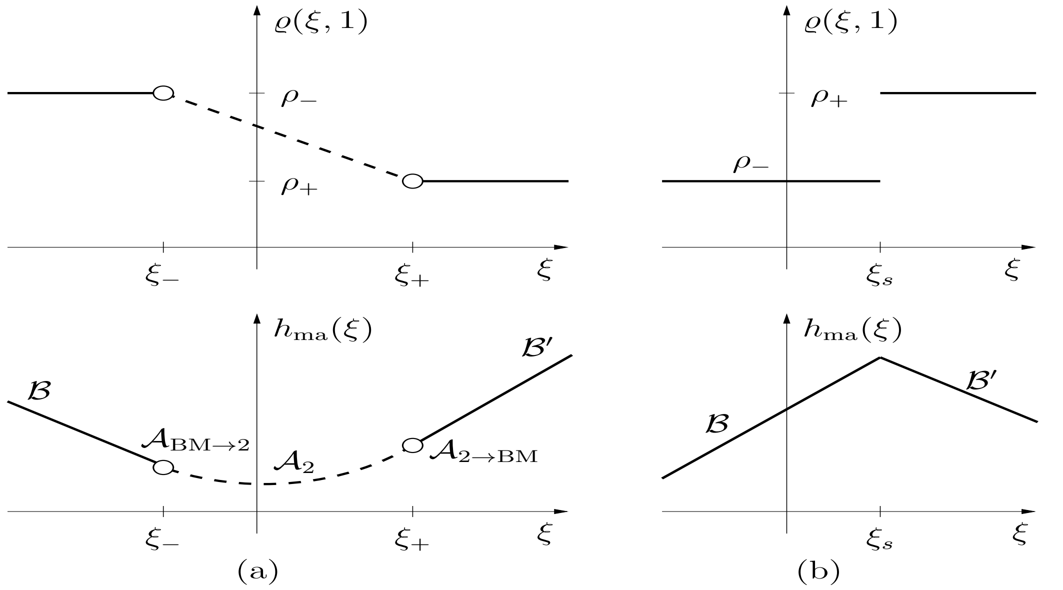
$X\Sigma X^*$ is a complex Wishart matrix whose joint eigenvalue density is well known.

Proof: explicit computation of the two distributions (matrix integrals Harisch-Chandra-Itzykson-Zuber, RSK correspondance for the LPP with geometric instead of exponentials).

If $\Sigma = Id$ then λ_{max} has Tracy-Widom fluctuations: step Tasep.

Some results

Summarizing picture for two-sided TASEP



The asymptotic density ϱ and the limit shape in the cases (a) $\rho_- > \rho_+$ and (b) $\rho_- < \rho_+$. Transitions happen at $\xi_\pm = 1 - 2\rho_\pm$ and shockwave at $\xi_s = 1 - (\rho_- + \rho_+)$.

Explanations: the two sided Tasep

"Competition" of two one source models.

Define $Q(a,b) = G((a,b),(x,y))$ to be the passage time from (a,b) to (x,y). Then

$$G(x,y) = \max\{Q(0,1), Q(1,0)\}.$$

$Q(0,1)$ cannot "see" the first line thus one source model: LPP with exponential r.v. with mean 1 except on the first column only.
$Q(0,1)$ has the same distribution as the largest eigenvalue of a well-chosen Wishart random matrix $X\Sigma X^*$.
A crucial role for $Q(0,1)$ and $Q(1,0)$ is played by the critical directions

$$\frac{y}{x} = \gamma_c(\rho) = \frac{\rho^2}{(1-\rho)^2}, \text{ with } \rho = \rho^{\pm}.$$

Gaussian/Tracy-Widom fluctuations

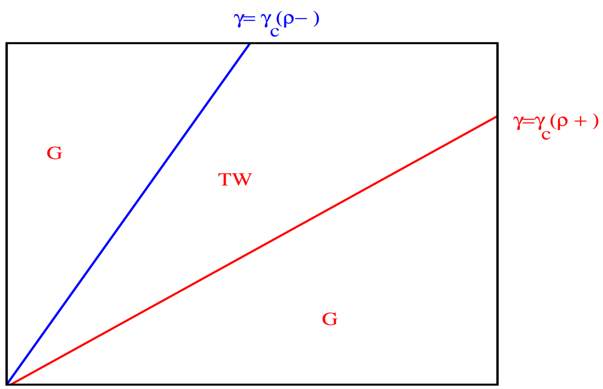
Set $x = \frac{N}{1+\gamma}$, $\quad y = \frac{N}{1+\gamma}\gamma$, $c_1 = \frac{\gamma}{1+\gamma}\left(\frac{1}{\rho_-} + \frac{1}{\gamma(1-\rho_-)}\right)$, $\quad c_1' = \frac{\gamma}{1+\gamma}(1+\frac{1}{\sqrt{\gamma}})^2$.

Theorem Baik-GBA-Peche (2005) There exist constants c_2, c_2' such that

$$\lim_{N\to\infty} \mathbb{P}(Q(1,0) \le c_1 N + c_2 s N^{1/2}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} dx\, e^{-x^2/2} \equiv \Phi(s), \text{ if } \gamma > \gamma_c(\rho^-),$$
$$\lim_{N\to\infty} \mathbb{P}(Q(1,0) \le c_1' N + c_2' s N^{2/3}) = F_{\mathrm{GUE}}(s), \text{ Tracy-Widom, if } \gamma < \gamma_c(\rho^-).$$

As $c_1' < c_1$ we get that if $\gamma > \gamma_c(\rho^-)$,

$$\lim_{N\to\infty} \mathbb{P}(G(x,y) \le c_1 N + c_2 s N^{1/2}) = \Phi(s).$$



Two sided Tasep II

Two sided TASEP with $0 < \rho_+ < \rho_- < 1$, the asymptotic macroscopic density is

$$\rho = \begin{cases} \rho_- & \text{if } \xi \leq 1 - 2\rho_-, \\ (1-\xi)/2 & \text{if } \xi \in [1-2\rho_-, 1-2\rho_+], \\ \rho_+ & \text{if } \xi \geq 1 - 2\rho_+. \end{cases}$$

Let $\xi \in [1-2\rho_-, 1-2\rho_+]$, then $h_{ma}(\xi) = (1+\xi^2)/2$. Set

$$X(\tau) = \lfloor \xi T + \tau (2(1-\xi^2))^{1/3} T^{2/3} \rfloor,$$
$$H(\tau, s) = \frac{1+\xi^2}{2} T + \xi \tau (2(1-\xi^2))^{1/3} T^{2/3} + (\tau^2 - s) \frac{(1-\xi^2)^{2/3}}{2^{1/3}} T^{1/3}.$$

Two sided Tasep III

Theorem Corwin-Ben Arous (2009) Corwin-Ferrari-Peche (2010)

Fix $m \in \mathbb{N}$, and any $\tau_1 < \tau_2 < \cdots < \tau_m$ and $s_1, \ldots, s_m$, we have:

(a1) If $\xi \in (1 - 2\rho_-, 1 - 2\rho_+)$, then

$$\lim_{T\to\infty} \mathbb{P}\left(\bigcap_{k=1}^{m} \{h_T(X(\tau_k)) \geq H(\tau_k, s_k)\}\right) = \mathbb{P}\left(\bigcap_{k=1}^{m} \{\mathcal{A}_2(\tau_k) \leq s_k\}\right).$$

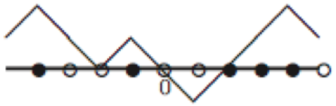
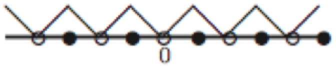
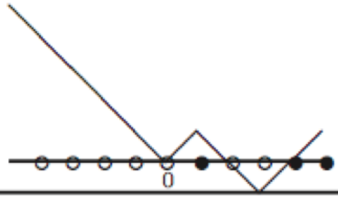
(a2) If $\xi = 1 - 2\rho_-$, then

$$\lim_{T\to\infty} \mathbb{P}\left(\bigcap_{k=1}^{m} \{h_T(X(\tau_k)) \geq H(\tau_k, s_k)\}\right) = \mathbb{P}\left(\bigcap_{k=1}^{m} \{\mathcal{A}_{\mathrm{BM}\to 2}(\tau_k) \leq s_k\}\right).$$

(a3) If $\xi < 1 - 2\rho_-$, then $\lim_{T\to\infty} \mathbb{P}\left(\bigcap_{k=1}^{m} \{h_{\theta_k T}(\xi_k \theta_k T) \geq h_{ma}(\xi_k)\theta_k T - 2 s_k T^{1/2}\}\right)$

$$= \mathbb{P}\left(\bigcap_{k=1}^{m} \{B(\theta_k(1 - 2\rho_- - \xi_k)(\rho_-(1 - \rho_-))) \leq s_k\}\right).$$

Other initial conditions

• Brownian 	$\bar{h}(T, TX) = T/2$	• One pt: F_0 [11, 67] • Multi pt: $\mathrm{Airy}_{\mathrm{stat}}$ [10]
• Flat 	$\bar{h}(T, TX) = T/2$	• One pt: F_{GOE} [12, 13, 68, 143] • Multi pt: Airy_1 [29, 30]
• Wedge→Brownian 	$\bar{h}(T, TX) = \begin{cases} -X & X < -1 \\ T\frac{1+X^2}{2} & X \in [-1, 0] \\ T/2 & X > 0 \end{cases}$	• One pt: $(F_{\mathrm{GOE}})^2$ [11, 8, 134, 17] • Multi pt: $\mathrm{Airy}_{2\to\mathrm{BM}}$ [88, 44]

Extensions

Slow decorrelation

To compute joint distributions (determinants formulae), needs the points $(x(\tau), y(\tau))$ to be on a line $y = \text{Cte}T$.

Nevertheless: multipoint fluctuation theorem for TASEP unchanged with

$$X(\tau,\theta) = \lfloor \xi(T+\theta T^{\nu}) + \tau(2(1-\xi^2))^{1/3}T^{2/3} \rfloor,$$

$$H(\tau,\theta,s) = \frac{1+\xi^2}{2}(T+\theta T^{\nu}) + \xi\tau(2(1-\xi^2))^{1/3}T^{2/3} + (\tau^2 - s)\frac{(1-\xi^2)^{2/3}}{2^{1/3}}T^{1/3}.$$

for any $\nu \in [0,1)$ and any real θ.

Meaning: Fluctuations then differ by a deterministic constant when $0 \longrightarrow \theta$.

The meaning of slow decorrelation

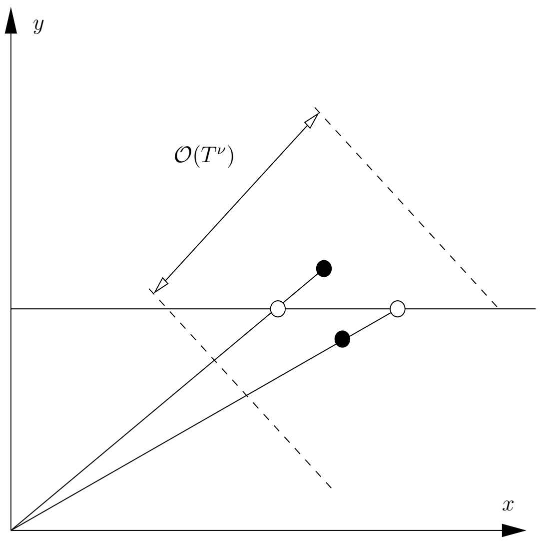
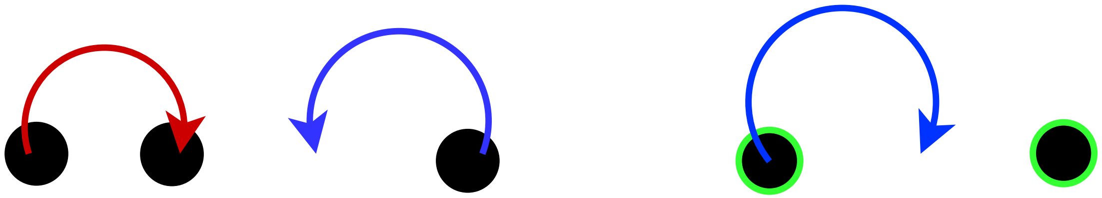


Figure 1: Assume that the black dots are $\mathcal{O}(T^\nu)$ for some $\nu < 1$ away from the line $y = \mathrm{Cte}T$. Then, the fluctuations of the passage time at the locations of the black dots are, on the $T^{1/3}$ scale, the same as those of their projection along the critical direction to the line $y = \mathrm{Cte}T$, the white dots.

Asep

Given η_o, the dynamics is defined as follows:
Particles can jump to the neighboring site only (Simple) provided that the site is empty (Exclusion).



Particles try to jump to the right with prob. p (resp. left with prob. q) and jump if allowed. Here $0 < p = 1 - q < 1$.

Jumps are governed by independent Poisson processes with rate 1: each particle has its own clock ringing after an exponential waiting time with mean one and reseting.

Asep cannot be mapped to a LPP model (surface not "growing" only).
Neither to queuing model (customers go back and forth).
No known connection to RMT...

Asep and the formulas of Tracy and Widom

T-W (2008) compute transition probability $P_Y(X,t)$ for N-Asep, i.e. $\mathbb{P}$ that N particles started at $Y = (Y_1 < Y_2 < \cdots < Y_N)$ are at $X = (X_1 < X_2 < \cdots < X_N)$ at time t.

Theorem: Tracy-Widom (2008)

$$\mathbb{P}_Y(X,t) = \sum_{\sigma \in S_N} \oint \cdots \oint A_\sigma(\xi) \prod_{i=1}^{N} \xi_i^{x_i - y_{\sigma(i)} - 1} e^{t f(\xi_i)},$$

with $f(\xi_i) = p/\xi_i + q\xi_i - 1$, and $A_\sigma = \prod_{\substack{i<j \\ \sigma(i)>\sigma(j)}} \left(-\frac{p+q\xi_{\sigma(i)}\xi_{\sigma(j)} - \xi_{\sigma(i)}}{p+q\xi_{\sigma(i)}\xi_{\sigma(j)} - \xi_{\sigma(j)}} \right)$.

For step Asep: obtain a Fredholm determinant expression for the rightmost particle!

This formula is obtained through a few magic formulas (Cauchy determinants identities....)
Not so simple formulas for the mth particle from the left $m \geq 1$.

Magic formulas

Theorem Tracy-Widom (2009)
Let $m = [\sigma t]$, $\gamma = p - q > 0$ fixed, then

$$\lim_{t \to \infty} \mathbb{P}\left(x_m(t/\gamma) \geq c_1(\sigma)t - c_2(\sigma)st^{1/3}\right) = F_2(s),$$

uniformly for σ in compact subsets of $(0, 1)$ where $c_1(\sigma) = 1 - 2\sqrt{\sigma}, c_2(\sigma) = \sigma^{-1/6}(1 - \sqrt{\sigma})^{2/3}$.

Asymptotic shape : Liggett (2005)
If $q = 0$ then one recovers Johansson's result. Speeded up by γ the same fluctuations as step Tasep.

TW (2009): fluctuations for step Bernoulli Asep.

Major corollary: Amir-Corwin-Quastel (2010), Sasamoto-Spohn (2010) The KPZ equation is in the KPZ universality class.

Conclusion

A lot of open questions:

- general initial condition:
  -connection to RMT?
  -explicit formulae?
  -determinantal formulae?

- Asep: same questions essentially...