Aspects of integrability in AdS/CFT duality

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Based on

- D, Fioravanti, G. Infusino, M. Rossi, *On the high spin expansion in the sl(2) $\mathcal{N} = 4$ SYM theory*, Nuclear Physics B 22 (2009) 467 - 492
- D, Fioravanti, G. Infusino, M. Rossi, *Functional relations and wrapping in high spin sl(2) SYM_4*, preprint

Key points

AdS/CFT duality - Maldacena conjecture [1998 Maldacena; 1998 Gubser et al.; 1998 Witten]

A String theory, where strings propagate on a background containing an AdS_{d+1} as a factor, is dual to a Conformal Field Theory in a flat $d-$dimensional spacetime.

Integrable system

A system with n degrees of freedom is called integrable if it has a number n of quantities Q_j, $j = 1, \cdots, n$ (including the Hamiltonian), which commute among themselves:

$$[Q_j, Q_k] = 0 \quad j, k = 1, \cdots, n$$

Outline

1. AdS/CFT duality
2. Integrability in AdS/CFT
   - Planar limit
   - SU(2) sector
   - Heisenberg spin chain integrability: the Bethe Ansatz
   - SL(2) sector
3. Non-linear Integral equation
   - General method
   - Application to Heisenberg model
4. Results
   - SL(2) sector - high spin at fixed twist
   - High spin and high twist limit

Motivations

- Superstring Theory: a way to understand Quantum Gravity?
- Gauge/String duality
- Maldacena conjecture
- The boundary M_d of an AdS_{d+1} spacetime is a flat $d-$dimensional spacetime where CFT is formulated
- For every string observable at the boundary of AdS_{d+1} there is a corresponding observable in the CFT on M_d whose values are expected to match
- Understand Quantum Gravity studying a more conventional Quantum Field Theory
- Not yet a formal proof!

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Strong/Weak duality

Parameters of the theories: weak/strong duality

Gauge side	String side	AdS/CFT	Perturbative regimes
$\lambda = g_{YM}^2 N_c$	$T = R^2/2\pi\alpha$	$\lambda = 4\pi^2 T^2$	$\lambda = 0$
N_c	g_{str}	$\frac{1}{N_c} = \frac{g_{str}}{\lambda}$	$g_{str} = 0\ ,\ \lambda \to \infty$

- Perturbative regime on String side $\rightarrow$ strong coupling regime in CFT (maybe in QCD?) and vice-versa
- Perturbative regimes do not overlap $\rightarrow$ no tests of validity!

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Aspects of Integrability on gauge side

- $\mathcal{N} = 4$ Super Yang-Mills $\leftrightarrow$ type IIB Superstring on $AdS_5 \times S_5$
- Anomalous dimensions $\Longleftrightarrow$ String states energies
- Planar limit [1974 t'Hooft]: $N_c \to \infty$, $\lambda = g_{YM} N_c$ finite
- Dilatation operators $\Longleftrightarrow$ Hamiltonians (or *Bethe-like equations*) of some integrable spin chains
- Integrability of planar limit of $\mathcal{N} = 4$ SYM $\to$ exact spectrum of anomalous dimensions at arbitrary λ
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$\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ Super Yang-Mills in $d = 4$

$$S = \frac{1}{g_{YM}^2} \int d^4x \mathrm{Tr} \left\{ -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi^2 - \sum_{i<j} [\phi_i, \phi_j]^2 + \right.$$
$$\left. + i\bar{\psi} \Gamma^\mu D_\mu \psi - \bar{\psi} \Gamma^i [\phi_i, \psi] \right\}$$

- Gauge field A_μ, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$
- Six massless real scalars ϕ_i, $i = 1, \cdots, 6$
- Four Majorana spinors ψ_α^A $(\bar{\psi}_{\dot{\alpha}}^A)$, $A = 1, \cdots, 4$
- $\Gamma^A = (\Gamma^\mu, \Gamma^i)$ ten 16x16 Dirac matrices
- $D_\mu(*) = \partial_\mu(*) - i[A_\mu, (*)]$

SU(2) sector integrability [2002, Minahan, Zarembo]

- $Z = \phi_1 + i\phi_2 \quad W = \phi_3 + i\phi_4$
- $O = \mathrm{Tr}(Z^{L-M} W^M + \text{permutations})$
- $\langle O_A(x) O_B(y) \rangle \to$ diverges (UV)
- $O_A^{ren} = Z_A^C O_C$
- $\langle O_A^{ren}(x) O_B^{ren}(y) \rangle \propto \dfrac{\delta_{AB}}{|x-y|^{2\Delta(\lambda)}}$
- $\Delta(\lambda) = L + \gamma(\lambda)$, $\gamma(\lambda)$ anomalous dimension
- Dilatation operator $\Gamma O = \gamma O$, $\Gamma = Z^{-1} \dfrac{dZ}{d \ln \Lambda}$, Λ UV cut-off

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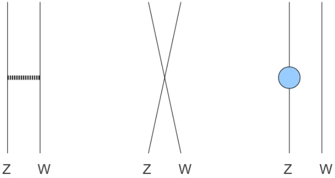
One loop contributions:the Heisenberg $XXX_{1/2}$ model



$$\Gamma = \frac{\lambda}{16\pi^2}\sum_{n=1}^{L}(1 - P_{n,n+1})$$

$$H = \frac{\lambda}{32\pi^2}\sum_{n=1}^{L}(1 - \vec{\sigma}_n \cdot \vec{\sigma}_{n+1})$$

$$O = \mathrm{Tr}(ZZZWWWZWZZ\cdots) \leftrightarrow |\psi\rangle = |\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\uparrow\downarrow\uparrow\uparrow\cdots\rangle$$

$$\Gamma O = \gamma O \leftrightarrow H|\psi\rangle = E|\psi\rangle$$

Integrability of the Heisenberg chain

A short overview [1994 Faddeev]

- $\mathcal{H} \otimes \mathcal{V}$, $\mathcal{H} = h^{\otimes L}$, $h = \mathbb{C}^2$, $\mathcal{V} = \mathbb{C}^2$
- Lax operator

$$L_{n,a}(u) = u(I_n \otimes I_a) + i(\vec{\sigma}_n \otimes \vec{\sigma}_a)$$

- $T_a(u)$ Monodromy matrix

$$T_a(u) = L_{L,a}(u) L_{L-1,a}(u) \cdots L_{1,a}(u)$$

- $F(u) = \mathrm{Tr}_a(T_a(u))$, polynomial in u at order L :

$$\text{(non-trivial)} \quad F(u) = 2u^L + \sum_{l=0}^{L-2} Q_l u^l \,, \; [F(u), F(v)] = 0$$

- $L-1$ commuting charges
- $H = \frac{i}{2}\frac{d}{du} \ln F(u)|_{u=i/2} - \frac{L}{2}$
- Adding a spin component, say S^3, one obtains L commuting charges

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The Bethe Ansatz [1931 Bethe]

Diagonalization of $F(u)$ → diagonalization of all charges

Bethe equations

$$\left(\frac{u_j - \frac{i}{2}}{u_j + \frac{i}{2}}\right)^L = \prod_{k=1,k\neq j}^{M} \frac{u_j - u_k - i}{u_j - u_k + i} \qquad j = 1, \cdots, M$$

- u_j Bethe roots
- $$E = \sum_{j=1}^{M} \frac{1}{u_j^2 + \frac{1}{4}} \qquad P = \sum_{j=1}^{M} \frac{i}{4} \ln\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)$$

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The SL(2) sector - Asymptotic Bethe Ansatz [2006 Beisert, Eden, Staudacher]

$O = \mathrm{Tr}(D^s Z^L + \text{permutations})$, long operators ($L \to \infty$)

Beisert - Staudacher equations for SL(2) sector

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L \left(\frac{1 + \frac{g^2}{2x^-(u_k)^2}}{1 + \frac{g^2}{2x^+(u_k)^2}}\right)^L = \prod_{\substack{j=1 \\ j \neq k}}^{s} \frac{u_k - u_j - i}{u_k - u_j + i} \left(\frac{1 - \frac{g^2}{2x^+(u_k)x^-(u_j)}}{1 - \frac{g^2}{2x^-(u_k)x^+(u_j)}}\right)^2 e^{2i\theta(u_k,u_j)}$$

$$x^{\pm}(u_k) = x(u_k \pm i/2)\,, \quad x(u) = \frac{u}{2}\left[1 + \sqrt{1 - \frac{2g^2}{u^2}}\right]\,, \quad \lambda = 8\pi^2 g^2$$

$\theta(u, v)$ dressing factor, function of g

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Wrapping corrections

- Self-interaction of the chains at higher loops
- Exact results only up to $L-1$ loops
- Short chains ($L = 2, 3$), up to six loops, wrapping affects $O\left(\frac{(\ln s)^2}{s^2}\right)$ terms [2008-2009 Bajnok, Janik, Lukowski, Rej, Velizhanin]
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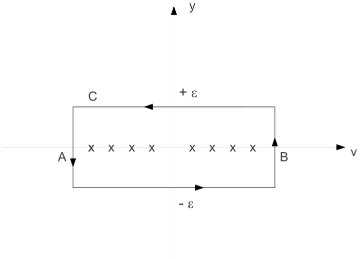
Non-linear Integral equation [1994 Destri, De Vega; 1997 Fioravanti et al.]

$$\sum_{k=1}^{M} O(u_k) \qquad u_k \text{ roots in } [A,B] \in \mathbb{R} \quad , \quad 2\pi i \sum_{k=1}^{M} O(u_k) = \oint_C dz O(z) f(z)$$



$f(z)$ poles in u_k, Res$[f(z)]$=1, $O(z)$ no poles in u_k

Definition: counting function $Z(z)$

$$f(z) = \frac{i\delta Z'(z)\exp[iZ(z)]}{1+\delta\exp[iZ(z)]} \quad \delta = \pm 1 \quad , \quad \exp[iZ(u_k)] = -\delta \text{ (Bethe equations)}$$

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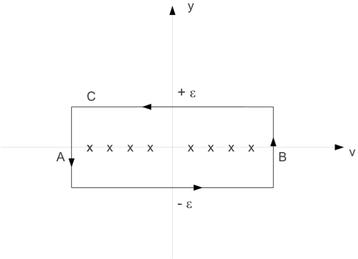

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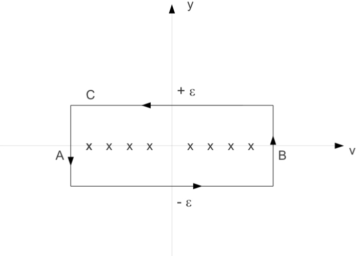

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Counting function for the Heisenberg model

$$\left(\frac{u_j - \frac{i}{2}}{u_j + \frac{i}{2}}\right)^L = \prod_{k=1, k\neq j}^{M} \frac{u_j - u_k - i}{u_j - u_k + i} \qquad j = 1, \cdots, M$$

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$$E=\sum_{k=1}^{L}\frac{1}{\frac{1}{4}+u_k^2}=2L\ln 2-\int_{-\infty}^{+\infty}\frac{dv}{\cosh \pi v}\frac{d}{dv}\mathrm{Im}\ln[1+(-1)^{\Delta}e^{iZ(v+i0^+)}]$$

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$$Z(u)=2L\arctan e^{\pi u}-\frac{\pi L}{2}+\int_{-\infty}^{+\infty}dvG(u-v)\mathrm{Im}\ln[1+(-1)^{\Delta}e^{iZ(v+i0^+)}]$$

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Evaluation of Non-linear term of the energy

$$\mathrm{NL}=-\sum_{k=0}^{\infty}\frac{(2\pi)^{2k+1}}{(2k+2)!}B_{2k+2}\left(\frac{1}{2}\right)\left[\left(\frac{1}{Z'(v)}\frac{d}{dv}\right)^{2k+1}\frac{1}{\cosh \pi v}\right]_{v=-\infty}^{v=+\infty}=\frac{\pi^2}{6L}+\cdots$$

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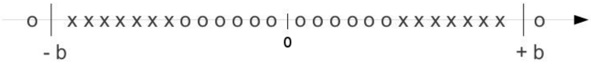
Main features of the counting function in SL(2) sector: ground state



- $\exp[iZ(u_k)] = (-1)^{L+1}$, s Bethe roots, L holes
- $L-2$ internal holes near the origin

$$Z(x_h) = \pi(2h+1-L)\,,\; h = 1, \cdots, L-2$$

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- Two external holes

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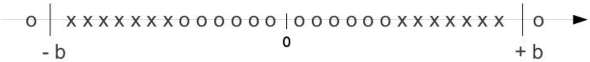

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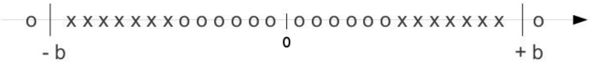

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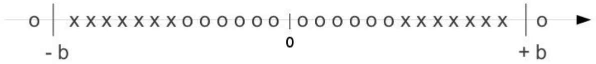

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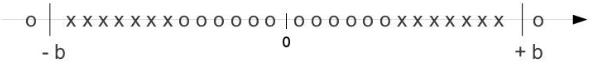

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SL(2) sector NLIE for $Z(u)$

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$$L(u) = \mathrm{Im}\ln[1 + (-1)^L\, e^{iZ(u-i0^+)}]$$

$$\Phi(u) = -2L\arctan 2u - iL\ln\left(\frac{1+\frac{g^2}{2x^-(u)^2}}{1+\frac{g^2}{2x^+(u)^2}}\right)$$

$$\phi(u,v) = 2\arctan(u-v) - 2i\left[\ln\left(\frac{1-\frac{g^2}{2x^+(u)x^-(v)}}{1-\frac{g^2}{2x^-(u)x^+(v)}}\right) + i\theta(u,v)\right]$$

Systematic study of anomalous dimension $\gamma(g, L, s)$

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- Subleading terms of high spin anomalous dimension

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Relations among $\gamma^{(n)}(g,L)$ and $\check{\gamma}^{(n)}(g,L)$

$$\check{\gamma}^{(-1)}(g,L) = \frac{1}{2}[f(g)]^2$$
$$\check{\gamma}^{(0)}(g,L) = \frac{1}{2}f(g)[L-1+f_{sl}(g,L)]$$
$$\check{\gamma}^{(1)}(g,L) = 0$$
$$\check{\gamma}^{(2)}(g,L) = -\frac{f(g)}{2}\gamma^{(2)}(g,L)$$
$$\check{\gamma}^{(3)}(g,L) = -f(g)\gamma^{(3)}(g,L) - (f_{sl}(g,L)+L-1)\gamma^{(2)}(g,L)$$
$$\check{\gamma}^{(4)}(g,L) = -\frac{3}{2}f(g)\gamma^{(4)}(g,L) - \frac{3}{2}(f_{sl}(g,L)+L-1)\gamma^{(3)}(g,L)$$
$$\check{\gamma}^{(5)}(g,L) = -2f(g)\gamma^{(5)}(g,L) - 2(f_{sl}(g,L)+L-1)\gamma^{(4)}(g,L) - \left(\gamma^{(2)}(g,L)\right)^2$$

Reciprocity and self-tuning [2007 Basso Korchemsky; 2008 Beccaria, Forini]

Self-tuning property

$$\gamma(g, L, s) = P\left(s + \frac{1}{2}\gamma(g, L, s)\right)$$

$$P(s) = \sum_{n=0}^{\infty} \frac{a_n(\ln C(s))}{C(s)^{2n}} \; , \; s \to \infty$$

Reciprocity

$$C(s)^2 = \left(s + \frac{L}{2} - 1\right)\left(s + \frac{L}{2}\right)$$

Reciprocity and self-tuning (2)

$$P(s) = f(g)\ln C(s) + f_{sl}(g,L) + \sum_{n=1}^{\infty} \frac{\gamma^{(n)}(g,L)}{(\ln C(s))^n} + O(1/C^2)$$

$$s \to \infty \Rightarrow C(s) = s + \frac{L-1}{2} + O(1/s)$$

Anomalous dimension - self-tuning prediction

$$\gamma(g,L,s) = f(g)\ln s + f_{sl}(g,L) + \sum_{n=1}^{\infty} \frac{\gamma^{(n)}(g,L)}{(\ln s)^n}$$

$$+ \frac{\ln s}{2s}[f(g)]^2 + \frac{1}{2s} f(g)(L-1+f_{sl}(g,L)) + \frac{f(g)}{2s}\sum_{n=1}^{\infty}\frac{\gamma^{(n)}(g,L)}{(\ln s)^n} -$$

$$-\sum_{n=1}^{\infty} n\frac{\gamma^{(n)}(g,L)}{2s(\ln s)^{n+1}}\left[f(g)\ln s + f_{sl}(g,L) + L - 1 + \sum_{m=1}^{\infty}\frac{\gamma^{(m)}(g,L)}{(\ln s)^m}\right] + O\left(s^{-1}(\ln s)^{-\infty}\right)$$

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High spin and high twist limit [2006 Belitsky, Gorsky, Korchemsky; 2007 Alday, Maldacena; Freyhult, Rej, Staudacher]

- $L \to \infty$, $s \to \infty$, $j \equiv \dfrac{L-2}{\ln s}$ fixed
- No Wrapping effect!
- Double expansion:

$$\gamma(g,j) = f(g,j)\ln s + f_{SL}(g,j) + \sum_{k=1}^{\infty} \gamma^{(k)}(g,j)(\ln s)^{-k} + O((\ln s)^{-\infty})$$

- Non-linear terms

$$\text{holes} \quad NL \propto (Z'(c))^{-1} \sim O\left(\frac{1}{(\ln s)}\right)$$

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Systematic study of $f(g,j)$ and $f_{SL}(g,j)$

j-expansion of $f(g,j)$ and $f_{SL}(g,j)$

$$f(g,j) = \sum_{n=0}^{\infty} f_n(g) j^n \quad f_{SL}(g,j) = \sum_{n=0}^{\infty} f_{SL,n}(g) j^n$$

Example of coefficients

$$f_{0,g^2} = 4\,,\ f_{1,g^2} = -4\ln 2\,,\ f_{3,g^2} = \frac{7\zeta(3)\pi^2}{24}$$

$$f_{SL,0,g^2} = 4\gamma_E\,,\ f_{SL,3,g^2} = -(2\ln 2 + \gamma_E)\frac{7\pi^2\zeta(3)}{12}$$

$$f_{0,g^4} = -\frac{2\pi^2}{3}\,,\ f_{1,g^4} = \frac{2\pi^2}{3}\ln 2 + 4\zeta 3\,,\ f_{3,g^4} = \frac{75\pi^4\zeta(3)}{144} - \frac{31\pi^2}{8}\zeta(5)$$

$$f_{SL,0,g^4} = -\frac{2\pi^2}{3}\gamma_E - 6\zeta(3)\,,\ f_{SL,3,g^4} = \frac{49\pi^2}{12}\zeta^2(3) - \frac{7\pi^4}{72}\zeta(3)(16\ln 2 + 5\gamma_E)$$