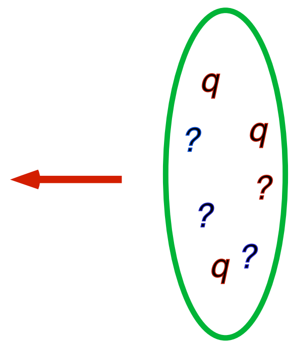
Factorization of hard processes, Part II

Orsay, June 4-8, 2012

II. Quantum field theory: finding where perturbation theory works

- A. Color and QCD
- B. Field Theory Essentials
- C. Infrared Safety
- D. Pinch surfaces, power counting and Ward identities

IIA. From Color to QCD



- **Enter the Gluon**
- **If $\phi_{q/H}(x)$ = probability to find q with momentum xp,**
- **then,**

$$M_q = \sum_q \int_0^1 dx\ x\ \phi_{q/H}(x) \quad = \quad \text{total fraction of momentum carried by quarks.}$$

- **Experiment gave**

$$M_q \sim 1/2$$

- **What else? Quanta of force field that holds H together?**
- **'Gluons' – but what are they?**

- **Where color comes from.**

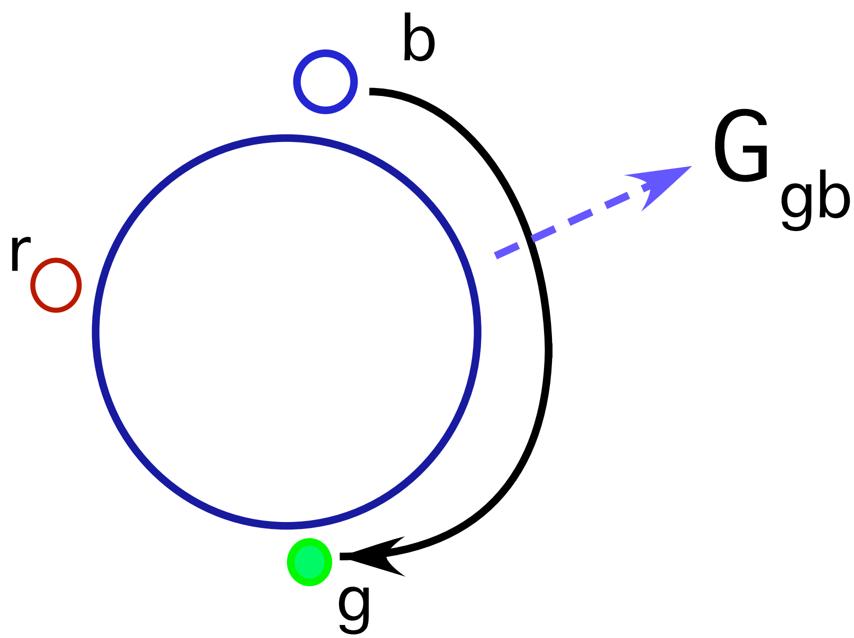
- **Quark model problem:**
  - $s_q = 1/2 \Rightarrow$ **fermion** $\Rightarrow$ **antisymmetric wave function, but**
  - (uud) **state symmetric in spin/isospin combination for nucleons and**
  - **Expect the lowest-lying** $\psi(\vec{x}_m, \vec{x}'_u, \vec{x}_d)$ **to be symmetric**
  - **So where is the antisymmetry?**

- **Solution: Han Nambu, Greenberg, 1968: Color**

- b, g, r, **a new quantum number.**

- **Here's the antisymmetry:** $\epsilon_{ijk}\psi(\vec{x}_u, \vec{x}'_u, \vec{x}_d)$, **(i,j,k)= (b,g,r)**

- **Quantum Chromodynamics: Dynamics of Color**
- **A globe with no north pole**



- **Position on 'hyperglobe' $\leftrightarrow$ phase of wave function**
  (Yang & Mills, 1954)
- **We can change the globe's axes at different points in space-time, and 'local rotation' $\leftrightarrow$ emission of a gluon.**
- **QCD: gluons coupled to the color of quarks**
  (Gross & Wilczek; Weinberg; Fritzsch, Gell-Mann, Leutwyler, 1973)

IIB. Field Theory Essentials

- **Fields and Lagrange Density for QCD**

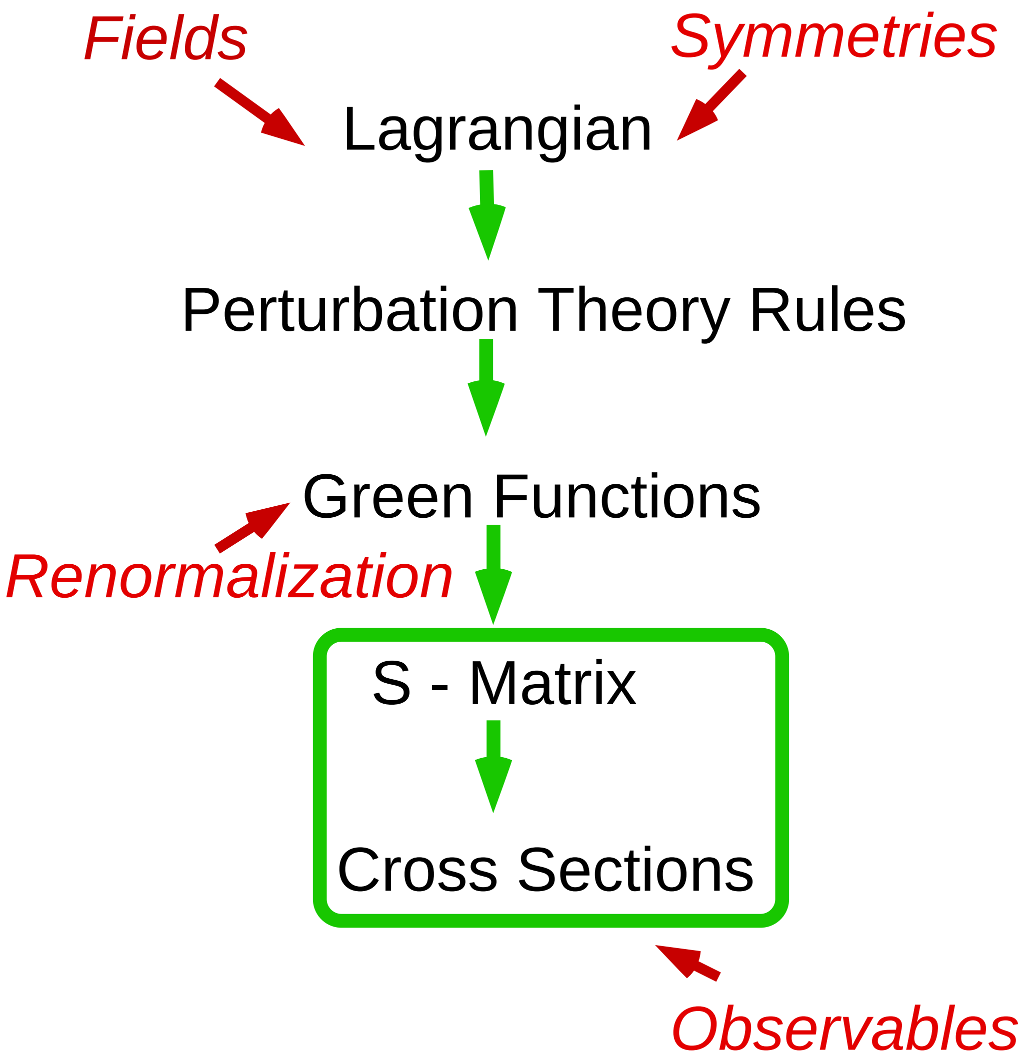
- $q_f(x)$, $f = u, d, c, s, t, b$: **Dirac fermions (like electron) but extra** $(i, j, k) = (b, g, r)$ **quantum number.**

- $A^\mu_a(x)$ **Vector field (like photon) but with extra** $a \sim (g\bar{b}\ldots)$ **quantum no. (octet).**

- $\mathcal{L}$ **specifies quark-gluon, gluon-gluon propagators and interactions.**

$$\mathcal{L} = \sum_f \bar{q}_f \left([i\partial_\mu - gA_{\mu a}T_a]\,\gamma^\mu - m_f \right) q_f - \frac{1}{4}\left(\partial_\mu A_{\nu a} - \partial_\nu A_{\mu a}\right)^2$$
$$-\frac{g}{2}\left(\partial_\mu A_{\nu a} - \partial_\nu A_{\mu a}\right) C_{abc} A^\mu_b A^\nu_c$$
$$-\frac{g^2}{4} C_{abc} A^\mu_b A^\nu_c C_{ade} A_{\mu d} A_{\nu e}$$

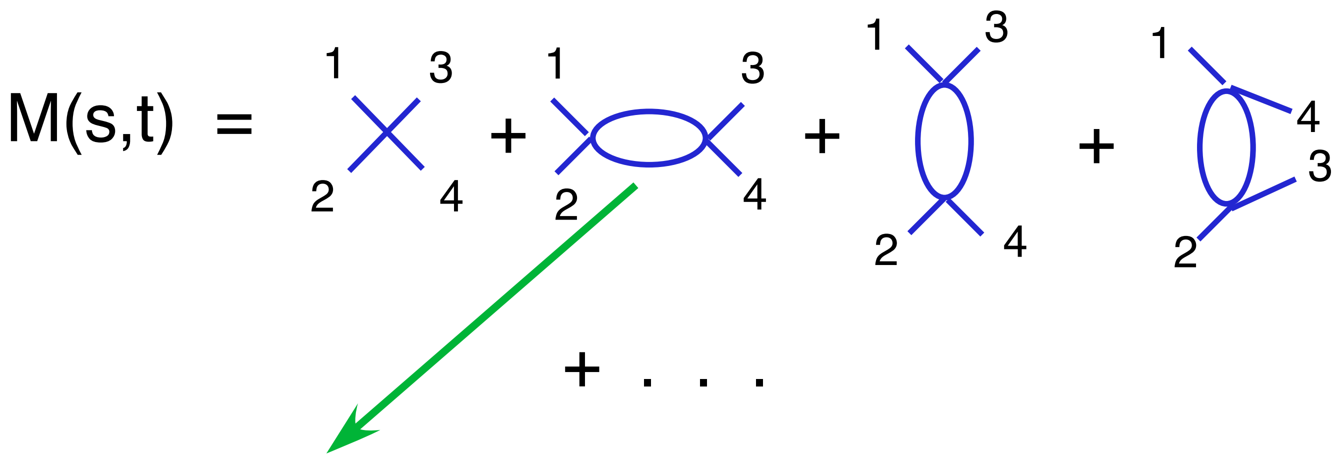
From a Lagrange density to observables, the pattern:



- **UV Divergences (toward renormalization & the renormalization group)**
- **Use as an example**

$$\mathcal{L}_{\phi^4} = \frac{1}{2}\left((\partial_\mu \phi)^2 - m^2\phi^2\right) - \frac{\lambda}{4!}\phi^4$$

- **The "four-point Green function":**

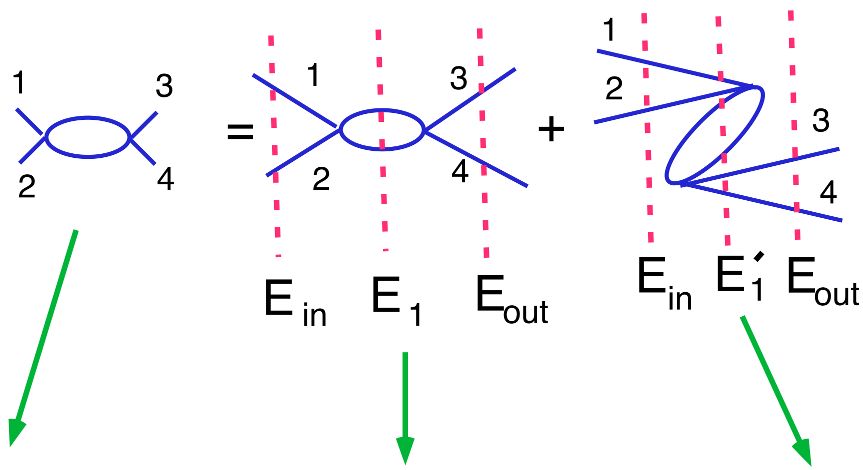


$$\int^\infty \frac{d^4k}{(k^2-m^2)((p_1+p_2-k)^2-m^2)} \sim \int^\infty \frac{d^4k}{(k^2)^2} \Rightarrow \infty$$

Interpretation: The UV divergence is due entirely states of high ‘energy deficit’,

$$E_{\rm in} - E_{\rm state\ S} = p_1^0 + p_2^0 - \sum_{i\ \in S} \sqrt{\vec{k}_i^2 + m^2}$$

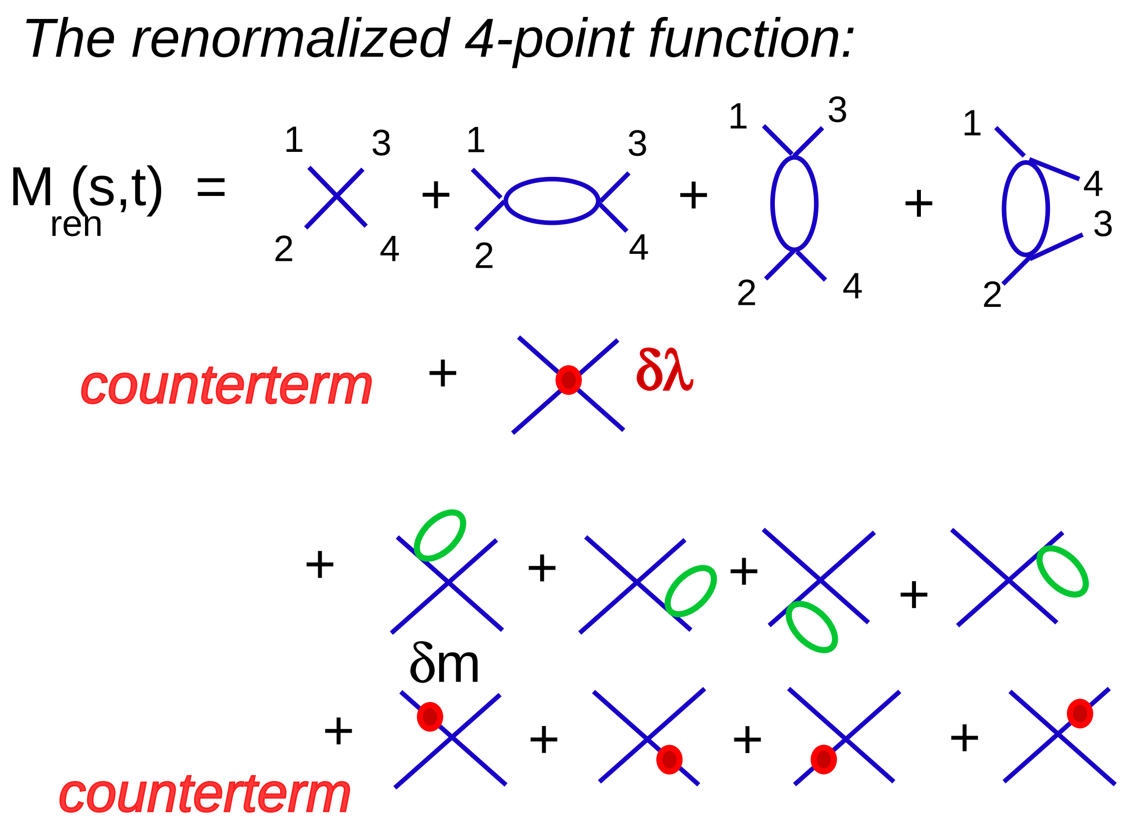
Made explicit in Time-ordered Perturbation Theory:



$$\int^{\infty} \frac{d^4k}{(k^2 - m^2)((p_1 + p_2 - k)^2 - m^2)} = \sum_{states} \left[\frac{1}{E_{\rm in} - E_1} + \frac{1}{E_{\rm in} - E_1'} \right]$$

Analogy to uncertainty principle $\Delta E \to \infty \Leftrightarrow \Delta t \to 0$.

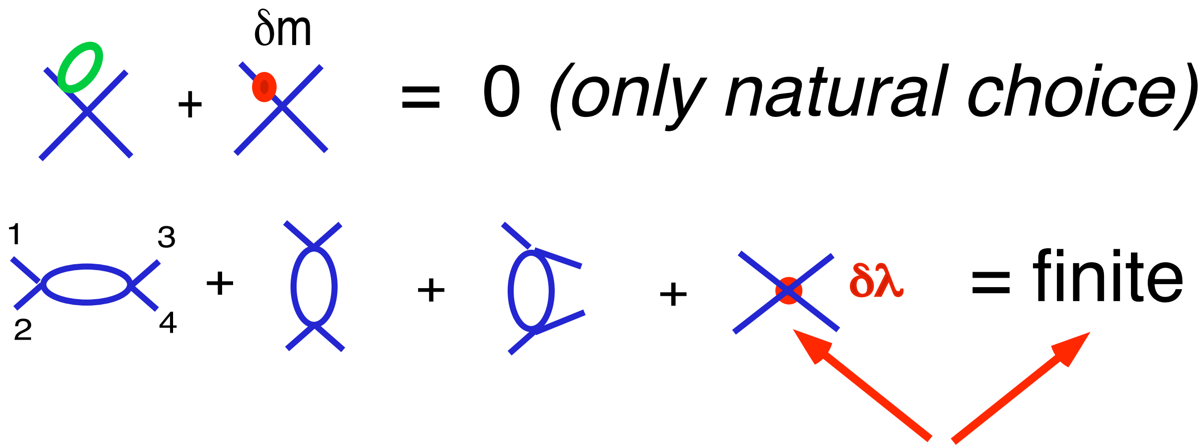
- **This suggests: UV divergences are ‘local’ and can be absorbed into the local Lagrange density. Renormalization.**
- **For our full 4-point Green function, two new “counterterms”:**



- **The combination is supposed to be finite.**

- **How to choose them? This is the renormalization "scheme"**

Renormalization:



But what should we choose for these?

A $\quad$ B $\quad$ C $\quad$ D

- **For example: define A+B+C by cutting off $\int d^4k$ at $k^2 = \Lambda^2$ (regularization). Then**

$$A + B + C = a \ln \frac{\Lambda^2}{s} + b(s, t, u, m^2)$$

- **Now choose:**

$$D = -\ a \ln \frac{\Lambda^2}{\mu^2}$$

so that

$$A + B + C + D = a \ln \frac{\mu^2}{s} + b(s, t, u, m^2)$$

independent of Λ.

- Criterion for choosing μ is a "renormalization scheme":
  MOM scheme: $\mu = s_0$, some point in momentum space.
  MS scheme: same μ for all diagrams, momenta

- But the value of μ is still arbitrary. $\mu =$ renormalization scale.

- Modern view (Wilson) We hide our ignorance of the true high-E behavior.

- All current theories are "effective" theories with the same low-energy behavior as the true theory, whatever it may be.

The Renormalization Group

- μ-dependence is the price we pay for working with an effective theory:

- As μ changes, mass m and coupling g have to change:
$m = m(\mu)$ $g = g(\mu)$ "*renormalized*" but ...

- Physical quantities can't depend on μ:

$$\mu \frac{d}{d\mu} \sigma \left(\frac{s_{ij}}{\mu^2}, \frac{m^2}{\mu^2}, g(\mu), \mu \right) = 0$$

- The 'group' is just the set of all changes in μ.

- 'RG' equation (Mass dimension $[\sigma] = d_\sigma$):

$$\left(\mu \frac{\partial}{\partial \mu} + \mu \frac{\partial g}{\partial \mu} \frac{\partial}{\partial g} + \mu \frac{\partial m}{\partial \mu} \frac{\partial}{\partial m} + d_\sigma \right) \sigma \left(\frac{s_{ij}}{\mu^2}, \frac{m^2}{\mu^2}, g(\mu), \mu \right) = 0$$

$$\text{The beta function :} \quad \beta(g) \equiv \mu \frac{\partial g(\mu)}{\partial \mu}$$

- **The Running coupling**
- **Consider <u>any</u> σ ($m = 0,\ d_\sigma = 0$) with kinematic invariants $s_{ij} = (p_i + p_j)^2$:**

$$\mu \frac{d\sigma}{d\mu} = 0 \quad \to \quad \mu \frac{\partial \sigma}{\partial \mu} = -\beta(g) \frac{\partial \sigma}{\partial g} \tag{1}$$

- **in PT:**

$$\sigma = g^2(\mu)\sigma^{(1)} + g^4(\mu)\left[\sigma^{(2)}\left(\frac{s_{ij}}{s_{kl}}\right) + \tau^{(2)} \ln \frac{s_{12}}{\mu^2}\right] + \ldots \tag{2}$$

- **(2) in (1) $\to$**

$$\begin{aligned} g^4\tau^{(2)} &= 2g\sigma^{(1)}\beta(g) + \ldots \\ \beta(g) &= \frac{g^3}{2}\frac{\tau^{(2)}}{\sigma^{(1)}} + \mathcal{O}(g^5) \equiv -\frac{g^3}{16\pi^2}\beta_0 + \mathcal{O}(g^5) \end{aligned}$$

- **In QCD:**

$$\beta_0 = 11 - \frac{2n_f}{3}$$

- $-\beta_0 < 0 \to g$ **decreases as μ increases.**

- **Asymptotic Freedom**: Solution for the QCD coupling

$$\begin{aligned}
\mu\frac{\partial g}{\partial \mu} &= -g^3\frac{\beta_0}{16\pi^2} \\
\frac{dg}{g^3} &= -\frac{\beta_0}{16\pi^2}\frac{d\mu}{\mu} \\
\frac{1}{g^2(\mu_2)} - \frac{1}{g^2(\mu_1)} &= -\frac{\beta_0}{16\pi^2}\ln\frac{\mu_2}{\mu_1} \\
g^2(\mu_2) &= \frac{g^2(\mu_1)}{1+\frac{\beta_0}{16\pi^2}g^2(\mu_1)\ln\frac{\mu_2}{\mu_1}}
\end{aligned}$$

- Vanishes for $\mu_2 \to \infty$. Equivalently,

$$\alpha_s(\mu_2) \equiv \frac{g^2(\mu_2)}{4\pi} = \frac{\alpha_s(\mu_1)}{1+\frac{\beta_0}{2\pi}\alpha_s(\mu_1)\ln\frac{\mu_2}{\mu_1}}$$

- **Dimensional transmutation: $\Lambda_{\rm QCD}$**

  – **Two mass scales appear in**

$$\alpha_s(\mu_2) = \frac{\alpha_s(\mu_1)}{1 + \frac{\beta_0}{2\pi}\alpha_s(\mu_1)\ln\frac{\mu}{\mu_1}}$$

  but the value of $\alpha_s(\mu_2)$ can't depend on choice of μ_1.

  – **Reduce it to one by defining $\Lambda \equiv \mu_1\, e^{-\beta_0/\alpha_s(\mu_1)}$, independent of μ_1. Then**

$$\alpha_s(\mu_2^2) = \frac{2\pi}{\beta_0 \ln\frac{\mu_2}{\Lambda}}$$

- **Asymptotic freedom strongly suggests a relationship to the parton model, in which partons act as if free at short distances. But how to quantify this observation?**

IIC. Infrared Safety

- **To use perturbation theory in QCD, would like to choose μ 'as large as possible' to make $\alpha_s(\mu)$ as small as possible.**

- **But how small is possible?**

- **A "typical" (dimensionless) cross section, define $Q^2 = s_{12}$ and $x_{ij} = s_{ij}/Q^2$,**

$$\sigma\left(\frac{Q^2}{\mu^2}, x_{ij}, \frac{m_i^2}{\mu^2}, \alpha_s(\mu)\right) = \sum_{n=1}^{\infty} a_n \left(\frac{Q^2}{\mu^2}, x_{ij}, \frac{m_i^2}{\mu^2}\right) \alpha_s^n(\mu)$$

with m_i^2 all fixed masses – external, quark, gluon (=0!)

- **Generically, the a_n depend logarithmically on their arguments, so a choice of large μ results in large logs of m_i^2/μ^2.**

- But if we could find quantities that depend on $m_i's$ only through powers, $(m_i/\mu)^p, p > 0$, the large-μ limit would exist.

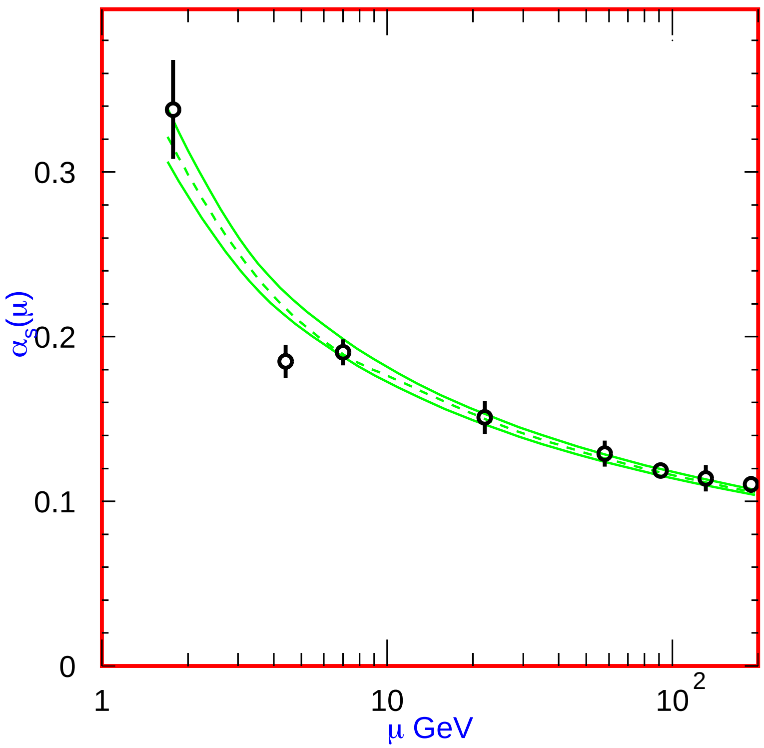
$$\sigma\left(\frac{Q^2}{\mu^2}, x_{ij}, \frac{m_i^2}{\mu^2}, \alpha_s(\mu)\right) = \sum_{n=1}^{\infty} a_n\left(\frac{Q}{\mu}, x_{ij}\right) \alpha_s^n(\mu) + \mathcal{O}\left(\left[\frac{m_i^2}{\mu^2}\right]^p\right)$$

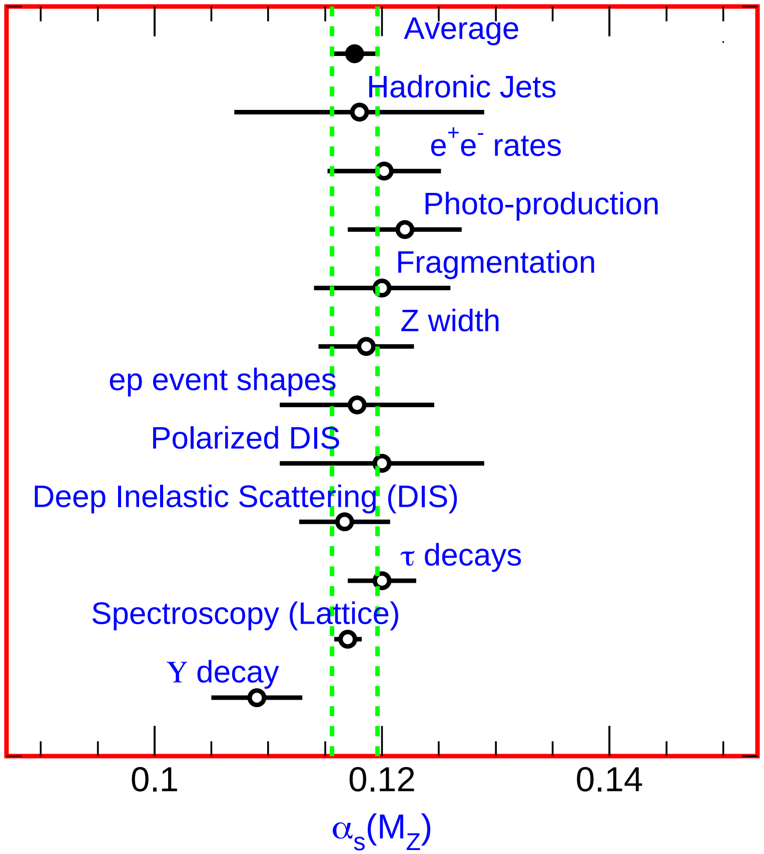
- Such quantities are called infrared (IR) safe.

- Measure $\sigma \to$ solve for α_s. Allows observation of the running coupling.

- Most of pQCD is the computation of IR safe quantities.

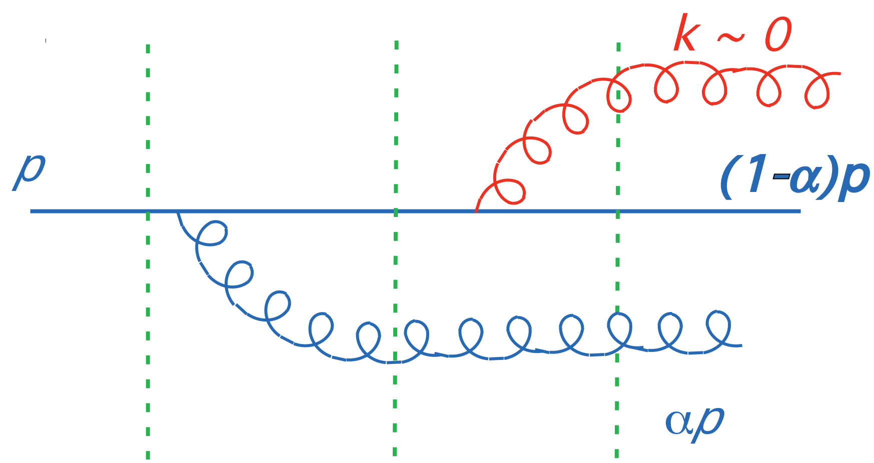
- **Consistency of $\alpha_s(\mu)$ found as above at various momentum scales**
  Each comes from identifying an IR safe quantity, computing it and comparing the result to experiment. **(Particle Data Group)**





- **To find IR safe quantities, need to understand where the low-mass logs come from.**

- To analyze diagrams, we generally think of $m \to 0$ limit in m/Q. Gives "IR" logs.
- Generic source of IR (soft and collinear) logarithms:



- IR logs come from degenerate states: Uncertainty principle $\Delta E \to 0 \Leftrightarrow \Delta t \to \infty$.
- After a while, noncollinear particles are too separated to interact. For soft emission and collinear splitting it's "never too late". But these processes don't change the flow of energy . . .
- This will follow from a general analysis, and leads to IR safety of jet cross sections.

- **For IR safety, sum over degenerate final states in perturbation theory, and don't ask how many particles of each kind we have. This requires us to introduce another regularization, this time for IR behavior.**

- **The IR regulated theory is like QCD at short distances, but is better-behaved at long distances.**

- **IR-regulated QCD <u>not the same as QCD except</u> for IR safe quantities.**

- **Similar considerations will apply to factorized cross sections and amplitudes.**

- **See how it works for the total $\mathrm{e}^+\mathrm{e}^-$ annihilation cross section to order α_s. Lowest order is $2 \to 2$, $\sigma_2^{(0)} \equiv \sigma_{\mathrm{LO}}$, σ_3 starts at order α_s.**

  – **Gluon mass regularization:** $1/k^2 \to 1/(k^2 - m_G)^2$

$$\sigma_3^{(m_G)} = \sigma_{\mathrm{LO}} \frac{4}{3}\frac{\alpha_s}{\pi}\left(2\ln^2\frac{Q}{m_g} - 3\ln\frac{Q}{m_g} - \frac{\pi^2}{6} + \frac{5}{2}\right)$$

$$\sigma_2^{(m_G)} = \sigma_{\mathrm{LO}}\left[1 - \frac{4}{3}\frac{\alpha_s}{\pi}\left(2\ln^2\frac{Q}{m_g} - 3\ln\frac{Q}{m_g} - \frac{\pi^2}{6} + \frac{7}{4}\right)\right]$$

  which gives

$$\sigma_{\mathrm{tot}} = \sigma_2^{(m_G)} + \sigma_3^{(m_G)} = \sigma_{\mathrm{LO}}\left[1 + \frac{\alpha_s}{\pi}\right]$$

  – **Pretty simple!** **(Cancellation of virtual (σ_2) and real (σ_3) gluon diagrams.)**

– Dimensional regularization: change the area of a sphere of radius R

$$4\pi R^2 \Rightarrow (4\pi)^{(1-\varepsilon)} \frac{\Gamma(1-\varepsilon)}{\Gamma(2(1-\varepsilon))} R^{2-2\varepsilon}$$

with $\varepsilon = 2 - D/2$ in D dimensions, and the coupling $g_s \to g_s \mu^{\varepsilon}$.

– Do the integrals this way, and get

$$\begin{aligned}
\sigma_3^{(\varepsilon)} &= \sigma_{\mathrm{LO}} \frac{4}{3} \frac{\alpha_s}{\pi} \left(\frac{(1-\varepsilon)^2}{(3-2\varepsilon)\Gamma(2-2\varepsilon)} \right) \left(\frac{4\pi\mu^2}{Q^2} \right)^{\varepsilon} \\
&\quad \times \left(\frac{1}{\varepsilon^2} - \frac{3}{2\varepsilon} - \frac{\pi^2}{2} + \frac{19}{4} \right) \\
\sigma_2^{(\varepsilon)} &= \sigma_{\mathrm{LO}} \left[1 - \frac{4}{3} \frac{\alpha_s}{\pi} \left(\frac{(1-\varepsilon)^2}{(3-2\varepsilon)\Gamma(2-2\varepsilon)} \right) \left(\frac{4\pi\mu^2}{Q^2} \right)^{\varepsilon} \right. \\
&\quad \left. \times \left(\frac{1}{\varepsilon^2} - \frac{3}{2\varepsilon} - \frac{\pi^2}{2} + 4 \right) \right]
\end{aligned}$$

which gives again

$$\sigma_{\mathrm{tot}} = \sigma_2^{(m_G)} + \sigma_3^{(m_G)} = \sigma_0 \left[1 + \frac{\alpha_s}{\pi} \right]$$

- This illustrates IR Safety: σ_2 and σ_3 depend on regulator, but their sum does not.

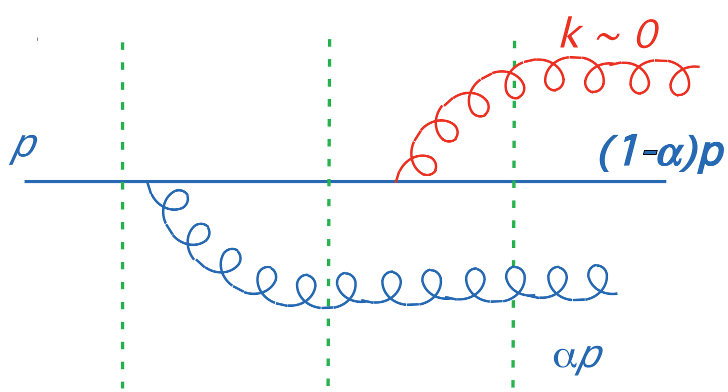
- **Generalized IR safety: sum over all states with the same flow of energy into the final state. Introduce** *IR safe weight "$e(\{p_i\})$"*

$$\frac{d\sigma}{de} = \sum_n \int_{PS(n)} |M(\{p_i\})|^2 \delta\left(e(\{p_i\}) - e\right)$$

with

$$e(\ldots p_i \ldots p_{j-1}, \alpha p_i, p_{j+1} \ldots) = e(\ldots (1+\alpha) p_i \ldots p_{j-1}, p_{j+1} \ldots)$$

- **Neglect long times in the initial state for the moment and see how this works in e^+e^- annihilation: event shapes and jet cross sections. Again . . .**



D1.The Transition to Perturbative QCD: Pinch surfaces, power counting and Ward identities

– **To prove IR Safety of jet & related of cross sections for $\mathrm{e^+e^-}$**

– **Provide a basis for factorization in ep, pp inclusive cross sections and exclusive amplitudes**

– **What we need (all orders):**

D1 Review of UV and IR divergences in dimensional regularization

D2 Method to identify infrared sensitivity in PT: "physical pictures"

D3 Method to identify IR finiteness/divergence: "power counting"

– **Our example: factorization for elastic amplitudes for parton-parton scattering in ϕ^4 and QCD. The roles of Ward identities and Wilson lines.**

– **Our purpose is to describe, not perform, explicit calculations and to describe the ultimate justification of factorization.**

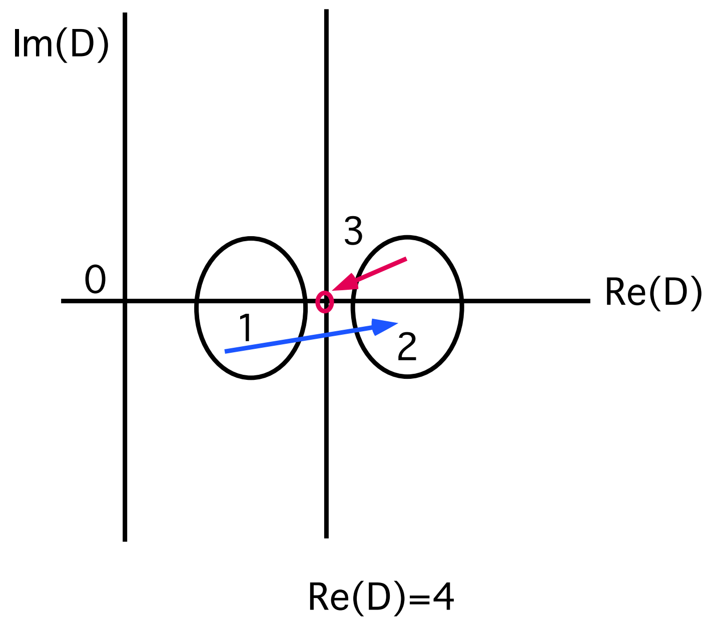
D1. Outline for Dimensional Regularization

$$
\begin{aligned}
\textbf{1.} \quad \mathcal{L}_{QCD} &\to G^{(reg)}(p_1, \dots p_n)\,, \quad D < 4 \\
&\to G^{(ren)}(p_1, \dots p_n)\,, \quad D < 4 + \Delta
\end{aligned}
$$

$$
\begin{aligned}
\textbf{2.} \quad &\to S^{(unphys)}(p_1, \dots p_n)\,, \quad 4 < D < 4 + \Delta \\
&\to \tau^{(unphys)}(p_1, \dots p_n)\,, \quad 4 < D < 4 + \Delta
\end{aligned}
$$

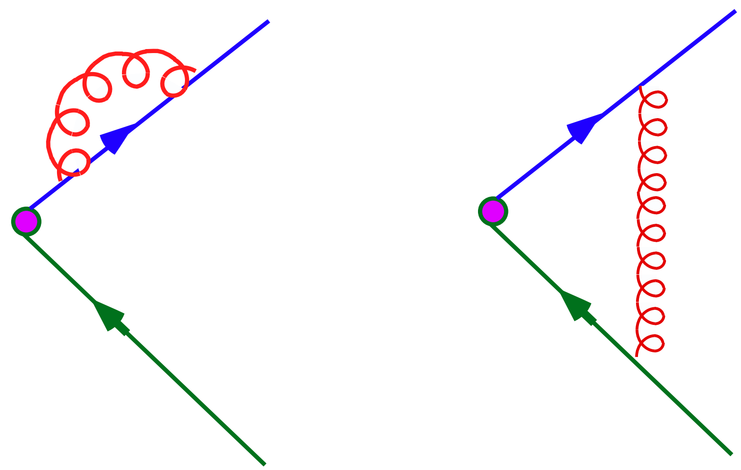
$$
\textbf{3.} \quad \to \tau^{(phys)}(p_1, \dots p_n)\,, \quad D = 4
$$

The D-plane



D2. Physical Pictures

– **Example: one-loop quark EM form factor**



$$\Gamma_\mu(q^2,\varepsilon) = -ie\mu^\varepsilon\ \bar{u}\ (p_1)\gamma_\mu v(p_2)\ \rho(q^2,\varepsilon)$$

$$\rho(q^2,\varepsilon) \ = \ -\frac{\alpha_s}{2\pi}C_F\left(\frac{4\pi\mu^2}{-q^2-i\epsilon}\right)^\varepsilon \ \frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}$$
$$\times\left\{\frac{1}{(-\varepsilon)^2}-\frac{3}{2(-\varepsilon)}+4\right\}$$

– **No UV counterterm necessary (QED Ward identity)**

– **Finding the IR $(1/(-\varepsilon))$ poles . . .**

– IR pole requires: (i) on-shell lines; (ii) pinch of momentum contours

– Also integral “singular enough” (power counting)

– Critereon for (i) and (ii): on-shell lines describe a physical process

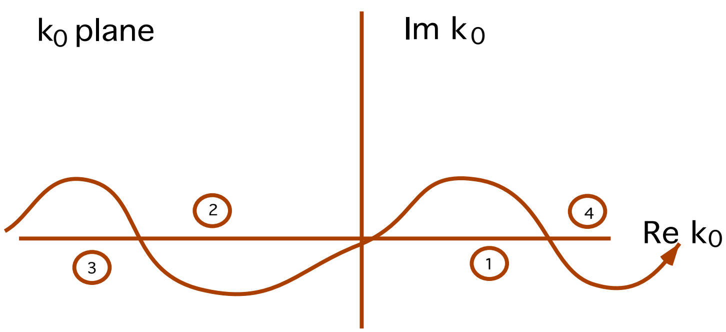
– Free propagation between vertices in space time

– Start with examples, then generalize . . .

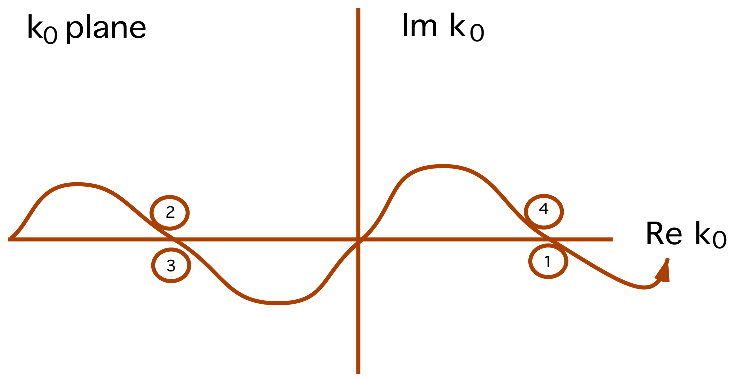
– **One-loop self energy with momentum p, when $p = (p_0, \vec{p}),\ p_0 = |\vec{p}|$:**

$$\begin{aligned}
\pi(p^2) &= \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(p-k)^2 - m^2 + i\epsilon} \\
&= \int \frac{d^3k}{(2\pi)^3} \int \frac{dk^0}{2\pi} \frac{1}{[(k^0 - |\vec{k}| + i\epsilon)(k^0 + |\vec{k}| - i\epsilon)]} \\
&\qquad \times \frac{1}{[(k^0 - p^0 - |\vec{k} - \vec{p}| + i\epsilon)(k^0 - p^0 + |\vec{k} - \vec{p}| - i\epsilon)]}
\end{aligned}$$

– **Label poles 1 . . . 4 in order. For generic $\vec{k}$, k^0 integral can avoid poles by continuation:**



– **But when $\vec{k} \to x\vec{p}$, pairs (1,4) and (2,3) pinch the contour at $k^0 = xp^0$:**



– **For example, pole 1 at**

$$k^0 = |x\vec{p}| - i\epsilon = xp^0 - i\epsilon$$

is "pinched" by pole 4:

$$k^0 = p^0 - |x\vec{p} - \vec{p}| + i\epsilon = p^0\ (1 - (1 - x)) = xp^0 + i\epsilon$$

– **Notice it only works for $0 < x < 1$, but now we're really sensitive to on-shell behavior. This particular point is (pretty obviously) "collinear".**

– **Next example: triangle, neglecting the numerator (using Feynman parameterization)**

$$I_\Delta \;=\; 2\int \frac{d^n k}{(2\pi)^n}\; \int_0^1 \; \frac{d\alpha_1 d\alpha_2 d\alpha_3 \, \delta(1 - \Sigma_{i=1}^3 \, \alpha_i)}{D^3} \tag{3}$$

where

$$D \;=\; \alpha_1 k^2 + \alpha_2 (p_1 - k)^2 + \alpha_3 (p_2 + k)^2 + i\epsilon$$

– D **quadratic: Solutions for each k^μ must coincide:**

$$\frac{\partial}{\partial k^\mu}\, D(\alpha_i, k^\mu, p_a) = 0$$

– **This gives the Landau equations . . .**

$$\alpha_1 k + \alpha_2 (p_1 - k) \;+\; \alpha_3 (p_2 + k) = 0$$

– **A “physical” reinterpretation . . .**

$$\alpha_1 k + \alpha_2(p_1 - k) \ + \ \alpha_3(p_2 + k) = 0$$
$$\Updownarrow \tag{4}$$
$$\Delta t_1 v_k + \Delta t_2 v_{p_1 - k} \ + \ \Delta t_3 v_{p_2 + k}$$

– **Interpretation of “times”** $\delta t_q = \alpha q^0$ **and “velocities”** $v_q^\mu = q^\mu / q^0$

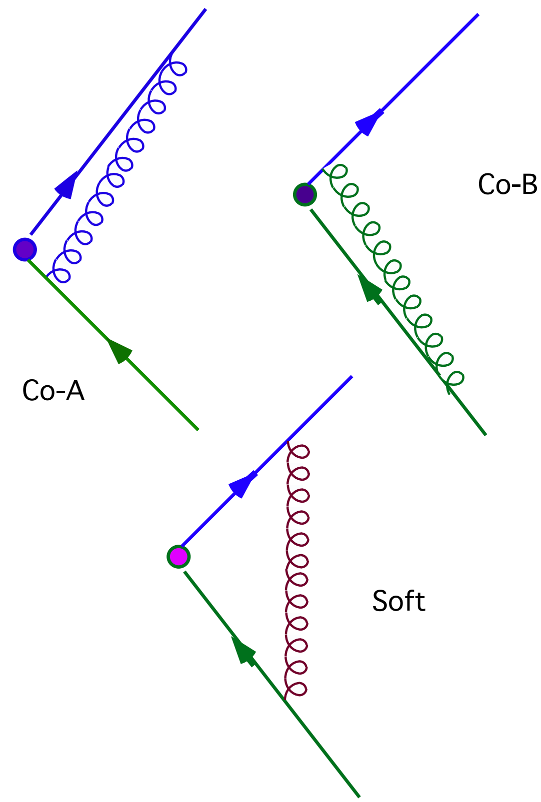
– **Solutions: “soft”:** $k^\mu = 0\,, \quad (\alpha_2/\alpha_1) = (\alpha_3/\alpha_1) = 0$

– **and “collinear-A” and “collinear-B”**

$$k \ = \ \zeta p_1\,, \quad \alpha_3 = 0\,, \quad \alpha_1 \zeta = \alpha_2(1 - \zeta)$$

$$k \ = \ -\zeta' p_2\,, \quad \alpha_2 = 0\,, \quad \alpha_1 \zeta' = \alpha_3(1 - \zeta')$$

Portraits of “Co-A”, “Co-B” and “soft”:

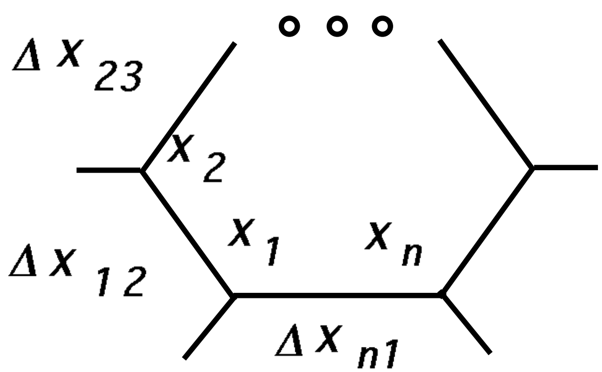


– **Generalization is easy:**

– **Landau equations & physical pictures for arbitrary diagrams**

$$\text{either } \ell_i^2 = m_i^2, \text{ or } \alpha_i \;=\; 0,$$

$$\text{and } \sum_{i \text{ in loop } s} \alpha_i \ell_i \epsilon_{is} \;=\; 0$$



$$\Delta x_{12} + \Delta x_{23} + \ldots + \Delta x_{n1} = 0$$

D3. Power Counting at Pinch Surfaces

– **Example: soft region for triangle with massless scalar quarks**

$$\Delta_{\rm soft} = \int_{\rm soft} d^D k \frac{(2p_1+k)^\mu(-2p_2+k)_\mu}{(2p_1^+ k^- - k_T^2 + i\epsilon)\,(-2p_2^- k^+ - k_T^2 + i\epsilon)\,(2k^+k^- - k_T^2 + i\epsilon)}$$

– **Rescale:** $k^\mu = \lambda\,\kappa^\mu$ **& insert unity**

$$1_{\rm soft} \equiv \int_0^{\lambda_{\rm max}} d\lambda^2\,\delta\left(\lambda^2 - \sum_\mu k_\mu^2\right)$$

– **Which gives . . .**

$$\begin{aligned}\Delta_{\rm soft} &= \int_0^{\lambda_{\rm max}} d\lambda^2\lambda^D \int\ d^n\kappa\ (4p_1^+p_2^- + \mathcal{O}(\lambda))\ \lambda^{-2}\,\delta\left(1 - \sum_\mu \kappa_\mu^2\right)\\ &\times\frac{1}{\lambda(2p_1^+\kappa^+ + \mathcal{O}(\lambda) + i\epsilon)\,\lambda(-2p_2^-\kappa^+ + \mathcal{O}(\lambda) + i\epsilon)\,\lambda^2(2\kappa^+\kappa^- - \kappa_T^2 + i\epsilon)}\\ &\sim\ 2\int_0^{\lambda_{\rm max}} \frac{d\lambda}{\lambda^{5-D}}\ \int d^D\kappa\ \frac{\delta\left(1-\sum_\mu\kappa_\mu^2\right)}{(\kappa^+ + i\epsilon)\,(-\kappa^- + i\epsilon)(2\kappa^+\kappa^- - \kappa_T^2 + i\epsilon)}\end{aligned}$$

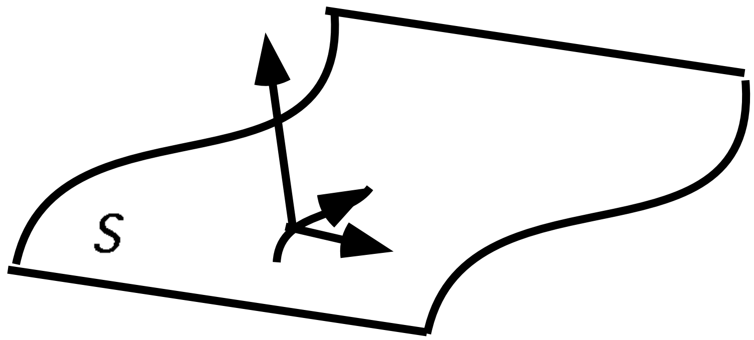
– λ **integral** $\to 1/(4-D) = 1/2\varepsilon$ **pole**

– **Remaining integral: pinches at** $\kappa^\pm = \kappa_T^2 = 0$ **(& hence** $\kappa^\mp = 1$**)**

– **Soft tail of collinear regions give the double poles we saw above.**

General pinch surface analysis

– **Consider a pinch surface γ of graph G: ℓ_b internal; κ_a normal**



– **Could be dimensionally regularized (D; $\varepsilon = 2 - D/2$)**

– **The integral near surface γ looks like**

$$G_\gamma(Q) = \int_{\mathcal{O}(Q)} \prod_b d\ell_b \int_\gamma \prod_{a=1}^{\mathcal{D}_\gamma(\varepsilon)} d\kappa_a \, \frac{n(\kappa_a, \ell_b, Q)}{\prod_j d_j(\kappa_a, \ell_b, Q)}$$

- **Where Q represents external momenta, and n numerator factors (things like $\not{k}$)**

– **Scaling:** $\boldsymbol{\kappa_a = \lambda_\gamma\, \kappa'_a}$

$$1_\gamma = \int_0^{\lambda_\gamma^{\max 2}} d\lambda^2\, \delta\left(\lambda_\gamma^2 - \sum_a \kappa_a^2\right)$$

$$\begin{aligned} n(\kappa_a, \ell_b, Q) &= \lambda^{N_n}\, \left[\bar{n}(\kappa'_a, \ell_b, Q) + \mathcal{O}(\lambda)\right] \\ d_j(\kappa_a, \ell_b, Q) &= \lambda^{N_j}\, \left[\bar{d}_j(\kappa'_a, \ell_b, Q) + \mathcal{O}(\lambda)\right] \end{aligned}$$

Gives . . .

$$G_\gamma(Q) = 2\int_0^{\lambda_\gamma^{\max}} d\lambda\, \lambda^{p_\gamma - 1}\, \Delta_\gamma(Q)$$

where the "homogeneous integral" is

$$\Delta_\gamma(Q) = \int_{\mathcal{O}(Q)} \prod_b d\ell_b \int_\gamma \prod_{a=1}^{\mathcal{D}_\gamma(\varepsilon)} d\kappa'_a\, \frac{\bar{n}(\kappa'_a, \ell_b, Q)}{\prod_j \bar{d}_j(\kappa'_a, \ell_b, Q)}\, \delta\left(1 - \sum_a {\kappa'_a}^2\right)$$

and the convergence/divergence is determined by

$$p_\gamma = \mathcal{D}_\gamma(\varepsilon) + N_n - \sum_j N_j$$

- **Simply the volume of the normal variable, plus numerator suppression minus denominator enhancement. If $p_\gamma > 0$, pQCD is applicable near this singular surface. We don't have to look for cancelation or factorization.**

Repeat for all γ of G

– **If pinch surfaces of $\Delta_\gamma(Q)$ are already counted, provides bounds on G and classifies IR poles in dimensional regularization**

– **This is the case for $\mathrm{e}^+\mathrm{e}^-$ annihilation with massless quarks**

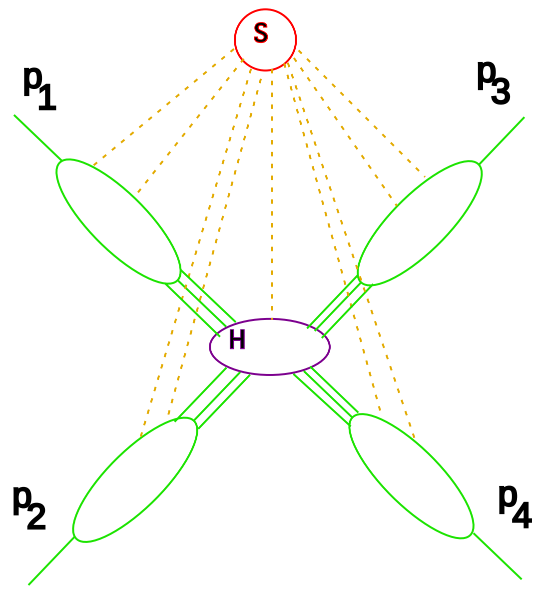
– **Massive case is similar, but more regions because mass provides an extra scale**

– **As long as $p_\gamma > 0$, integrals are infrared safe.**

– **The search for long-distance behavior is the search for $p_\gamma = 0$.**

– Of special interest: elastic scattering $2 \to n$, here $2 \to 2$, with Q c.m. energy and θ^* the c.m. scattering angle

– The most general pinch surface with a connected set of off-shell (hard) lines



– Sets of collinear, finite-energy on-shell lines ("jets") connected to the hard scattering, joined by sets of zero momentum lines.

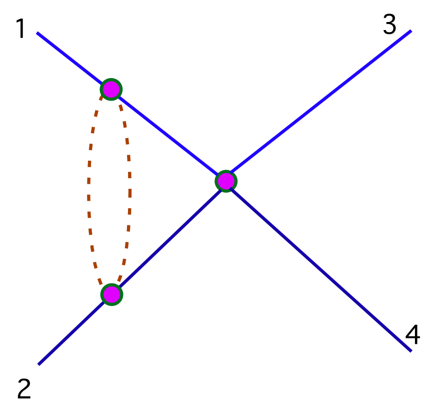
– But which of these will give logarithmic power counting?

- **Normal coordinates:**

  - **2 per loop collinear to each external momentum p: $l_\perp^2$ and $p \cdot l$ for each loop l. Then all propagators in the p-jet $\sim \lambda$.**

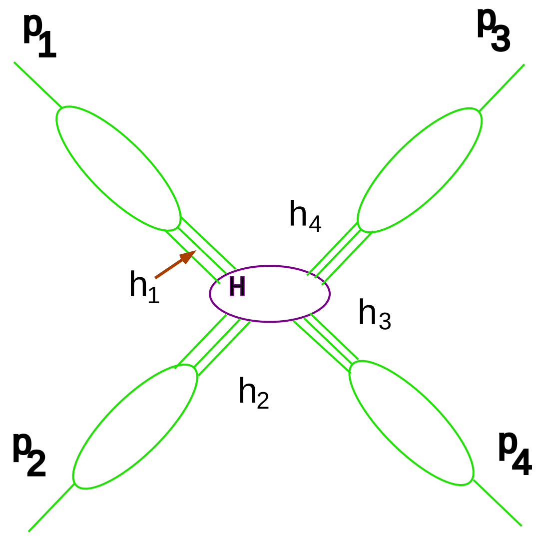
  - **4 per loop for "soft" momenta, $l^0 \ldots l^3$: Then all propagators in $S \sim \lambda^2$**

- **A ϕ^4 example with two soft lines:**



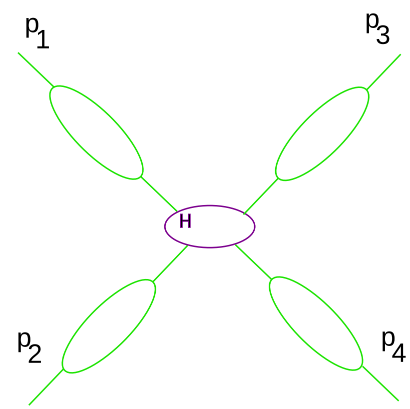
  - **Power counting: two soft loops, two soft lines and two jet lines: $p = 2(4) - 2(2) - 2(1) = 2$, finite.**
  - **Readily generalizes to all loop order in ϕ^4.**

- So for ϕ^4 can use collinear lines alone ... forget about "S"



- Collinear power counting for ϕ^4 jet

$$\begin{aligned} p &= 2L - N = 2(L-N) + N \\ L &= N - (v_4 + 1) + 1 \\ 2N &= 4v_4 - 1 + h \\ &\Downarrow \\ p &= \left(\frac{h-1}{2}\right) \quad \text{only self energies give } p = 0! \end{aligned}$$



- **A factorized S-matrix amplitude. The jets are all identical:**

$$\boldsymbol{A(Q,\theta^*,m) = H(Q/\mu,\theta^*,g(\mu))\ \ [\,J(m/\mu,g(\mu))\,]^4}$$

- **Each J can be extended to the full matrix element:**

$$\boldsymbol{J} = \langle 0|\boldsymbol{\phi}(0)|\boldsymbol{p}\rangle$$

- **Now invoke $\mu dA/d\mu = 0$ and use separation of variables and chain rule:**

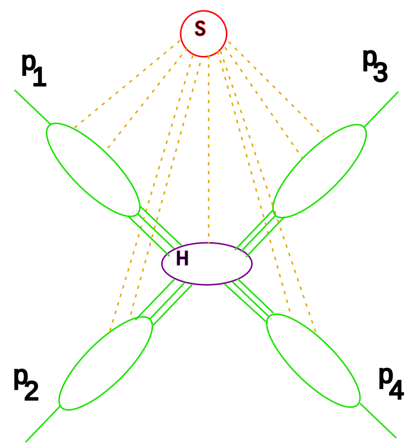
$$Q\frac{d\ln H}{dQ} = -\,\gamma(g(\mu)) = 4\mu\frac{d\ln J}{d\mu} \tag{5}$$

because renormalization scale μ is the only variable held in common. $\gamma(a)$ is IR safe because it can be found from the hard scattering.

- **Lesson: factorization determines energy dependence.**

Elastic scattering of gauge theory partons

- Certainly soft and collinear singularities (recall 1-loop example) so we need the general case



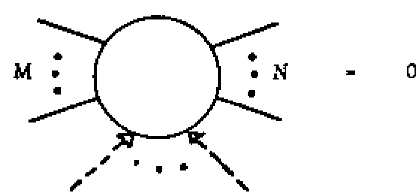
- Let's do power counting with the same normal variables

- **For gauge theories, the scaling of numerators is important**
  - **Any 3-gluon vertex internal to a jet gives a vector l^μ, order λ^0 near pinch surface.**
  - **Any fermion line in the jet gives a Dirac matrix $\not{l}$ order λ^0 near pinch surface.**
  - **Rule: The number of factors of λ^0, "jet" momentum in the numerator equals the number of 3-point vertices in the jet function in that jet.**
  - **The scalar product of two such momenta is order λ**
- **Observation: because the amplitude is a Lorentz scalar, pairs of jet momenta appear only scalar products with each other – unless they are contracted gluon propagators in S or they are contracted with vertices in H.**
- **For jet i: $N_n^{(i)} = (1/2)\left(v_3^{(i)} - n_L^{(i)} - n_s^{(i)}\right)$ where $v_3^{(i)}$ is the number of 3-point vertices in the jet subdiagram.**

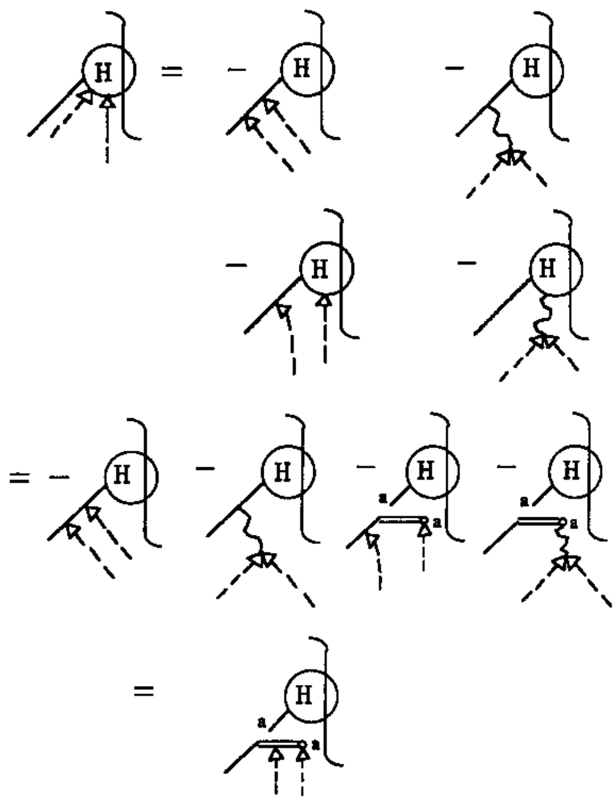
- **For simplicity, pure collinear power counting for gauge theory jet**

$$
\begin{aligned}
p &= 2L - N + \frac{v_3 - v_L}{2} = 2(L-N) + N + \frac{v_3 - v_L}{2} \\
L &= N - (v_4 + v_3 + 1) + 1 \\
2N &= 4v_4 + 3v_3 - 1 + h \\
&\Downarrow \\
p &= \left(\frac{(h - v_L) - 1}{2}\right)
\end{aligned}
$$

- **Result: only one parton connecting any jet to the hard part is NOT a gluon that is contracted with the jet momentum. Such gluons are longitudinally- or scalar-polarized. This is an unphysical polarization. As such we know a lot about them . . .**
- **The basic Ward identity that decouples longitudinally polarized gluons from amplitudes involving physical partons, states 'M' and 'N' here:**



- The Ward identity result requires a sum over all diagrams. For the leading pinch surfaces we don't have all diagrams, but we can prove that the result is insensitive to the details of the hard scattering. In fact, longitudinally-polarized gluons in one jet don't even know the directions of the other jets. Here's how it works. Basic steps of the inductive proof (Collins, Soper, S. 1989). . .



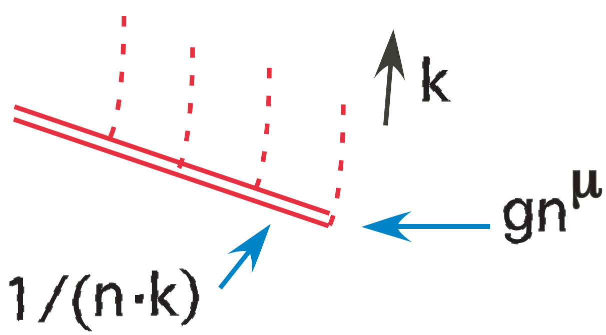
- The double line is a "gauge link", in this case from the position of the physical parton to infinity.

- **The gauge link [a.k.a. Wilson line, path ordered expential, nonabelian phase, eikonal line] in x^- direction $(n^\mu = \delta_{\mu -})$ is defined by**

$$W_n^{(A)}(\infty, x^-) = P \exp\left[-ig \int_0^\infty n \cdot A^{(adj)}\left((x^- + \lambda)n\right)\right]$$

- **To the jet, all that's left of the rest of the world is a gluon source!**

- **The vector n^μ is arbitrary so long as it is not proportional to the jet momentum.**



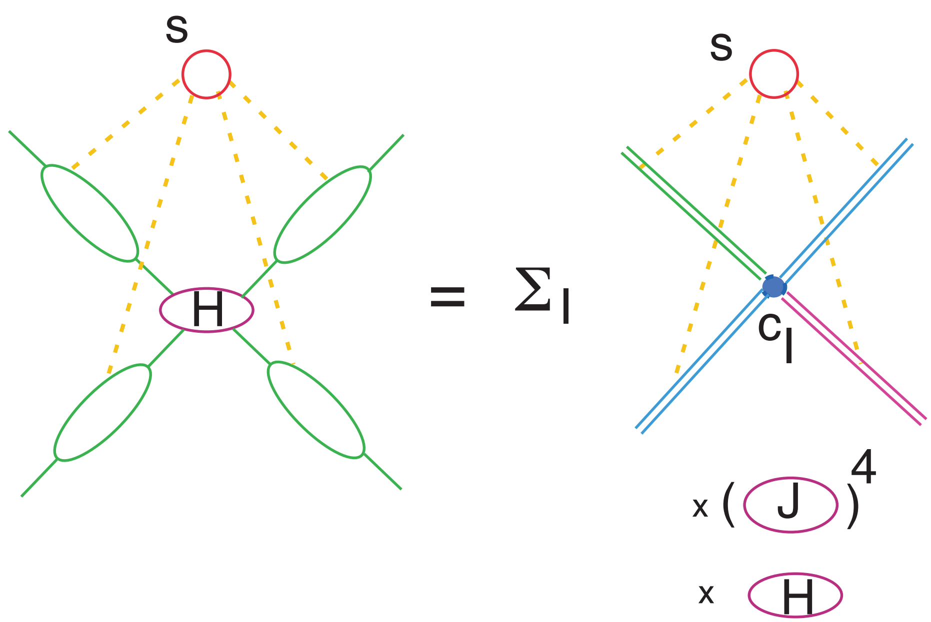
- **The matrix element of the jet function:**

$$\langle 0|\Phi^{(A)}(\infty, 0)\phi(0)|p\rangle$$

- **A common feature of all factorized long-distance functions, including parton distributions.**

- **The complete amplitude also requires the soft gluons, which remember only the directions and charges of the jets:**

Factorization of soft gluons:



- **In a full treatment, we need to carefully avoid double counting between the soft gluon function and the jet functions. As applied to cross sections, this specifies whether the eikonals that define jet functions point toward the past or future.**