Region of Interest Tomography

Review

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Notations

In parallel geometry, 2D tomography aims at reconstructing the density function $f(x)$ from its line integrals, (the Radon transform RT) :

$$p(\omega, s) = \int_{\Re} dt\ f(s\omega^{\perp} + t\omega)$$

A projection is the 1D function $p(\omega,)$ for a fixed angular position ω. All the projections form a sinogram.

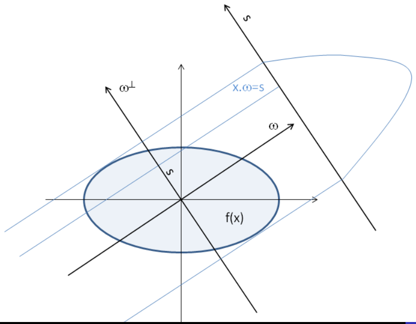

Figure 1: The function is integrated along lines of direction $\omega = (\cos(\phi), \sin(\phi))^t$ at a distance s of the origin.

Filter Back Projection reconstruction

If $p(\omega, s)$ is the RT of $f(x)$ then :

$$f(x) = \int_0^{\pi} d\phi \; p_r(\omega, x \cdot \omega^{\perp}) \text{ with } \omega = (\cos\phi, \sin\phi)^t \tag{1}$$

where

$$p_r(\omega, s) = \int_{\Re} ds' p(\omega, s') r(s - s') \text{ and the FFT of } r \text{ is } \hat{r}(\sigma) = |\sigma| \tag{2}$$

Step described by (1) is called the backprojection whereas (2) is the filtering step. The filter r is called the ramp-filter.

The backprojection step

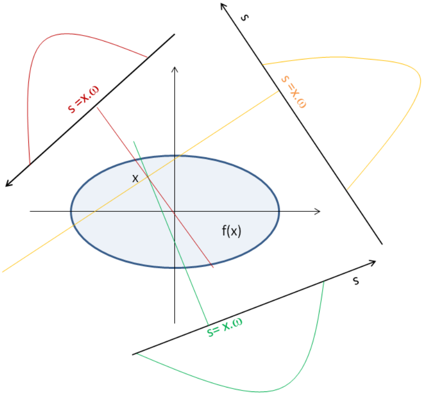

Figure 2: Backprojecting at position x consists in integrating all the values $p(\omega, s)$ for which x had contributed.

Non locality of the ramp filter

In the Fourier domain, the ramp filter can also be written : $\hat{r}(\sigma) = \sigma sign(\sigma)$ which corresponds, in the direct domain, to the derivation of the Hilbert transform (HT)

$$\begin{aligned} p_r(\omega, s) &= \frac{\partial}{\partial s} p_H(\omega, s) \text{ with} \\ p_H(\omega, s) &= \int_{\Re} ds' p(\omega, s') h(s - s') \text{ where } h(s) = \frac{1}{\pi s} \end{aligned} \tag{3}$$

Thus, as the HT is not local (if values of $p(\omega, s)$ are missing for some s, there is no s for which $p_H(\omega, s)$ can been calculated), so is it for the ramp filter. Indeed the ramp filter support is infinite.

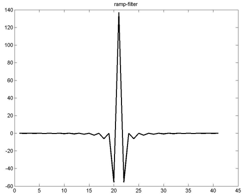

Figure 3: The ramp filter in the direct space with Fourier regularization.

Incomplete data

In various situations, only part of the sinogram can be measured.

- The object of interest can be too wide respect to the detector size: data are said "truncated".
- In electron tomography, the source detector system is fixed while the object is tilting. Due to the mechanic, the tilt angle is typically $\pm 70°$: the problem is said "limited angle" and has no solution.

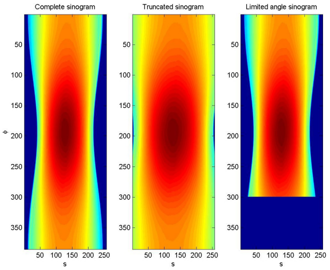

Figure 4: Two cases of incomplete data: complete sinogram (left), truncated sinogram (middle), limited angle sinogram (right).

Until 2002, it was believed that due to the non locality of the filter step, no region could be exactly recovered in any case of incomplete data.

Which ROI?

Since 2002, methods have been proposed to solve the incomplete data problem. The aim is to exactly reconstruct the biggest possible ROI. **Which ROI can we expected to be reconstructible?**

Let Ω be the support of $f(x)$ and region A, the region of Ω for which all the line integrals have been measured. Region $B = \Omega \setminus A$.

Region B can not been recovered because of the limited-angle problem. The reconstructible ROI belongs to A (but is not necessary fully A).

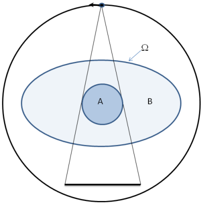

Figure 5: Region A is where all the line integrals have been measured : $ROI \subset A$.

Two different approaches, that are not equivalent, have been developed:

- The Virtual Fan-Beam (VBP) method
- The Differentiated Backprojection (DBP) method

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The Fan-Beam geometry

Let us note $g(v_\lambda, \omega)$ the fan-beam data defined by :

$$g(v_\lambda, \omega) = \int_0^\infty dt\ f(v_\lambda + t\omega) \text{ with } \omega = (\cos\phi, \sin\phi)^t$$

where v_λ is the vertex, taken outside the convex hull of the function $f(x)$. Example: for a circular trajectory of radius r, $v_\lambda = r(\cos\phi, \sin\phi)^t$.

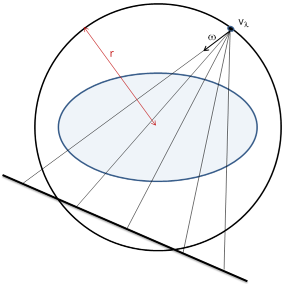

Figure 6: The fan-beam geometry.

The parallel/fan-beam Hilbert projection equality

As for the parallel geometry (it was (3)) we define the Hilbert transform for the fan-beam geometry :

$$g_H(v_\lambda, \omega) \quad = \quad \int_0^{2\pi} d\phi' \; g(v_\lambda, \omega') h(\sin(\phi - \phi')) \text{ with still } h(s) = \frac{1}{\pi s}$$

The parallel /fan-beam Hilbert equality is :

$$p_H(\omega, s) = g_H(v_\lambda, \omega) \textbf{ for } s = v_\lambda \cdot \omega^\perp \qquad (4)$$

The parallel/fan-beam Hilbert projection equality

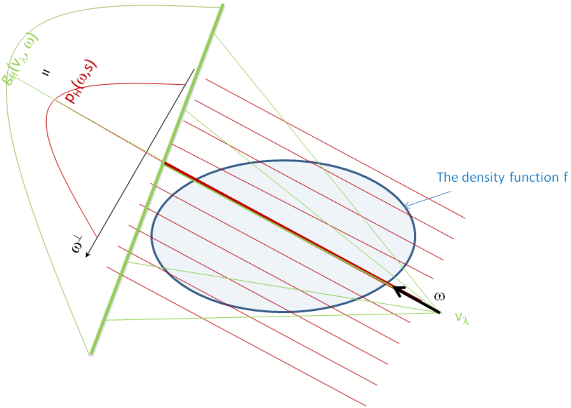

Figure 7: The parallel /fan-beam Hilbert equality :

$$p_H(\omega, s) = g_H(v_\lambda, \omega) \text{ for } s = v_\lambda \cdot \omega^\perp$$

The Virtual Fan Beam method (VBP)

Remember that

$$f(x) = \int_0^{\pi} d\phi \, \frac{\partial}{\partial s} p_H(\omega, s = x \cdot \omega^{\perp})$$

The VBP method is based on using in the above equation, the Hilbert equality (4) when $p_H(\omega, s)$ can not be calculated whereas $g_H(v_\lambda, \omega)$ can be calculated.

VFB example

Let the support of the density function f be ellipse-shaped with axis size a and b, and assume that all line-integrals for lines that cross the circle of radius r with $b < r < a$, have been measured : some projections are truncated (others are not).

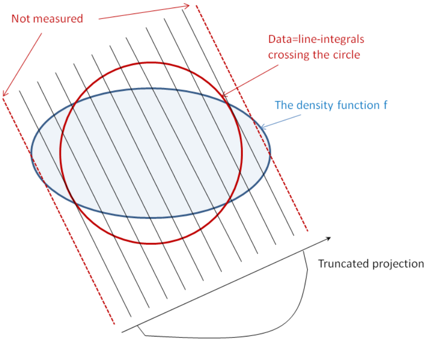

Figure 8: VFB example: truncated projection

VFB example

Let the support of the density function f be ellipse-shaped with axis size a and b, and assume that all line-integrals for lines that cross the circle of radius r with $b < r < a$, have been measured : some projections are truncated (others are not).

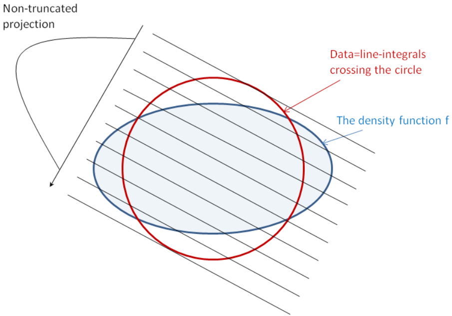

Figure 9: VFB example: not truncated projection

VFB example

For some ϕ angles,

- $p_H(\omega, s)$ can be calculated from the parallel projections $p(\omega, s)$,
- $p_H(\omega, s)$ can only be calculated using a virtual fan-beam projections $g(v_\lambda, \omega)$ and the Hilbert equality $p_H(\omega, s) = g_H(v_\lambda, \omega)$

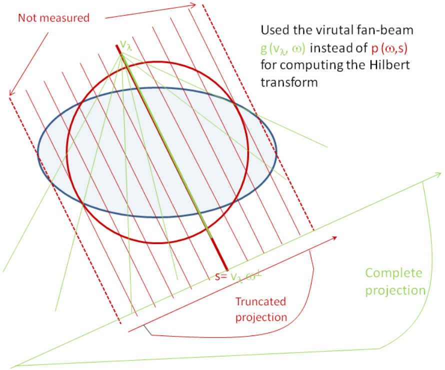

Figure 10: VFB example: when the parallel projection is truncated, use a virtual fan-beam

VFB example

Algorithm:

For all the ϕ angles, compute $p_H(\omega, s)$ using
  if (the parallel projection $p(\omega, s)$ is not truncated)
    equation (3) based on $p(\omega, s)$
  else
    the parallel-Hilbert equality (4) based on a virtual-fan beam $g(v_\lambda, \omega)$
  end
end

For all the ϕ angles compute
  $p_r(\omega, s) = \frac{\partial}{\partial s} p_H(\omega, s)$
end

Compute $f(x)$ by backprojecting $p_r(\omega, s)$

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Hilbert transform and its inversion

Let $k(t)$ be a 1D function with support I (i.e. $k(t) = 0$ for $t \notin I$). Its Hilbert transform (HT)

$$Hk(t) = \int_{\Re} dt' k(t') h(t - t') \text{ with } h(t) = \frac{1}{\pi t}$$

has infinite support. However even if $Hk(t)$ is only known for $t \in [L, R] \supset I$, $k(t)$ can be recovered using

$$k(t) = \frac{-1}{\sqrt{(t-L)(R-t)}} \left(\int_R^L dt' \, \sqrt{(t'-L)(R-t')} \frac{Hk(t')}{\pi(t-t')} - C \right)$$

where C is a constant.

Hilbert transform and its inversion

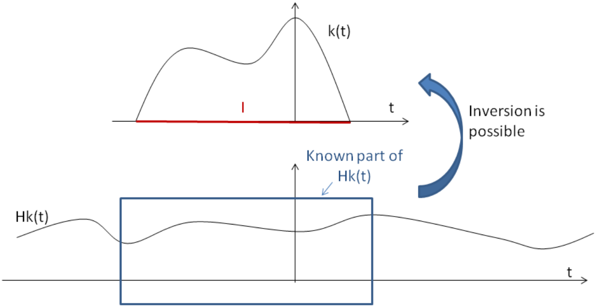

Figure 11: The 1D function $k(t)$ of finite support I can be recovered from its HT if known on a support including I.

Hilbert transform of a 2D function

Let $f(x)$ be a 2D function. We define $H_\phi f(x)$ the HT along lines of direction ω:

$$H_\phi f(x) = \int_{\Re} dt f(x - t\omega) h(t) \tag{5}$$

For instance, if $\phi = 0$, noting $x = (x_1, x_2)$ then $H_0 f(x_1, x_2)$ is the 1D HT on horizontal lines:

$$H_0 f(x_1, x_2) = \int_{\Re} dt\ f(x_1 - t, x_2) h(t)$$

Differentiated Backprojection

Let define the Differentiated Backprojection (DBP) :

$$b_\phi(x) = \frac{1}{\pi} \int_{\phi}^{\phi+\pi} d\phi \, \frac{\partial}{\partial s} p(\phi, s = x \cdot \omega^\perp) \tag{6}$$

NB : remark the integral bounding limits.

DBP and HT

It can be shown that

If $p(\phi, s)$ is the parallel Radon transform of f, then

$$b_\phi(x) = 2H_\phi f(x) \tag{7}$$

with b_ϕ defined in equation (6) and H_ϕ defined in equation (5).

DBP example

Again assume that the support of f is ellipse-shaped and that truncated data had been measured, as previously. Then if we choose $\phi = \pi/2$, $H_{\pi/2} f(x_1, x_2) = \int_{\Re} dt\ f(x_1, x_2 - t) h(t)$ can be calculated by backprojecting $\frac{\partial}{\partial s} p(\omega, s)$ from $\pi/2$ to $3\pi/2$. The density function is recovered by inverting the truncated HT along the vertical lines.

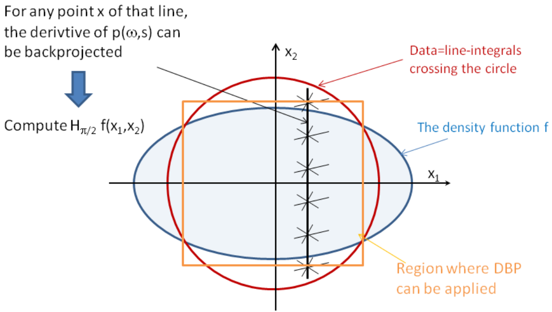

Figure 12: The DBP can be applied on the vertical lines that lie inside the orange region.

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VFB simulation

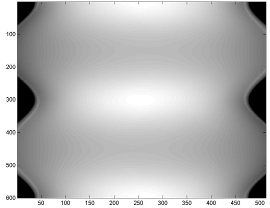

Figure 13: Truncated sinogram

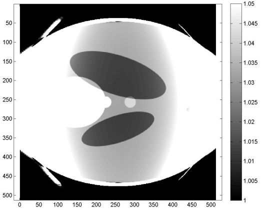

Figure 14: FBP reconstruction

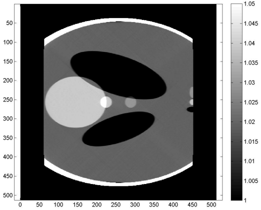

Figure 15: VFB reconstruction

VFB simulation

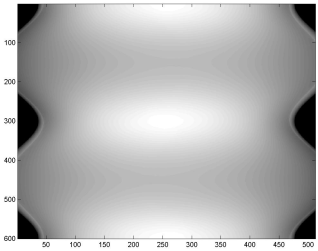

Figure 16: Truncated sinogram

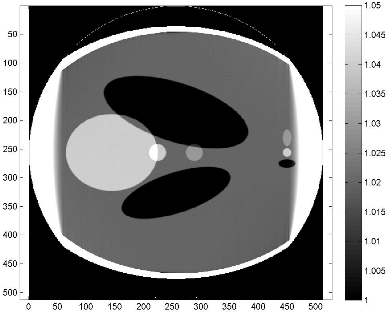

Figure 17: DBP reconstruction along vertical lines

VFB versus DBP

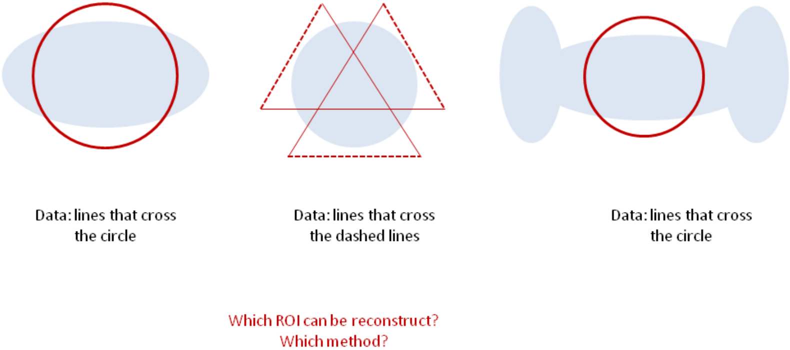

Figure 18: Three different situations of incomplete data.

VFB versus DBP

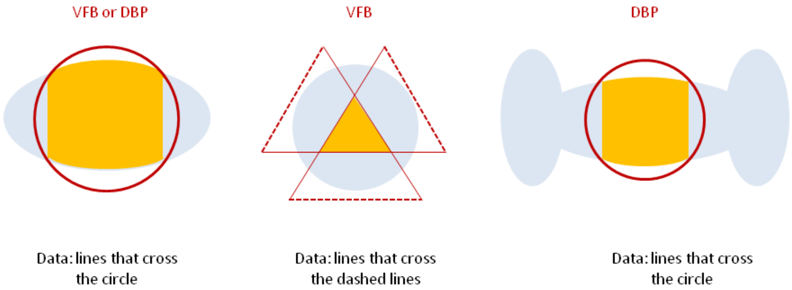

Figure 19: The two approaches are not equivalent. Even for case 1, reconstructions are not the same from noisy data.

Conclusion

We have presented the Virtual Fan-Beam and the Differentiated Backprojection methods, to solve some truncation problems.

Some ROI problems remain open like "which minimum data to measure in order to handle the interior problem?". Completing the theory of ROI reconstruction in 2D and 3D, is a future challenge.