NON-PARAMETRIC REGULARIZATION OF TOMOGRAPHIC PROBLEMS

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Inverse Problems: Notations

We try to identify structures that cannot be directly observed. Modelling of hidden physical systems $\Longrightarrow (\mathbb{M}, \mathbb{D}, \mathbf{g})$

- $\mathbb{M}$: the set of modelling parameters describing the hidden physical system
- $\mathbb{D}$: the set of observable parameters, or data, that are related to modelling parameters through $\mathbf{g}$:
- $\forall \mathbf{m} \in \mathbb{M}, \quad \mathbf{m} \longrightarrow \mathbf{d} = \mathbf{g}(\mathbf{m}) \in \mathbb{D}$

Forward problem : given $\mathbf{m}$ and $\mathbf{g}$, compute $\mathbf{d} = \mathbf{g}(\mathbf{m})$

Inverse problem : given $\mathbf{d}$ and $\mathbf{g}$, infer the manifold of models $\mathbf{m}$ corresponding to data $\mathbf{d}$, up to measurement errors

Stochastic Approach of Inverse Problem

In this approach, at each step of knowledge, the information on parameters is quantified by a measure

At prior step, the measures represent :

- errors in physical measurements for data $\mathbf{d} \in \mathbb{D}$
- the information that can be a priori gathered on modelling parameters $\mathbf{m} \in \mathbb{M}$, independently of data

These measures are usually defined through probability density functions (*pdf*) for finite dimensionnal data or model space :

$$\rho_{obs}(.) \;\; \textit{pdf} \text{ of } \mathbf{d} \text{ over } \mathbb{D}, \text{ and } \rho_{prior}(.) \;\; \textit{pdf} \text{ of } \mathbf{m} \text{ over } \mathbb{M}$$

For instance, $P(A) = \int_A \rho_{obs}(\mathbf{v})d\mathbf{v}$ represents the probability that the true data vector $\mathbf{d}$ belongs to the measurable set A in $\mathbb{D}$

$\mathbf{m}$ and $\mathbf{d}$ can be considered as independent random vectors, although there are not results of random experiments

Conditional Probability and Inverse Problem

$$\mathbf{m} \in \mathbb{M} \qquad \mathbf{d} \in \mathbb{D} \qquad \mathbf{d} = \mathbf{g}(\mathbf{m}) \qquad \mathbf{G_m}$$

- $\mathbf{m}$ and $\mathbf{d}$ are assumed to be independent random vectors with *pdf* $\rho_{prior}(\mathbf{m})$, and $\rho_{obs}(\mathbf{d})$
- $\mathbf{t} = \mathbf{d}$ - $\mathbf{g}(\mathbf{m})$ a priori is a random vector; but in fact, in virtue of the physical theory : $\mathbf{t} = \mathbf{0}$
- we define : $\rho_{post}(\mathbf{m}) = \rho(\mathbf{m}|\mathbf{t} = \mathbf{0})$

Consider $(\mathbf{d}, \mathbf{m}) \longrightarrow (\mathbf{t} = \mathbf{d} - \mathbf{g}(\mathbf{m}), \mathbf{m})$, the Jacobian of which is

$$\begin{vmatrix} \mathbf{I}_d & -\mathbf{G_m} \\ 0 & \mathbf{I}_d \end{vmatrix} = 1, \text{ hence : } \rho(\mathbf{t}, \mathbf{m}) = \rho_{obs}(\mathbf{t} + \mathbf{g}(\mathbf{m}))\rho_{prior}(\mathbf{m})$$

- and : $\rho_{post}(\mathbf{m}) \propto \rho_{prior}(\mathbf{m})\rho_{obs}(\mathbf{g}(\mathbf{m})) \propto \exp(\text{- } E(\mathbf{m}))$

The Gaussian Linear Case

$$\mathbf{d}\ :\ \mathbf{d}_{obs},\ \mathbf{C}_d\ ; \quad \mathbf{m}\ :\ \mathbf{m}_{prior},\ \mathbf{C}_m\ ; \quad \mathbf{g} \equiv \mathbf{G}$$

- $2\,\mathrm{E}(\mathbf{m}) = (\mathbf{C}_d^{-1}(\mathbf{Gm}-\mathbf{d}_{obs})|\mathbf{Gm}-\mathbf{d}_{obs}) + (\mathbf{C}_m^{-1}(\mathbf{m}-\mathbf{m}_{prior})|\mathbf{m}-\mathbf{m}_{prior})$

$\mathbf{G}^*$ adjoint of $\mathbf{G}$: $\quad \forall(\mathbf{v},\mathbf{u}) \in \mathbb{M}\times\mathbb{D} \quad (\mathbf{Gv}|\mathbf{u})_{\mathbb{D}} = (\mathbf{v}|\mathbf{G}^*\mathbf{u})_{\mathbb{M}}$

- $2\,\mathrm{E}(\mathbf{m}) = ((\mathbf{C}_m^{-1} + \mathbf{G}^*\mathbf{C}_d^{-1}\mathbf{G})(\mathbf{m}-\mathbf{m}_{post})|\mathbf{m}-\mathbf{m}_{post})$
  $-((\mathbf{C}_m^{-1} + \mathbf{G}^*\mathbf{C}_d^{-1}\mathbf{G})\mathbf{m}_{post}|\mathbf{m}_{post}) + (\mathbf{C}_m^{-1}\mathbf{m}_{prior}|\mathbf{m}_{prior}) + (\mathbf{C}_d^{-1}\mathbf{d}_{obs}|\mathbf{d}_{obs})$

with : $\quad \mathbf{m}_{post} = (\mathbf{C}_m^{-1} + \mathbf{G}^*\mathbf{C}_d^{-1}\mathbf{G})^{-1}(\mathbf{G}^*\mathbf{C}_d^{-1}\mathbf{d}_{obs} + \mathbf{C}_m^{-1}\mathbf{m}_{prior})$

Hence : $\quad \rho_{post}(\mathbf{m}) \propto \exp\left\{-\frac{1}{2}(\mathbf{C}^{-1}(\mathbf{m}-\mathbf{m}_{post})|\mathbf{m}-\mathbf{m}_{post})\right\}$

The posterior *pdf* of $\mathbf{m}$ is Gaussian with expectation $\mathbf{m}_{post}$ and covariance $\mathbf{C} = (\mathbf{C}_m^{-1} + \mathbf{G}^*\mathbf{C}_d^{-1}\mathbf{G})^{-1}$

Gaussian Linear Case: Expressions of posterior Expectation and Covariance

$$\mathbf{m}_{post} = \mathbf{C}(\mathbf{G}^*\mathbf{C}_d^{-1}\mathbf{d}_{obs} + \mathbf{C}_m^{-1}\mathbf{m}_{prior}), \quad \mathbf{C} = (\mathbf{C}_m^{-1} + \mathbf{G}^*\mathbf{C}_d^{-1}\mathbf{G})^{-1}$$

Making use of the two basic formulae :

- $\mathbf{C}_m\mathbf{G}^*(\mathbf{C}_d + \mathbf{G}\mathbf{C}_m\mathbf{G}^*)^{-1} = (\mathbf{C}_m^{-1} + \mathbf{G}^*\mathbf{C}_d^{-1}\mathbf{G})^{-1}\mathbf{G}^*\mathbf{C}_d^{-1}$
- $\mathbf{C}_m - \mathbf{C}_m\mathbf{G}^*(\mathbf{C}_d + \mathbf{G}\mathbf{C}_m\mathbf{G}^*)^{-1}\mathbf{G}\mathbf{C}_m = (\mathbf{C}_m^{-1} + \mathbf{G}^*\mathbf{C}_d^{-1}\mathbf{G})^{-1}$

we obtain : $\mathbf{m}_{post} =$

$$(\mathbf{C}_m^{-1} + \mathbf{G}^*\mathbf{C}_d^{-1}\mathbf{G})^{-1}\{\mathbf{G}^*\mathbf{C}_d^{-1}\mathbf{d}_{obs} + (\mathbf{C}_m^{-1} + \mathbf{G}^*\mathbf{C}_d^{-1}\mathbf{G} - \mathbf{G}^*\mathbf{C}_d^{-1}\mathbf{G})\mathbf{m}_{prior}\}$$

and :

- $$\begin{cases} \mathbf{m}_{post} = \mathbf{m}_{prior} + \mathbf{C}_m\mathbf{G}^*(\mathbf{C}_d + \mathbf{G}\mathbf{C}_m\mathbf{G}^*)^{-1}(\mathbf{d}_{obs} - \mathbf{G}\mathbf{m}_{prior}) : \\ \mathbf{C} = (\mathbf{C}_m^{-1} + \mathbf{G}^*\mathbf{C}_d^{-1}\mathbf{G})^{-1} = \mathbf{C}_m - \mathbf{C}_m\mathbf{G}^*(\mathbf{C}_d + \mathbf{G}\mathbf{C}_m\mathbf{G}^*)^{-1}\mathbf{G}\mathbf{C}_m \end{cases}$$

The latter equality indicates the reduction in variance of the *pdf* since :

$$\forall \mathbf{v} \in \mathbf{M},\ (\mathbf{C}\mathbf{v}|\mathbf{v}) = (\mathbf{C}_m\mathbf{v}|\mathbf{v}) - ((\mathbf{C}_d + \mathbf{G}\mathbf{C}_m\mathbf{G}^*)^{-1}\mathbf{G}\mathbf{C}_m\mathbf{v}|\mathbf{G}\mathbf{C}_m\mathbf{v}) \le (\mathbf{C}_m\mathbf{v}|\mathbf{v})$$

Estimation versus Stochastic Approach

- stochastic : a priori, $\mathbf{d}$ and $\mathbf{m}$ are Gaussian with expectations $\mathbf{d}_{obs}$ and $\mathbf{m}_{prior}$, and covariances $\mathbf{C}_d$ and $\mathbf{C}_m$. A posteriori, $\mathbf{m}$ is Gaussian with expectation $\mathbf{m}_{post}$, and covariance $\mathbf{C} = (\mathbf{C}_m^{-1} + \mathbf{G}^*\mathbf{C}_d^{-1}\mathbf{G})^{-1}$

- estimation : a priori, $\mathbf{d}_{obs}$ and $\mathbf{m}_{prior}$ are Gaussian estimators with expectations $\mathbf{d} = \mathbf{Gm}$ and $\mathbf{m}$ (unbiased), and covariances $\mathbf{C}_d$, $\mathbf{C}_m$. A posteriori, we search for the unbiased estimator $\mathbf{m}_{post}$ which depends linearly on the prior estimators : $\mathbf{m}_{post} = \mathbf{Lm}_{prior} + \mathbf{Kd}_{obs}$ and makes minimum $\mathcal{E}\|\mathbf{m}_{post} - \mathbf{m}\|^2 = \text{tr}\ \{\mathbf{Cov}(\mathbf{m}_{post})\}$ (BLUE)

unbiased $\Longrightarrow \mathbf{m} = \mathbf{Lm} + \mathbf{KGm} \Longrightarrow \mathbf{L} = \mathbf{I}_d - \mathbf{KG}$

$$\mathbf{Cov}(\mathbf{m}_{post}) = (\mathbf{I}_d - \mathbf{KG})\mathbf{C}_m(\mathbf{I}_d - \mathbf{KG})^* + \mathbf{KC}_d\mathbf{K}^*$$

$$\text{tr}\ \{\mathbf{Cov}(\mathbf{m}_{post})\}\ \text{minimum} \Longrightarrow \begin{cases} \mathbf{K} = \mathbf{C}_m\mathbf{G}^*(\mathbf{C}_d + \mathbf{GC}_m\mathbf{G}^*)^{-1} \\ \mathbf{Cov}(\mathbf{m}_{post}) = \mathbf{C} \end{cases} \quad \text{as for}$$

the expectation and covariance in the stochastic approach

Functional Regularization

- $dim\,\mathbb{D} = n, \quad \mathbb{M} = \mathbf{L}^2(\mathbf{V}), \quad \mathbf{V} \subseteq \mathbb{R}^d, \quad d = 2$ or 3
- We suppose, for the sake of simplicity, that $\mathbb{M}$ corresponds to a single physical parameter $\mathrm{m}(\mathbf{r})$ distributed over $\mathbf{V}$
- data can be GPS, or InSAR displacements that we want to invert for the total displacement field over a source domain, as a fault system, or overpressures over a dike, by using an elastic propagator. They can also be travel time, attenuation or opacity over rays in a given domain.
- $\mathbf{G}$ bounded $\Longrightarrow \quad \forall i \in \{1, ..., n\} \quad \mathrm{d}^i = (\mathbf{Gm})^i = \int_V \mathrm{h}^i(\mathbf{r})\mathrm{m}(\mathbf{r})\ \mathrm{d}V$ the n functions $\mathrm{h}^i \in \mathbf{L}^2(V)$ are kernels related to data
- $\mathbf{G}^*\mathbf{d} = \sum_{i=1}^n \mathrm{d}^i\mathrm{h}^i$

$$(\mathbf{Gm}|\mathbf{d}) = \sum_{i=1}^{n} \mathrm{d}_i \int_V \mathrm{h}^i(\mathbf{r})\mathrm{m}(\mathbf{r})\ \mathrm{d}V = (\mathbf{m}|\sum_{i=1}^{n} \mathrm{d}^i\mathrm{h}^i)_{L^2}$$

Gaussian Random Functions, Covariance Functions

- A random function is a set of random variables m($\mathbf{r}$) depending on the position $\mathbf{r}$ within a domain $\mathbf{V} \subseteq \mathbf{R}^d$. The random function is Gaussian if for any integer n and any positions $\mathbf{r}_1$, $\mathbf{r}_2$,..., $\mathbf{r}_n$, the joint *pdf* of the variables m($\mathbf{r}_1$),m($\mathbf{r}_2$),...,m($\mathbf{r}_n$) is Gaussian.

- the covariance function C($\mathbf{r}$,$\mathbf{r}$') is the covariance of m($\mathbf{r}$) and m($\mathbf{r}$') when $\mathbf{r}$ and $\mathbf{r}$' vary within $\mathbf{V}$.

- A covariance function is symmetric with respect to ($\mathbf{r}$,$\mathbf{r}$'), and is characterized by the fact it is a positive definite function :

$$\forall n \in \mathbf{N},\ \forall(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_n) \in \mathbf{V}^n,\ \forall(v_1, v_2, ..., v_n) \in \mathbf{R}^n\ \sum_{i,j=1}^{n} C(\mathbf{r}_i, \mathbf{r}_j) v_i v_j \geq 0$$

- if C is a covariance function, C($\mathbf{r}$,$\mathbf{r}$')f($\mathbf{r}$)f($\mathbf{r}$') is also a covariance function, as well as the restriction of C to any subdomain of $\mathbf{V}$

- we can only considered correlation function ranging in (-1,1) over $\mathbf{R}^d$

Characterization of Covariance functions and Covariance Operators

- Covariance Function : $\begin{cases} C(\mathbf{r},\mathbf{r}') = \sigma(\mathbf{r})\sigma(\mathbf{r}')\,\phi(\mathbf{r}-\mathbf{r}'),\ \phi(0)=1 \\ |\phi| \leq 1,\ \phi \in \mathbf{L}^1(\mathbb{R}^d) \text{ even and continuous} \end{cases}$

- $\mathbf{C}\ :\ \forall f \in \mathbf{L}^2(\mathbf{R}^d),\ f \longrightarrow \mathbf{C}(f)(\mathbf{r}) = \int_{\mathbf{R}^d} \phi(\mathbf{r}-\mathbf{r}')f(\mathbf{r}')\ \mathrm{d}V(\mathbf{r}') \in \mathbf{L}^2(\mathbb{R}^d)$ is bounded in $\mathbf{L}^2(\mathbb{R}^d)$

- ϕ definite positive $\Longleftrightarrow \mathbf{C} \geq 0$

$$\forall f \in \mathbf{L}^2(\mathbb{R}^q) \quad (\mathbf{C}f|f)_{\mathbf{L}^2} = \int_{\mathbf{R}^d\times\mathbf{R}^d} \phi(\mathbf{r}-\mathbf{r}')f(\mathbf{r})f(\mathbf{r}')\ \mathrm{d}V(\mathbf{r})\ \mathrm{d}V(\mathbf{r}') \geq 0$$

- ϕ definite positive $\Longleftrightarrow \mathcal{F}(\phi) \geq 0$ (Bochner, Khintchine)

$$\mathbf{C}(f) = \phi * f, \quad (\mathbf{C}f|f)_{\mathbf{L}^2} = (\phi * f|f)_{\mathbf{L}^2} = \mathcal{F}(\phi)\mathcal{F}(f)\overline{\mathcal{F}(f)} = \mathcal{F}(\phi)|\mathcal{F}(f)|^2$$

Exemple of Homogeneous Covariance Functions

$$\mathrm{C}(\mathbf{r},\mathbf{r}') = \phi(\tfrac{\mathbf{r}-\mathbf{r}'}{\xi}) \qquad \xi : \text{ correlation length}$$

- dimension $d = 1$

$$\phi(r) = e^{-|r|}, \qquad \hat{\phi}(k) \propto \tfrac{2}{1+k^2}$$

$$\phi(r) = (1+|r|)e^{-|r|}, \qquad \hat{\phi}(k) \propto \tfrac{4}{(1+k^2)^2}$$

$$\phi(r) = 2^{1-\nu}/\Gamma(\nu)\, r^{\nu}\, \mathrm{K}_{\nu}(r), \qquad \hat{\phi}(k) \propto \tfrac{2}{(1+k^2)^{\nu+1/2}}$$

$$\phi(r) = (cos(r/\sqrt{2}) + sin(|r|/\sqrt{2})e^{-|r|/\sqrt{2}}, \qquad \hat{\phi}(k) \propto \tfrac{2\sqrt{2}}{(1+k^4)}$$

$$\phi(r) = e^{-r^2}, \qquad \hat{\phi}(k) \propto \sqrt{\pi}e^{-k^2/4}$$

$$\phi(r) = \tfrac{1}{ch(r)}, \qquad \hat{\phi}(k) \propto \tfrac{1/2}{ch(k\pi/2)}$$

- $d \geq 1$: $\quad \phi(\mathbf{r}) = \prod_{i=1}^{d} \phi_i(r_i)$

Homogeneous Isotropic Covariance Functions

$$\mathrm{C}(\mathbf{r},\mathbf{r}') = \phi(\tfrac{\mathbf{r}-\mathbf{r}'}{\xi}) = \psi(\|\tfrac{\mathbf{r}-\mathbf{r}'}{\xi}\|) \quad \xi : \text{correlation length}$$

- $d \leq 3$: $\phi(\mathbf{r}) = \psi(\|\mathbf{r}\|_d)$: ψ continuous, even, $r^2\psi \in \mathbf{L}^1(\mathbb{R})$, $\mathcal{F}(\psi)(\mathrm{k})$ decreasing for $\mathrm{k} \geq 0$

- $d = \infty$ (for any d): $\psi(r) = \int_0^\infty exp(-sr^2)\mu(\,\mathrm{d}s)$ where μ is a finite positive measure (Schoenberg, 1938)

- Can be generalized to $\|(\mathbf{r}\|^2 = (\mathbf{\Sigma r}|\mathbf{r})$

A posteriori Expectation

- formula of the finite dimensional case generalize into :

$$\mathrm{m}_{post} - \mathrm{m}_{prior} = \mathbf{G}^{\bullet}(\mathbf{C}_d + \mathbf{G}\mathbf{G}^{\bullet})^{-1}(\mathbf{d}_{obs} - \mathbf{G}\mathbf{m}_{prior})$$

where $\mathbf{G}^{\bullet}(= \mathbf{C}_m\mathbf{G}^*)$ denotes the adjoint of G : $\mathbf{R}(\mathbf{C}_m^{1/2}) \rightarrow \mathbb{D}$

and $\mathbf{R}(\mathbf{C}_m^{1/2})$ is a RKHS when endowed with $(\mathbf{C}_m^{-1/2}.|\mathbf{C}_m^{-1/2}.)$

- it yields :

$$\mathrm{m}_{post}(\mathbf{r}) - \mathrm{m}_{prior}(\mathbf{r}) = \sigma(\mathbf{r}) \sum_{i=1}^{n} v_i \int_{\mathbf{V}} \phi((\mathbf{r}-\mathbf{r}')/\xi)\, \sigma(\mathbf{r}')\mathrm{h}^i(\mathbf{r}')\ \mathrm{d}V'$$

where $(\mathrm{v}_i)_{i=1,n}$ verifies: $\sum_{j=1}^{n} \mathrm{M}^{ij} v_j = \mathrm{d}^i_{obs} - \int_{\mathbf{V}} \mathrm{h}^i(\mathbf{r})\, \mathrm{m}_{prior}(\mathbf{r})\ \mathrm{d}V$

with : $\quad \mathrm{M}^{ij} = \mathbf{C}_d^{ij} + \int_{\mathbf{V}\times\mathbf{V}} \sigma(\mathbf{r})\mathrm{h}^i(\mathbf{r})\, \phi((\mathbf{r}-\mathbf{r}')/\xi)\, \sigma(\mathbf{r}')\mathrm{h}^j(\mathbf{r}')\ \mathrm{d}V\ \mathrm{d}V'$

Resolution Analysis

$$(\mathrm{m}_{post}-\mathrm{m}_{prior})(\mathbf{r}) = \underbrace{\int_{\mathbf{V}} \mathrm{R}(\mathbf{r},\mathbf{r}')(\mathrm{m}-\mathrm{m}_{prior})(\mathbf{r}')\mathrm{d}V'}_{\mathbf{R}=\mathbf{G}\mathbf{C}_m\mathbf{M}^{-1}\mathbf{G}=\mathbf{K}\mathbf{G}} - \overbrace{\sigma(\mathbf{r})\sum_{i=1}^{n}\mathrm{s}^i(\mathbf{r})(\mathrm{d}^i-\mathrm{d}^i_{obs})}^{noise:\mathbf{K}(\mathbf{d}-\mathbf{d_{obs}})}$$

$$\text{with :}\quad \mathrm{R}(\mathbf{r},\mathbf{r}') = \sigma(\mathbf{r})\sum_{i=1}^{n}\mathrm{s}^i(\mathbf{r})\mathrm{h}^i(\mathbf{r}')$$

$\mathrm{R}(\mathbf{r},\mathbf{r}')$ centred near $\mathbf{r}=\mathbf{r}'$, with few negative lobes. Thinking of it as density :

- width : $w(\mathbf{r}) = \left(\frac{\int_{\mathbf{V}}\mathrm{R}(\mathbf{r},\mathbf{r}')\|\mathbf{r}-\mathbf{r}'\|^2\ \mathrm{d}V'}{\int_{\mathbf{V}}\mathrm{R}(\mathbf{r},\mathbf{r}')\ \mathrm{d}V'}\right)^{1/2}$
- averaging index : $I(\mathbf{r}) = \int_{\mathbf{V}}\mathrm{R}(\mathbf{r},\mathbf{r}')\ \mathrm{d}V' = \sigma(\mathbf{r})\sum_{i=1}^{n}\mathrm{s}^i(\mathbf{r})\int_{\mathbf{V}}\mathrm{h}^i(\mathbf{r}')\ \mathrm{d}V'$
  low index $\Longrightarrow$ poor inference

A priori Information and Regularization

- $\mathbf{C_0}$: physical covariance operator $\begin{cases} \xi_0 \text{ : reference physical length} \\ \sigma_{phys}(\mathbf{r}) \end{cases}$

How to specify $\mathbf{C}_m$?

- $1 : (\mathrm{m}_{post} - \mathrm{m}_{prior})(\mathbf{r}) = \int_{\mathbf{V}} \underbrace{\sigma(\mathbf{r})\phi((\mathbf{r}-\mathbf{r}')/\xi)\,\sigma(\mathbf{r}')}_{regularization} \overbrace{\left\{ \sum_{i=1}^{n} v_i \,\mathrm{h}^i(\mathbf{r}') \right\}}^{data\ \ influence} \mathrm{d}V'$

- $2 : \mathbf{C}_m(f)(\mathbf{r}) \underset{\xi \to 0}{\sim} \xi^d \sigma^2(\mathbf{r}) f(\mathbf{r})$

$\Longrightarrow$ put $\sigma(r) = w(r,\xi)\,\sigma_{phys}(r)$ with :

- $w(\mathbf{r}) \int_{\mathbf{V}} \phi((\mathbf{r}-\mathbf{r}')/\xi)\,w(\mathbf{r}')\ \mathrm{d}V' = \xi_0^d \int_{\mathbf{R}^d} \phi(\mathbf{r})\ \mathrm{d}V$

$w = 1$ for $\xi \sim \xi_0$ in order to recover $\sigma = \sigma_{phys}$

Regularization and scaling

$$\sigma(r) = w(r)\,\sigma_{phys}(r)$$

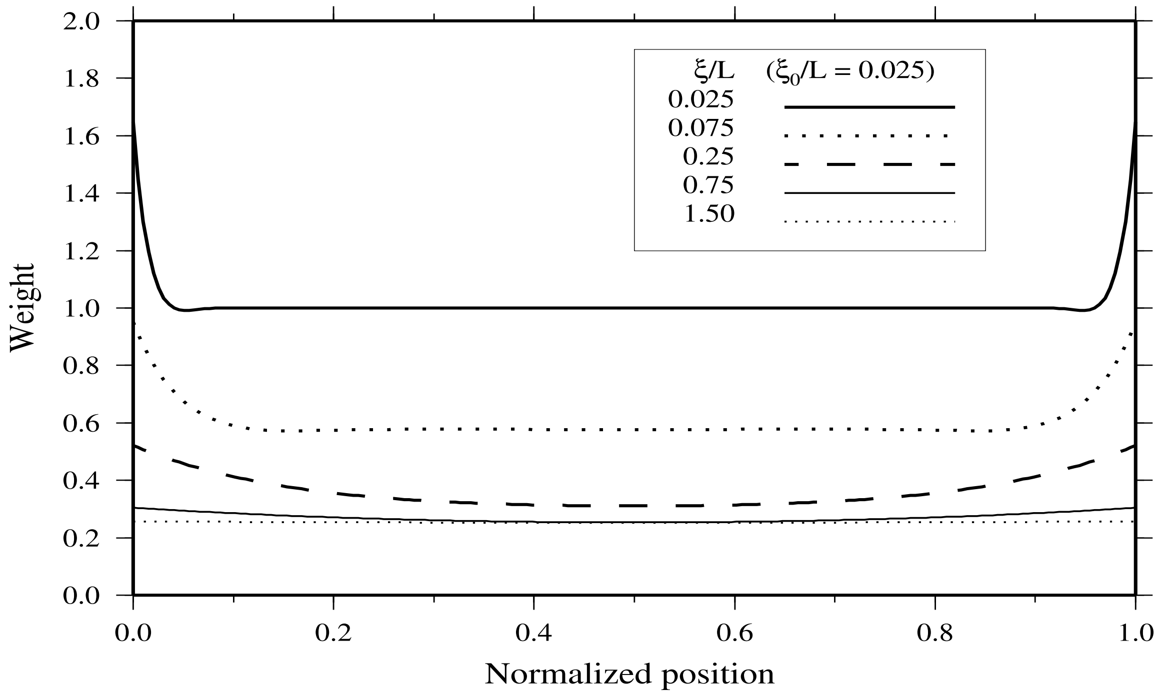
- implicit equation : $w(\mathbf{r}) = \xi_0^d \int_{\mathbb{R}^d} \phi(\mathbf{r})\ \mathrm{d}V / \int_{\mathbf{V}} \phi((\mathbf{r}-\mathbf{r}')/\xi)\, w(\mathbf{r}')\ \mathrm{d}V'$

$$\sigma^2(\mathbf{r}) \simeq \frac{\xi_0^d |\mathbf{V}| \int_{\mathbf{R}^d} \phi(\mathbf{r})\ \mathrm{d}V}{\int_{\mathbf{V}\times\mathbf{V}} \phi((\mathbf{r}-\mathbf{r}')/\xi)\ \mathrm{d}V\ \mathrm{d}V'}\,\sigma^2_{phys}(\mathbf{r}) \quad \text{exept at boundaries}$$

$$\sigma^2(\mathbf{r}) \underset{\xi\to 0}{\sim} \left(\frac{\xi_0}{\xi}\right)^d \sigma^2_{phys}(\mathbf{r}), \qquad \sigma^2(\mathbf{r}) \underset{\xi\to\infty}{\sim} \frac{\xi_0^d}{|\mathbf{V}|}\,\sigma^2_{phys}(\mathbf{r}) \int_{\mathbb{R}^d} \phi(\mathbf{r})\ \mathrm{d}V$$

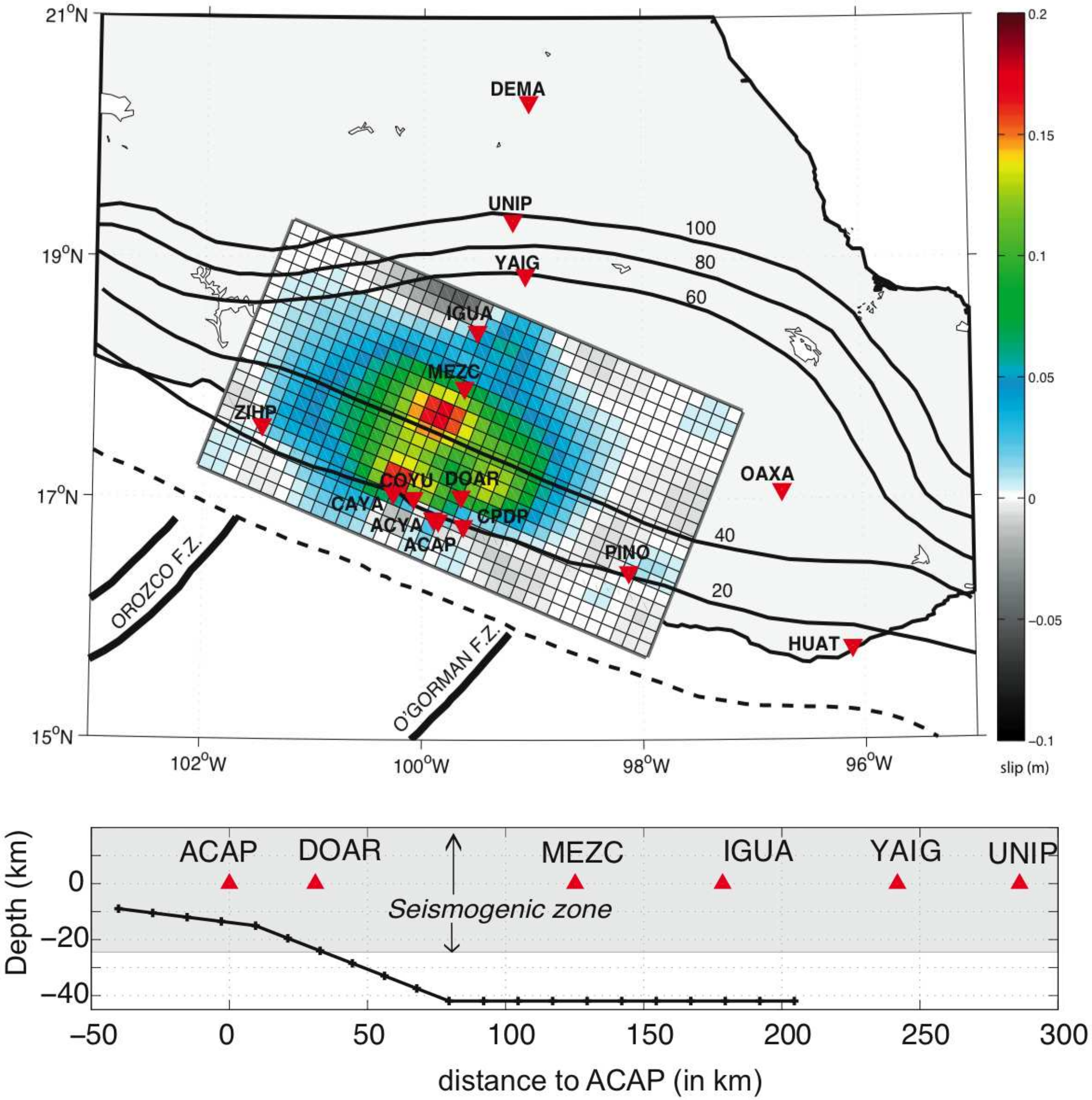
where : $|\mathbf{V}| = \int_{\mathbf{V}}\ \mathrm{d}V$

- Aside the a priori physical standard deviation, we make constant the variances of parameter averaged over $\mathbf{V}$, whatever ξ is



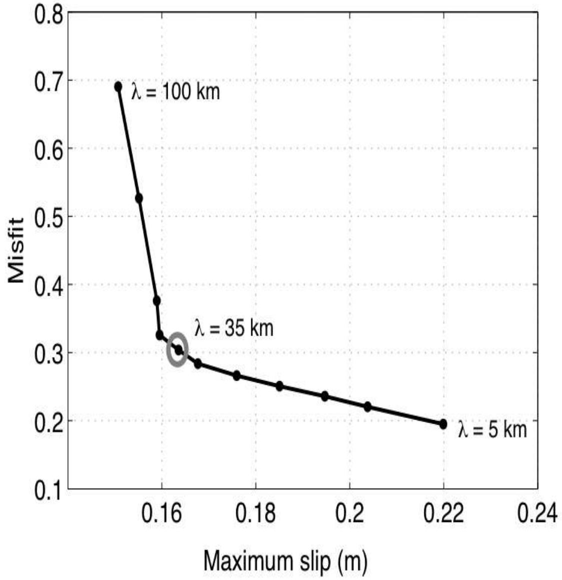
ξ/L (ξ0/L = 0.025)
0.025
0.075
0.25
0.75
1.50
Weight
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
Normalized position

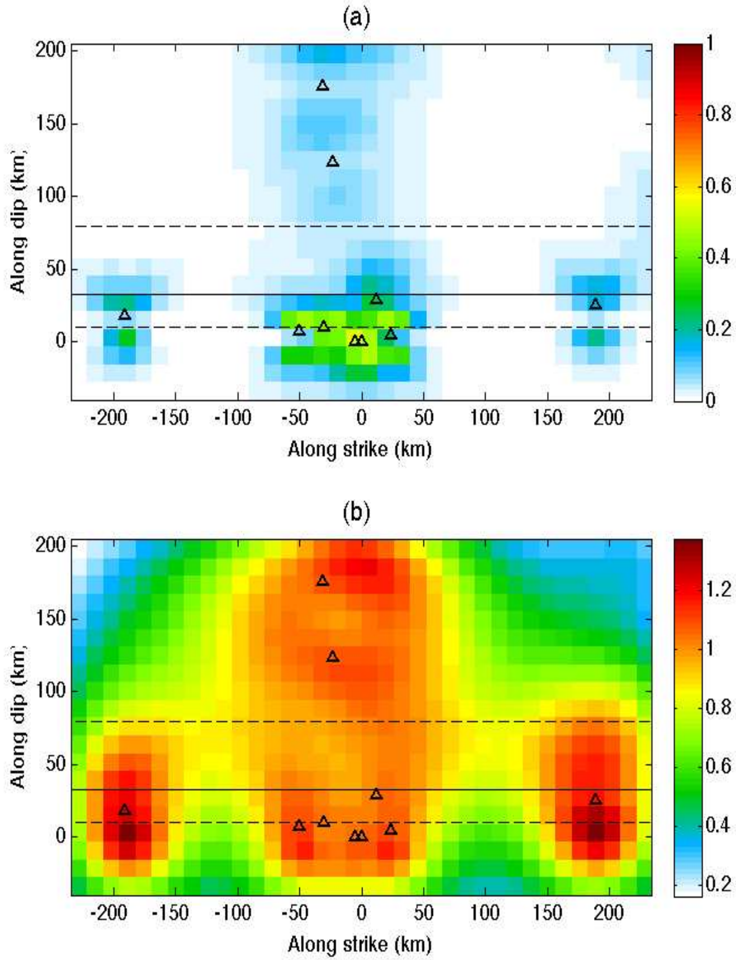
MNR 2012



The 2006 Guerrero slow slip event, Radiguet et al., Geophys. J. Int. 2011

How to choose ξ ? L-curve (left) and averaging index (right, b) !





Radiguet et al., Geophys. J. Int. 2011, see Vergely et al. A&A. 2010 for an example in astrophysics

Exemple of Linear Tomography in 2-d

$$\mathrm{d}^i = \int_{S_i} \mathrm{n}(\mathbf{r})\ \mathrm{d}s(\mathbf{r}) \quad i = 1, ..., n = 107$$

$$\mathrm{n}_{post}(\mathbf{r}) - \mathrm{n}_{prior}(\mathbf{r}) = \sigma(\mathbf{r}) \sum_{i=1}^{n} v_i \int_{\mathbf{S_i}} \phi((\mathbf{r} - \mathbf{r}')/\xi)\, \sigma(\mathbf{r}')\ \mathrm{d}s'$$

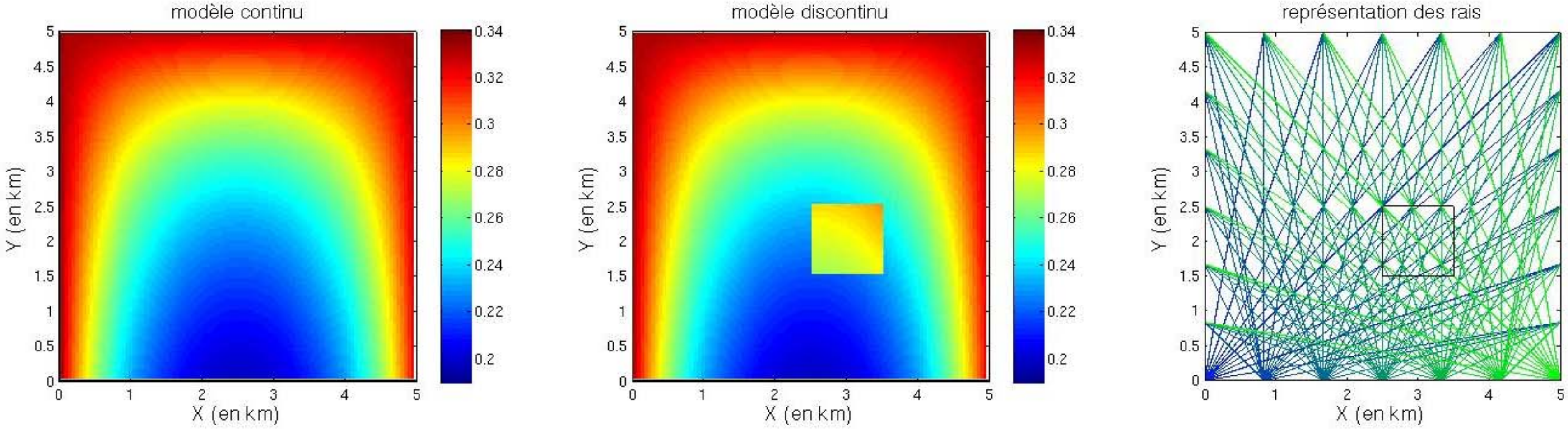
where $(\mathrm{v}_i)_{i=1,n}$ verifies:

$$\sum_{j=1}^{n} \mathrm{M}^{ij} v_j = \mathrm{d}^i_{obs} - \int_{\mathbf{S_i}} \mathrm{h}^i(\mathbf{r})\, \mathrm{n}_{prior}(\mathbf{r})\ \mathrm{d}s$$

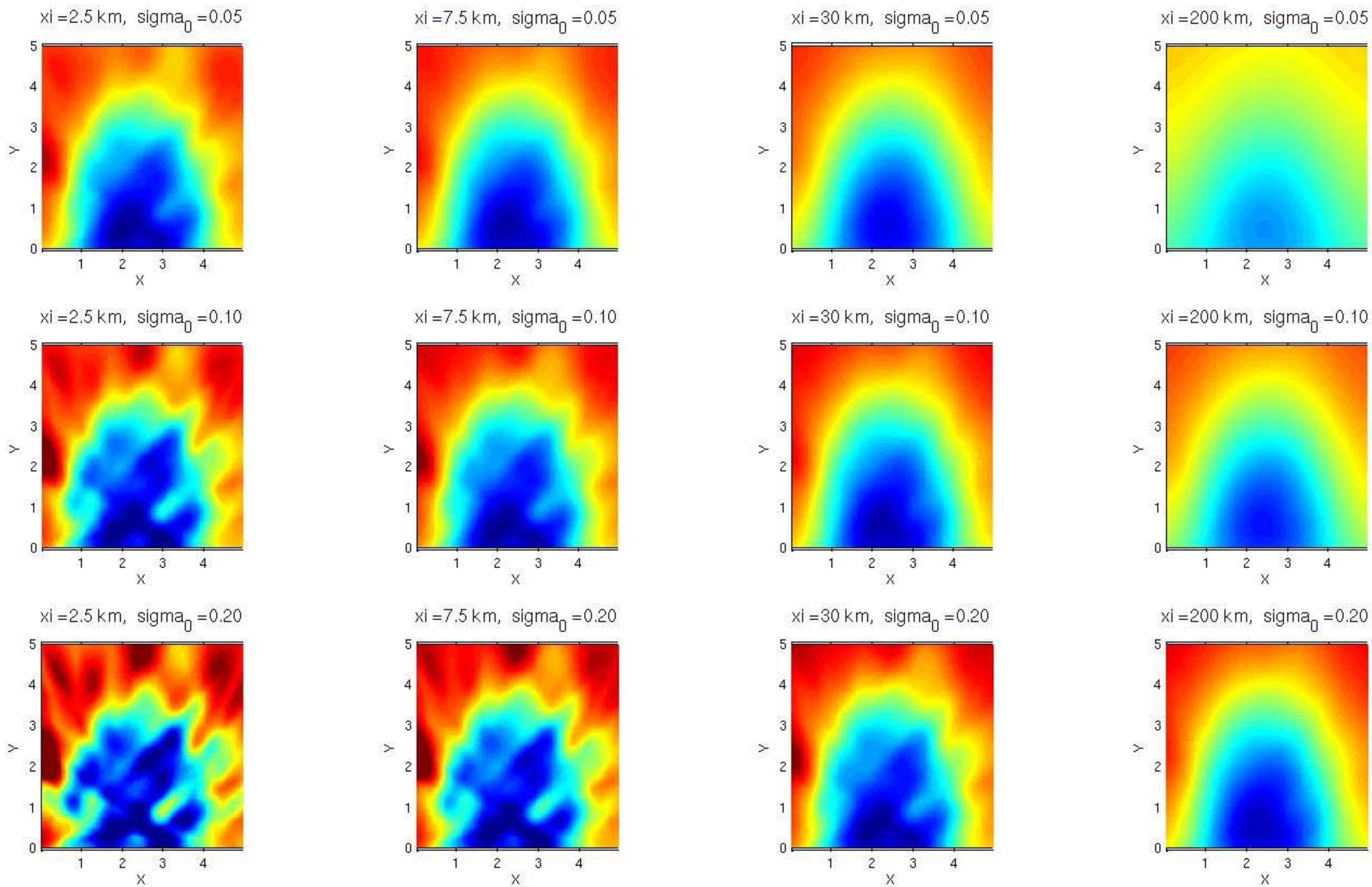
$$\text{with :} \quad \mathrm{M}^{ij} = \mathbf{C}_d^{ij} + \int_{\mathbf{S_i} \times \mathbf{S_j}} \sigma(\mathbf{r})\, \phi((\mathbf{r} - \mathbf{r}')/\xi)\, \sigma(\mathbf{r}')\ \mathrm{d}s\ \mathrm{d}s'$$

$$\text{and :} \quad \phi(\mathbf{r}) = exp(-\|\mathbf{r}\|)\ , \quad \sigma(\mathbf{r}) = \left(\frac{\xi_0}{\xi}\right)^{d/2} \sigma_0$$

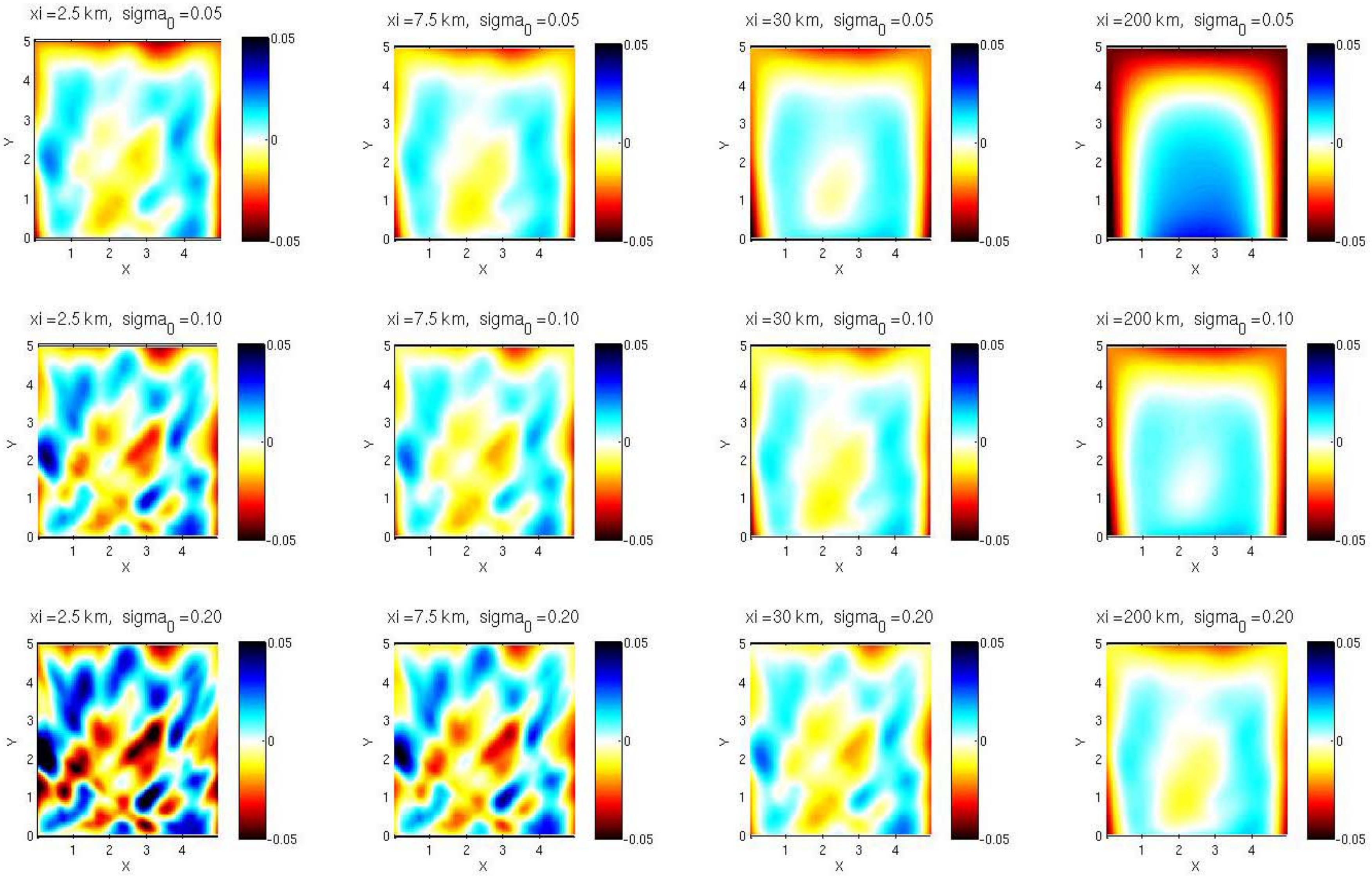
7 sources, 17 receptors, 107 rays
σd = 0.01s, ξ0 = 50 m



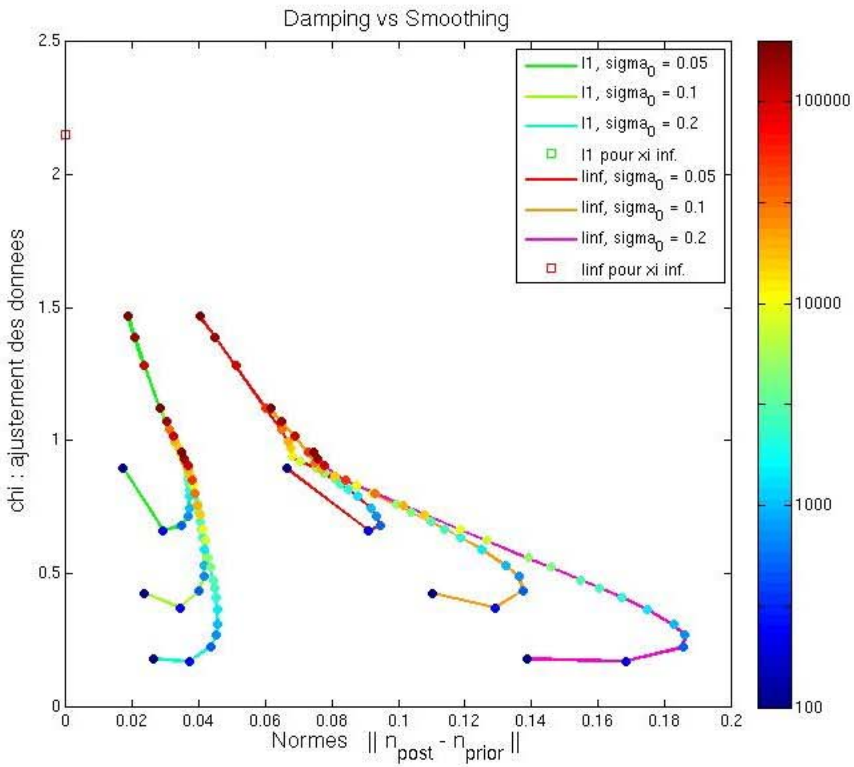
Damping versus Smoothing

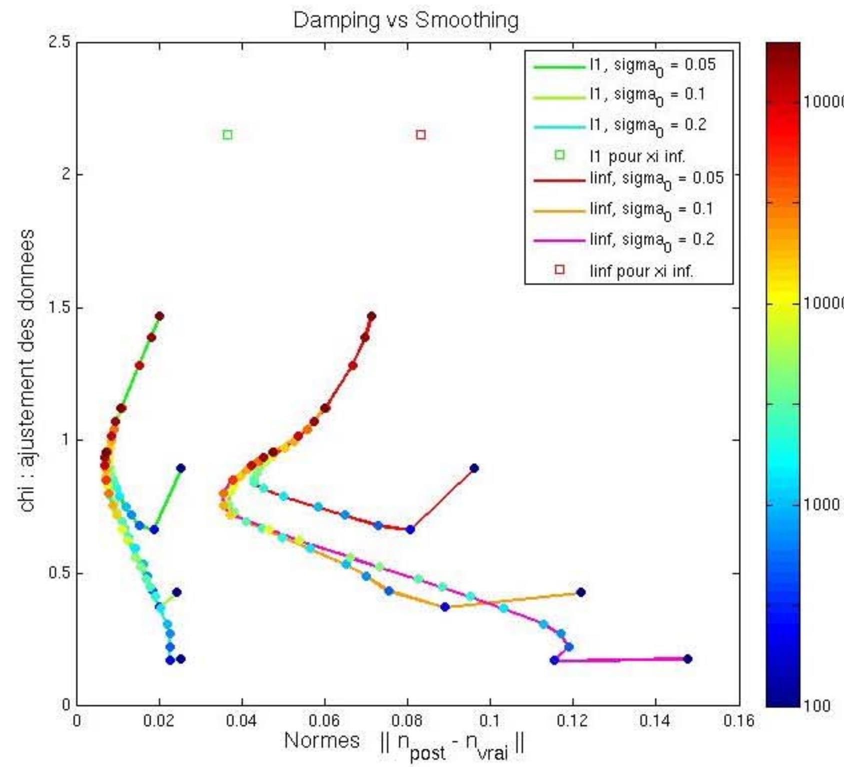


Damping versus Smoothing Comparison to true model

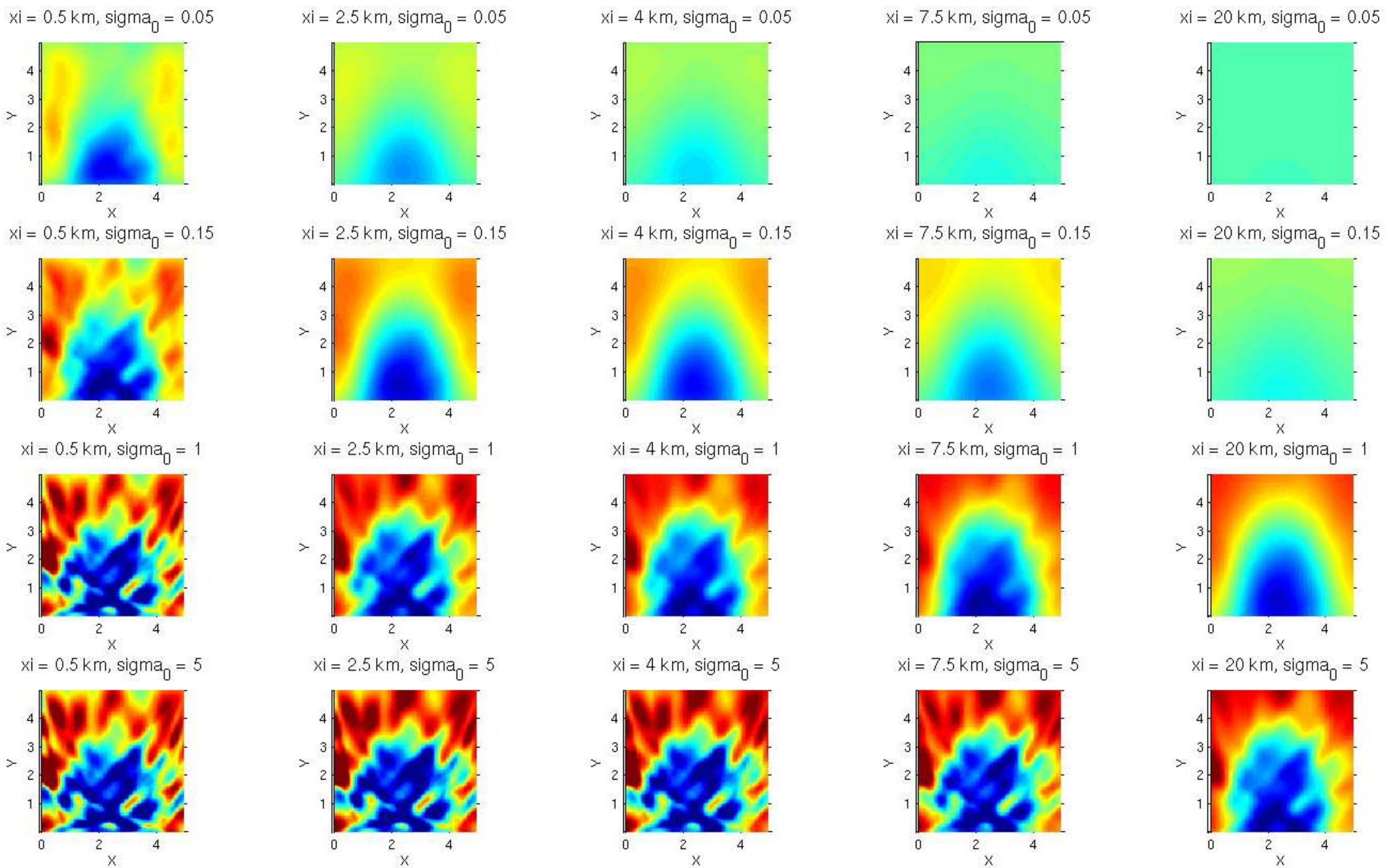


‘L-Curve’ and true model

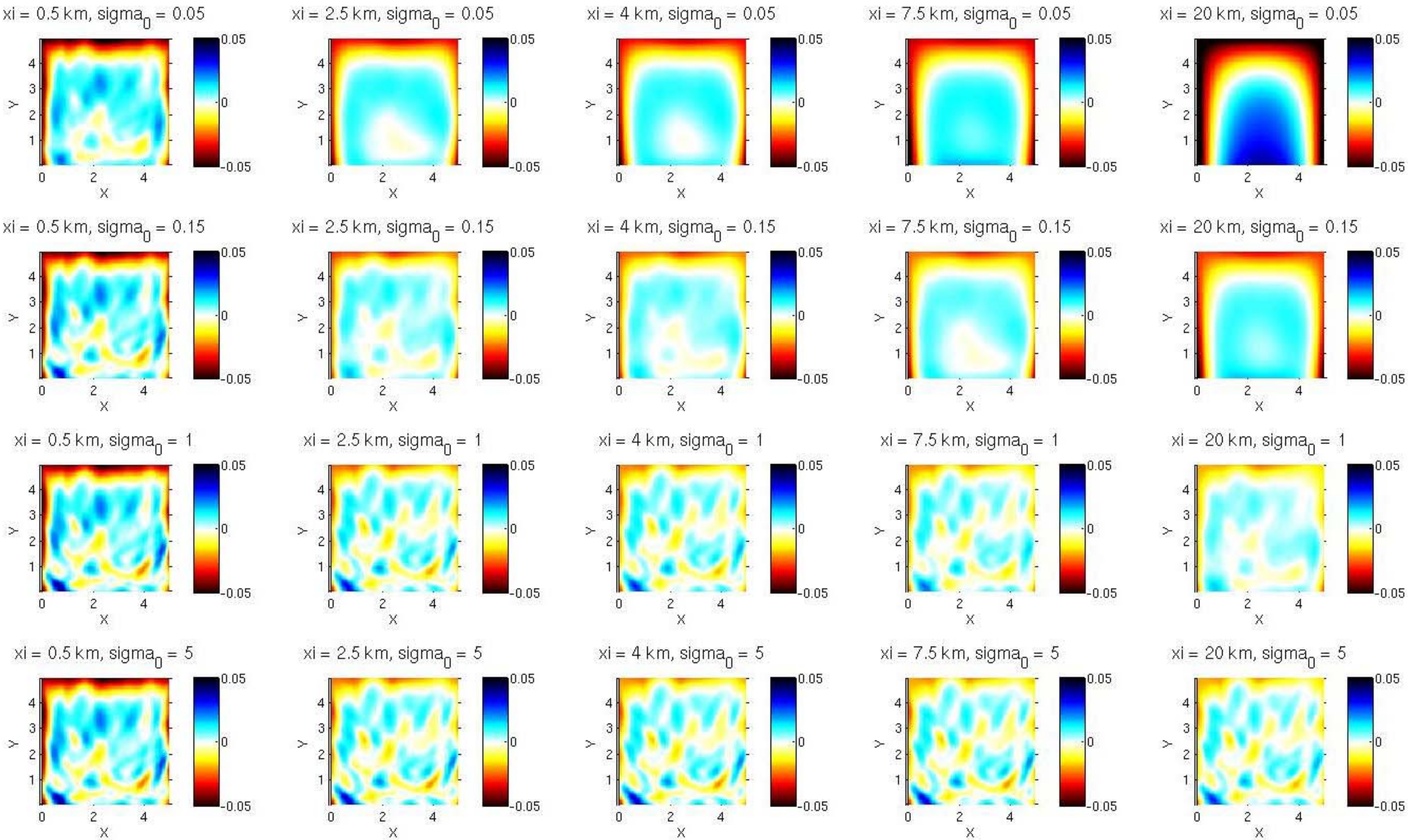




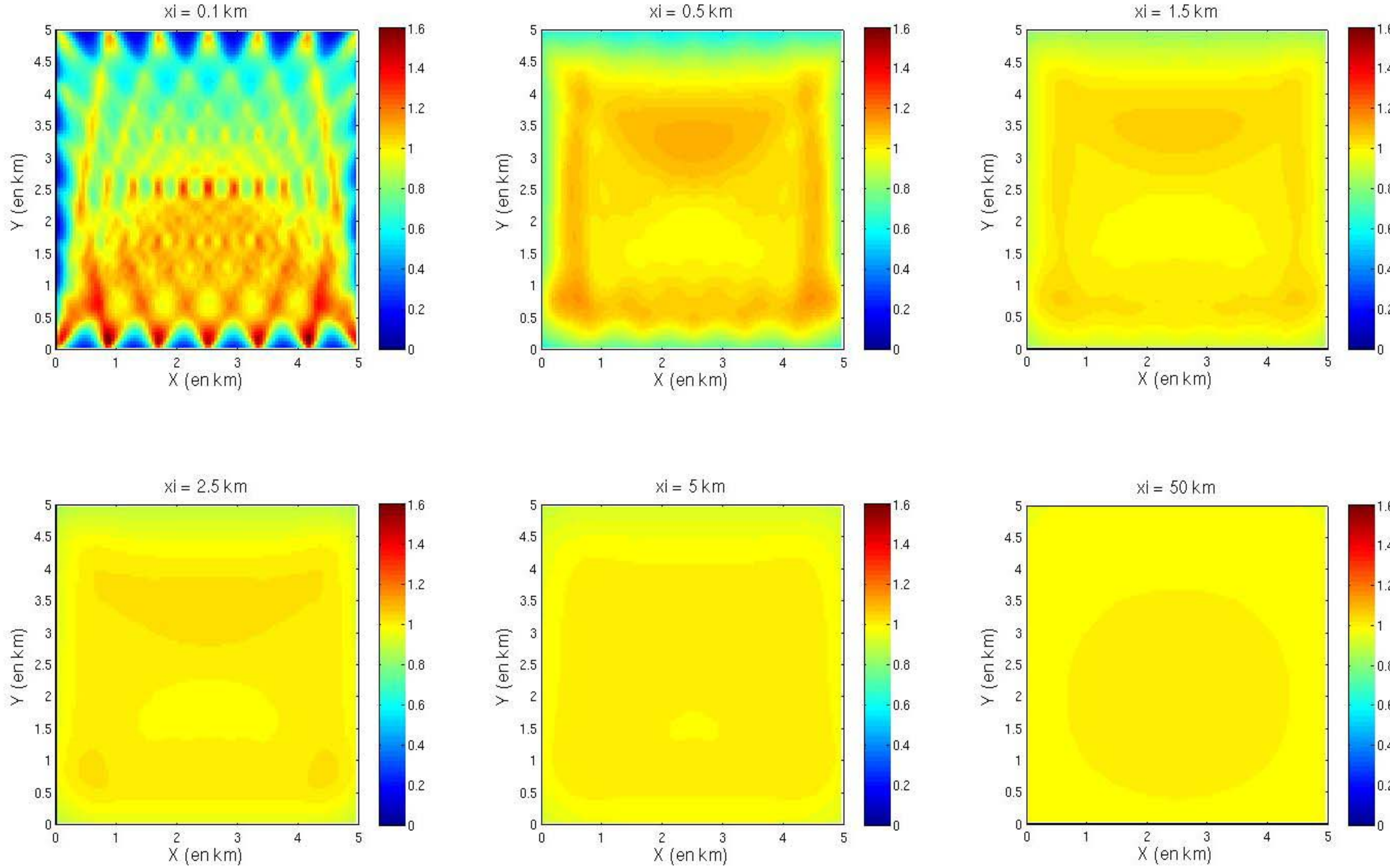
Damping versus Smoothing d=1



d=1


xi = 0.5 km, sigma0 = 0.05
xi = 2.5 km, sigma0 = 0.05
xi = 4 km, sigma0 = 0.05
xi = 7.5 km, sigma0 = 0.05
xi = 20 km, sigma0 = 0.05
xi = 0.5 km, sigma0 = 0.15
xi = 2.5 km, sigma0 = 0.15
xi = 4 km, sigma0 = 0.15
xi = 7.5 km, sigma0 = 0.15
xi = 20 km, sigma0 = 0.15
xi = 0.5 km, sigma0 = 1
xi = 2.5 km, sigma0 = 1
xi = 4 km, sigma0 = 1
xi = 7.5 km, sigma0 = 1
xi = 20 km, sigma0 = 1
xi = 0.5 km, sigma0 = 5
xi = 2.5 km, sigma0 = 5
xi = 4 km, sigma0 = 5
xi = 7.5 km, sigma0 = 5
xi = 20 km, sigma0 = 5

Averaging Index



Stochastic Approach provides efficient tools to regularize under-constrained tomographic problems

Thank You