Interpolating between Gauge and String Theory

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Spectral problem and AdS/CFT correspondence

Spectral problem

- Starting point: $\mathcal{N} = 4$ Super-Yang-Mills theory is a conformal field theory (CFT)
- It depends on two dimensionless parameters: the 't Hooft coupling $\lambda \equiv g_{YM}^2 N_c$ and the number of colors N_c
- Important observables: spectrum of scaling dimensions Δ of (local) conformal operators $\mathcal{O}$

AdS/CFT correspondence

(Planar) $\mathcal{N} = 4$ SYM theory is equivalent to (free) type IIB superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$ background

$$\text{string tension} \equiv 2g = \sqrt{\lambda}/2\pi \qquad \text{string coupling} \sim 1/N_c$$

Dictionnary

Spectrum of (planar) scaling dimensions = spectrum of energies of (free) string

Provides us with a geometrically picture/understanding of the gauge dynamics

Spectral problem and integrability

Main difficulty: How to confront the gauge and string theory?

- Gauge theory is tractable at weak coupling: $\lambda \ll 1$
- String theory is tractable at strong coupling: $\lambda \gg 1$

In most cases, to test the correspondence we need control on the weak/strong coupling interpolation $\rightarrow$ need non-perturbative methods

Important recent progress: Discovery of integrable structures (in the planar limit)

[Minahan,Zarembo'02],[Beisert,Staudacher'03'05]
[Lipatov'98],[Braun,Derkachov,Korchemsky,Manashov'98],[Belitsky'99]
[Bena,Polchinski,Roiban'03],[Kazakov,Marshakov,Minahan,Zarembo'04]
[Gromov,Kazakov,Vieira'09],[Gromov,Kazakov,Kozak,Vieira'09],
[Bombardelli,Fioravanti,Tateo'09],[Arutyunov,Frolov'09]

$\rightarrow$ Complete solution to spectral problem in the planar limit

Motivations:

- Solving the four-dimensional gauge theory (at least in the planar limit)
- Quantizing the string theory on the curved background
- Testing the AdS/CFT correspondence

Probing the correspondence

Probe: consider (local) operators in the so-called $\mathfrak{sl}(2)$ sector

$$\mathcal{O}(0) = \mathrm{tr}\, D^S Z^J(0) + \text{mixing}$$

with

- Z a complex scalar field in the adjoint representation of the gauge group
- $D \equiv n^\mu D_\mu$ a light-cone covariant derivative $n^2 = 0$

They carry spin S and twist (R-charge) J

Spectrum of scaling dimensions

$$\Delta \equiv \Delta_{S,J}(\lambda)$$

from Bethe ansatz equations (for any coupling λ)

Outline

Large spin limit

- The string theory point of view: excitations around long rotating GKP string
- The gauge theory perspective: excitations around large spin twist-two operator
- Interpolation via Bethe ansatz equations: all-loop dispersion relations

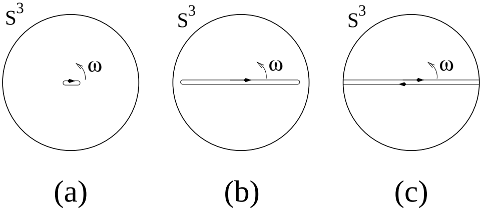
Small spin limit

- Exact formula for the slope of the (minimal) twist-J scaling dimension

Large spin limit

The GKP string

Folded string rotating in $AdS_3 \subset AdS_5$ with spin S [Gubser,Klebanov,Polyakov'02]



- (a) Short string : $S \sim 0 \longrightarrow$ length $\sim S^{1/2} \sim 0$
- (c) Long string : $S \sim \infty \longrightarrow$ length $= 2\log S \gg 1$ + worldsheet homogeneous

Energy of long GKP string : (string tension) $2g = \sqrt{\lambda}/2\pi \gg 1$

$$E \equiv \Delta - S = 4g \log S + O(\log^0 S) \longrightarrow 2\Gamma_{\text{cusp}}(g) \log S + \ldots$$

Spectrum of excitations from string theory I

Quadratic fluctuations (relativistic spectrum) [Frolov,Tseytlin'02]

- 5 massless bosons for $\perp$ fluctuations in S^5
- 2 bosons with mass $\sqrt{2}$ for $\perp$ fluctuations in AdS_5/AdS_3
- 1 boson with mass 2 for $\perp$ fluctuation in AdS_3
- 8 fermions with mass 1

Symmetries [Alday,Maldacena'07]

- All supersymmetries are broken (complicated background)
- SO(2) transverse symmetry
- SO(6) symmetry broken down spontaneously to SO(5) at the perturbative level

Spectrum of excitations from string theory II

Higher-loop correction?... recently: one-loop correction [Giombi,Ricci,Roiban,Tseytlin'10]

$$E(p) = \sqrt{p^2 + m(g)^2}\left[1 - c\,\frac{p^2}{g} + O(1/g^2)\right]$$

- Non relativistic correction controlled by single coefficient c
- Coefficient c depends on flavor of excitation... as correction to mass $m(g)$
- Spectrum undisturbed otherwise.... though presence of bound states [Zarembo,Zieme'11]

Non-perturbatively? [Alday,Maldacena'07]

- Low energy effective dynamics: 2D O(6) non-linear sigma model
- Restoration of SO(6) symmetry $\longrightarrow$ 6 scalars in O(6) vector multiplet with mass (dimensional transmutation)

$$m \sim \mathrm{e}^{-\pi g}$$

Gauge theory picture

Vacuum

Long GKP string = large spin twist-two operator $\sim \operatorname{tr} Z D^{S} Z$

Vacuum energy: $E_{\text{vacuum}} = \Delta - S = 2\Gamma_{\text{cusp}}(g) \log S + \dots$

Excitation

Insertion of operators in the background of covariant derivatives

One-particle state = length-three operator

$$\operatorname{tr} Z D^{k_1} \Phi D^{k_2} Z$$

with spin $S \sim k_1 + k_1 \gg 1$

Fundamental (lightest) excitations

Analogy with BMN operators

$$\longrightarrow \text{a fundamental excitation } \Phi \text{ has twist} = (\Delta - S)_{\text{tree-level}} = 1$$

Energy: $\Delta - S = E_{\text{vacuum}} + E_{\Phi}(p)$

Mass: $E_{\Phi}(p = 0) = \text{twist} + O(g^2)$

Spectrum of excitations

Flavors of twist-one excitations

- 6 scalars Z
- 8 fermions Ψ
- 2 field strength components $F = F_{+\perp} \sim \partial_+ A_\perp \longrightarrow$ gauge field excitation

Bound states...

... of gauge fields: embedded as length $= 1$, high-twist operators

$$D_\perp^{\ell-1} F_{+\perp} \qquad \text{twist} = \ell$$

with $\ell = 1, 2, 3, \ldots$

Spectrum of masses

Masses	Weak coupling	Strong coupling
Scalar	1	$\sim e^{-\pi g}$
Fermion	1	1
Gauge field	1	$\sqrt{2}$
?	2	2

$$\text{`?'} \quad = \quad D_{-}, F_{+-}, \text{ or two-fermion state ?}$$

Interpolation?

Spectral problem

Large spin operators with $J-2$ insertions:

$$\mathcal{O} = \mathrm{tr}\, Z D^{k_1} \Phi \ldots D^{k_{J-2}} \Phi D^{k_{J-1}} Z$$

Elementary excitations: (twist-1 partons)

$$\Phi \;=\; Z \;\; (\text{scalars}), \quad \Psi \;\; (\text{fermions})\,, \quad F_{+\perp} \;\; (\text{gauge fields})$$

Mixing problem $\rightarrow$ Spectrum of scaling dimensions at large spin $S \sim \sum_i k_i \gg 1$?

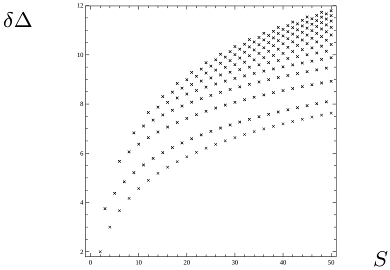
Solution:

$$E_{\text{eigenstate}} = \Delta - S = E_{\text{twist-two}} + \sum_{i \in \text{excitations}} E_i(p_i) + \ldots$$

Spectrum of high-spin operators

Illustration:

One-loop spectrum of anomalous dimensions of twist-3 operators $\mathcal{O} = \mathrm{tr}\, Z D^{k_1} Z D^{k_2} Z$



Goal: parameterize anomalous dimensions trajectories close to the minimal one...

$$\delta\Delta - \delta\Delta_{\text{twist-two}} = E(p) \qquad p \sim 1/\log S$$

... and extract the dispersion relation (here for a scalar Z)

Kinematics

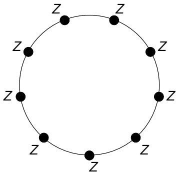
Operators

$$\mathcal{O}_{\{k_m\}} = \mathrm{tr}\, D^{k_1} Z ... D^{k_J} Z$$

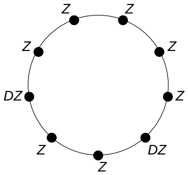
- $\mathrm{tr}\, Z...Z...Z \rightarrow$ vacuum state of the spin chain
- $\mathrm{tr}\, Z...DZ...Z \rightarrow$ one-particle state of the spin chain (magnon)

Quantum numbers

- Twist $J \rightarrow$ spin chain length
- Lorentz spin $S = k_1 + ... + k_J \rightarrow$ number of excitations (magnons) over the vacuum



Spin Chain (Ferromagnetic) Vacuum



Two−Magnon State

Tool : Integrability

Kinematics

Operators

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Dynamics

Callan-Symanzik equation

$$\mu \frac{\partial}{\partial \mu} \mathcal{O}_{\{k_m\}} = -\delta\mathbb{D} \cdot \mathcal{O}_{\{k_m\}}$$

- Dilatation operator $\delta\mathbb{D} \rightarrow$ Hamiltonian of the spin chain
- Spectrum of anomalous dimensions $\delta\Delta \rightarrow$ spectrum of energies of the spin chain

One-loop example

Mapping with $\mathfrak{sl}(2)$ integrable Heisenberg spin chains

[Lipatov'97],[Braun,Belitsky,Derkachov,Korchemsky,Manashov'98]
[Minahan,Zarembo'02],[Beisert,Staudacher'03]

Kinematics : spin-chain Hilbert space $\mathcal{H} = V_s^{\otimes J}$

- scalar : conformal spin $s = 1/2$
- fermion : conformal spin $s = 1$
- gauge field : conformal spin $s = 3/2$

Dynamics : $\delta\mathbb{D}$ = Hamiltonian of XXX_s $\mathfrak{sl}(2)$ Heisenberg spin chain

- System with J degrees of freedom... and J commuting conserved charges

  Liouville definition of a completely integrable system

- The complete family of conserved charges can be diagonalized simultaneously with $\delta\mathbb{D}$ by means of the algebraic Bethe ansatz

Bethe ansatz solution

Solution to mixing problem (here for scalar)

- Bethe ansatz equations

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^J = \prod_{j \neq k}^{S} \frac{u_k - u_j - i}{u_k - u_j + i}$$

- S magnons $\leftrightarrow$ S rapidities u_k

- One-loop scaling dimension

$$\Delta = J + S + 2g^2 \sum_{k=1}^{S} \frac{1}{u_k^2 + 1/4} + O(g^4)$$

Large spin limit

Continuum limit of the Bethe ansatz equations

[Korchemsky'95],[Belitsky,Gorsky,Korchemsky'06]
[Freyhult,Rej,Staudacher'07]

- Continuous distribution of Bethe roots described by density $\rho(u)$
- Bethe ansatz equations turn into integral equation for $\rho(u)$

$$2\pi\rho(u) = \frac{4J}{1+4u^2} - \sum_{l=1}^{J-2} \frac{2}{(u-\hat{u}_l)^2+1} + 2\int dv \, \frac{\rho(v)}{(u-v)^2+1}$$

- What are the $\hat{u}_l$'s?... they are the holes' rapidities

Particle/hole transformation : trading dynamics of magnons D on spin chain of Z's for dynamics of Z's through the background of D's

One hole $\leftrightarrow$ one Z insertion

Dispersion relation

Solve integral equation for given set of holes rapidities and compute energy

$\rightarrow$ vacuum energy + energy of excitation carrying hole rapidity

Vacuum energy: (twist-two scaling dimension) [Korchemsky'89],[Korchemsky,Marchesini'92]

$$E_{\text{vacuum}} = \Delta - S\big|_{\text{twist-two}} = 2\Gamma_{\text{cusp}}(g)\log S + O(\log^0 S)$$

with $\Gamma_{\text{cusp}}(g) = 4g^2 + O(g^4)$

Dispersion relation: (for a hole with rapidity u) [Korchemsky'95], [Belitsky,Gorsky,Korchemsky'06]

- Energy

$$E(u) = \Delta - S\big|_{\text{above vacuum}} = 1 + 2g^2\left(\psi(s+iu) + \psi(s-iu) - 2\psi(1)\right) + O(g^4)$$

- Momentum? Look at quantity conjugated to length '$2\log S$' in effective equations

$$p(u) = 2u + O(g^2)$$

Masses

Direct application to computation of the mass $= E(p=0)$

General formula:

$$m_s = 1 + 4g^2(\psi(s) - \psi(1)) + O(g^4)$$

Elementary (twist-1) excitations:

$$\begin{aligned} m_{\text{scalar}} &= 1 - 8g^2 \log 2 + O(g^4) && < 1 \\ m_{\text{fermion}} &= 1 - \mathbf{0} \cdot g^2 + O(g^4) && = 1 \\ m_{\text{gauge-field}} &= 1 + 8g^2(1 - \log 2) + O(g^4) && > 1 \end{aligned}$$

In qualitative agreement with smooth interpolation with string theory!

Higher-loop integrability in a nutshell

- BMN vacuum $= \mathrm{tr}\, Z^J$ with energy $\equiv \Delta - J = 0$
- Fundamental excitations: BMN mass $\equiv \Delta - J = 1$, e.g., insertion of a derivative D
- Symmetries: (centrally extended) $SU(2|2) \times SU(2|2)$ [Beisert'06]
- Factorizable scattering $\rightarrow$ Asymptotic Bethe Ansatz (ABA) equations [Staudacher'04],[Beisert,Staudacher'05]

$$\mathrm{e}^{-ip_k J} = \prod_{j \neq k}^{S} S(p_k, p_j)$$

with $S(p_k, p_j)$ the two-by-two scattering S-matrix

- Scaling dimension:

$$\Delta - J = \sum_{k=1}^{S} E(p_k)$$

- Dispersion relation: [Beisert,Dippel,Staudacher'05]

$$E(p) = \sqrt{1 + 16g^2 \sin^2\left(\frac{p}{2}\right)}$$

Asymptotic Bethe Ansatz equations for (planar) dilatation operator (all loops)

- Proposal for the $\mathfrak{sl}(2)$ sector [Beisert,Staudacher'05],[Beisert'05]

$$\left(\frac{x_k^+}{x_k^-}\right)^J = \prod_{j\neq k}^{S} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\,\frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+}\,\exp\left(2i\theta(u_k, u_j)\right)$$

with the deformed spectral parameter $u_k \pm i/2 = x_k^\pm + g^2/x_k^\pm$ [Beisert,Dippel,Staudacher'05]

and with the dressing phase $\theta(u_k, u_j)$ $\left(= O\left(g^6\right)\right)$ [Beisert,Eden,Staudacher'06] [Beisert,Hernández,López'06]

- All-loop 'asymptotic' scaling dimension

$$\Delta = J + S + 2g^2 \sum_{j=1}^{S}\left[\frac{i}{x_j^+} - \frac{i}{x_j^-}\right]$$

All-loop analysis

First step: Classify large-spin solutions of ABA equations [BB'10]

- fundamental excitations
- isotopic roots implementing symmetries
- bound states

Some help: existing results from spectroscopy of large-spin scaling dimensions [Beisert,Bianchi,Morales,Samtleben'04],[Freyhult,Rej,Zieme'09]

Second step: Derive linear integral equation (schematically)

$$\rho + \mathcal{K} \star \rho = \mathcal{I}$$

with

- $\rho =$ large-spin density of roots (the unknown)
- $\mathcal{K} =$ kernel (known)
- $\mathcal{I} =$ inhomogeneous term (known)

Generalization of the Beisert-Eden-Staudacher equation for the cusp anomalous dimension

Third and last step: Apply methodology developed for solving this type of problem and compute the energy (dispersion relation)

All-loop cusp anomalous dimension from BES equation

Weak coupling expansion $g \ll 1$ [Beisert,Eden,Staudacher'06]

$$\Gamma_{\text{cusp}}(g) = 4g^2 - \frac{4\pi^2}{3}g^4 + \frac{44\pi^4}{45}g^6 - \left(\frac{292\pi^6}{315} + 32\zeta_3^2\right) + O(g^{10})$$

Strong coupling expansion $g \gg 1$ [Kotikov,Lipatov'06],[Benna,Benvenuti,Klebanov,Scardicchio'07], [Alday,Arutyunov,Benna,Eden,Klebanov'07],[Kostov,Serban,Volin'07],[Beccaria,DeAngelis,Forini'07] [BB,Korchemsky,Kostanski'07],[Kostov,Serban,Volin'08],[BB,Korchemsky'08'09] [Casteill,Kristjansen'07],[Belitsky'07],[Gromov'08]

$$\Gamma_{\text{cusp}}(g) = 2g - \frac{3\log 2}{2\pi} - \frac{\mathrm{K}}{8\pi^2 g} + O(1/g^2)$$

In agreement with direct gauge/string theory calculations at weak/strong coupling [Kotikov,Lipatov,Onishchenko,Velizhanin'04], [Bern,Czakon,Dixon,Kosower,Smirnov'06],[Cachazo,Spradlin,Volovich'06] [Gubser,Klebanov,Polyakov'02],[Frolov,Tseytlin'02],[Roiban,Tseytlin'07]

Remark: At intermediate coupling we can analyze the interpolation numerically [Benna,Benvenuti,Klebanov,Scardicchio'06]

All-loop dispersion relation

Example for scalar [BB'10]

$$E(u) = 1 + \int_0^\infty \frac{dt}{t} \frac{\mathrm{e}^{t/2} \cos(ut) - J_0(2gt)}{\mathrm{e}^t - 1} \gamma(-2gt)$$

$$p(u) = 2u - \int_0^\infty \frac{dt}{t} \frac{\mathrm{e}^{t/2} \sin(ut)}{\mathrm{e}^t - 1} \gamma(2gt)$$

The function $\gamma(t)$ solves the BES equation, known explicitely at both weak/strong coupling

[Kotikov,Lipatov'06],[Benna,Benvenuti,Klebanov,Scardicchio'07], [Alday,Arutyunov,Benna,Eden,Klebanov'07],[Kostov,Serban,Volin'07],[Beccaria,DeAngelis,Forini'07] [BB,Korchemsky,Kostanski'07],[Kostov,Serban,Volin'08],[BB,Korchemsky'08'09]

Strong coupling regimes (scalar)

- Non-perturbative regime: $E \sim p \sim m$

$$E = \sqrt{p^2 + m^2}\left[1 - c(g)p^2 + O(m^4, m^2 p^2, p^4)\right]$$

Mass is exponentially small [Fioravanti,Grinza,Rossi'08],[BB,Korchemsky'08]]

$$m = k\, g^{1/4}\, \mathrm{e}^{-\pi g}\, (1 + O(1/g))$$

Expression for constant k agrees with string theory prediction [Alday,Maldacena'07],[BB,Korchemsky'08]]

- Perturbative regime: $E \sim p \sim 1$

$$E = p\left[1 - c\,\frac{p^2}{g} + O(1/g^2)\right]$$

- Other regimes: Near-flat space $E \sim p \sim g^{1/4}$ and Giant hole $E \sim p \sim g$

Comparison with string theory

Perturbative regime: $E \sim p \sim 1$

$$E = \sqrt{p^2 + m^2(g)}\left[1 - c\,\frac{p^2}{g} + O(1/g^2)\right]$$

Agreement with string theory computations for most of the excitations...

[Giombi,Ricci,Roiban,Tseytlin'10]

- gauge field: $m(g) = \sqrt{2} - 1/(4\sqrt{2}g) + O(1/g^2)$ $\qquad c = \frac{1}{8}$
- fermion: $m(g) = 1$ $\qquad c = \frac{1}{4}$

... except for scalar

- scalar: $m(g) = O(1/g^\infty)$ OK!... but

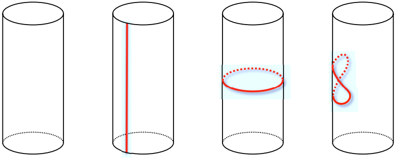
$$c_{\text{ABA}} = \frac{\Gamma(\frac{1}{4})^4}{8(12)^{5/4}\pi^2} \neq \frac{7}{24\pi} = c_{\text{string}}$$

Recent explanation was proposed in [Zarembo,Zieme'11]

Some applications I

Application to computation of subleading large-spin corrections to twist-two scaling dimensions [BB,Belitsky'11]

- Twist-two scaling dimension = vacuum energy for a string with length $R \sim 2\log S \gg 1$
- $\rightarrow$ Vacuum energy receives finite-size corrections due to exchange of virtual excitations



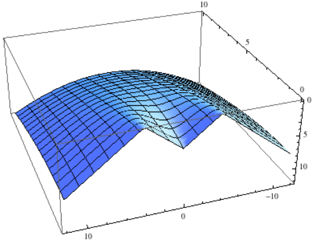
(a) (b) (c) (d)

The finite-size corrections (c) are qualitatively different from those (d) that contribute to the leading (bulk) energy $\Delta - S \sim 2\Gamma_{\text{cusp}}(g)\log S$

It leads to an interesting (but intricate) interpolation between gauge and string theory which can be analyzed by means of integrability

Some applications II

Application to computation of scattering amplitudes in the near-collinear limit
[Alday,Gaiotto,Maldacena,Sever,Vieira'10-11]



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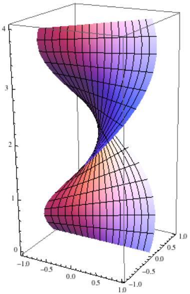


Bottom line: scattering amplitudes = light-like (cusped) Wilson loops, related to GKP string
[Alday,Maldacena'07],[Drummond,Korchemsky,Sokatchev'07],
[Drummond,Henn,Korchemsky,Sokatchev'07],[Brandhuber,Heslop,Travaglini'07]

Small spin limit

Small spin expansion

- Consider minimal scaling dimension Δ of operators

$$\mathcal{O} \sim \mathrm{tr}\, D^S Z^J + \text{mixing}$$

- Δ is defined for physical operators (integer spin)

$$\Delta \equiv \Delta_J(S)$$

as a function of spin S, twist (R-charge) J, and 't Hooft coupling λ

- Perform analytical continuation in the spin S and expand around $S = 0$ (BPS point)

$$\Delta = J + \alpha_J(\lambda) S + O(S^2)$$

- The slope $\alpha_J(\lambda)$ is a function of J and λ only, computable at weak and strong coupling

Interpolation?

Illustration

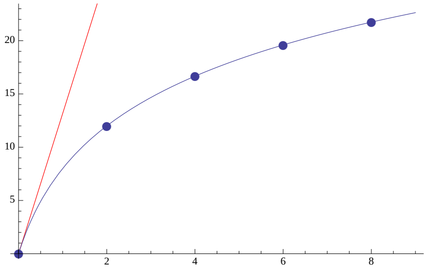
- Consider twist-two operator $(J = 2)$

$$\mathcal{O} = \mathrm{tr}\, D^S Z^2 + \text{mixing}$$

- Its scaling dimension is given up to one loop as

$$\Delta_{\text{twist-two}} = 2 + S + \frac{\lambda}{2\pi^2}(\psi(S+1) - \psi(1)) + O(\lambda^2)$$

with ψ the logarithmic derivative of Euler Gamma function



- Straigthforward expansion at small spin yields [Kotikov,Lipatov,Onishchenko,Velizhanin]

$$\alpha_{J=2}(\lambda) = \left.\frac{d\Delta_{\text{twist-two}}}{dS}\right|_{S=0} = 1 + \frac{\lambda}{12} - \frac{\lambda^2}{576} + \frac{\lambda^3}{17280} + O(\lambda^4)$$

Proposal

Exact slope in planar $\mathcal{N} = 4$ SYM theory [BB'11]

$$\alpha_J(\lambda) = \frac{\sqrt{\lambda}}{J} \frac{I'_J(\sqrt{\lambda})}{I_J(\sqrt{\lambda})} = 1 + \frac{\sqrt{\lambda}}{J} \frac{I_{J+1}(\sqrt{\lambda})}{I_J(\sqrt{\lambda})}$$

Expressed in terms of the modified Bessel's function $I_J(x)$ (and its derivative $I'_J(x) \equiv dI_J(x)/dx$)

Proposal: Formula is correct for any twist J and 't Hooft coupling λ

Immediate checks

- Weak coupling expansion

$$\alpha_J(\lambda) = 1 + \frac{\lambda}{2J(J+1)} - \frac{\lambda^2}{8J(J+1)^2(J+2)} + O(\lambda^3)$$

OK with previous twist-two expression for $J = 2$!

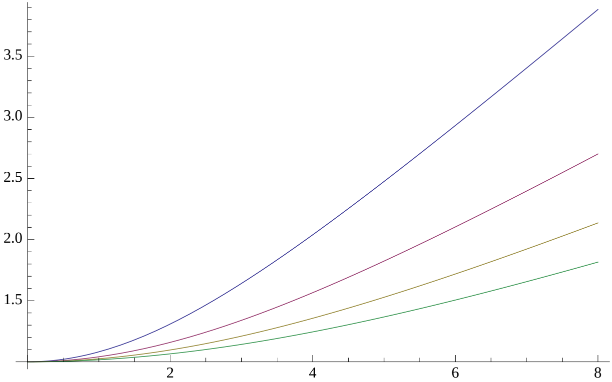
- At large J (and for any λ)

$$\alpha_J(\lambda) = 1 + \frac{\lambda}{2J^2} + O(1/J^2)$$

Correct BMN limit!

Numerical interpolation

Plot of the slope $\alpha_J(\lambda)$ as a function of the coupling $\sqrt{\lambda}$



for $J = 2$ (blue) to $J = 5$ (green)

Strong coupling expansion

Let us reformulate the proposal as

$$\Delta^2 = J^2 + \beta_J(\lambda)S + O(S^2)$$

where

$$\beta_J(\lambda) \equiv 2J\alpha_J(\lambda) = 2\sqrt{\lambda}\frac{I'_J(\lambda)}{I_J(\lambda)}$$

Motivation: remember the flat-space string theory result

$$\Delta^2 = J^2 + 2\sqrt{\lambda}S$$

Here we find that at strong coupling $\sqrt{\lambda}$ (i.e., large string tension)

$$\beta_J(\lambda) = 2\sqrt{\lambda} - 1 + \frac{J^2 - 1/4}{\sqrt{\lambda}} + \frac{J^2 - 1/4}{\lambda} + O(1/\lambda^{3/2})$$

- Correct flat-space limit!
- Correct one-loop correction!

[Gromov,Serban,Shenderovitch,Volin'11],
[Roiban,Tseytlin'11],[Vallilo,Mazzucato'11]

Further check: consider the semiclassical string regime where $\mathcal{J} \equiv J/\sqrt{\lambda}$ is fixed, then

$$\beta_J(\lambda) = 2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2} - \frac{1}{1+\mathcal{J}^2} + O(1/\sqrt{\lambda})$$

Comment: it is in perfect agreement with classical and one-loop string prediction

[Frolov,Tseytlin'02],[Gromov,Valatka'11]

Physical application I

Apply the formula

$$\Delta^2 = J^2 + \beta_J(\lambda)S + \gamma_J(\lambda)S^2 + \delta_J(\lambda)S^3 + O(S^4)$$

to physical operators (i.e., for finite spin) at strong coupling

Assumption: coefficients of higher spin powers are suppressed by higher powers of $1/\sqrt{\lambda}$, e.g.,

$$\beta_J(\lambda) = O(\sqrt{\lambda})\,, \qquad \gamma_J(\lambda) = O(1)\,, \qquad \delta_J(\lambda) = O(1/\sqrt{\lambda})\,, \qquad \cdots$$

Further assumption: coefficients of small spin expansion can be directly matched against those predicted by the semiclassical string computation

Comments:

- Non-trivial claim since the semiclassical analysis produces an expansion at small semiclassical spin $\mathcal{S} \equiv S/\sqrt{\lambda}$ (possible order of limit issue)
- So far these assumptions have been found to be in good agreement with exact (numerical) predictions from Y-system [Gromov,Serban,Shenderovitch,Volin'11],[Gromov,Valatka'11]

Physical application II

Under the battery of assumptions

$$\Delta^2 = J^2 + \beta_J(\lambda)S + \gamma_J(\lambda)S^2 + \delta_J(\lambda)S^3 + \dots$$

applies to physical operators (i.e., for finite spin) at strong coupling, with (up to two loops)

- $\beta_J(\lambda) = 2\sqrt{\lambda} - 1 + (J^2 - 1/4)/\sqrt{\lambda}$
- $\gamma_J(\lambda) = 3/2 - b/\sqrt{\lambda}$
- $\delta_J(\lambda) = -3/(8\sqrt{\lambda})$

Missing piece for two-loop prediction: the one-loop semiclassical coefficient b was unknown...

... up to recent work of [Gromov,Valatka'11] who found that

$$b = \frac{3}{8} - 3\zeta_3$$

Complete two-loop prediction for minimal scaling dimension at strong coupling... observed to be in good agreement with Y-system (numerical) result

In particular: For the Konishi scaling dimension, i.e., for $S = J = 2$, ones find [Gromov,Valatka'11]

$$\Delta = 2\lambda^{1/4} + \frac{2}{\lambda^{1/4}} + \frac{1/2 - 3\zeta_3}{\lambda^{3/4}} + O(1/\lambda^{5/2})$$

Some interesting features

The expression from the slope hints that

- Weak coupling expansion is convergent

  Radius of convergency is finite and fixed by the first non-trivial zero of Bessel's function $I_J(\sqrt{\lambda})$

- Strong coupling expansion is asymptotic and non-Borel summable

  Strong coupling series determines the exact expression up to exponentially small contributions $\sim \exp(-2\sqrt{\lambda})$ only

Similar to the situation for the cusp anomalous dimension (as predicted from the BES equation) [Beisert,Eden,Staudacher'06],[Basso,Korchemsky,Kotanski'07]

Summary and outlook

Examples of interpolation between gauge and string theory based on integrability

- Solution to spectrum of excitations over the GKP string at any coupling
- Exact representation for dispersion relations
- Interpolation of finite size corrections to twist-two scaling dimension
- Formula for the slope of minimal scaling dimension

Extensions

- Scattering amplitudes
- Spectrum of short strings