QFT applications of (multi-dimensional) Mellin-Barnes representation:
Asymptotic expansions, analytic continuation and more

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QFT applications of Mellin-Barnes representation

Mellin-Barnes (MB) representation may be of use for at least three domains in QFT:

1. **Perturbative contributions:** Feynman diagrams with several scales, etc. (and related quantities).
   1 st lecture (19 th Nov.) and 3 rd lecture (15 th Dec.).
2. **Non-perturbative contributions:** Exponentially suppressed terms.
   2 nd lecture (13 th Dec.).
3. **Reconstruction of non-analytic functions** from partial information in several limits: Form factors, Green functions...
   2 nd lecture (13 th Dec.).

❶ Perturbative contributions

1^{st} lect. (19^{th} Nov.) and 3^{rd} lect. (15^{th} Dec.)

S. Friot, D. Greynat and E. de Rafael, Phys. Lett. B 628, 73 (2005)

J.-Ph. Aguilar, D. Greynat and E. de Rafael, Phys. Rev. D 77, 093010 (2008)

S. Friot and D. Greynat, arXiv:1107.0328 - accepted in J. Math. Phys.

From the scaling property of Mellin transform

$$\mathcal{M}\left[f(ax)\right](s) = a^{-s}\mathcal{M}\left[f(x)\right](s)$$

one can easily separate the *physical* information (masses, momenta,...) from the topology of a given diagram. It appears as a very powerful tool:

- in any kind of Feynman diagrams in the Standard Model and beyond Ex.: 4 and 5 loops corrections to the muon $g-2$ in QED.
- in other theories (*e.g.* for Wilson loops in $\mathcal{N}=4$ SYM).

What is the MB Representation?

A typical integral: V. Del Duca, C. Duhr and V. A. Smirnov, JHEP 1005 (2010)

$$\int_{-i\infty}^{+i\infty} \cdots \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \wedge \cdots \wedge \frac{dz_8}{2\pi i} \; X_1{}^{z_1} X_2{}^{z_2} X_3{}^{z_3} X_4{}^{z_4} X_5{}^{z_5} X_6{}^{z_6} X_7{}^{z_7} X_8{}^{z_8}$$
$$\times \Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(-z_5)\Gamma(-z_6)\Gamma(-z_7)\Gamma(-z_8)$$
$$\times \Gamma(\epsilon - z_1 - z_2 - z_7 - 1)\Gamma(2\epsilon - z_2 - z_4 - z_5 - z_6 - z_7 - z_8 - 1)$$
$$\times \Gamma(2\epsilon - z_1 - z_2 - z_3 - z_5 - z_7 - z_8 - 1)\Gamma(-\epsilon + z_1 + z_2 + z_5 + z_6 + z_7 + z_8 + 2)$$
$$\times \Gamma(\epsilon - z_5 - z_6 - z_8)\Gamma(-2\epsilon + z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 + z_8 + 2)$$
$$\times \frac{\Gamma(z_1 + z_5 + z_7 + 2)\Gamma(z_2 + z_4 + z_7 + 1)\Gamma(z_2 + z_6 + z_8 + 1)\Gamma(z_3 + z_5 + z_8 + 1)}{\Gamma(2\epsilon - z_1 - z_2 - z_7)\Gamma(2\epsilon - z_5 - z_6 - z_8)\Gamma(z_1 + z_2 + z_5 + z_6 + z_7 + z_8 + 3)}$$

(appears in the evaluation of the 2-loop hexagon Wilson loop in $\mathcal{N} = 4$ SYM)

Evaluation steps

1. Solve the ϵ singularity.
2. Choose a good kinematics to reduce the number of parameters x_j to only three combinations of them.
3. Close contours when achievable (Barnes' lemma).
4. Evaluate the twofold (and threefold) unreducible (master) Mellin-Barnes integrals depending on the combinations of parameters.

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Main purpose of this talk

S. Friot and D. Greynat, arXiv:1107.0328 - accepted in J. Math. Phys.

❷ Non-perturbative contributions: Hyperasymptotic expansions and exponentially suppressed terms

2^{nd} lect. (13^{th} Dec.)

S. Friot and D. Greynat, SIGMA 6 (2010) 079

Motivation

In QFT, one deals with divergent (supposed asymptotic) series. How to extract non-perturbative informations from perturbative expansions?

Example of a "functional" integral for ϕ^4 theory in 0 dimension

For $\lambda \in \mathbf{C}$ and $\mathrm{Re}\,\lambda > 0$

$$Z(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}\phi \; \mathrm{e}^{-\frac{1}{2!}\phi^2 - \frac{\lambda}{4!}\phi^4} \underset{\lambda\to 0}{=} 1 - \frac{1}{8}\lambda + \frac{35}{384}\lambda^2 - \frac{385}{3072}\lambda^3 + \mathcal{O}(\lambda^4)$$

$$= \frac{1}{\sqrt{\pi}} \int_{c+i\,\mathbf{R}} \frac{\mathrm{d}\,s}{2i\pi} \left(\frac{\lambda}{6}\right)^{-s} \Gamma(s)\,\Gamma\left(\frac{1}{2} - 2s\right)$$

Using the inverse factorial expansion and the Borel resummation, **one builds the hyperasymptotic expansion of $Z(0)$**

- Numerical analysis: $\lambda = 1/3$

		$Z(0)$	S_n	S_m	$S_{m'}$	$S_{m''}$
Mathematica		0.965560481..				
Pertur. expa.		0.96555187 ± 0.001410990				
Hyp. expa.	1	0.965562911	0.9696	-0.0040		
	2	0.965560477	1.0573	-0.0917	0.0000061	
	3	0.965560486	-27.696	28.662	-0.0001292	9×10^{-9}

❸ Form factors and Green functions reconstruction

2^{nd} lect. (13^{th} Dec.)

D. Greynat and S. Peris, Phys.Rev. D82 (2010) 034030

D. Greynat, P. Masjuan and S. Peris, hep-ph 1104.3425, accepted in Phys.Rev. D

Motivation

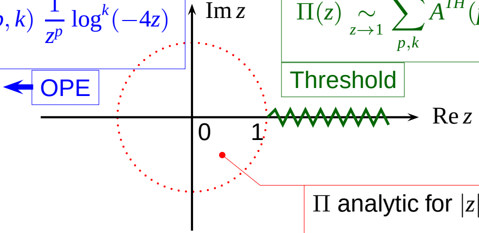
Reconstruct non-analytic functions from only fragmented information: form factors, Green functions, etc.

$$\Pi(z) \underset{z\to-\infty}{\sim} \sum_{p,k} B^{OPE}(p,k)\,\frac{1}{z^p}\,\log^k(-4z)$$

OPE

$$\Pi(z) \underset{z\to 1}{\sim} \sum_{p,k} A^{TH}(p,k)\,(1-z)^p\,\log^k(1-z)$$

Threshold



Π analytic for $|z|<1$:

$$\Pi(z) \underset{|z|<1}{=} \sum_{n=0}^{\infty} C(n)\,z^n$$

Previous reconstructions

One can use Padé approximants, but only if the imaginary part is strictly positive, otherwise there is no alternative.

New reconstruction method

- It is an analytic reconstruction of the function.
- It is systematic and convergent order by order.
- It is a controlled approximation.
- All is based on a mathematical theorem (*Converse Mapping* Theorem).

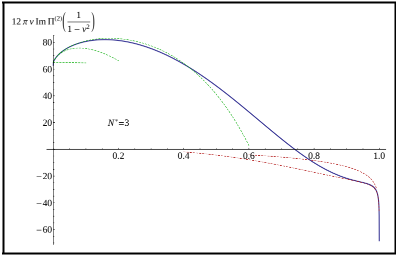
A perfect application example: Heavy-Quark Correlators

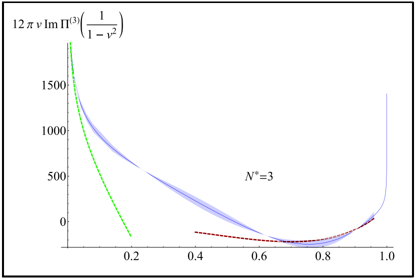
The vacuum vector-vector polarization $\Pi(q^2)$ (in the massive case) is

$$\left(g_{\mu\nu}q^2 - q_\mu q_\nu\right)\,\Pi(q^2) = -i\int d^4x\, e^{iqx}\,\langle 0\,|\mathrm{T}\, j_\mu(x)\, j_\nu(0)|\,0\,\rangle\ ,$$

It may be decomposed as

$$\Pi(q^2) = \Pi^{(0)}(q^2) + \left(\frac{\alpha_s}{\pi}\right)\Pi^{(1)}(q^2) + \left(\frac{\alpha_s}{\pi}\right)^2\Pi^{(2)}(q^2) + \left(\frac{\alpha_s}{\pi}\right)^3\Pi^{(3)}(q^2) + \mathcal{O}(\alpha_s^4)$$





A new Mellin-Barnes method to evaluate Feynman diagrams and other quantities

J.-Ph. Aguilar, D. Greynat and E. de Rafael, Phys. Rev. D 77, 093010 (2008)

S. Friot and D. Greynat, arXiv:1107.0328 - accepted in J. Math. Phys.

- Systematic method to extract several convergent series from Mellin-Barnes integrals.
- Evaluation of their convergence domains before calculation.
- Analytic continuation appears.

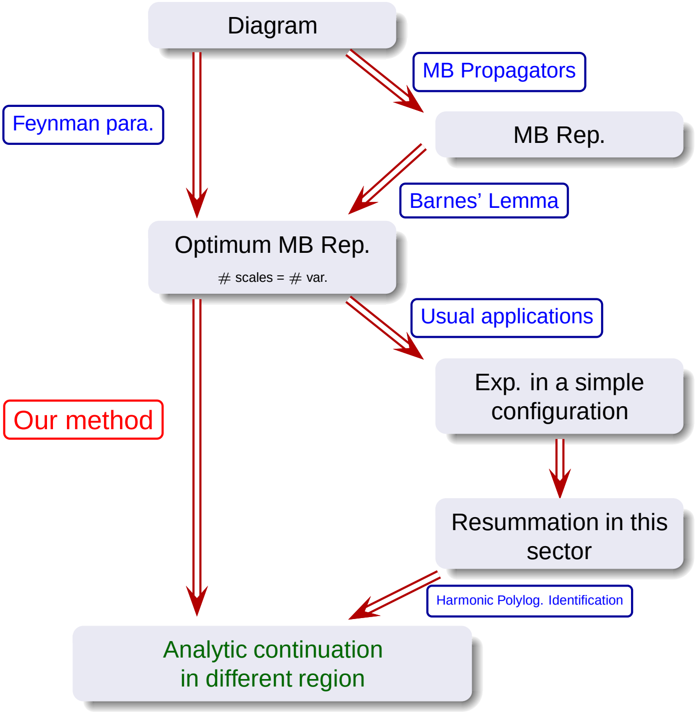


Diagram
MB Propagators
MB Rep.
Feynman para.
Barnes' Lemma
Optimum MB Rep.
scales = # var.
Usual applications
Exp. in a simple configuration
Our method
Resummation in this sector
Harmonic Polylog. Identification
Analytic continuation in different region

The Mellin transform of a function f and its inverse transform are defined as

$$\mathcal{M}[f(x)](s) \doteq \int_0^\infty \mathrm{d}x\, x^{s-1} f(x) \qquad \longleftrightarrow \qquad f(x) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\mathrm{d}s}{2i\pi}\, x^{-s} \mathcal{M}[f(x)](s)$$

If and only if

$$\gamma \doteq \mathrm{Re}\, s \in]\alpha, \beta[\qquad \text{written} \qquad \langle \alpha, \beta \rangle \qquad \textit{Fundamental strip}$$

It corresponds to the behaviours

$$f(x) \underset{x\to 0^+}{=} \mathcal{O}(x^{-\alpha}) \qquad \& \qquad f(x) \underset{x\to +\infty}{=} \mathcal{O}(x^{-\beta})$$

$$(1+x)^{-\nu} \qquad \longleftrightarrow \qquad \frac{\Gamma(\nu-s)\Gamma(s)}{\Gamma(\nu)} \qquad \langle 0, \mathrm{Re}\,\nu \rangle$$

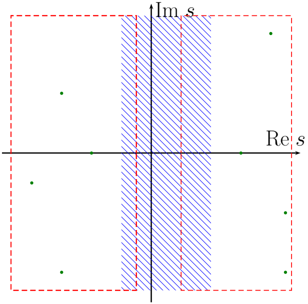
$$\ln(1+x) \qquad \longleftrightarrow \qquad \Gamma(1-s)\Gamma(1+s) \qquad \langle -1, 0 \rangle$$

Onefold Mellin-Barnes integrals

Let us consider a one (complex) scale x integral, it can be written as

$$I(x) = \int_{\gamma + i\mathbf{R}} \frac{\mathrm{d}s}{2i\pi}\, x^{-s}\, \frac{\prod_{j=1}^{m} \Gamma(a_j s + b_j)}{\prod_{k=1}^{n} \Gamma(c_k s + d_k)}$$

$$\boxed{\begin{aligned} \Delta &\doteq \sum_{j=1}^{m} a_j - \sum_{k=1}^{n} c_k \\ \alpha &\doteq \sum_{j=1}^{m} |a_j| - \sum_{k=1}^{n} |c_k| \end{aligned}}$$



- If $\Delta > 0$
  - left closing: convergent series for any value of x
  - right closing: divergent asymptotic expansion
- If $\Delta < 0$
  - left closing: divergent asymptotic expansion.
  - right closing: convergent series for any value of x
- If $\Delta = 0$
  left and right closing give two convergent series and if $\alpha > 0$, they are the analytic continuation of the other

Generalization in the case of twofold Mellin-Barnes integrals

Zhdanov and Tsikh, Siberian Mathematical Journal 39 (1998)

$$I(x,y) = \int_{\gamma+i\,\mathbf{R}^2} \frac{\mathrm{d}\,z_1}{2i\pi} \wedge \frac{\mathrm{d}\,z_2}{2i\pi}\, x^{-z_1}\, y^{-z_2}\, \frac{\prod_{j=1}^{m} \Gamma(a_j z_1 + b_j z_2 + e_j)}{\prod_{k=1}^{p} \Gamma(c_k z_1 + d_k z_2 + f_k)}$$

$$\mathbf{\Delta} = \begin{pmatrix} \sum_{j=1}^{m} a_j - \sum_{k=1}^{p} c_k \\ \sum_{j=1}^{m} b_j - \sum_{k=1}^{p} d_k \end{pmatrix}$$

- $\mathbf{\Delta} \neq (0,0)$ and there is no j so that $(a_j, b_j) \propto \mathbf{\Delta}$
  Absolute convergence of the (double) series in the whole double complex plane (except at the origins)
- $\mathbf{\Delta} = (0,0)$: degenerate case
- $\mathbf{\Delta} \neq (0,0)$ and there exists j so that (a_j, b_j): degenerate case

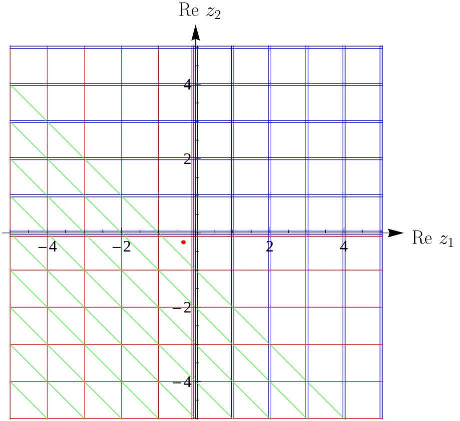
$\mathbf{\Delta} = (0,0)$ several convergent (double) series expansions coexist and they are analytic continuations of each other.

Geometrical way to find the double series representations for $\Delta = (0,0)$

A simple example: (where $c = -\frac{1}{3}$ and $d = -\frac{1}{4}$)

$$R(u_1,u_2) = \int_{c+i\mathbf{R}} \int_{d+i\mathbf{R}} \frac{dz_1}{2i\pi} \wedge \frac{dz_2}{2i\pi}\, u_1^{z_1}\, u_2^{z_2}\, \Gamma^2(-z_1)\,\Gamma(z_1)\,\Gamma^2(-z_2)\,\Gamma(z_2)\,\Gamma(z_1+z_2+1)$$

$$\Gamma(z) \underset{z\to -n}{\sim} \frac{(-1)^n}{n!}\,\frac{1}{z+n}$$



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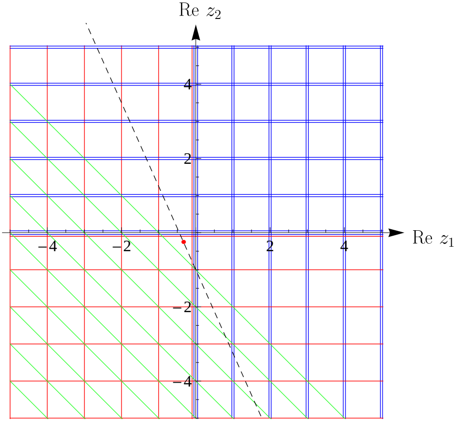

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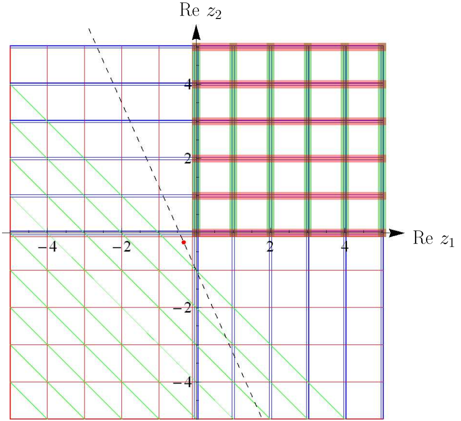

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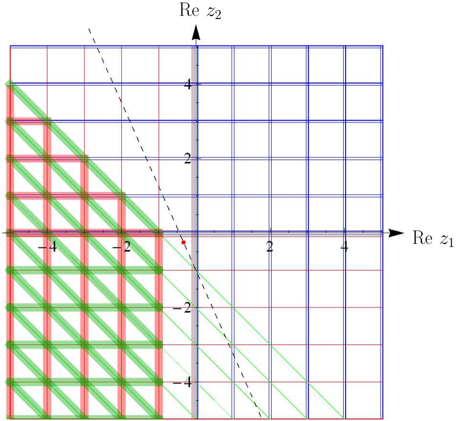
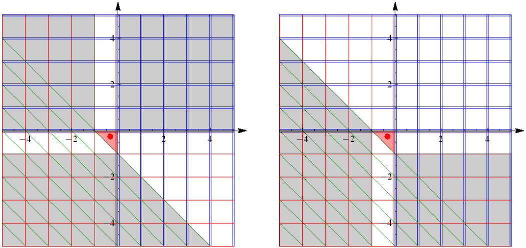
$$\Gamma(z) \underset{z\to -n}{\sim} \frac{(-1)^n}{n!}\,\frac{1}{z+n}$$



Systematic way to find all convergent (complete or partial) double series representations of $R(u_1, u_2)$



Two kinds of singularities may appear in a given region (cone)

J.-Ph. Aguilar, D. Greynat and E. de Rafael, Phys. Rev. D 77, 093010 (2008)

- Only vertical and horizontal lines intersect each other: Cauchy singularity
- intersection with at least one oblique divisor: Transformation law singularity

Where do each series in u_1 and u_2 converge ?

A quick look at Horn series

Horn series

$f(x,y) = \sum_{n,m} a_{m,n} x^m y^n$ is a Horn series if the two functions

$$\mathrm{f}(m,n) \doteq \frac{a_{m+1,n}}{a_{m,n}} \doteq \frac{\mathrm{P}(m,n)}{\mathrm{R}(m,n)} \ , \qquad \mathrm{g}(m,n) \doteq \frac{a_{m,n+1}}{a_{m,n}} \doteq \frac{\mathrm{Q}(m,n)}{\mathrm{S}(m,n)} \ ,$$

are purely rational, then $\mathrm{P}, \mathrm{Q}, \mathrm{R}$ and S are polynomial of degree p, q, r and s.

$$\mathrm{F}(m,n) = \overline{\lim_{\eta\to+\infty}}\, \mathrm{f}(\eta\, m, \eta\, n) \qquad\qquad \mathrm{G}(m,n) = \overline{\lim_{\eta\to+\infty}}\, \mathrm{g}(\eta\, m, \eta\, n)\ .$$

Conditions	Domain of convergence
$\mathrm{p} > \mathrm{r}$ or $\mathrm{q} > \mathrm{s}$	$\lvert x\rvert = 0$ and $\lvert y\rvert = 0$
$\mathrm{p} < \mathrm{r}$ and $\mathrm{q} < \mathrm{s}$	$(\lvert x\rvert, \lvert y\rvert) \in \mathbf{R}_+^2$
$\mathrm{p} < \mathrm{r}$ and $\mathrm{q} = \mathrm{s}$	$\lvert x\rvert \in \mathbf{R}_+$ and $\lvert y\rvert < \frac{1}{\lvert\mathrm{G}(0,1)\rvert}$
$\mathrm{p} = \mathrm{r}$ and $\mathrm{q} < \mathrm{s}$	$\lvert x\rvert < \frac{1}{\lvert\mathrm{F}(1,0)\rvert}$ and $\lvert y\rvert \in \mathbf{R}_+$
$\mathrm{p} = \mathrm{r}$ and $\mathrm{q} = \mathrm{s}$	$(\lvert x\rvert, \lvert y\rvert) \in \mathcal{D} \cap \mathcal{C}$

$$\mathcal{D} = \left\{ (|x|, |y|) \ \middle|\ 0 \leqslant |x| < \frac{1}{|\mathrm{F}(1,0)|} \ \text{ and } \ 0 \leqslant |y| < \frac{1}{|\mathrm{G}(0,1)|} \right\}$$

and

$$\mathcal{C} = \left\{ (|x|, |y|) \ \middle|\ 0 \leqslant |x| \ \text{ and } \ 0 \leqslant |y| < \partial\mathcal{C} \right\}$$

with $\partial\mathcal{C}$ is the parametric curve given by the equation

$$\partial\mathcal{C} : (|x|, |y|) \longmapsto \begin{cases} |x| = \frac{1}{|\mathrm{F}(m,n)|} \\ \\ |y| = \frac{1}{|\mathrm{G}(m,n)|} \end{cases}$$

Until $a_{m,n}$ is a ratio of Γ functions, the series are Horn type.

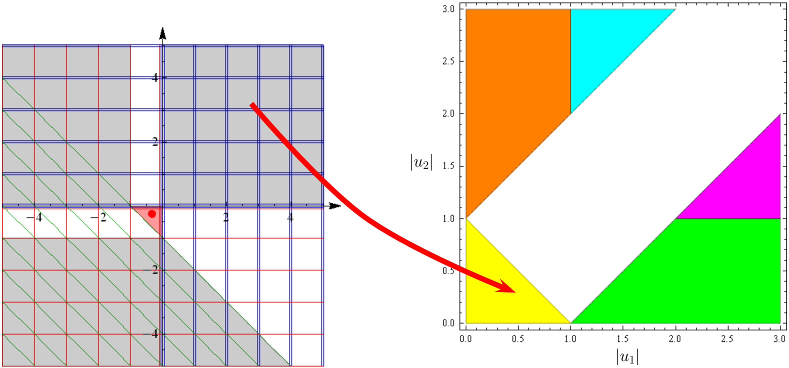
Unfortunately, one often has to compute derivatives of Γ functions, then $a_{m,n}$ contains $\psi^{(n)}$ functions and the series is not Horn type anymore.

Improvement

We proved that the $\psi^{(n)}$ do not change the domain of convergence of the corresponding Horn series

`S. Friot and D. Greynat, arXiv:1107.0328 - accepted in J. Math. Phys.`

Evaluation steps



1. Enumerate the different types of singularities ($m, n \geq 0$ are integers)
   - $(\text{Re } z_1, \text{Re } z_2) = (1+m, 1+n)$
   - $(\text{Re } z_1, \text{Re } z_2) = (1+m, 0)$
   - $(\text{Re } z_1, \text{Re } z_2) = (0, 1+m)$
   - $(\text{Re } z_1, \text{Re } z_2) = (0, 0)$
2. Perform a c.o.v. to bring the singularity to the origin.
3. Evaluate the associated domain of convergence.

Evaluation of the associated domain of convergence

The poles are at $(\mathrm{Re}\, z_1, \mathrm{Re}\, z_2) = (1+m, 1+n)$, $R(u_1,u_2) = \mathrm{Res}\left[\frac{h_1(z_1,z_2)}{z_1^2\ z_2^2}\right]$ where:

$$h_1(z_1,z_2) \doteq u_1^{z_1+1+m}\, u_2^{z_2+1+n}\ \frac{\Gamma(1+z_1)^2\Gamma(1-z_1)^2\Gamma(z_1+1+m)\Gamma(1+z_2)^2\Gamma(1-z_2)^2\Gamma(z_2+1+n)\Gamma(z_1+z_2+3+m+n)}{\Gamma(z_1+m+2)^2\Gamma(z_2+n+2)^2}$$

We proved that we just have to consider $h(0,0)$,

$$h(0,0) = u_1^{1+m}\, u_2^{1+n}\ \ a_{m,n} \qquad \text{with} \qquad a_{m,n} = \frac{\Gamma(1+m)\,\Gamma(1+n)\,\Gamma(3+m+n)}{\Gamma^2(m+2)\,\Gamma^2(n+2)}$$

One has the degenerate case

$$\mathrm{f}(m,n) = \frac{(1+m)(2+m+n)^2}{(2+m)^2(3+m+n)} \qquad \longrightarrow \qquad \frac{1}{\mathrm{F}(m,n)} = \frac{m}{m+n}$$

$$\mathrm{g}(m,n) = \frac{(2+m+n)^2}{(1+n)(3+m+n)} \qquad \longrightarrow \qquad \frac{1}{\mathrm{G}(m,n)} = \frac{n}{n+m}$$

that lead to the domain of convergence (the yellow domain)

$$|u_1| + |u_2| < 1$$

Evaluation of the contribution

Now one can evaluate the contribution directly by application of the Cauchy formula:

$$\text{Res}\left[\frac{h(z_1,z_2)}{z_1^n\, z_2^m}\, dz_1 \wedge dz_2\right] = \frac{1}{(n-1)!(m-1)!}\, \frac{\partial^{n+m-2}\, h(z_1,z_2)}{\partial z_1^{n-1} \partial z_2^{m-1}}\bigg|_{(0,0)}$$

Then,

$$R(u_1,u_2)\bigg|_{Type1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \text{Res}\left[\frac{h_1(z_1,z_2)}{z_1^2\, z_2^2}\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\partial^2 h_1(z_1,z_2)}{\partial z_1 \partial z_2}\bigg|_{(0,0)}$$

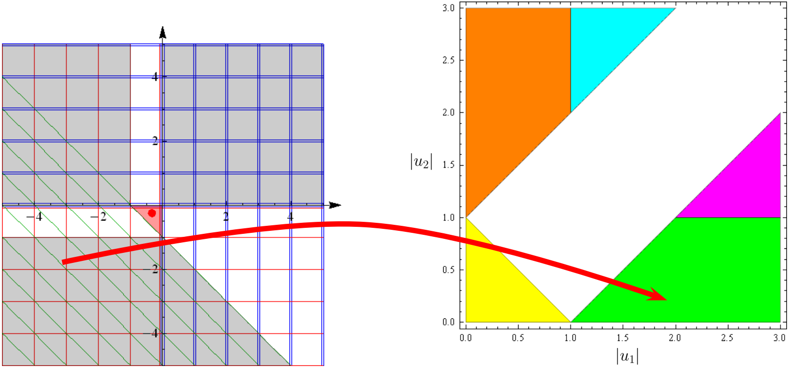
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{u_1^{m+1} u_2^{n+1}\Gamma(3+m+n)}{\Gamma(m+3)\Gamma(n+3)} \left\{\left[-\frac{2}{m+1} - H_m + H_{2+m+n} + \ln u_1\right]\right.$$

$$\times\left[-\frac{2}{n+1} - H_n + H_{2+m+n} + \ln u_2\right]$$

$$\left. + \zeta_2 - H^{(2)}_{2+m+n}\right\}$$

where $H_m = \sum_{i=1}^m \frac{1}{i}$ and $H^{(k)}(m) = \sum_{i=1}^m \frac{1}{i^k}$ are the harmonic numbers and their generalizations.

In an other region: Transformation law singularities



1. Enumerate the different types of singularities ($m, n \geq 0$ are integers)
   $\bullet(\mathrm{Re}\, z_1, \mathrm{Re}\, z_2) = (-1-m, -1-n)$
   $\bullet(\mathrm{Re}\, z_1, \mathrm{Re}\, z_2) = (0, -1-m)$
   $\bullet(\mathrm{Re}\, z_1, \mathrm{Re}\, z_2) = (1+m, -2-m-n)$
2. Perform a c.o.v. to bring the singularity to the origin.
3. Evaluate the associated domain of convergence.
4. Apply the Transformation law and the Cauchy formula.

If we consider the contributions from $(\mathrm{Re}z_1, \mathrm{Re}z_2) = (-1-m, -1-n)$, one has

$$R(u_1, u_2)\big|_{\mathrm{Type2}} = \mathrm{Res}\left[\frac{h_2(z_1, z_2)}{z_1 z_2 (z_1 + z_2)}\right]$$

where

$$h_2(z_1, z_2) \doteq u_1^{z_1-1-m} u_2^{z_2-1-n} \frac{\Gamma^2(1+m-z_1)\Gamma(1+z_1)\Gamma(1-z_1)\Gamma^2(1+n-z_2)\Gamma(1+z_2)\Gamma(1-z_2)\Gamma(1+z_1+z_2)\Gamma(1-z_1-z_2)}{\Gamma(2+m-z_1)\Gamma(2+n-z_2)\Gamma(2+m+n-z_1-z_2)}$$

It is impossible to apply the Cauchy theorem to this form.

One needs **the Transformation Law** (Global Residue Theorem).

The Transformation law (Global Residue Theorem)

The Transformation law applied to Mellin-Barnes integrals

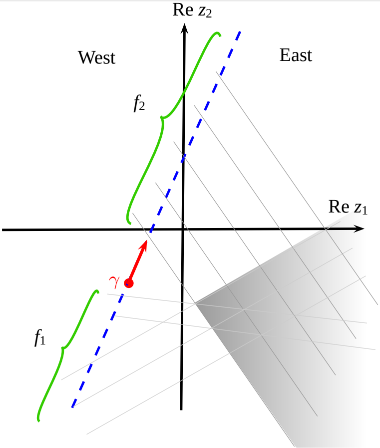
Let take $\boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $\boldsymbol{g} = \begin{pmatrix} z_1^n \\ z_2^m \end{pmatrix}$
where f_1 is the product of a *specific* subset of $\{(z_1 + a_1 z_2)^{p_1}$, ..., $(z_1 + a_N z_2)^{p_N}$, z_1^r, $z_2^s\}$ while f_2 is its complementary,

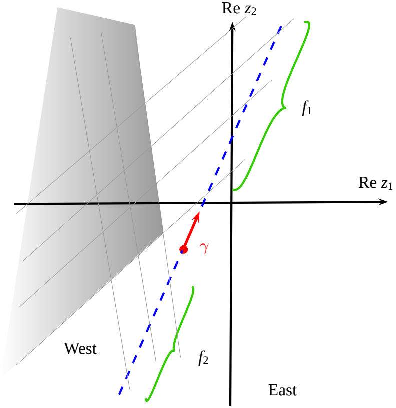
If $\boldsymbol{f}^{-1}(0,0) = \boldsymbol{g}^{-1}(0,0) = (0,0)$ and if it exists an analytic 2×2 matrix A such that $A\,\boldsymbol{f} = \boldsymbol{g}$ then

$$\text{Res}\left[\frac{h(z_1,z_2)}{\prod_{j=1}^{N}(z_1 + a_j z_2)^{p_j}\ z_1^r\ z_2^s}\ dz_1 \wedge dz_2\right] = \text{Res}\left[\frac{h(z_1,z_2)\det A}{z_1^n\ z_2^m}\ dz_1 \wedge dz_2\right]$$

A and $\boldsymbol{g}$ are not unique in general.

The Transformation law (Global Residue Theorem)





All singular lines which cross the dotted line on one of its side "contribute" to f_1, all others "contribute" to f_2

$$R(u_1, u_2)\Big|_{\text{Type2}} = \text{Res}\left[\frac{h_2(z_1, z_2)}{z_1 z_2 (z_1 + z_2)}\right]$$

One can choose $f = (z_1(z_1 + z_2), z_2)^T$, $g = (z_1^2, z_2)^T$ and

$$A = \begin{pmatrix} 1 & -z_1 \\ z_2 & 1 - z_1 z_2 - z_1^2 \end{pmatrix}$$

whose determinant is $\det A = 1 - z_1^2$, one then finds

$$\text{Res}\left[\frac{h_2(z_1, z_2)}{z_1 z_2 (z_1 + z_2)}\, dz_1 \wedge dz_2\right] = \text{Res}\left[\frac{h_1(z_1, z_2)}{z_1^2\, z_2}\, dz_1 \wedge dz_2 - \frac{h_2(z_1, z_2)}{z_2}\, dz_1 \wedge dz_2\right]$$

$$R(u_1,u_2)\Big|_{\text{Type2}} = \sum_{m=0}^{\infty}\sum_{N=0}^{\infty} \frac{\partial h_2(z_1,z_2)}{\partial z_1}\Bigg|_{(0,0)}$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{u_1^{-m-1}u_2^{-n-1}\Gamma(m+1)\Gamma(n+1)}{(m+1)(n+1)\Gamma(2+m+n)}\left[\frac{1}{m+1} - H_m + H_{1+m+n} + \ln u_1\right]$$

Remark: with $g = (z_1^3, z_2)^T$ and

$$A = \begin{pmatrix} z_1 - z_2 & z_1 z_2 \\ z_2 & 1 - z_1 z_2 - z_1^2 \end{pmatrix}$$

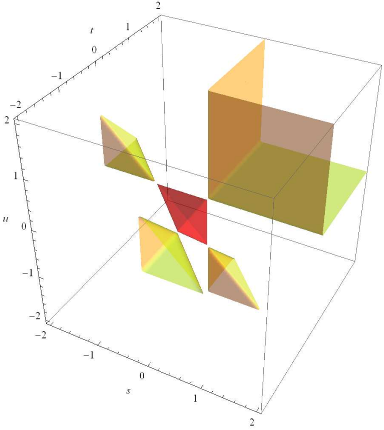
($\det A = z_1 - z_1^3 - z_2$) we get

$$\text{Res}\left[\frac{h_2(z_1,z_2)}{z_1 z_2 (z_1+z_2)}\, dz_1 \wedge dz_2\right] = \text{Res}\left[\frac{h_2(z_1,z_2)}{z_1^2\, z_2}\, dz_1 \wedge dz_2 - \frac{h_2(z_1,z_2)}{z_2}\, dz_1 \wedge dz_2 \right.$$
$$\left. - \frac{h_2(z_1,z_2)}{z_1^3}\, dz_1 \wedge dz_2\right]$$

Extension of the formalism to Mellin-Barnes integral of higher dimension

$$I(u_1,u_2,u_3) = \int_{\gamma+i\mathbf{R}^3} \frac{\mathrm{d}\mathbf{z}}{(2i\pi)^3} \; u_1{}^{z_1}\, u_2{}^{z_2}\, u_3{}^{z_3} \Gamma^2(-z_1)\; \Gamma^2(-z_2)\; \Gamma^2(-z_3)\; \Gamma^2(1+z_1+z_2+z_3)$$

where $\gamma = (-\frac{1}{8}, -\frac{1}{9}, -\frac{1}{10})^T$.



- Cone 1:
  $\mathrm{Re}\ z_1 > 0$, $\mathrm{Re}\ z_2 > 0$ and $\mathrm{Re}\ z_3 > 0$
- Cone 2:
  $\mathrm{Re}\ z_1 > 0$, $\mathrm{Re}\ z_2 > 0$ and $\mathrm{Re}\ (1+z_1+z_2+z_3) < 0$
- Cone 3:
  $\mathrm{Re}\ z_1 > 0$, $\mathrm{Re}\ z_3 > 0$ and $\mathrm{Re}\ (1+z_1+z_2+z_3) < 0$
- Cone 4:
  $\mathrm{Re}\ z_2 > 0$, $\mathrm{Re}\ z_3 > 0$ and $\mathrm{Re}\ (1+z_1+z_2+z_3) < 0$

In this case, on the cone 1 with the parametrisation (m, n, p) we get

$$I(u_1, u_2, u_3)\Big|_{\text{Cone 1}} = -\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{\partial^3 h(z_1, z_2, z_3)}{\partial z_1 \partial z_2 \partial z_3}\Big|_{(0,0,0)}$$

where

$$h(z_1, z_2, z_3) = u_1^{z_1+m}\, u_2^{z_2+n}\, u_3^{z_3+p}\, \frac{\Gamma^2(1+z_1)\,\Gamma^2(1-z_1)}{\Gamma^2(z_1+m+1)}\, \frac{\Gamma^2(1+z_2)\,\Gamma^2(1-z_2)}{\Gamma^2(z_2+n+1)} \times \frac{\Gamma^2(1+z_3)\,\Gamma^2(1-u)}{\Gamma^2(z_3+p+1)}\Gamma^2(z_1+z_2+z_3+m+n+p+1)\ .$$

converging on

$$\sqrt{|u_1|} + \sqrt{|u_2|} + \sqrt{|u_3|} < 1$$

Numerical analysis

$$I(10, 20, 30)\big|_{\text{Integral Representation}} \simeq 6.669804656$$

$$I(10, 20, 30)\big|_{\text{Cone 1, first 6 terms}} \simeq 6.669804658$$

Conclusions

- The Mellin-Barnes representation is a very powerful tool to obtain asymptotic expansions of ”Feynman diagrams”-type integrals depending on one or several scales.
- Infinite convergent multiple series and therefore exact representations may be easily obtained in the case of a large class of twofold Mellin-Barnes integrals by the systematic and simple method we presented.
- We proved that one may obtain the domain of convergence of associated double series without their full expression.
- In general several series coexist as analytic continuations of each other.
- With this approach it is not necessary to know a complete series expansion to get another one related by analytic continuation.
- Integrals of higher dimension are currently under study.