Ternary algebras and groups

Michel Rausch de Traubenberg

Université de Strasbourg
IPHC-DRS-Physique Théorique des Hautes Énergies

March 27, 2012

1. Motivations
2. Lie algebras of order F
   - Definition
   - Infinite dimensional $F-$Lie algebras – Anyons
   - Finite dimensional Lie algebras – Extension of the Poincaré algebra
   - Representation of Lie algebras of order three
   - Lie algebras of order three and Noether Theorem
   - Cubic extension of the Poincaré algebra
3. Formal study of Lie algebras of order F
   - Ternary supergroup
   - Ternary superspace
4. Conclusion

1. Motivations
2. Lie algebras of order F
   - Definition
   - Infinite dimensional $F-$Lie algebras – Anyons
   - Finite dimensional Lie algebras – Extension of the Poincaré algebra
   - Representation of Lie algebras of order three
   - Lie algebras of order three and Noether Theorem
   - Cubic extension of the Poincaré algebra
3. Formal study of Lie algebras of order F
   - Ternary supergroup
   - Ternary superspace
4. Conclusion

1. Motivations
2. Lie algebras of order F
   - Definition
   - Infinite dimensional $F-$Lie algebras – Anyons
   - Finite dimensional Lie algebras – Extension of the Poincaré algebra
   - Representation of Lie algebras of order three
   - Lie algebras of order three and Noether Theorem
   - Cubic extension of the Poincaré algebra
3. Formal study of Lie algebras of order F
   - Ternary supergroup
   - Ternary superspace
4. Conclusion

1. Motivations

ALGEBRAS

* classify elementary particles
* properties of elementary particles

FIELDS

bosons
fermions

QUANTISATION →

SPIN-STATISTIC THEOREM

$$[b^a, b^b] = 0 \quad [b^\dagger{}_a, b^\dagger{}_b] = 0 \quad [b^a, b^\dagger{}_b] = \delta^a{}_b$$

$$\{f^i, f^j\} = 0 \quad \{f^\dagger{}_i, f^\dagger{}_j\} = 0 \quad \{f^i, f^\dagger{}_j\} = \delta^i{}_j$$

↓ CONSERVED CHARGED

NOETHER THEOREM

$$B = \sum_{a,b} b_a^\dagger (K_{bb})^a{}_b \, b^b + \sum_{i,j} f_i^\dagger (K_{ff})^i{}_j f^j$$

$$F = \sum_{a,i} b_a^\dagger (K_{bf})^a{}_i f^i + \sum_{i,a} f_i^\dagger (K_{fb})^i{}_a b^a$$

← GENERATORS

LIE (SUPER-) ALGEBRAS

$$[B,B] \sim B$$
$$[B,F] \sim F$$
$$\{F,F\} \sim B$$

↑ SYMMETRY

LIE ALGEBRAS

1. fundamental interactions = internal symmetries
   $\rightarrow$ the standard model $\mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$
2. Space-time symmetries (Poincaré algebra)
   $\rightarrow$ $\mathfrak{iso}(1,3) = \mathfrak{so}(1,3) \ltimes \mathbb{R}^{1,3}$

LIE SUPERALGEBRAS

1. supersymmetry, supergravity
   $\rightarrow$ $\Big(\mathfrak{iso}(1,3) \times \mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1)\Big) \oplus \Big(\mathbb{C}^2 + \overline{\mathbb{C}^2}\Big)$

Motivation: symmetry in field theory

- field theory with other symmetries?

  1 extend the algebra

  2 deform the algebra

  3 exceptional dimensions

- Lie (super)algebras quadratic relations

$\rightarrow$ define algebras close with higher order product

  1 ternary algebras

  2 $n-$ary algebras

TERNARY ALGEBRAS $\Rightarrow$ TERNARY GROUPS

Motivations: symmetry in field theory

- Construction of mathematical structure which extends Lie super-algebras

a priori in contradiction with principles of physics

Noether theorem

spin-statistics theorem

Higher order algebras (ternary, $n-$arry)

Appear recently in

1. Multiple M_2-branes (Bagger-Lamber-Gustavsson models, Filippov algebra, fully antisymmetric)
2. Nambu algebras (volume preserving diffeomorphism in brane *etc.*)
3. Higher order extensions of the Poincaré algebra $V^n \sim P$

2. Lie algebras of order F

2.1 Definition

Lie algebras of order F : Definition ...

Higher order algebras (MRT, Slupinski, JMP 00, 02)

$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \{X_i, i = 1, \cdots, \dim \mathfrak{g}_0\} \oplus \{Y_a, a = 1, \cdots, \dim \mathfrak{g}_1\}$ is a Lie algebra of order F if:

1. $\mathfrak{g}_0$ is a (real) Lie algebra $[X_i, X_j] = f_{ij}{}^k X_k$;
2. $\mathfrak{g}_1$ is a (real) representation of $\mathfrak{g}_0$ $[X_i, Y_a] = R_{ia}{}^b Y_b$;
3. We have the $F-$order brackets
   $\{Y_{a_1}, \cdots, Y_{a_F}\} = \sum_{\sigma \in \Sigma_F} Y_{a_{\sigma(1)}} \cdots Y_{a_{\sigma(F)}} = Q_{a_1 \cdots a_F}{}^i X_i$;
4. Fundamental identities
   $\left[\{Y_{a_2}, \cdots, Y_{a_{F+1}}\}, Y_{a_1}\right] + \cdots + \left[\{Y_{a_1}, \cdots, Y_{a_F}\}, Y_{a_{F+1}}\right] = 0.$

- the algebra is partially quadratic/of order F
- for matrix rep. the identities are trivially satisfied
- Generalisation Lie algebras and Lie superalgebras.

2. Lie algebras of order F

2.2 Infinite dimensional Lie algebras – Anyons

$F-$Lie algebras: applications in small dimensions

$D = 1, 1+1$ et $D = 1+2$ exceptional

1. $D = 1$ fractional quantum mechanics $Q^F = H$
2. $D = 1+1$ applications in string theory
3. $D = 1+2$ new symmetry between anyons.

$F-$Lie algebras: applications in small dimensions $D = 1 + 2$

- Lorentz algebra $\mathfrak{so}(1,2)$ exceptional

$L_0 = -iJ_0, L_\pm = -iJ_1 \mp J_2$ $[L_0, L_\pm] = \pm L_\pm$, $[L_+, L_-] = -2L_0$

$\pi_1(SO(1,2)) = \mathbb{Z} \Rightarrow$ exists states of spin $s \in \mathbb{R}$ anyons = discrete series of Bargmann $Q = L_0^2 + \frac{1}{2}(L_+L_- + L_-L_+) = s(s+1)$

$$\mathcal{D}_s^+ : \quad \begin{aligned} L_{s0}|s_+, n\rangle &= (s+n)|s_+, n\rangle \\ L_{s+}|s_+, n\rangle &= \sqrt{(2s+n)(n+1)}|s_+, n+1\rangle \\ L_{s-}|s_+, n\rangle &= \sqrt{(2s+n-1)n}|s_+, n-1\rangle, \end{aligned}$$

$$\mathcal{D}_s^- : \quad \begin{aligned} L_{s0}|s_-, n\rangle &= -(s+n)|s_-, n\rangle \\ L_{s+}|s_-, n\rangle &= -\sqrt{(2s+n-1)n}|s_-, n-1\rangle \\ L_{s,-}|s_- n\rangle &= -\sqrt{(2s+n)(n+1)}|s_-, n+1\rangle. \end{aligned}$$

$\mathcal{D}_s^\pm$ states of spin $\pm(s+n)$

unitary and et exponentiable if $s \geq 0$ relat. eq. (Jackiw & Nair, Plyushchay)

$F-$Lie algebras: applications in small dimensions $D = 1 + 2$

Non-trivial extensions of the Poincaré algebra in three-dimensions (MRT, Slupinski MPLA97)

non-unitary representation of spin $-1/F$ infinite number of generators $\left\{Q^{\pm}_{-1/F+n}, n \in \mathbb{N}\right\} = \mathcal{D}^{\pm}_{-1/F}$

1. Complicated infinite dimensional algebra $(Q^{\pm}_{-1/F})^F = P_{\mp}, \cdots$
2. study of representations unitary
3. symmetry on anyons

$$|\pm s\rangle \xrightarrow{Q^{\pm}} \left|\pm s \mp \frac{1}{F}\right\rangle \xrightarrow{Q^{\pm}} \cdots \xrightarrow{Q^{\pm}} \left|\pm s \mp \frac{F-1}{F}\right\rangle \xrightarrow{Q^{\pm}} |\pm s\rangle$$

Lie algebra of order F in arbitrary dimension

$Q^F = P \longrightarrow$ two problems :

1. momentum spin 1 Q is *a priori* of spin $1/F$

$$Q^F = P \Longrightarrow 1 = F \times \frac{1}{F}$$

contradiction with group theory

2. How can we not contradict the spin statistics and the Noether theorems

Finite dimensional Lie algebras of order F

2. Lie algebras of order F

2.3 Finite dimensional Lie algebra

Lie algebras of order three: Induction Theorem

- Basically **two types of algebras**:
  1. associated to an induction theorem
  2. defined by matrix rep.
- **No general classification**

Induction theorem (MRT, Slupinski, JMP 02)

Let $\mathfrak{g}_0$ be a Lie algebra and $\mathfrak{g}_1$ a representation of $\mathfrak{g}_0$ such that
(i) $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie algebra of order $F_1 > 1$;
(ii) $\mathfrak{g}_1$ admits a $\mathfrak{g}_0-$equivariant symmetric form of order $F_2 > 1$.
Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ admits a Lie algebra of order $F_1 + F_2$ structure.

This theorem can be extended to $F_1 = 1$

CONSEQUENCE

Lie (super)algebras $\Rightarrow$ Lie algebras of order F

Examples constructed from Lie algebras

Ternary extension of Lie algebras

$\mathfrak{g}_3 = \mathfrak{g}_0 \oplus \mathrm{ad}(\mathfrak{g}_0) = \langle J_a \rangle \oplus \langle A_a \rangle, \qquad g_{ab} = \mathrm{Tr}(A_a A_b)$

$$[J_a, J_b] = f_{ab}{}^c J_c, \ [J_a, A_b] = f_{ab}{}^c A_c,$$
$$\{A_a, A_b, A_c\} = g_{ab} J_c + g_{bc} J_a + g_{ca} J_b,$$

Ternary extension of Poincaré algebras

$\mathfrak{iso}_3(1, D-1) = \mathfrak{iso}(1, D-1) \oplus \mathcal{T} = \langle L_{\mu\nu}, P_\mu \rangle \oplus \langle V_\mu \rangle$

$$\begin{aligned}
[L_{\mu\nu}, L_{\rho\sigma}] &= \eta_{\nu\sigma} L_{\rho\mu} - \eta_{\mu\sigma} L_{\rho\nu} + \eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\rho} L_{\nu\sigma}, \\
[L_{\mu\nu}, P_\rho] &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu, \ [P_\mu, P_\nu] = 0 \\
[L_{\mu\nu}, V_\rho] &= \eta_{\nu\rho} V_\mu - \eta_{\mu\rho} V_\nu, \ [P_\mu, V_\nu] = 0, \\
\{V_\mu, V_\nu, V_\rho\} &= \eta_{\mu\nu} P_\rho + \eta_{\mu\rho} P_\nu + \eta_{\rho\nu} P_\mu .
\end{aligned}$$

- **The ternary extension** $\mathfrak{so}_3(1,4)$ is related by Inönü-Wigner contraction and deformations to $\mathfrak{iso}_3(1,3)$

Defined by matrix representation

$$\mathfrak{gl}(m_1, m_2, m_3) = \left\{ \begin{pmatrix} a_0 & b_1 & 0 \\ 0 & a_1 & b_2 \\ b_0 & 0 & a_2 \end{pmatrix} \right\} = \langle X \rangle \oplus \langle Y \rangle$$

$$\begin{aligned} [X, X] &= X, \\ [X, Y] &= Y \\ \{Y, Y, Y\} &= X \end{aligned}$$

$$\mathfrak{g}_0 = \mathfrak{gl}(m_1) \oplus \mathfrak{gl}(m_2) \oplus \mathfrak{gl}(m_3)$$

2. Lie algebras of order F

2.2 Representations

Representation of Lie algebras of order F

• are strongly related to Clifford algebras of polynomials $\mathcal{C}_P$ (Roby CRAS, '69)

1. $P(x^1, \cdots, x^k) = x^{i_1} \cdots x^{i_F} g_{i_1 \cdots i_F}$
2. $\left\{ g_{i_1}, \cdots, g_{i_F} \right\} = F! g_{i_1 \cdots i_F} \to (x^1 g_1 + \cdots + x^k g_k)^n = P(x)$
3. Infinite dimensional algebra
4. there exist many non-faithful inequivalent (even of the same dimension)
5. systematic way to obtain representation (Fleury, MRT, JMP92)

• Application to Lie algebras of order F example $\mathfrak{iso}_3(1,3)$. In the little group for the massive case the non-vanishing brackets are

$$\left\{ V_0, V_0, V_0 \right\} = 3m, \quad \left\{ V_0, V_i, V_j \right\} = -m\delta_{ij}.$$

representations associated to the polynomial $P(x^\mu) = 3x^0(x^\mu x_\mu)$.

2. Lie algebras of order F

2.5 Lie algebras of order three and Noether Theorem

Higher order algebras and Noether Theorem

- Noether and spin-statistics theorems seems to forbid higher order algebras
- Φ representation of $\mathfrak{iso}_3(1,3)$ $\delta_\mu \Phi = V_\mu \Phi$;
- $\mathcal{L}(\Phi)$ Lagrangian $\hat{V}_\mu = -i \int d^3x \Pi V_\mu \Phi$ conserved charges
- After quantisation $[\Pi(\vec{x},t), \Phi(\vec{y},t)]_\pm = i\delta^3(\vec{x}-\vec{y})$

$$\Longrightarrow \delta_\mu \Phi = [\hat{V}_\mu, \Phi] \equiv V_\mu \Phi$$

- The algebra is realised through multiple commutators

$$[\hat{V}_\mu[\hat{V}_\nu[\hat{V}_\rho, \Phi]]] + \text{perm.} = \eta_{\mu\nu}[\hat{P}_\rho, \Phi] + \eta_{\nu\rho}[\hat{P}_\mu, \Phi] + \eta_{\rho\nu}[\hat{P}_\mu, \Phi].$$

- We cannot realise the algebra in an Φ independent way weak application of the Noether theorem new structures in QFT

2. Lie algebras of order F

2.6 Cubic extension of the Poincaré algebra

Cubic extension of the Poincaré algebra

Some results (in any spacetime dimension)

1. Representations
   - Clifford algebras of the Polynomial $x_0 x_\mu x^\mu$
   - multiplet $=$ $p-$forms
     $A_{\mu_1 \cdots \mu_p} \to F_{\mu_1 \cdots \mu_p \mu_{p+1}}$ fully antisymmetric
   - natural transformations on $p-$forms
     $\longrightarrow$ wedge and inner products
2. construction of invariant actions
   $\longrightarrow$ duality properties (different from electromagnetic duality)
3. non-propagating $D-$ and $D-1-$forms
   $\longrightarrow$ like in IIA et IIB superstring
4. coupling to $p-$branes ???

Negatif points

1. problem with quantisation
   $\longrightarrow$ gauge fixing condition ghosts ??
2. In $D = 1 + 3$ and for a given multiplet
   $\longrightarrow$ NO interacting terms

3. Lie algebras of order F

3.1 Ternary supergroup

Formal study

Problem to have interesting physical models $\longrightarrow$ formal study
transformations generated by V

* **infinitesimal**
* Is there **fine transformations**?

A priori no

in group **the product of two elements is always defined**

the product of three V is defined in Lie algebras of order 3

Formal study

Construction of group by steps

1. define **universal enveloping algebra** =
   algebra of polynomial in the generators
2. endow this structure of a **Hopf algebra** structure
3. the benefit is double
   a. parameters of the transformation
   b. group associated two order three Lie algebras

Formal study

- **parameters:** 3−exterior algebra $\Lambda_3(\mathbb{R}^{1.4})$ generated by θ_μ

1. in vector representation of $\mathfrak{so}(1,3)$;
2. satisfying

$$\theta_\mu\theta_\nu\theta_\rho+\theta_\nu\theta_\rho\theta_\mu+\theta_\rho\theta_\mu\theta_\nu+\theta_\mu\theta_\rho\theta_\nu+\theta_\nu\theta_\mu\theta_\rho+\theta_\rho\theta_\nu\theta_\mu=0$$

cubic extension of Grassmann algebra

- infinite dimensional algebra
- $\mathbb{Z}_3$−graded algebra: $\Lambda_3=\Lambda_0\oplus\Lambda_1\oplus\Lambda_2$

Group of order 3

THEOREM

Let $A(\Lambda_0) \in \mathcal{M}_k(\Lambda_0)$ s.t
$A(\Lambda_0) = A_0 + A_1{}^{abc}\theta_a\theta_b\theta_c + \cdots$ (finite sum)
$A(\Lambda_0)$ is invertible *iff* A_0 is invertible. We define $GL_f(n, \Lambda_0)$ to be the set of invertible matrices s.t. $A(\Lambda_0)$ and $A^{-1}(\Lambda_0)$ finite sum.

Matricial group

Define the following matrices

1. $k \times k$ matrices $\mathcal{M}_k(\Lambda_0) = \mathcal{M}_k(\mathbb{C}) \otimes \Lambda_3(\mathbb{C}^m)_0$
2. $k \times \ell$ matrices $\mathcal{M}_{k,\ell}(\Lambda_1) = \mathcal{M}_{k,\ell}(\mathbb{C}) \otimes \Lambda_3(\mathbb{C}^m)_1$
2. $k \times \ell$ matrices $\mathcal{M}_{k,\ell}(\Lambda_2) = \mathcal{M}_{k,\ell}(\mathbb{C}) \otimes \Lambda_3(\mathbb{C}^m)_2$

Set $GL(m_1, m_2, m_3) = \left\{ \begin{pmatrix} A_0(\Lambda_0) & B_1(\Lambda_1) & C_2(\Lambda_2) \\ C_0(\Lambda_2) & A_1(\Lambda_1) & B_2(\Lambda_1) \\ B_0(\Lambda_1) & C_1(\Lambda_2) & A_2(\Lambda_0) \end{pmatrix} \right\}$ such that

$A_0(\Lambda_0) \in GL_f(m_1, \Lambda_0), \cdots$
$B_1(\Lambda_1) \in \mathcal{M}_{f\, m_1, m_2}(\Lambda_1), \cdots$
$C_2(\Lambda_1) \in \mathcal{M}_{f\, m_1, m_3}(\Lambda_2), \cdots$

then $GL(m_1, m_2, m_3)$ is a group

3. Lie algebras of order F

3.2 Ternary superspace

Ternary superspace

In supersymmetry there exists a

1. **superspace**
2. **superfields**
3. **invariant actions**

Formal study two-fold

TERNARY SUPERSPACE

$$[[\theta^\mu,\theta^\nu],\theta^\rho]=0, \qquad [[\theta^\mu,\theta^\nu],\partial_\rho]=-\delta^\mu{}_\rho\theta^\nu+\delta^\nu{}_\rho\theta^\mu$$
$$[[\theta^\mu,\partial_\nu],\theta^\rho]=\delta_\nu{}^\rho\theta^\mu, \qquad [[\theta^\mu,\partial_\nu],\partial_\rho]=-\delta^\mu{}_\rho\partial_\nu,$$
$$[[\partial_\mu,\partial_\nu],\theta^\rho]=-\delta_\mu{}^\rho\partial_\nu+\delta_\nu{}^\rho\partial_\mu \qquad [[\partial_\mu,\partial_\nu],\partial_\rho]=0.$$

$$\begin{aligned}
\{\theta^\mu,\theta^\nu,\theta^\rho\} &= 0,\\
\{\theta^\mu,\theta^\nu,\partial_\rho\} &= 2\delta^\mu{}_\rho\theta^\nu+2\delta^\nu{}_\rho\theta^\mu,\\
\{\theta^\mu,\partial_\nu,\partial_\rho\} &= 2\delta^\mu{}_\nu\partial_\rho+2\delta^\mu{}_\rho\partial_\nu,\\
\{\partial_\mu,\partial_\nu,\partial_\rho\} &= 0.
\end{aligned}$$

$\Longrightarrow$ Ternary superfields

4. Conclusion

- Symmetries $\rightarrow$ properties of particles
- Possible symmetries of the space-time
- Lie algebras of order F $\rightarrow$ higher order extensions of the Poincaré algebra
- Only free theories have been constructed need interacting theories
- Share some similarities with Lie superalgebras and Lie supergroups
- Some difficulties to construct realistic models superspace (parafermions) associated quadratic structure
- Interesting relationship between somequartic extensions of the Poincaré algebra and $N = 2$ supersymmetry.