GPDs on the lattice

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Plan

- List of difficult points
- Short reminder of lattice QCD
- GPDs lattice teams
- Special problems with GPDs lattice calculations
- Results review
- where we are and where we should go

List of difficult points

- continuum limit $a \rightarrow 0$
- Infinite volume limit
- Zero temperature limit (excited state mixing)
- Chiral extrapolation $m_\pi \rightarrow 140MeV$
- Disconnected contractions

Continuum theory

$$\mathcal{L}_{QCD} = \bar{\psi}(i\gamma_\mu D_\mu - M)\psi - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a$$

- fermions part

$$D_\mu = \partial_\mu + igA^a_\mu \frac{\lambda_a}{2}$$

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- $\mathcal{L}_{QCD}$ invariant under local gauge transformations

$$\begin{aligned} \psi(x) &\rightarrow G(x)\psi(x), \\ A_\mu(x) &\rightarrow G(x)(A_\mu(x) - ig\partial)G(x)^{-1} \end{aligned}$$

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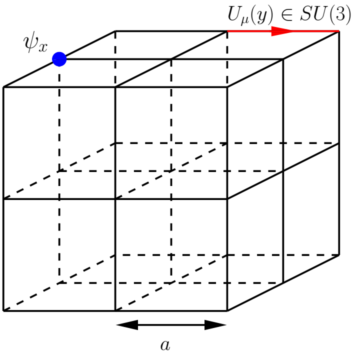
- any approximation should respect this invariance because it is the source of the interactions.

Discrétization

- 4D lattice



- $U_\mu(x) = e^{igaA_\mu(x)}$
- Volume : $(24^3 \times 48)$ up to $(48^3 \times 96)$
- Lattice spacing : $a \simeq 0.05 \div 0.1$ fm

- If S_{QCD} is discretization of the QCD action

$$\langle O[U,\psi,\bar{\psi}]\rangle = \frac{1}{Z}\int \prod_{\text{sommets}} [d\bar{\psi}][d\psi] \prod_{\text{liens}} [dU]\, \mathcal{O}[U,\psi,\bar{\psi}] e^{-S_{QCD}[U,\psi,\bar{\psi}]}$$

Principe of lattice calculation

- A typical corrélator

$$\langle \mathcal{O}(t+\tau)\mathcal{O}(t)\rangle, \tau \geqslant 0$$

- can be evaluated on the lattice as

$$\langle \mathcal{O}(t+\tau)\mathcal{O}(t)\rangle = \frac{1}{Z}\int [d\psi d\bar{\psi} dU]\, \mathcal{O}(t+\tau)\mathcal{O}(t) e^{-S[U,\psi,\bar{\psi}]}$$

- on the other hand (if one work in Euclidian space-time)

$$\begin{aligned} \langle \mathcal{O}(t+\tau)\mathcal{O}(t)\rangle &= \langle 0|\mathcal{O} e^{-H\tau}\mathcal{O}|0\rangle \\ &= \sum_n |\langle 0|\mathcal{O}|n\rangle|^2 e^{-M_n\tau} \end{aligned}$$

- More generally the physical information is extracted from the time dependance of some correlator.

Fermion integration and Monte Carlo

Fermions are not ordinary numbers. They cant be integrated numerically but they can be integrated analytically :

$$\langle O[U,\psi,\bar{\psi}]\rangle = \frac{1}{Z}\int \prod_{\text{sommets}} [d\bar{\psi}][d\psi] \prod_{\text{liens}} [dU]\, e^{-S_{QCD}[U,\psi,\bar{\psi}]}\, \mathcal{O}[U,\psi,\bar{\psi}]$$

$$= \frac{1}{Z}\int \prod_{\text{liens}} [dU]\ \det \mathcal{D}[U] e^{-S_g[U]}$$

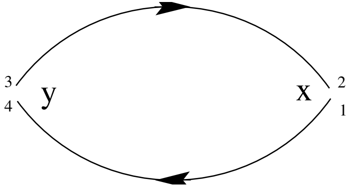
$$\times \sum_{\textit{all contractions in } \mathcal{O}} \Big[\text{propagateurs}\Big]_{\text{U}}$$

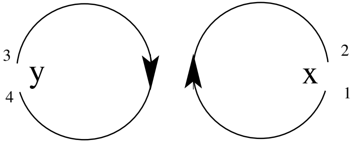
- $\det \mathcal{D}[U]$ is evaluated with pseudo-fermions method
- U is sampled with the hybrid Monte Carlo method (molecular dynamics)
- propagators are evaluated by inverting Dirac equation, a large sparse linear system

Example

This operator $\langle q_3(y)\bar{q}_4(y)\ \ q_1(x)\bar{q}_2(x)\rangle$ involves contractions

like this



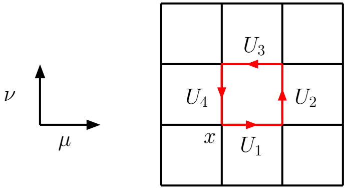


comment

- Disconnected contractions are one of the hard points (cost).
- Generally neglected.
- Do not contribute to $u - d$ combination

Gluon action

- Plaquette



$$U(x,\mu,\nu) = U_4 U_3 U_2 U_1$$

- Wilson action(1974)

$$S_g = \sum_{\text{Plaquettes}} \beta \left[1 - \frac{1}{3} \mathrm{Re}\,\mathrm{Tr}[U(x,\mu,\nu)] \right]$$

- with $\beta = 6/g^2$ then

$$S_g \xrightarrow[a \to 0]{} S_{YM} = \frac{1}{4} \int d^4x F^a_{\mu\nu} F^{\mu\nu}_a$$

Fermion action and active GPDs groups

Discretization

- discretize $\bar{\psi}(i\gamma_\mu D_\mu - M)\psi$ in a consistent way
- several solutions according to chiral symmetry implementation
  - chiral symmetry exists **at finite lattice spacing** : Ginsparg-Wilson, domain wall, overlap
  - partial chiral symmetry exists **at finite lattice spacing** : staggered quarks
  - chiral symmetry exists only **at zero lattice spacing** : Wilson, Wilson clover, twisted mass

GPDs active groups

- LHPC : staggered quarks in the sea, domain wall in valence **arXiv :1001.3620**
- QCDSF : Wilson clover **arXiv :1101.2326v1**
- ETMC : twisted mass **arXiv :1104.1600v1**

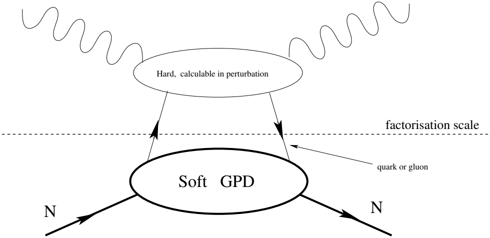
Continuum limit

- what we compute is (for instance) $a * M = f(g)$ because there is no mass scale in the action
- g must be a function of a in order that $M = \frac{f(g)}{a}$ be finite when $a \to 0$
- Asymptotic freedom tells that $g(a) \to 0$ when $a \to 0$
- so the continuum theory is at $\beta = \frac{6}{g^2} \to \infty$.

Continuum limit

Observables must be independent of β when $\beta \to \infty$.

GPDs



generic GPD is

$$F(x, \xi, \mu^2 = Q^2) = \int d\lambda e^{ix\lambda} \langle N(p')|\bar{q}(-\lambda n/2)[\cdots]q(\lambda n/2)|N(p)\rangle$$

where n is a light-like vector along $p + p'$.

$$GPD \sim \int d\lambda e^{ix\lambda} \langle N(p')|\bar{q}(-\lambda n/2)[\cdots]q(\lambda n/2)|N(p)\rangle \quad (1)$$

Space-time is Euclidian : $n^2 = 0 \Rightarrow n = 0$.

Operators separated by light-like distance cannot be computed directly on the lattice.

Way out

Expand $q(\pm\lambda n/2)$ in power serie. Operators to compute on the lattice are then ($\partial \to D$ for gauge inv.)

$$O^{\mu\nu\cdots} = \bar{q}(0)[\cdots D^{\mu} D^{\nu} \cdots]q(0)$$

Note

In the serie only the symmetric traceless part of $O^{\mu\nu\cdots}$ is in play. This is implicit below

$P = (p + p')/2 \quad \Delta = p' - p$

Generalized form factors.

$$\begin{aligned}
\langle p'|\bar{q}(0)\gamma^\mu q(0)|p\rangle &= \bar{u}(p')[A_1(t)\gamma^\mu + i\frac{B_1(t)}{2m}\sigma^{\mu\alpha}\Delta_\alpha]u(p) \\
\langle p'|\bar{q}(0)\gamma^\mu D^\nu q(0)|p\rangle &= \\
\bar{u}(p')[A_2 P^\mu \gamma^\mu &+ i\frac{B_2}{2m}P^\mu \sigma^{\nu\alpha}\Delta_\alpha + \frac{C_2}{m}\Delta^\mu \Delta^\nu]u(p) \quad \color{red}{etc...}
\end{aligned}$$

Note that $A_2(t=0) =< x >$ and $A_2(t=0) + B_2(t=0) = J$

Mellin moments

$$A_1(t) = \int dx H(x,t,\xi), \quad B_1(t) = \int dx E(x,t,\xi)$$

$$A_2 + 4\xi^2 C_2 = \int dx x H, \quad B_2 - 4\xi^2 C_2 = \int dx x E$$

Operator renormalization 1

- On the lattice we replace the operator A by a discretized version $A(a)$
- what we (can) compute is $\lim_{a\to 0}\langle A(a)\rangle$
- In general $\lim_{a\to 0}\langle A(a)\rangle \neq \langle \lim_{a\to 0} A(a)\rangle = \langle A\rangle$

Operator renormalisation relates

- what we want : $\langle A\rangle$
- to what we compute $\lim_{a\to 0}\langle A(a)\rangle$

Operator renormalization 2

Wilson analysis

the continuum limit is achieved by using a renormalized operator $A_R(\mu)$ such that

$$A_R(\mu) = \lim_{a \to 0} Z(a\mu, \beta) A(a) + \sum_i Z^i(a\mu, \beta) A^i(a)$$

- The A^i are limited by their dimensions and their symmetry.
- We consider only operators which have **no mixing** (multiplicative renormalization)

$$A_R(\mu) = \lim_{a \to 0} Z(a\mu, \beta) A(a)$$

- This what limits the number of moments that we can compute.

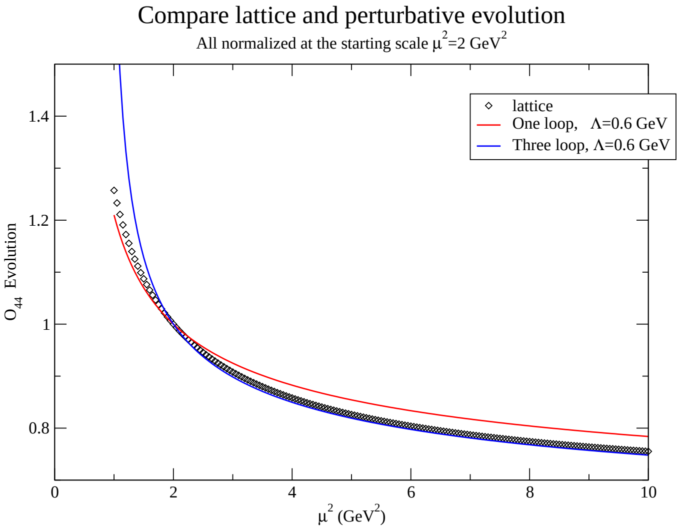
Z calculation

RI-MOM scheme

- The Zs are computed non perturbatively
- by imposing normalisations conditions to matrix elements between (off-shell) quark states of virtuality $p^2 = \mu^2$ (MOM)
- Need to fix the gauge.
- Need a good control of hypercubic lattice artefacts.

At large enough μ the $Z(RI - MOM)$ are related to $Z(\overline{MS})$ through perturbation theory.

Operator evolution



Some results

Limitations

- Disconnected contractions are NOT included
- Only first few moments are calculated (operator mixing)

3 extrapolations involved

- Continuum limit
- Infinite volume
- $m_\pi \to 140 MeV$

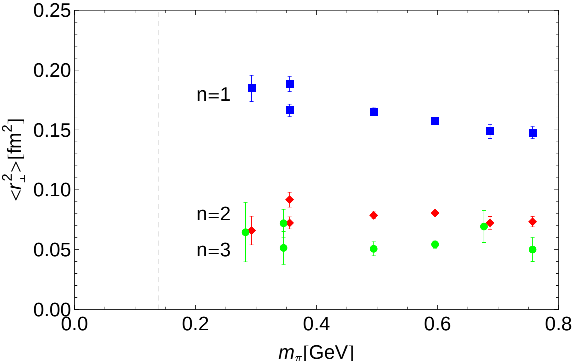
Transverse size vs x (LHPC)

Up to terms of order ξ^2 $\quad A_n(t) \sim \int dx x^{n-1} H(x,t,\xi)$

so large x in H corresponds to large n in A_n

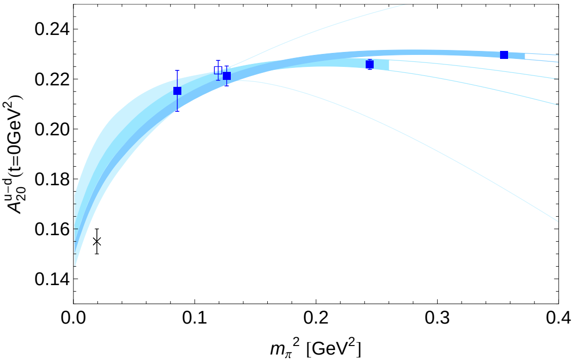
Define effective transverse radius

$$A_n(t) = A_n(0)\left(1 + \langle r_\perp^2 \rangle t/6\right)$$



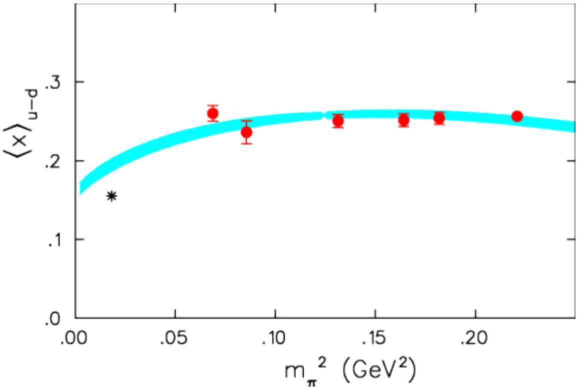
Momentum fraction from LHPC

$$< x >= A_2(0)$$



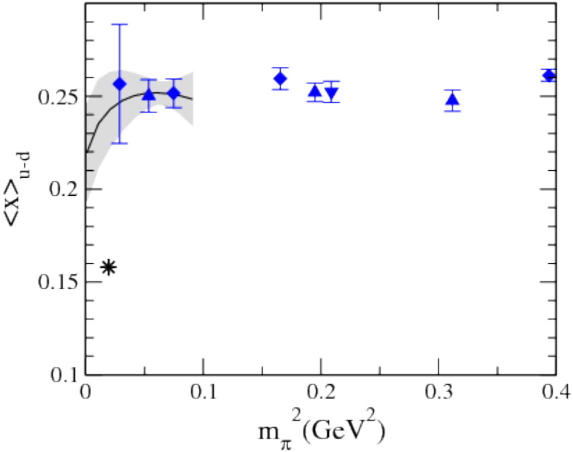
Momentum fraction from ETMC

$$< x >= A_2(0)$$

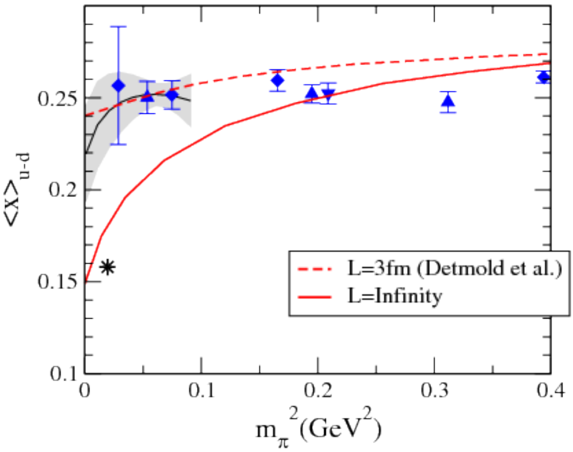


Momentum fraction from QCDSF

$< x >= A_2(0)$ smallest pion mass : 170MeV !



Finite volume effect

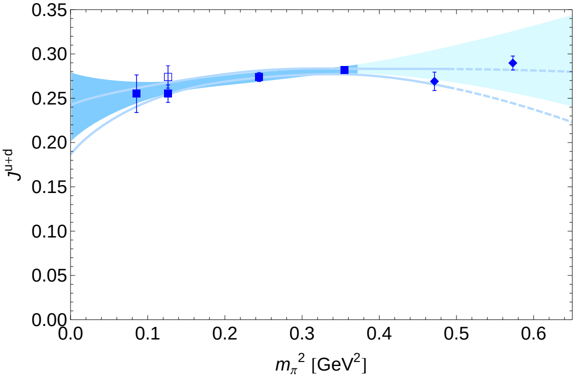


Finite volume effect from W. Detmold, W. Melnitchouk, A.W. Thomas arXiv :hep-lat/0310003v1

Angular momentum, LHPC

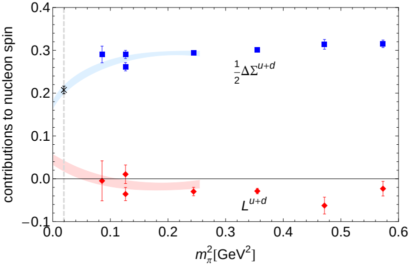
Similar results from ETMC.
No result from QCDSF.

$$J_q = \frac{1}{2}[A_2(0) + B_2(0)]_{\mu=2GeV}$$



Orbital angular momentum, LHPC

Write $L_q = J_q - \frac{1}{2}\Sigma_q$ and get Σ_q from $\langle\gamma_5\gamma_\mu\rangle$



- $L_q \sim 0$ at scale $\mu = 2GeV$
- This is $u + d$. The role of disconnected contractions is unknown.
- This is the gauge invariant orbital momentum $r \times (\nabla - A)$

List of difficult points

- continuum limit $a \to 0$ not bad
- continuum limit (renormalisation, operator mixing) still a lot to do
- Infinite volume limit critical ! Next generation of simulation (and computer)
- Zero temperature limit (excited state mixing) not too bad, anyway will be fixed by previous point
- Chiral extrapolation soon the end of it (oh yes ! ! ! ! ! ! ! ! ! ! !)
- Disconnected contractions everyone hopes they are negligible. Hard to motivate

End

additions

$$m_\pi = 140\ \mathrm{MeV}, m_\pi L = 3.5, a = 0.05\ \mathrm{fm} \to L/a \sim 100$$