Quantum integrable models with boundaries and q-Onsager algebras.

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Quantum integrable models

Quantum integrable models and hidden symmetries

The Quantum Integrability is a mimic of the classical Liouville's definition :

Definition

A quantum model with Hamiltonian H is said Integrable if we can find N algebraic independent operators H_i such that H and H_i form an abelian algebra. N is the number of degrees of freedom of H.

Quantum integrability ⟷ possible to calculate exactly quantities of physical interest

Hidden symmetry (HS) $=$ non-abelian algebra with $\{H, H_i\}$ as abelian subalgebra

Different families of the quantum integrable models

- One-dimensional Spin chains (XXX, XXZ, XYZ , Hubbard model, generalizations,...)
- Two-dimensional Quantum field theories (Conformal models, Chiral principal models, affine Toda models, sigma models,...)
- Two-dimensional Statistical models (Ising, Chiral-Potts, 6-8 vertex, ...)

All these models are related by the R-matrix solution of the Yang Baxter equation (Star-triangle equation, factorization equation), (Yang 1967, Baxter 1971-73) :

$$R_{12}(u,v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u,v).$$

- One-dimensional Spin chains : $l(u) = tr_0(L_0(u)) = tr_0(R_{01}(u)\dots R_{0N}(u))$ are generating functions of Halmitonian (Leningrad school 1979).
- Two-dimensional Quantum field theories : scattering amplitudes $S(u) \propto R(u)$ (Zamolodchikov and Zamolodchikov 1979).
- Two-dimensional Statistical models : Boltzmann weights are related to $R(u)$ (Baxter 1982).

Quantum integrable models with boundary

For Quantum integrable models with boundary we need one more object, the K-matrix, solution of the Reflection equation ($\tilde{R}, \hat{R}, \bar{R}$ are related to R and depends of the type of boundary) (Cherednik 1984, Sklyanin 1988) :

$$R_{12}(u,v)K_1(u)\bar{R}_{12}(u,v)K_2(v) = K_2(v)\tilde{R}_{12}(u,v)K_1(u)\hat{R}_{12}(u,v).$$

- One dimensional Spin chains : $K(u)$ encodes the boundary of the spin chain and appears in the construction of the generating functions of the hamiltonian (Sklyanin 1988).
- Two dimensional Quantum field theories : boundary scattering amplitudes $B(u) \propto K(u)$ (Cherednik 1984).
- Two dimensional Statistical models : boundary Boltzmann weights are related to $K(u)$.

$\Rightarrow$ **Solution of Yang Baxter equation and Reflection equation are central for integrable models but both are nonlinear equations hard to solve directly.**

Quantum affine Toda field theory

The Quantum affine Toda field theory on the line are defined by the Euclidean action :

$$A_{bulk} = \frac{1}{4\pi}\int dxdt\Big(\partial\phi\bar{\partial}\phi + \frac{\lambda}{2\pi}\sum_{j=0}^{n} n_j \exp\big(-i\hat{\beta}\frac{1}{|\alpha_j|^2}\alpha_j\cdot\phi\big)\Big)$$

where $\phi(x,t)$ is an $n-$component bosonic field, $\{\alpha_j\}$ and n_j are the simple roots and the Kac labels of $\widehat{g}$, λ is related with the mass scale and $\widehat{\beta}$ is the coupling constant.

Remark

- **non-local conserved currents** *close on the algebra* $U_q(\widehat{g})$ *which is the symmetry algebra, (Bernard and Leclair 1991)*

From this symmetry algebra we can compute the scattering matrix S up to a scalar function **from a linear equation**. Unitarity and Crossing symmetry constrain the scalar function.

Quantum affine Toda field theories on the half-line

The Quantum affine Toda field theory on the half-line (soliton non-preserving) are defined by the Euclidean action :

$$A_{boundary} = S_{bulk}|_{x<0} + \frac{\lambda_b}{2\pi} \int dt \sum_{j=0}^{n} \epsilon_j \exp\left(- i\frac{\hat{\beta}}{2}\alpha_j \cdot \phi(0,t)\right)$$

λ_b is related with the mass scale and $\{\epsilon_j\}$ are the boundary parameters or operators.

Remark

- **non local conserved currents** *are in* $B \subset U_q(\widehat{g})$ *(Mezincescu and Nepomechie 1998, Delius and MacKay 2002).*

We can compute the boundary scattering matrix K up to a scalar function **from linear equation**. Unitarity and Crossing symmetry constraint the scalar function.

Quantum algebras and their subalgebras

Different presentation of the Quantum algebra $U_q(\widehat{g})$

Ding and Frenkel (1993)

RLL algebra
Leningrad School (1979,1989)

Current algebra
Drinfel'd (1988)

Reshetikhin and
Semenov-Tian-Shansky (1990)

$U_q(\widehat{g})$

Jing (1998)
Beck (1994)

q-Serre-Chevalley algebra
Drinfel'd (1986) Jimbo (1986)

q-Serre-Chevalley presentation

Let $\{a_{ij}\}$ be the extended Cartan matrix of the an affine Lie algebra $\widehat{g}$ and integers d_i are such that $d_i a_{ij}$ is symmetric. $U_q(\widehat{g})$ is an associative algebra over $\mathbb{C}$ with unit 1 generated by the elements $\{x_i^{\pm}, q_i^{\pm\frac{h_i}{2}}\}$, $i \in 0 \dots n$ subject to the relations (Drinfeld 1986, Jimbo 1986) :

$$q_i^{\pm\frac{h_i}{2}} q_i^{\mp\frac{h_i}{2}} = 1, \qquad q_i^{\frac{h_i}{2}} q_j^{\frac{h_j}{2}} = q_j^{\frac{h_j}{2}} q_i^{\frac{h_i}{2}} ,$$

$$q_i^{\frac{h_i}{2}} x_j^{\pm} q_i^{-\frac{h_i}{2}} = q_i^{\pm\frac{a_{ij}}{2}} x_j^{\pm}, \quad [x_i^{+}, x_j^{-}] = \delta_{ij} \frac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}} ,$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \left[\begin{array}{c} 1 - a_{ij} \\ r \end{array} \right]_{q_i} (x_i^{\pm})^{1-a_{ij}-r} x_j^{\pm} (x_i^{\pm})^r = 0 .$$

with $q_i = q^{d_i}$, $\left[\begin{array}{c} b \\ r \end{array} \right]_q = \frac{[b]_q!}{[r]_q![b-r]_q!}$, $[b]_q! = [b]_q[b-1]_q \dots [1]_q$ and $[b]_q = \frac{q^b - q^{-b}}{q - q^{-1}}$.

- **Hopf structure** : $\Delta(x_i^{\pm}) = x_i^{\pm} \otimes q_i^{-\frac{h_i}{2}} + q_i^{\frac{h_i}{2}} \otimes x_i^{\pm}, \quad \Delta(q_i^{\pm\frac{h_i}{2}}) = q_i^{\pm\frac{h_i}{2}} \otimes q_i^{\pm\frac{h_i}{2}}$.

RLL presentation

$U_q(\widehat{g})$ is a quasi-triangular Hopf algebra, i.e. there is an R-matrix such that (intertwiner equation) :

$$R_{12}(u/v)(\pi_u \otimes \pi_v)\Delta(x) = (\pi_u \otimes \pi_v)\,\sigma \circ \Delta(x) R_{12}(u/v), \quad x \in \mathcal{U}_q(\widehat{g})$$

with $\sigma(a \otimes b) = b \otimes a$. For $\pi_u \otimes \pi_v$ an irreducible representation, R is unique and is a solution of the Yang-Baxter Equation :

$$R_{12}(u/v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u/v).$$

The algebra $U_q(\widehat{g})$ can be formulate with generators $L(u) = \sum_{ij}\sum_{n=0}^{\infty} E_{ij} L_{ij}^{(n)} u^{-n}$ subject to the relations (Leningrad school 80'ies) :

$$R_{12}(u/v)L_1(u)L_2(v) \quad = \quad L_2(v)L_1(u)R_{12}(u/v),$$

- **Hopf structure** : $\Delta(L(u)) = L(u)\dot{\otimes}L(u)$

Applications of different presentations

- q-Serre Chevalley presentation : construction of the R-matrices from the intertwiner equation (Jimbo 1986), study of the finite dimensional representations (Chary Presley 1991),...

- RLL presentation : for $c = 0$ construction of the abelian algebra $l(u) = tr(L(u))$ with $[l(u), l(v)] = 0$, diagonalisation of $l(u)$ using the algebraic Bethe ansatz method, calculation of correlation functions (for $\widehat{g} = \widehat{sl}_2$ Lyon group's 1998),...

- Current presentation : Infinite dimensional representation (Frenkel and Jing 1988), Vertex operators and Correlation functions of infinite XXZ spin chains (Jimbo and all 1992), Scalar products of the eigenvectors of the $U_q(\widehat{sl}_3)$ spin chains (Belliard, Ragoucy and Pakuliak 2010),...

Reflection algebras

There are two families of reflection algebras, $B_q(\widehat{g})$ and $B_q^*(\widehat{g})$, generated by the generators $\mathcal{K}(u)$ and $\mathcal{K}^*(u)$, respectively, subject to the relations (Cherednik 1984, Sklyanin 1988) :

$$
\begin{aligned}
R_{12}(u/v)\mathcal{K}_1(u)R_{21}(uv)\mathcal{K}_2(v) &= \mathcal{K}_2(v)R_{12}(uv)\mathcal{K}_1(u)R_{21}(u/v) \\
R_{12}(u/v)\mathcal{K}_1^*(u)R_{12}^{t_1}(1/uv)\mathcal{K}_2^*(v) &= \mathcal{K}_2^*(v)R_{12}^{t_1}(1/uv)\mathcal{K}_1^*(u)R_{12}(u/v)
\end{aligned}
$$

These algebras are two subalgebras of $\mathcal{U}_q(\widehat{g})$:

$$\mathcal{K}(u) = L(u)K(u)L(u^{-1})^{-1}, \quad \mathcal{K}^*(u) = L(u)K^*(u)L(u^{-1})^t.$$

- **Coideal structure**, $\delta : B_q^{(*)}(\widehat{g}) \to \mathcal{U}_q(\widehat{g}) \otimes B_q^{(*)}(\widehat{g})$.

$\delta(\mathcal{K}(u)) = L(u)\mathcal{K}(u)L(u^{-1})^{-1}$, $\delta(\mathcal{K}^*(u)) = L(u)\mathcal{K}^*(u)L(u^{-1})^t$.

Remark :

- Automorphisms : $\mu_1(L(u)) = L(u^{-1})^{-1}$ and $\mu_2(L(u)) = L(u^{-1})^t$.
- If the matrix R satisfy the crossing symetry $R_{12}(u) = M_2 R_{12}^{t_1}(u^{-1}a)M_2$ with $M^2 = \mathbb{I}$ then $\mathcal{K}^*(u) = \mathcal{K}(u\sqrt{a})M$.

$\Rightarrow$ **Other presentations of the reflection algebras ?**

Other presentations of the reflection algebras

- In the case of finite Lie algebra g, reflection algebras appear in the study of quantum symmetric spaces Noumi and Sugitani (1995) :
- $\mathcal{K}$: deformation of $AIII = SU(n+p)/(SU(n) \times SU(p))$.
- $\mathcal{K}^*$: deformation of $AI = SU(n)/SO(n)$ or $AII = SU(2n)/Sp(2n)$.

- For the case infinite Lie algebra $\widehat{g}$, the generator of the symmetry algebra of the quantum Toda field theory on the half-line are given by :

$$A_i = c_i(x_i^+ q_i^{h_i/2} + x_i^- q_i^{h_i/2}) + w_i q_i^{h_i}.$$

Remark :

- The operator A_i belong to a subalgebra of $U_q(\widehat{g})$ invariant by the automorphism : $\eta(x_i^{\pm}) = x_i^{\mp}, \quad \eta(h_i) = -h_i, \quad \eta(q_i) = q_i^{-1}$ (i.e. $\eta(A_i) = A_i$).
- Coideal structure : $\Delta(A_i) = c_i(x_i^+ q_i^{h_i/2} + x_i^- q_i^{h_i/2}) \otimes 1 + q_i^{h_i} \otimes A_i$.

$\Rightarrow$ **On which algebra these generator close ?**

q-Onsager algebras

Definition (Baseilhac and Belliard 2010)

Let $\{a_{ij}\}$ be the extended Cartan matrix of the an affine Lie algebra $\widehat{g}$. The generalized $q-$Onsager algebra $\mathcal{O}_q(\widehat{g})$ is an associative algebra with unit 1, elements A_i and scalars $\rho_{ij}^k \in \mathbb{C}$ with $i,j \in \{0,1,...,n\}$, $k \in \{0,1,...,[-\frac{a_{ij}}{2}]-1\}$ [1] and $l \in \{0,1,...,-a_{ij}-1-2\,k\}$ (k and l are positive integer). The defining relations are (the γ_{ij}^{kl} are known) :

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \left[\begin{array}{c} 1-a_{ij} \\ r \end{array} \right]_{q_i} \mathsf{A}_i^{1-a_{ij}-r} \mathsf{A}_j \, \mathsf{A}_i^r =$$

$$\sum_{k=0}^{[-\frac{a_{ij}}{2}]-1} \rho_{ij}^k \sum_{l=0}^{-2\,k-a_{ij}-1} (-1)^l \, \gamma_{ij}^{kl} \, \mathsf{A}_i^{-2\,k-a_{ij}-1-l} \mathsf{A}_j \, \mathsf{A}_i^l \ ,$$

- **Coideal structure**, $\delta : \mathcal{O}_q(\widehat{g}) \to \mathcal{U}_q(\widehat{g}) \otimes \mathcal{O}_q(\widehat{g})$.

$$\delta(A_i) = c_i(x_i^+ q_i^{h_i/2} + x_i^- q_i^{h_i/2}) \otimes 1 + q_i^{h_i} \otimes A_i$$

with relations between the c_i and the ρ_{ij}.

1. $[a]$ means the nearest higher integer of a with [1/2]=1.

Some direct applications

- Admissible integrable scalar conditions ϵ_i for quantum affine Toda field theories on the half-line are given by the one dimensional representation $\mathcal{E}$ of the q-Onsager algebra. In the classical limit $q \to 1$ we recover the known results (Corrigan and col. 1994-1995).
- Construction of the scalar K-matrices from intertwiner relations :

$$K^*(u)\left(\pi_u \otimes \mathcal{E}\right) \circ \delta(A_i) = \left(\bar{\pi}_{u^{-1}} \otimes \mathcal{E}\right) \circ \delta(A_i)\, K^*(u)$$

  for $a_n^{(1)}$ and $d_n^{(1)}$ (Delius and MacKay 2002-2003).
- Admissible integrable algebraic "dynamical" conditions for quantum affine Toda field theories on the half-line :
  $\epsilon_i = W_i = \psi(A_i)$ with ψ an homomorphism of q-Onsager algebras (Bazhanov, Hibberd and Khoroshkin 2002, Baseilhac and Koizumi 2003).
- Construction of Dynamical K-matrices from intertwiner relations

$$K^*(u)\left(\pi_u \otimes \psi\right) \circ \delta(A_i) = \left(\bar{\pi}_{u^{-1}} \otimes \psi\right) \circ \delta(A_i)\, K^*(u)$$

  case $a_2^{(2)}$ and $a_2^{(1)}$ (Belliard and Fomin, in preparation)

Example : the case of $\mathcal{O}_q(a_2^{(2)})$

- related physical models : Bullough-Dodd, Mikhailov-Zhiber-Shabat or Izergin-Korepin models.

The algebra $\mathcal{O}_q(a_2^{(2)})$ is given by the relations :

$$
\begin{aligned}
&[A_0,[A_0,A_1]_{q^4}]_{q^{-4}} - \rho A_1 = 0,\\
&[A_1,[A_1,[A_1,[A_1,[A_1,A_0]_{q^4}]_{q^{-4}}]_{q^2}]_{q^{-2}},\ A_0]\\
&\qquad -[A_1,\bar\rho\,(A_1\,A_1\,A_0 - w\,A_1\,A_0\,A_1 + A_0\,A_1\,A_1) + \tilde\rho\,A_0] = 0,\\
&w = \frac{(q-1+q^{-1})(q^4+2\,q^2+4+2\,q^{-2}+q^{-4})}{q^4+3+q^{-4}}.
\end{aligned}
$$

With the coaction :

$$
\begin{aligned}
\delta(\mathsf{A}_1) &= c_1(x_1^+ q^{h_1/2} + x_1^- q^{h_1/2}) \otimes 1 + q^{h_1} \otimes \mathsf{A}_1\ ,\\
\delta(\mathsf{A}_0) &= c_0(x_0^+ q^{2h_0} + x_0^- q^{2h_0}) \otimes 1 + q^{4h_0} \otimes \mathsf{A}_0\ .
\end{aligned}
$$

Dynamical K-matrix

- A finite dimensional subalgebra is given by the Zhedanov or Askey-Willson algebra (Zhedanov 1992) :

$$[\mathsf{W}_1, [\mathsf{W}_1, \mathsf{W}_0]_{q^4}]_{q^{-4}} = \mu \mathsf{W}_0 + C, \qquad [\mathsf{W}_0, [\mathsf{W}_0, \mathsf{W}_1]_{q^4}]_{q^{-4}} = \bar{\mu} \mathsf{W}_1$$

The map $\psi(\mathsf{A}_i) \rightarrow W_i$ defines an algebra homomorphism (with $\rho, \bar{\rho}$ and $\tilde{\rho}$ function of q, μ and $\bar{\mu}$).

- For this case, the two family of reflection algebra are equivalent (crossing symmetrie). We consider the intertwiner relation :

$$K^d(u) \left(\pi_u \otimes \psi\right) \circ \delta(\mathsf{A}_i) = \left(\pi_{u^{-1}} \otimes \psi\right) \circ \delta(\mathsf{A}_i) \, K^d(u).$$

If $K^d(u)$ existe and the space is irreducible then $K^d(u)$ is unique (up to a scalar function) and is a solution of the reflection equation :

$$R_{12}(u/v) K_1^d(u) R_{12}(uv) K_2^d(v) = K_2^d(v) R_{12}(uv) K_1^d(u) R_{12}(u/v).$$

A solution of the intertwiner relation is given by :

$$K(u) = K^{(1)}(u) + K^{(2)}(u)$$

with

$$K^{(1)}(u) = \begin{pmatrix} h_1(u,C,Q) & 0 & h(u,C) \\ 0 & h_2(u,C,Q) & 0 \\ h(u,C) & 0 & h_3(u,C,Q) \end{pmatrix},$$

and

$$K^{(2)}(u) =$$
$$\begin{pmatrix} \alpha\beta g\, u\, W_1^2 + \frac{q}{\beta(q+q^{-1})} W_0 & q^{-\frac{7}{2}}[W_1, W_0]_{q^{-4}} + q^{\frac{7}{2}} \alpha\, u\, W_1 & \beta\, [[W_0, W_1]_{q^4}, W_1]_{q^4} \\ q^{-\frac{7}{2}}[W_0, W_1]_{q^{-4}} + q^{\frac{7}{2}} \alpha\, u\, W_1 & -\frac{q^4}{\beta} W_0 & q^{\frac{7}{2}}[W_0, W_1]_{q^4} + q^{-\frac{7}{2}} \alpha\, u^{-1}\, W_1 \\ \beta\, [W_1, [W_1, W_0]_{q^4}]_{q^4} & q^{\frac{7}{2}}[W_1, W_0]_{q^4} + q^{-\frac{7}{2}} \alpha\, u^{-1}\, W_1 & -\alpha\beta \frac{g}{q^8\, u} W_1^2 + \frac{q^7}{\beta(q+q^{-1})} W_0 \end{pmatrix}.$$

Dynamical boundary amplitudes

The Dynamical boundary amplitudes of the Dynamical boundary Toda field theory $a_2^{(2)}$ with imaginary coupling is given by :

$$B(u) = k(u)K^d(u)$$

with satisfy the axiom of :

- Unitarity :

$$B(u)B(u^{-1}) = B(u^{-1})B(u) = 1$$

- Crossing symmetrie :

$$tr_2\Big(S_{12}(u^2)M_1^{t_1}B^{t_1}(-q^{-6}u^{-1})M_1P_{12}\Big) = B_1(u)$$

It fixes $B(u)$ up to CDD factor $b(u)$ such that $b(u)b(u^{-1}) = 1$ and $b(u) = b(-q^{-6}u^{-1})$.

Conclusion

The results and work in progress

- The symmetry algebra of the affine Toda field theories on the half-line with non preserving boundary conditions has been obtained and allowed to classify admissible boundary condition (scalar, dynamic) : $O_q(\widehat{g})$. (Baseilhac and Belliard 2010)
- For the case $O_q(a_1^{(1)})$ a systematic derivation of the integrable hierarchy has been obtained. (Baseilhac and Belliard 2010)
- For the case $O_q(a_2^{(2)})$ and $O_q(a_2^{(1)})$ the dynamical K-matrices are obtained (Belliard and Fomin 2011).
- Dynamical amplitudes for quantum Toda fields theories related to $O_q(a_2^{(2)})$ and $O_q(a_2^{(1)})$, weak strong duality ? (Belliard, in preparation).
- Same result could be obtain for the Principal Chiral model : $\mathcal{Y}^t(a_1)$ (Belliard and Crampé, in preparation).

Some open problems and direction of research

- Finite dimensional representation theory of $\mathcal{O}_q(\widehat{g})$?
  -Case $O_q(\widehat{sl}_2)$: Tridiagonal pairs (Terwilliger and Ito 2008, Baseilhac 2006)
  -Case $O_q(\widehat{sl}_n)$: evaluation to the Gavrilik and Klimyk algebra $= O_q(sl_n)$ and use the this representation theory (Gavrilik and Iorgov 1997)

  $\Rightarrow$ application to the diagonalisation of finite spin chains (Baseilhac and Kozumi (2007) for XXZ spin chain), Orthogonal polynomials,...

- Infinite dimensional representation theory of $\mathcal{O}_q(\widehat{g})$, Vertex operators.
  -Case $O_q(\widehat{sl}_2)$: Baseilhac and Belliard (2011)

  $\Rightarrow$ application to semi-infinite spin chains (Jimbo and all), construction of boundary Q Baxter operator,...

- "q-Serre-Chevalley" formulation for $\mathcal{Y}^t(g)$, case other families of reflection equations, clarify the link with affine symmetric spaces,...