

Charmed-Meson Decay Constants from Improved QCD Sum Rules

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Incentive: improving QCD sum rules [1–14]

Within the method of QCD sum rules, the concept of quark–hadron duality is implemented by [assuming](#) that above a certain ‘continuum threshold’ the contributions to suitably defined correlators at the level of the QCD degrees of freedom equal those at the level of hadronic bound states. Here we seek to quantify the uncertainty induced by this [approximation](#) and to improve the accuracy of predictions by allowing the threshold to depend on the involved momenta and parameters introduced upon applying Borel transformations.

Test area: nonrelativistic potential models

This idea is best tested in a situation where the outcome for all bound-state characteristics is known [exactly](#). So, we study a quantum-mechanical model defined by a nonrelativistic Hamiltonian with harmonic-oscillator potential

$$H = \frac{\mathbf{p}^2}{2m} + \frac{m\omega^2 r^2}{2}, \quad r \equiv |\mathbf{x}|.$$

For this problem, the exact solutions may be given even analytically: energy E_g , decay constant $\sqrt{R_g}$, and form factor $F_g(q)$ of its ground state (g) read

$$E_g = \frac{3}{2}\omega, \quad R_g \equiv |\psi_g(\mathbf{0})|^2 = \left(\frac{m\omega}{\pi}\right)^{3/2}, \quad F_g(q) = \exp\left(-\frac{q^2}{4m\omega}\right);$$

for potentials not possessing analytic solutions, numerical integration of the Schrödinger equation by well-known routines[15] provides the exact results.

Quantum mechanics: correlation functions

For harmonic-oscillator potentials all correlators are given [analytically](#), e.g.,

- the Borelized 2-point vacuum correlator (polarization operator) [1–3,6],

$$\Pi(T) \equiv \langle \mathbf{x}_f = \mathbf{0} | \exp(-H T) | \mathbf{x}_i = \mathbf{0} \rangle \stackrel{\text{HO}}{=} \left[\frac{m \omega}{2 \pi \sinh(\omega T)} \right]^{3/2},$$

- the Borelized 3-current correlator for some current operators $J(\mathbf{q})$ [4,7],

$$\Gamma(T_2, T_1, q) \equiv \underset{T_{1,2} \rightarrow \infty}{\longrightarrow} R_g \exp[-E_g(T_1 + T_2)] F_g(q);$$

for equal (Euclidean) Borel ‘times’ $T_1 = T_2 = \frac{T}{2}$ the latter simplifies to

$$\begin{aligned} \Gamma(T, q) &\stackrel{\text{HO}}{=} \Pi(T) \exp \left[-\frac{q^2}{4 m \omega} \tanh \left(\frac{\omega T}{2} \right) \right] \\ &\stackrel{T \rightarrow \infty}{\longrightarrow} R_g \exp(-E_g T) F_g(q) \equiv \Gamma_g(T, q). \end{aligned}$$

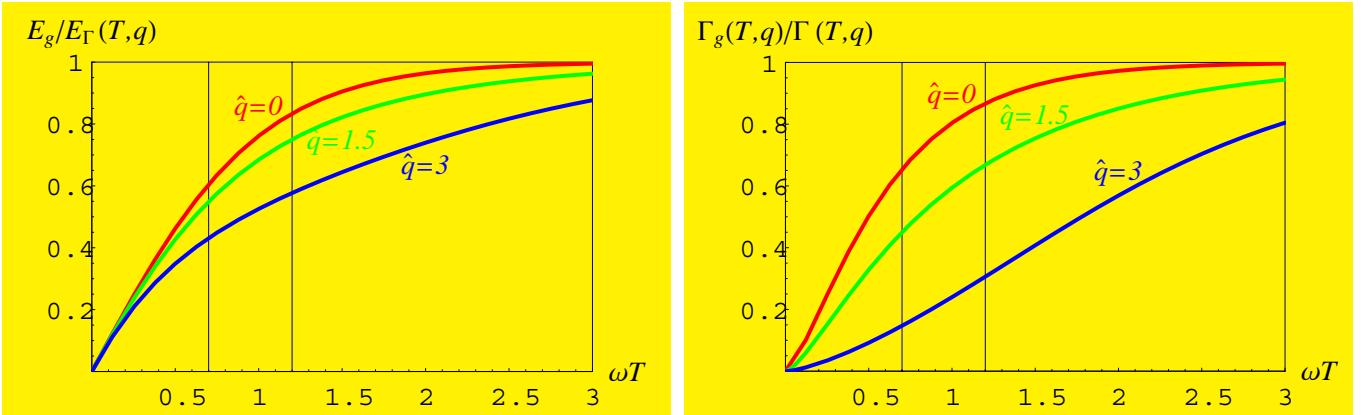
Ground-state features ensue from these objects. For Γ , the [average](#) energy is

$$E_\Gamma(T, q) \equiv -\frac{\partial}{\partial T} \log \Gamma(T, q) \stackrel{\text{HO}}{=} \frac{3}{2} \omega \coth(\omega T) + \frac{q^2}{4 m [1 + \cosh(\omega T)]}.$$

Two evident requirements determine the range (or [working Borel ‘window’](#)) of reasonable Borel-parameter values (illustrated below for the lower limit):

- sufficiently large ground-state contributions yield its lower boundaries,
- sufficient accuracy of the QCD descriptions (achieved by suppression of higher perturbative and power corrections) yields its upper boundaries.

Relative ground-state contributions to average energy $E_\Gamma(T, q)$ and 3-point correlator $\Gamma(T, q)$ at fixed dimensionless momentum transfer $\hat{q} \equiv q/\sqrt{m \omega}$



Quark–hadron duality: improved accuracy

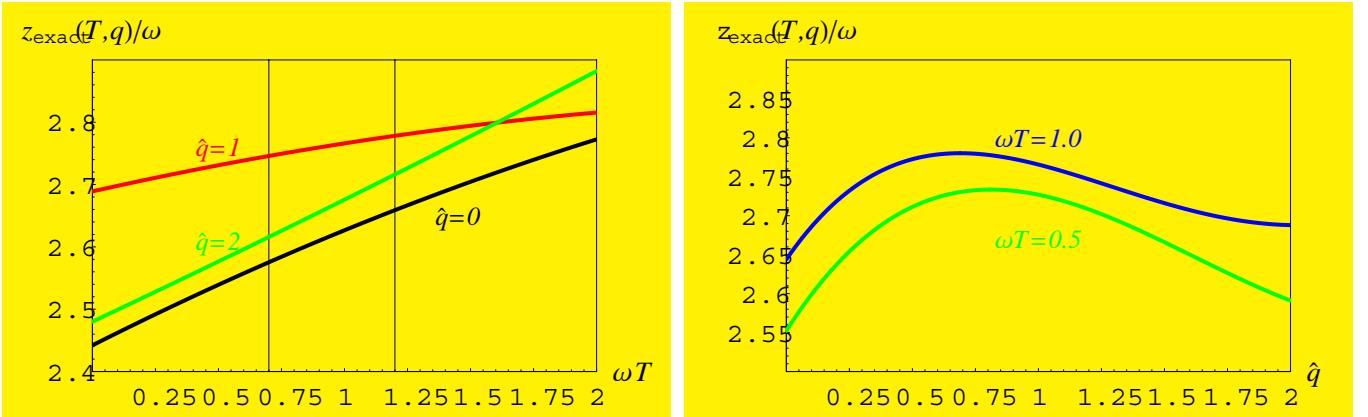
The art of constructing dispersive sum rules involves several (crucial) steps:

- Derive the operator product expansion (OPE) of the non-local product of currents in the correlator; in quantum mechanics this corresponds to the expansion of the correlator in powers of the Borel parameter T [1,6].
- Represent the perturbative part of the correlator as dispersion integral.
- By quark–hadron duality, skip any contribution to the correlator above its T - (perhaps q -) dependent effective continuum threshold $z_{\text{eff}}^{\Pi,\Gamma}(T, q)$.

Elevation of all effective thresholds from constants to functions of T enables us to introduce ‘dual correlators’, defined by the requirement that the QCD contribution with spectral integration truncated at some effective threshold $z_{\text{eff}}^{\Pi,\Gamma}(T, q)$ exactly counterbalances the hadronic ground-state contribution:

$$\begin{aligned} \Pi_g(T) &\equiv R_g \exp(-E_g T) \\ &\stackrel{\text{SR}}{=} \Pi_{\text{dual}}(T, z_{\text{eff}}^{\Pi}(T)) \equiv \int_0^{z_{\text{eff}}^{\Pi}(T)} dz \exp(-z T) \rho_0(z) + \Pi_{\text{power}}(T) , \\ \Gamma_g(T, q) &\equiv R_g \exp(-E_g T) F_g(q) \stackrel{\text{SR}}{=} \Gamma_{\text{dual}}(T, q, z_{\text{eff}}^{\Gamma}(T, q)) \\ &\equiv \int_0^{z_{\text{eff}}^{\Gamma}(T, q)} dz_1 \int_0^{z_{\text{eff}}^{\Gamma}(T, q)} dz_2 \exp\left(-\frac{z_1 + z_2}{2} T\right) \Delta_0(z_1, z_2, q) + \Gamma_{\text{power}}(T, q) . \end{aligned}$$

Exact effective continuum threshold $z_{\text{exact}}^{\Gamma}(T, q)$ of 3-point function $\Gamma(T, q)$ vs. T , resp. \hat{q} : $\Gamma(T_2, T_1, 0) = \Pi(T_1 + T_2)$ implies $z_{\text{exact}}^{\Gamma}(T, q = 0) = z_{\text{exact}}^{\Pi}(T)$



From these and similar plots it is evident that the exact effective continuum thresholds $z_{\text{exact}}^{\Pi,\Gamma}$, as arising from our notion of dual correlator, do depend on the Borel parameter T , and on all involved momenta q : $z_{\text{exact}}^{\Pi,\Gamma} = z_{\text{exact}}^{\Pi,\Gamma}(T, q)$.

Thresholds $z_{\text{eff}}(T)$ from bound-state mass

One's (theoretical or experimental) knowledge of the ground state's mass or binding energy may be exploited for a determination of the functional form $z_{\text{eff}}(T, q)$ of the effective continuum threshold in use. As its T dependence is just needed in limited Borel windows, simple polynomial Ansätze suffice [8]:

$$z_{\text{eff}}^{(n)}(T, q) = \sum_{j=0}^n z_j^{(n)}(q) (\omega T)^j , \quad n = 0, 1, 2, \dots .$$

The minimization of the deviation of the **dual energy**, defined for $\Gamma(T, q)$ by

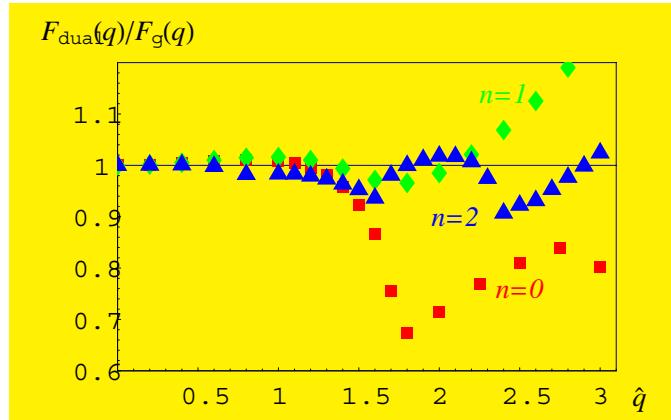
$$E_{\text{dual}}^{\Gamma}(T, q) \equiv -\frac{d}{dT} \log \Gamma_{\text{dual}}(T, q, z_{\text{eff}}^{\Gamma}(T, q)) ,$$

from the exact energy E_g then fixes the behaviour of the function $z_{\text{eff}}^{(n)}(T, q)$:

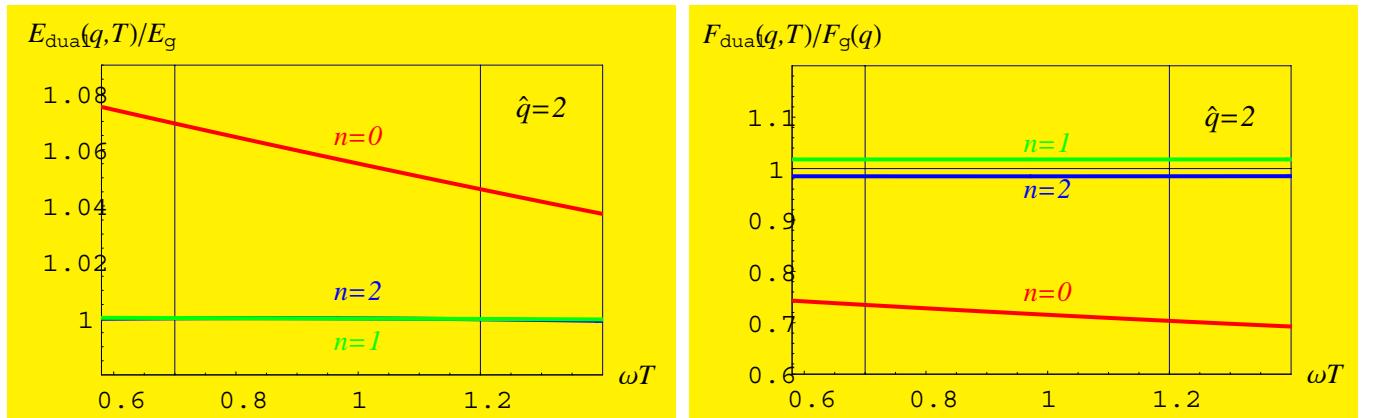
$$\chi^2 \equiv \frac{1}{N} \sum_{i=1}^N \left[E_{\text{dual}}^{\Gamma,A}(T_i, q) - E_g \right]^2 \quad \text{with } T_{i \leq N} \in \text{Borel window} .$$

Form factors $F(q)$ in quantum mechanics

Dual form factor $F_{\text{dual}}(q)$ from 3-point correlator $\Gamma(T, q)$ vs. $\hat{q} \equiv q/\sqrt{m\omega}$



Fitted dual energy, $E_{\text{dual}}^{\Gamma}(T, q)$, and form factor, $F_{\text{dual}}(T, q)$, from 3-current correlator $\Gamma(T, q)$, for dimensionless momentum transfer $\hat{q} \equiv q/\sqrt{m\omega} = 2$



Bridging the gap: QCD \approx potential model

In order to scrutinize the applicability of the developed improved sum-rule algorithms to the real-life case of QCD [9,10], we confront our extractions of ground-state decay constants predicted by nonrelativistic potential models,

$$f_{QM} \equiv \sqrt{R_g} = |\psi_g(\mathbf{0})| ,$$

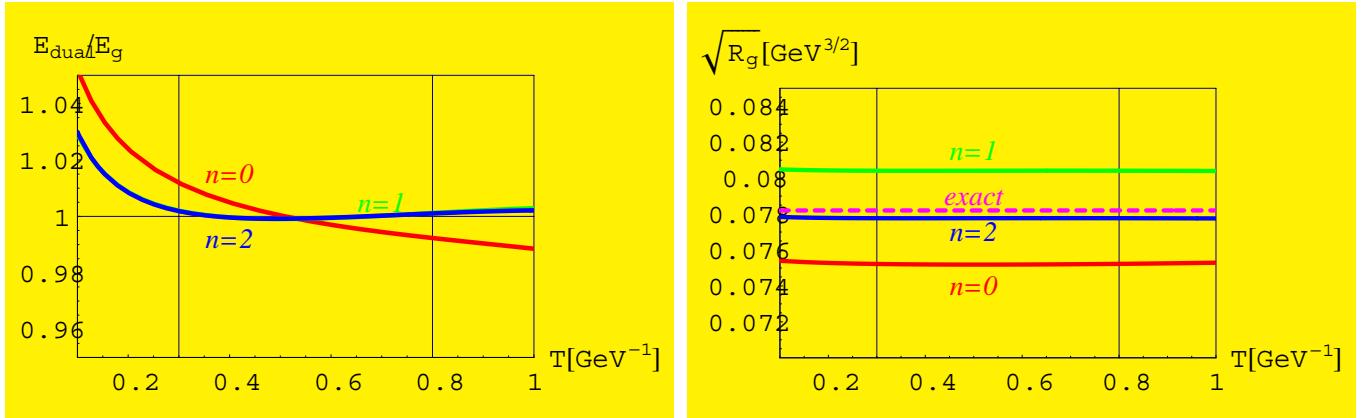
with like extractions of heavy pseudoscalar-meson decay constants in QCD.

Decay constants from quantum-mechanical potential models

‘Funnel’ potential: linear confinement + attractive Coulomb interaction [16]

$$H = \frac{\mathbf{p}^2}{2m} + \sigma r - \frac{\alpha}{r} , \quad r \equiv |\mathbf{x}| , \quad \sigma > 0 , \quad \alpha > 0 .$$

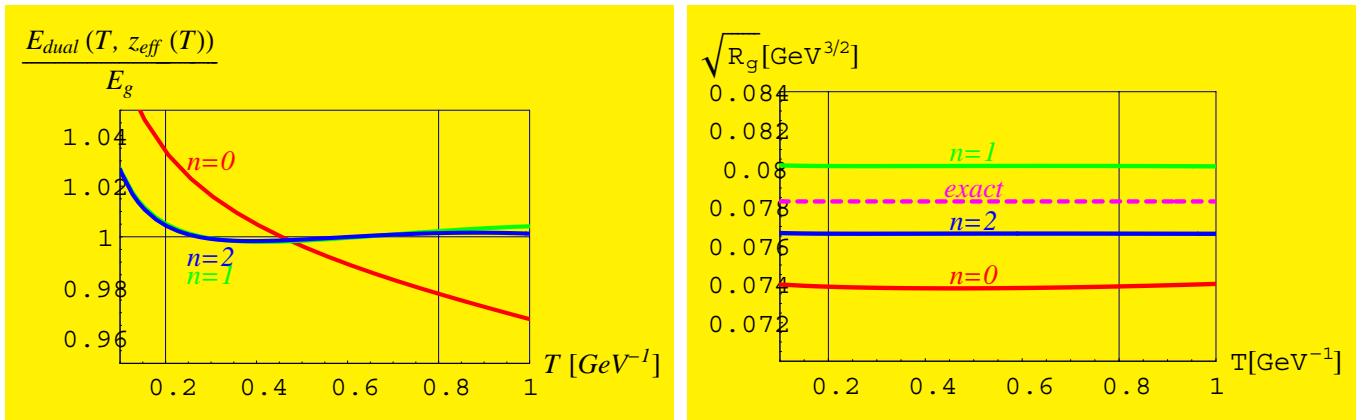
Dual energy E_{dual} and decay constant $\sqrt{R_g}$ for linear confining interaction



Confining harmonic-oscillator + attractive Coulomb-like interactions [9,10]

$$H = \frac{\mathbf{p}^2}{2m} + \frac{m\omega^2 r^2}{2} - \frac{\alpha}{r} , \quad r \equiv |\mathbf{x}| , \quad \alpha > 0 .$$

Dual energy $E_{\text{dual}}(T)$ and decay constant $\sqrt{R_g}$ from harmonic confinement



By inspecting these plots, we estimate the intrinsic sum-rule uncertainty for f_{QM} to be given by the range delimited by our $n = 1$ and $n = 2$ predictions.

Decay constants f_Q of pseudoscalar heavy mesons $P \equiv (Q \bar{q})$

We discuss the correlator $\Pi(p^2) \equiv i \int d^4x \exp(i p \cdot x) \langle 0 | T(j_5(x) j_5^\dagger(0)) | 0 \rangle$ of two pseudoscalar currents, $j_5 \equiv (m_Q + m_q) \bar{q} i \gamma_5 Q$, and its Borel transform

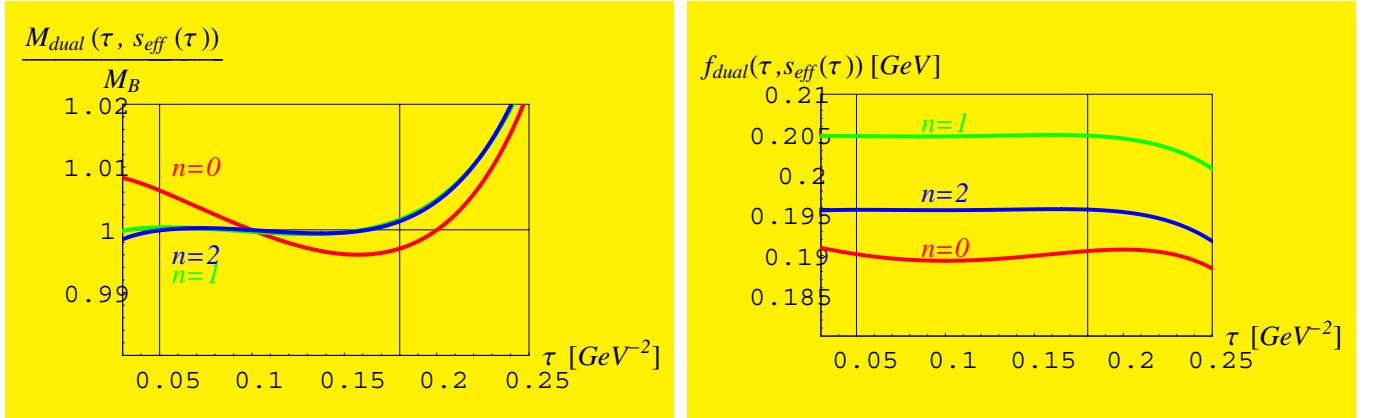
$$\begin{aligned} \Pi(\tau) &= \int_{(m_Q+m_q)^2}^{\infty} ds \exp(-s\tau) \rho_{\text{pert}}(s, \mu) + \Pi_{\text{power}}(\tau, \mu) \\ &\stackrel{\text{SR}}{=} f_Q^2 M_Q^4 \exp(-M_Q^2 \tau) + \dots \xrightarrow{\tau \rightarrow \infty} f_Q^2 M_Q^4 \exp(-M_Q^2 \tau) \equiv \Pi_g(\tau), \end{aligned}$$

involving the perturbative spectral density $\rho_{\text{pert}}(s, \mu)$ which may be derived as expansion in powers of the strong coupling α_s at renormalization scale μ . The P -meson decay constant f_Q is defined according to $\langle 0 | j_5 | P \rangle = f_Q M_Q^2$. Exploiting the assumption of quark–hadron duality, we obtain the sum rule

$$\Pi_g(\tau) \stackrel{\text{SR}}{=} \Pi_{\text{dual}}(\tau, \mu, s_{\text{eff}}(\tau)) \equiv \int_{(m_Q+m_q)^2}^{s_{\text{eff}}(\tau)} ds \exp(-s\tau) \rho_{\text{pert}}(s, \mu) + \Pi_{\text{power}}(\tau, \mu).$$

- Insert in the **dual mass** defined by $M_{\text{dual}}(\tau) \equiv -\frac{d}{d\tau} \log \Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau))$
- a polynomial Ansatz for the effective threshold [9] $s_{\text{eff}}^{(n)}(\tau) = \sum_{j=0}^n s_j^{(n)} \tau^j$,
- fixed by minimizing $\chi^2 \equiv \frac{1}{N} \sum_{i=1}^N [M_{\text{dual}}(\tau_i \in \text{Borel window}) - M_Q^2]^2$,
- to find the **dual decay constant** $f_{\text{dual}}(\tau) \equiv \frac{\exp(M_Q^2 \tau)}{M_Q^4} \Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau))$.

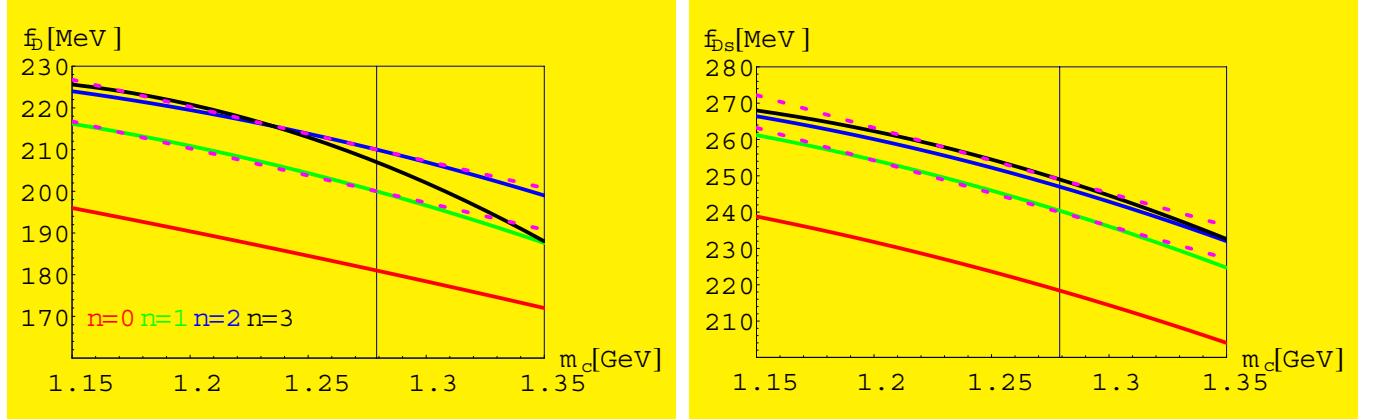
Illustrative example: dual mass M_{dual} and decay constant f_{dual} of B meson



The resemblance of the extraction procedures of bound-state parameters in quantum mechanics and QCD gives us strong confidence that our proposed alterations of the sum-rule techniques may be reasonably applied to hadron phenomenology; there our first goal is $D_{(s)}$ and $B_{(s)}$ decay constants [12,14].

Decay constants $f_{D(s)}$ of the $D(s)$ mesons

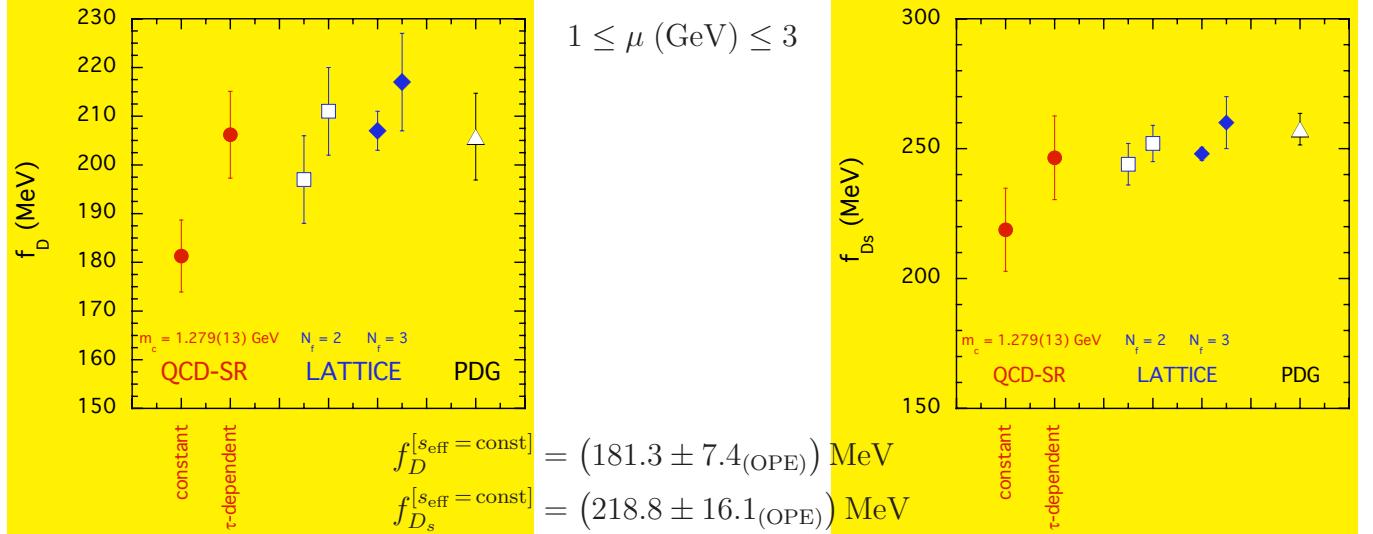
Dual decay constants $f_{D(s)}$ of D and D_s as functions of $\overline{\text{MS}}$ c -quark mass m_c



$$f_D = (206.2 \pm 7.3_{\text{(OPE)}} \pm 5.1_{\text{(syst)}}) \text{ MeV} \quad @ m_c = (1.279 \pm 0.013) \text{ GeV}$$

$$f_{D_s} = (245.3 \pm 15.7_{\text{(OPE)}} \pm 4.5_{\text{(syst)}}) \text{ MeV} \quad \text{where } m_c \equiv \overline{m}_c(\overline{m}_c)$$

Sum-rule results vs. lattice findings (N_f dynamical flavours) & experiment



Observations, results & conclusions [11–14]

- Exact effective (continuum) thresholds depend on Borel parameter and on relevant momenta; they are not universal (i.e., vary with correlator).
- Our proposed techniques raise dramatically the accuracy of traditional sum-rule predictions and yield solid estimates of intrinsic uncertainties.
- The blatant quantitative similarity of our hadron-parameter extraction procedures in potential models and in QCD calls for QCD applications.
- Our sophisticated approach reconciles QCD sum rules and experiment.

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