The transverse gauge links in \textsc{scet}

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M.G.E., A.I., I.S. Phys.Rev.D84, 011502 (2011) and work in progress
Outline

- SCET and its building blocks
- Gauge invariance for covariant gauges
- Gauge invariance for singular gauges (Light–cone gauge)
- A new Wilson line in SCET: T
- The origin of T–Wilson lines in SCET Lagrangian: gauge conditions for different sectors
- Phenomenology
- Conclusions
SCET, an effective theory of QCD

- SCET (soft collinear effective theory) is an effective theory of QCD
- SCET describes interactions between low energy, "soft" partonic fields and collinear fields (very energetic in one light–cone direction)
- SCET and QCD have the same infrared structure: matching is possible
- SCET helps in the proof of factorization theorems and identification of relevant scales
## SCET: Kinematics

Bauer, Fleming, Pirjol, Stewart, ’00

### Light-cone coordinates

\[
\mathbf{p} = (\pm, -, \perp)
\]

### Soft modes

\[
\begin{align*}
np & \sim Q \\
\mathbf{p}_\perp & \sim \lambda Q \\
np & \sim \lambda^2 Q
\end{align*}
\]

Do not interact with (anti) collinear or \(\text{u-soft}\) in covariant gauge

### Expressions

\[
\begin{align*}
n^n = (1, \bar{n}), & \quad \bar{n}^\mu = (1, -\bar{n}) \\
\bar{n}^2 = 1, & \quad n^2 = 0, \quad \bar{n}^2 = 0
\end{align*}
\]

\[
\psi(x) = \sum_n \sum_p e^{-ipx} \psi_{n,p}(x)
\]

\[
\psi = \left(\frac{n\bar{n}}{4} + \frac{\bar{n}n}{4}\right) \psi = \xi + \phi
\]

Integrated out with EOM

### Table

<table>
<thead>
<tr>
<th>SCET [(\lambda \sim m/Q \ll 1)]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n)-collinear ((\xi_n, A_n^\mu))</td>
</tr>
<tr>
<td>(\bar{n})-collinear ((\xi_{\bar{n}}, A_{\bar{n}}^\mu))</td>
</tr>
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</table>

Crosstalk: soft \((q_s, A_s^\mu)\) \(p_s^\mu \sim Q(\lambda^2, \lambda^2, \lambda^2)\)

\(\text{U-soft}\)
Leading order Lagrangian (n-collinear)

\[ \mathcal{L}_{c,n} = \bar{\xi}_{n,p'} \left\{ i n \cdot D + g n \cdot A_{n,q} + \left( \mathcal{P}_\perp + g A_{n,q}^\perp \right) \right\} W \frac{1}{\mathcal{P}} W^\dagger \left( \mathcal{P}_\perp + g A_{n,q}^\perp \right) \frac{\eta^\dagger}{2} \xi_{n,p} \]

\[ W_n(x) = \bar{P} \exp \left( -ig \int_0^\infty ds \, \bar{n} \cdot A_{n}(\bar{n}s + x) \right) \]

\[ Y_n(x) = \bar{P} \exp \left( -ig \int_0^\infty ds \, \bar{n} \cdot A_{us}(\bar{n}s + x) \right) \]

The new fields do not interact anymore with u-soft fields
Factorization Theorem For DIS

$$F_1(x, Q^2) = \sum_f \int_x^1 \frac{dy}{y} C_f \left( \frac{x}{y}, \frac{Q^2}{\mu^2} \right) q_f(y, \mu^2),$$

- PDF In Full QCD

$$q(x) = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle P|\bar{\psi}(\lambda n) \gamma^\mu e^{-ig} f_0 d\lambda' A(\lambda' n) \psi(0)|P \rangle,$$

- Factorization In SCET

$$F_1^N(Q^2) = H(Q^2/\mu^2) \phi_N(Q^2/\mu^2) S_N(Q^2/\mu^2) J_N(Q^2/\mu^2),$$

- PDF In SCET:

$$\phi(x, \mu_I^2) = \left\langle P \left| \bar{\xi}_n W_n \delta \left( x - \frac{n \cdot P_+}{n \cdot p} \right) \frac{\slashed{p}}{\sqrt{2}} W^n_+ \xi_n \right| P \right\rangle$$

[Neubert et.al, Manohar]

[Stewart et.al]

$$\phi(x, \mu_f^2)$$ is gauge invariant because each building block is gauge invariant.
Factorization Theorem For SIDIS in QCD: Covariant gauge

- In Full QCD And At Low Transverse Momentum:

\[ F(x_B, z_h, P_{h\perp}, Q^2) = \sum_{q=u,d,s,...} e_q^2 \int d^2 k_\perp d^2 p_\perp d^2 \ell_\perp \]
\[ \times q(x_B, \mu^2, x_B \xi, \rho) \hat{q}_T(z_h, p_{\perp}, \mu^2, \xi/z_h, \rho) S(\ell_\perp, \mu^2, \rho) \]
\[ \times H(Q^2, \mu^2, \rho) \delta^2(z_h k_\perp + p_\perp + \ell_\perp - P_{h\perp}) , \]

- "Naïve" Transverse Momentum Dependent PDF (TMDPDF):

\[ q \approx Q/S \]

\[ Q(x, k_\perp, \mu, x\xi) = \frac{1}{2} \int \frac{d\xi^-}{2\pi} e^{-ix\xi-P^+} \int \frac{d^2 \tilde{b}_\perp}{(2\pi)^2} e^{i\tilde{b}_\perp \cdot k_\perp} \left\langle P \left| \bar{\psi}_q(\xi^-, 0, \tilde{b}_\perp) \mathcal{L}^+_{\nu}(\infty; \xi^-, 0, \tilde{b}_\perp) \gamma^+ \mathcal{L}_{\nu}(\infty; 0) \psi_q(0) \right| P \right\rangle \]

\[ \mathcal{L}_{\nu}(\infty; \xi) = \exp \left( -ig \int_0^\infty d\lambda v \cdot A(\lambda v + \xi) \right) . \]

This result is true only in "regular" gauges: Here all fields vanish at infinity
Transverse Gauge Link in QCD

For gauges not vanishing at infinity [Singular Gauges] like the Light–Cone gauge (LC) one needs to introduce an additional Gauge Link which connects \((\infty, \vec{0}_\perp)\) with \((\infty, \vec{b}_\perp)\) to make it Gauge Invariant \(A^{\mu\perp}(r_\perp)\)

In LC Gauge This Gauge Link Is Built From The Transverse Component Of The Gluon Field:

\[
\mathcal{F}_{i/h}(x, k_\perp) = \frac{1}{2} \int \frac{d\xi^- d^2 \xi_\perp}{(2\pi)^2} e^{-i k^+ \xi^- + i k_\perp \cdot \xi_\perp} \left\langle h | \bar{\psi}_i(\xi^-, \xi_\perp) [\xi^-, \xi_\perp; \infty^-, \xi_\perp; \infty^-, \xi_\perp]^+_n [\infty^-, \xi_\perp; \infty^-, \xi_\perp]_n \psi_i(0^-, 0_\perp) | h \right\rangle 
\]

\[
[\infty^-, \xi_\perp; \infty^-, \xi_\perp]_n \equiv \mathcal{P} \exp \left[ ig \int_0^\infty dt \mathbf{l} \cdot A_a t^a (\xi_\perp + l \mathbf{r}) \right]
\]
Are TMDPDF fundamental matrix elements in SCET?

Are SCET matrix elements gauge invariant?

Where are transverse gauge link in SCET?

The SCET Lagrangian is formed by gauge invariant building blocks. Gauge Transformations in covariant gauge for $W_n^+ \xi$

$\xi \rightarrow U \xi$

$W_n^+ \rightarrow W_n^+ U^+$

$W^+ \xi \xrightarrow{LC gauge} \xi$
Gauge invariance of SCET building blocks

We calculate $\langle 0 | W_n^\dagger \xi_n | q \rangle$ at one-loop in Feynman Gauge and in LC gauge.

In Feynman Gauge

\[
I_w = \frac{\alpha_s}{4\pi} C_F i\beta_0 \left[ \frac{1}{eUV} + 1 - \ln \frac{-p^2}{\mu^2} \right],
\]

\[
I_{\xi,Fey} = -2g^2 C_F \mu^{2\varepsilon} i \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + i0)(k^+ + i0)} \left( \frac{p^+ + k^+}{(p + k)^2 + i0} \right)
\]

\[
W_{n,y} = P \exp \left( ig \int_{-\infty}^{y} ds \, \bar{n} \cdot A_n(s\bar{n}) \right)
\]

\[
I_n = -\frac{g^2}{8\pi^2} C_F \gamma^\mu \left[ -\frac{1}{e^2} - \frac{1}{e} + \frac{1}{e} \ln \frac{-p_1^2}{\mu^2} \right. \\
\left. - \frac{1}{2} \ln^2 \frac{-p_1^2}{\mu^2} + \ln \frac{-p_1^2}{\mu^2} - 2 + \frac{\pi^2}{12} \right].
\]
Gauge invariance of SCET building blocks

We calculate \( \langle 0 | W_{\bar{n}}^\dagger \xi_{\bar{n}} | q \rangle \) at one-loop in Feynman Gauge and in LC gauge.

In LC Gauge

\[
A^+ = 0 \rightarrow W_{\bar{n}} = W_{\bar{n}}^\dagger = 1
\]

\[
\tilde{D}_{\mu\nu}(k) = -\frac{i}{k^2 + i0} \left( g_{\mu\nu} - \frac{n_\mu k_\nu + n_\nu k_\mu}{k^+} \right)
\]

[Bassetto, Lazzizzera, Soldati]
Canonical quantization imposes ML prescription

<table>
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<tr>
<td>+i0</td>
<td>1/(k^+ + i0)</td>
</tr>
<tr>
<td>-i0</td>
<td>1/(k^+ - i0)</td>
</tr>
<tr>
<td>PV</td>
<td>1/2(1/(k^+ + i0) + 1/(k^+ - i0))</td>
</tr>
<tr>
<td>ML</td>
<td>1/(k^+ + i0Sgn(k^-))</td>
</tr>
</tbody>
</table>
We calculate $\langle 0 | W_n^\dagger \xi_n | q \rangle$ at one-loop in **Feynman Gauge** and in **LC** gauge.

In **LC Gauge**

$$\tilde{D}_{\mu\nu}(k) = \frac{-i}{k^2 + i0} \left( g_{\mu\nu} - \frac{n_\mu k_\nu + n_\nu k_\mu}{k^+} \right)$$

$$\Sigma_{LC}(p) = \Sigma_{Fey}(p) + \Sigma_{Ax}^{(Pres)}(p) = \left( I_{w,Fey}(p) + I_{w,Ax}^{(Pres)}(p) \right) \frac{ip^2}{p^+} \frac{\hbar}{\sqrt{2}}$$
Gauge invariance of SCET building blocks

We calculate $\langle 0|W_{\bar{n}}^\dagger \xi_{\bar{n}}|q\rangle$ at one-loop in Feynman Gauge and in LC gauge.

In LC Gauge

$\Sigma_{LC}(p) = \Sigma_{Fey}(p) + \Sigma_{Ax}^{(ML)}(p) = (I_{w,Fey}(p) + I_{w,Ax}^{(ML)}(p)) \frac{ip^2}{p^+} \frac{\eta}{\sqrt{2}}$

$I_{w,Ax}^{(ML)} = 4ig^2C_F\alpha_s^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + i0)((p+k)^2 + i0)[k^+]}$

$\frac{1}{[k^+]} = \frac{\theta(k^-)}{k^+ + ip^+\eta} + \frac{\theta(-k^-)}{k^+ - ip^+\eta}$

The result of this is independent of $\eta$ and has got only a single pole. Zero-bin subtraction is nul in ML.
The SCET matrix element \( \langle 0 | W_{\tilde{n}}^\dagger \xi_{\tilde{n}} | q \rangle \) is not gauge invariant. Using LC gauge we have different result (moreover the result of the one-loop correction depends on the used prescription).
Gauge invariance in SCET

In order to restore gauge invariance we have to introduce a new Wilson line, T, in SCET matrix elements

\[ T_n^+(x^+, x_\perp) = P \exp \left[ ig \int_0^\infty d\tau l_\perp \cdot A_\perp (\infty^-, x^+; l_\perp \tau + x_\perp) \right] \]

And the new gauge invariant matrix element is

\[ \langle 0 | T_n^+ W_n^+ \xi_n | q \rangle \]
In covariant gauges $T = T^\dagger = 1$, so we recover the SCET results $\langle 0 | W_n^\dagger \xi_n | q \rangle$

In LC gauge

$\langle 0 | T_n^\dagger \xi_n | q \rangle$
The T–Wilson Line

\[ I_{T,Ax}^{(ML)} = 2C_F g^2 \mu^2 \varepsilon i \int \frac{d^d k}{(2\pi)^d} \frac{p^+ + k^+}{(k^2 + i0)((p + k)^2 + i0)} \left[ \frac{C^{(ML)}_\infty}{k^+ - i0} - \frac{C^{(ML)}_\infty}{k^+ + i0} \right] \]

\[ C^{(ML)}_\infty = \theta(-k^-) \]

All prescription dependence cancels out and gauge invariance is restored no matter what prescription is used.
Where does the T–Wilson Line come from?

Is there a way to understand the T–Wilson lines from the SCET Lagrangian?

An example, the quark form factor: from QCD to SCET in LCG

The T–Wilson line is born naturally in One loop matching.
Where does the T–Wilson Line come from?

\((\bar{n}A, nA, A_\perp) \sim Q(1, \lambda^2, \lambda)\)

\(\bar{n}A = 0\)

In the canonical quantization of the gauge field (Bassetto et al.)

\[ A^a_\mu(k) = T^a_\mu(k)\delta(k^2) + \bar{n}_\mu \frac{\delta(\bar{n}k)}{k_\perp^2} \Lambda^a(nk, k_\perp) + \frac{ik_\mu}{k_\perp^2} \delta(\bar{n}k) U^a(nk, k_\perp) \]

\(\bar{n}^\mu T^a_\mu(k) = 0; \quad k^\mu T^a_\mu(k) = 0;\)

We define

\[ A^{(\infty)}(x^+, x_\perp) \overset{\text{def}}{=} A(x^+, \infty^-, x_\perp) \]

\[ \tilde{A}(x^+, x^-, x_\perp) \overset{\text{def}}{=} A(x^+, x^-, x_\perp) - A^{(\infty)}(x^+, x_\perp) \]

And we can show

\[ iD^\mu_\perp \overset{\text{def}}{=} iD_\perp + gA^{(\infty)}_\perp \]

\[ iD^\mu_\perp = TiD^\mu_\perp T^\dagger \]

\[ T^\dagger = P \exp \left[ -ig \int_0^\infty dt_\perp A^{(\infty)}_\perp(x^+, x_\perp - l_\perp t) \right] \]
The T–Wilson Lines in SCET–I

In SCET–I only collinear and u–soft fields. The first step to obtain the SCET Lagrangian is integrating out energetic part of spinors

\[ \mathcal{L} = \bar{\xi}_n \left( i nD + i \partial_\perp \frac{1}{i \bar{m}D} \right) \frac{\bar{\nu}}{2} \xi_n \]

And then applying multipole expansion, \( x_n \sim 1/Q(1,1/\lambda^2,1/\lambda) \)
\( x_{us} \sim 1/Q(1/\lambda^2,1/\lambda^2,1/\lambda^2) \)

\[ \mathcal{L}_n = \bar{\xi}_n \left( i n\partial_n + gnA_{us}(x^+) + i \partial_{n\perp} W_n^T \frac{1}{i \bar{m}\partial} W_n^{T+} i \partial_{n\perp} \right) \frac{\bar{\nu}}{2} \xi_n \]

Where \( W_n^T = T_n W_n \)

U–soft field does not give rise to any transverse gauge link!!
There are no transverse u–soft fields and they cannot depend on transverse coordinates!!
The T–Wilson Lines in SCET–II

Now the degrees of freedom are just collinear and soft

\[(\bar{n}A_n, nA_n, A_{n\perp}) \sim Q(1, \eta^2, \eta); \quad (\bar{n}p_n, np_n, p_{n\perp}) \sim Q(1, \eta^2, \eta);
\]

\[(\bar{n}A_s, nA_s, A_{s\perp}) \sim Q(\eta, \eta, \eta); \quad (\bar{n}p_s, np_s, p_{s\perp}) \sim Q(\eta, \eta, \eta);\]

No interaction is possible for on-shell states

\[\mathcal{L}_{III} = \bar{\xi}_n \left( in\slashed{D}_n + i\slashed{D}_{n\perp}W_n \frac{1}{in\slashed{D}}W_n^\dagger i\slashed{D}_{n\perp} \right) \frac{\bar{n}}{2} \xi_n\]

Is this true in every gauge?
The T–Wilson Lines in SCET–II

\[(\vec{n}A_n, nA_n, A_{n\perp}) \sim Q(1, \eta^2, \eta); \quad (\vec{n}p_n, np_n, p_{n\perp}) \sim Q(1, \eta^2, \eta);\]
\[(\vec{n}A_s, nA_s = 0, A_{s\perp}) \sim Q(\eta, \emptyset, \eta); \quad (\vec{n}p_s, np_s, p_{s\perp}) \sim Q(\eta, \eta, \eta);\]

The gauge ghost however acts only on some momentum components

\[
\prod_i \phi^i_n(x) A^\infty_{s\perp}(x^-, x^\perp) \rightarrow \prod_i \phi^i_n(x) A^\infty_{s\perp}(0, x^\perp)
\]

Thus the covariant derivative is

\[iD^\mu = i\partial^\mu + gA^\mu_n(x) + gA^{(\infty)\mu}_{s\perp}(0^-, x^-)\]

The decoupling of soft fields requires

\[A^{(0)\mu}_n(x) = T_{sn}(x_-) A^\mu_n(x) T^\dagger_{sn}(x_-)
\]

\[T_{sn} = \bar{P} \exp \left[ ig \int_0^\infty d\tau l_{\perp} \cdot A^{(\infty)}_{s\perp}(0^-, x_- - l_{\perp} \tau) \right] \]
The T–Wilson Lines in SCET–II

The new SCET–II Lagrangian is

\[
\mathcal{L}_{\Xi\Xi} = \bar{\xi}_n^{(0)} \left( i n D_n^{(0)} + i \mathcal{D}_{n\perp} W_n^{T(0)} \frac{1}{i n \partial} W_n^{T(0)\dagger} i \mathcal{D}_{n\perp} \right) \frac{\bar{\nu}}{2} \xi_n^{(0)}
\]

\[
A_n^{(0)\mu}(x) = T_{sn} (x_\perp) A_n^{\mu}(x) T_{sn}^\dagger (x_\perp)
\]

\[
D_n^{(0)\mu} = i \partial^\mu + g A_n^{(0)\mu}
\]

\[
\xi_n^{(0)} = T_{sn} (x_\perp) \xi_n (x)
\]

\[
T_{sn} = \bar{P} \exp \left[ i g \int_0^\infty d\tau l_\perp \cdot A_{s\perp}^{(\infty)} (0^-, x_\perp - l_\perp \tau) \right]
\]
Applications

TMDPDF

Drell-Yan at low Pt [Becher, Neubert]

Higgs production at low Pt [Mantry, Petriello]

Beam functions [Jouttenus, Stewart, Tackmann, Waalewijin]

Heavy Ion physics – Jet Broadening [Ovanesyan, Vitev ★]
We Can Define A Gauge Invariant TMDPDF In SCET (And Factorize SIDIS)

\[ \chi_{\bar{n}}(y) \equiv T_{\bar{n}}^\dagger(y^+, y_\perp) W_{\bar{n}}^\dagger(y) \xi_{\bar{n}}(y) \]

\[ \phi_{q/P} = \langle \bar{P}_{\bar{n}} | \chi_{\bar{n}}(y) \delta\left(x - \frac{nP}{np}\right) \delta^{(2)}(p_\perp - \mathcal{P}_\perp) \frac{\eta}{\sqrt{2}} \chi_{\bar{n}}(0) | P_{\bar{n}} \rangle \]
Drell–Yan At Low $p_T$

$$d\sigma = \frac{4\pi\alpha^2}{3q^2s} \frac{d^4q}{(2\pi)^4} \int d^4x \ e^{-iq\cdot x} \ (-g_{\mu\nu}) \ \langle N_1(p) N_2(\bar{p}) | J^{\mu\dagger}(x) J^{\nu'}(0) | N_1(p) N_2(\bar{p}) \rangle$$

$$J^\mu \to C_V (-q^2 - i\varepsilon, \mu) \sum_q \left( g_L \bar{\chi}_{hc} S_n^\dagger \gamma^\mu \frac{1 - \gamma_5}{2} S_n \chi_{hc} + g_R \bar{\chi}_{hc} S_n^\dagger \gamma^\mu \frac{1 + \gamma_5}{2} S_n \chi_{hc} \right)$$

$$\chi_{hc} = W_{hc}^\dagger \xi_{hc}$$

Introduce Gauge Invariant Quark Jet:

$$\chi_{\bar{n}}(y) \equiv T_{\bar{n}}^\dagger(y^+, y_\perp) W_{\bar{n}}^\dagger(y) \xi_{\bar{n}}(y),$$

$$\phi_{q/P} = \langle P_{\bar{n}}|\chi_{\bar{n}}(y)\delta \left( x - \frac{n\mathcal{P}}{np} \right) \delta^{(2)}(p_\perp - \mathcal{P}_\perp) \frac{\gamma^\prime}{\sqrt{2}} \chi_{\bar{n}}(0)|P_{\bar{n}}\rangle$$

The TMDPDF Is Indeed Gauge Invariant.
• Application To Heavy–Ion Physics

\[
\sum_{m=1, n=1}^{\infty} \frac{d^2 A_{nm}}{d^2 k_\perp} = \frac{\sqrt{2}}{L^3 N_c} \int dy^+ dy_\perp dy'_\perp e^{-ik_\perp \cdot (y_\perp - y'_\perp)} \left\langle \text{Tr} \left[ \left( W_F^+ [y^+, y_\perp] - 1 \right) \left( W_F [y^+, y_\perp] - 1 \right) \right] \right\rangle
\]

\[W_F [y^+, y_\perp] \equiv P \left\{ \exp \left[ ig \int_{y_0}^{y_L} dy^- A^+(y^+, y^-, y_\perp) \right] \right\}\]

In LC Gauge The Above Quantity Is Meaningless. If We Add To It The T–Wilson line Then We Get A Gauge Invariant Physical Entity.

D´Eramo, Liu, Rajagopal
Conclusions

The usual SCET building blocks have to be modified introducing a New Gauge Link, the T-Wilson line.

Using the new formalism we get gauge invariant definitions of non-perturbative matrix elements. In particular the T is compulsory for matrix elements of fields separated in the transverse direction. These matrix elements are relevant in semi-inclusive cross sections or transverse momentum dependent ones.

It is possible that the use of LC gauge helps in the proofs of factorization. The inclusion of T is so fundamental. Work in progress in this direction.
Conclusions

It is definitely possible to understand the origin of T-Wilson lines in a Lagrangian framework for EFT.

Every sector of the SCET can be appropriately written in LCG.

The LCG has peculiar property for loop calculation and can avoid the introduction of new ad-hoc regulators.

There is a rich phenomenology to be studied... so a lot of work in progress!!

THANKS!
Production Current: $Q \gg m$

\[ \bar{\psi} \Gamma^\mu \psi \rightarrow (\bar{\xi}_n W_n)\omega \Gamma^\mu (W_n^\dagger \xi_{\bar{n}})\bar{\omega} = (\bar{\xi}_n W_n)\omega Y_n^\dagger \Gamma^\mu Y_{\bar{n}} (W_n^\dagger \xi_{\bar{n}})\bar{\omega} \]

\[ J_\mu^i (0) = \int d\omega \ d\bar{\omega} \ C(\omega, \bar{\omega}, \mu) J_i^{(0)\mu} (\omega, \bar{\omega}, \mu) \]
The T–Wilson Line in other prescriptions

Let us consider the pole part of the interesting integral with \( 1/[k^+] = 1/(k^+ \pm i\eta) \) or the PV prescription. The result is

\[
I_{\pm i\eta} = -2 \frac{\alpha_s}{4\pi} \frac{g^2}{4\pi^2} C_F \frac{1}{\varepsilon} \int_0^1 dz \frac{(1-z)^{1-\varepsilon} z^{-\varepsilon}}{z \mp i\eta} = - \frac{g^2}{4\pi^2} C_F \frac{1}{\varepsilon} [1 + \ln \mp i\eta] + \text{finite}
\]

And in PV the result does not have any imaginary part. The gauge invariance is restored either with the T with a prescription dependent factor

<table>
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<tr>
<td>+i0</td>
<td>0</td>
</tr>
<tr>
<td>-i0</td>
<td>1</td>
</tr>
<tr>
<td>PV</td>
<td>1/2</td>
</tr>
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</table>

OR with zero-bin Subtraction!!

The values of this constant depend also on the convention for inner/outer moments.