Finite energy for a Gravitational Potential Falling Slower than 1/r

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Berezhiani-Comelli-Nesti-LP JHEP 0807 130 (2008)
Blas-Comelli-Nesti-LP PRD D80, 044025 (2009)
Comelli-Nesti-LP PRD83 084042 (2011)

Comelli-Nesti-LP arXiv 1105.3010
Potential falling slower than $1/r$?

Suppose we have a solution of Einstein equations with a static potential $\phi$, 

$$ g_{tt} = -1 - 2\phi $$

that, at large distances, falls off slower than $1/r$.

- The total energy of the system would infinite. According Newton, source’s total mass is $\sim$ flux of $\nabla \phi$

$$ E = \frac{1}{4\pi G} \int_{S_2} d^2 x \, \vec{\nabla} \phi \cdot \vec{n} $$

Finite $E$ only if $\phi \sim 1/r$

- No such a solution in perturbative GR: Green function goes as $1/r$

- Modify gravity, Why do we need a non-Newtonian potential?
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Suppose we have a solution of Einstein equations with a static potential $\phi$, 

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Einstein’s GR

A 90 year-long successful theory
a single free parameter and it works great

- Equivalence principle $10^{-12}$ level
- Solar system tests (weak field) $10^{-4}$ level
- Binary pulsar (nonlinear) $10^{-3}$ level

- however ....
- CMB + Supernovae data require Dark energy
  $p = \rho \cdot w$, $w < 0$. Expanded acceleration
- Perhaps just a tiny (??) cosmological constant, $w = -1$, $\Lambda \sim (10^{-4} \text{eV})^4$ or a bizarre fluid?
- Is GR an isolated theory? How rigid is GR?
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**Degrees of Freedom**

- **GR**
  \[ M_{\text{pl}}^2 E_{\mu\nu}^{(1)} = T_{\mu\nu}^{(1)}, \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]
  
  DOF \ 10 - 2 \times 4 = 2 \quad 4 \text{ gauge modes} \quad \delta h_{\mu\nu} = \partial_{\mu} \xi_{\nu} + \partial_{\nu} \xi_{\mu} 

- Massive gravitons (Minkowski) have 5 DOF, more DOF needed

- Give up gauge symmetry. Fierz-Pauli theory (1939)
  
  \[ L_{\text{FP}} = M_{\text{pl}}^2 L_{\text{grav}}^{(2)} + M_{\text{pl}}^2 m^2 \left( a h_{\mu\nu} h^{\mu\nu} + b h^2 \right) \]
  
  \[ E_{\mu\nu}^{(1)} - \frac{1}{4} m^2 \left( a h_{\mu\nu} + b h \eta_{\mu\nu} \right) = M_{\text{pl}}^{-2} T_{\mu\nu}^{(1)} \quad \partial^\nu E_{\mu\nu}^{(1)} = 0 \]
  
  4 constraints \quad DOF \ 10 - 4 = 6 = 5 + 1

- The sixth mode is a ghost (Boulware-Deser).
  
  Absent in flat space when \( a + b = 0 \) (FP theory)
  present in curved space and at the nonlinear level

- When the ghost is projected out, light bending badly contradicts experiments (van Dam, Veltman, Zakharov) vdBZ discontinuity
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Beyond the Linear Level

Bottom line

Massive gravity around Minkowski background is problematic

- Linear FP theory suffers from vDVZ discontinuity
- Nonlinear effects are dominated by unknown UV physics
- Very low cutoff, solar system scale physics cannot be trusted if \( m \sim H \)

Giving up Minkowski background can help

- At linearized level there is no ghost, no vDVZ discontinuity
  FP tuning not needed
- Also propagation around curved background looks better

Rubakov, Dubowsky, Comelli-Nesti-Pilo
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Step 1: Recasting the mass term

- $g^{\mu \nu} = \eta^{\mu \nu} - h^{\mu \nu} + h^{\mu \alpha} h^{\nu}_{\alpha} + \cdots \Rightarrow g^{\mu \nu} \eta_{\mu \nu}$ represents a mass term
- To recover diff (gauge) invariance replace $\eta_{\mu \nu}$ by a dynamical (Stuckelberg) extra metric field $\tilde{q}_{\mu \nu}$
  $\eta_{\mu \nu} \rightarrow q_{\mu \nu}$
- New tensor from the two metric $X^\mu_{\nu} = g^{\mu \alpha} q_{\alpha \nu}$
- Typical mass terms are made out $\tau_n = \text{Tr}(X^n)$
  $a (\tau_1 - 4)^2 + b (\tau_2 - 2\tau_1 + 4) = (a h_{\mu \nu} h^{\mu \nu} + b h^2) + \cdots$
The Stuckelberg Trick in Massive GR: Bigravity I

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Step 2: Stuckelberg Dynamics

- The extra metrics is turned into a dynamical field

\[ S_{MGR} = \int d^4 x \left[ \sqrt{g} M_{pl}^2 R(g) + \kappa M_{pl}^2 \sqrt{\tilde{g}} R(\tilde{g}) - 4(\tilde{g}g)^{1/4} V(X) \right] \]

- Matter couples only to \( g_{\mu\nu} \)
- Gauge symmetry: Diff
- When \( \kappa \to \infty \), \( \tilde{g}_{\mu\nu} \) gets non-dynamical and flat: \( \tilde{g}_{\mu\nu} = e^a_\mu e^b_\nu \tilde{\eta}_{ab} \)
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In the bigravity unitary gauge and $\kappa \to \infty$, the Stuckelberg fields are $\phi^a$

- Powerful formalism to treat in unified way both the Lorentz preserving and Lorentz breaking cases
- $X_{|bkg} = \text{Diag}(1, 1, 1, 1)$ Lorentz preserving (LI) background
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  - only rotational symmetry is present
- For any $V$ the LI background is always present

Modified Einstein equations (Bigravity Unitary gauge)

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M_{pl}^2 E^\mu_\nu + [\text{Det}(X)]^{1/4} \left[ V \delta^\mu_\nu - 4(V'X)^\mu_\nu \right] = T^\mu_\nu
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The Stuckelberg Trick in Massive GR: Bigravity III

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Modifying Schwarzschild I

- Spherically symmetric ansatz

\[ ds^2 = -J(r) \, dt^2 + K(r) \, dr^2 + r^2 \, d\Omega^2 \]

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Finding all solutions if very hard. Consider solutions with \( D \neq 0 \):

Potential independent analysis: \( g_{\mu\nu} \) is diagonal \( \Rightarrow \) \( E^\mu_\nu \) diagonal

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Solution

\[ J = \left( 1 - \frac{2Gm_1}{r} \right) + 2GSr^\gamma, \quad KJ = 1 \]

\[ C = c^2\omega^2 \left( 1 - \frac{2Gm_2}{\kappa r} \right) - \frac{2G}{c\omega^2\kappa}Sr^\gamma, \quad D^2 + AC = c^2\omega^4 \]

\[ B = \omega^2r^2, \quad A = \ldots \]

- Integration constants: \( m_1, m_2 \) and \( S \). Determined by the parameters in \( V: c, \omega \)
- When \( \gamma < 2 \), for \( r \to \infty \)
  \[ g \to \text{diag}(-1, 1, 1, 1) \text{ and } g \to \omega^2 \text{diag}(-c^2, 1, 1, 1) \]
  Lorentz Breaking asymptotics for \( c \neq 1 \)
- When \( S \neq 0 \) nontrivial modification but still flat at infinity when \( \gamma < 2 \). When \(-1 < \gamma < 2\) the large \( r \) behaviour is modified!
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To be physical the solution must have finite total energy

Energy in GR is tricky

- Equivalence principle forbids localization of gravitational energy
- Hypothetical EMT of gravity: $T_{GR}(x_0) \sim \mathcal{F}(\partial g)|_{x_0}$. But at each $x_0$
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- Energy cannot be taken apart but must be considered as whole
- Locally there is no gravity!
- Energy in GR is the conserved charge associated with an arbitrary translation in time, diff generated by a timelike vector
- Equivalently, given a solution, its ADM energy is the value of the Hamiltonian
- Needed: a splitting of spacetime in space + time
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Energy as a Noether Charge

- Consider the Noether charge associated to timelike translations: 
  \[ x^\mu \rightarrow x^\mu + \xi^\mu, \text{ with } \xi^2 < 0 \]

- Choose a set boundary condition for dynamical variables, adjust boundary terms in the action so that the charge is a scalar (coordinate independent). NB a reference metric is needed. We use flat space.

- Fixing the induced metric on the 2-surface \( t = \text{const}, \ r = \bar{r} \) with \( \bar{r} \) large, we get the Nester expression for the energy:

\[
E = \frac{1}{32\pi G} \int_{S_t} d^2 z \epsilon_{\rho\sigma\mu\nu} \left( \xi^\tau \Pi^{\beta\lambda} \Delta \Gamma^\alpha_{\beta\gamma} \delta^{\mu\nu\gamma}_{\alpha\lambda\tau} + \tilde{\nabla}_{\beta} \xi^\alpha \Delta \Pi^{\beta\lambda} \delta^{\mu\nu}_{\alpha\lambda} \right) \frac{\partial x^\rho}{dz^1} \frac{\partial x^\sigma}{dz^2},
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- For Schwarzschild, \( E = M \), even in Painleve coordinates. Actually does not depend on coordinates! Ideal tool for us.
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Computation of the energy

- Boundary terms come only from the kinetic parts; the potential has no role here.
- Contribution of $R(g)$: $E = M - S \bar{r}^{\gamma+1}$
- Contribution of $R(\tilde{g})$: $\tilde{E} = \tilde{M} c^2 + S \bar{r}^{\gamma+1}$.
- Total energy, finite even when $\bar{r} \to \infty$:
  \[ E_{\text{tot}} = E + \tilde{E} = M + \tilde{M} c^2 \]

- Beware! Consider a the frozen $\tilde{g}$ theory, equivalent to $\kappa \to \infty$. The solution for $g$ is similar, but there is no $\tilde{E}$ contribution. Energy is infinite!

- No decoupling effects of “heavy modes” of $\tilde{g}$, needed to account for all energy budget.

- Effective field theories are tricky in gravity when energy is concerned, heavy modes warp spacetime and sometime cannot be neglected.
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- Bigravity is a great tool for studying massive deformation of GR.
- No dDVZ discontinuity in bigravity massive deformation.
- Spherically symmetric solution featuring:
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- Full canonical analysis of bigravity. What does propagate?
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Massive Deformed GR

Not an easy task!

**Challenge**

Build a version of GR modified at large distances such that

- It is consistent with experiments in the solar system
- Ideally valid up to the scale \( \Lambda_2 = (M_{pl} m)^{1/2} \)
  
  As for broken gauge theories, gauge boson mass \( \sim m \)
  
  \[ \Lambda_2 = m g^{-1} = m \left( \frac{\Lambda_2}{M_{pl}} \right)^{-1} \Rightarrow \Lambda_2^2 = m M_{pl}^2 \]

From GR valid up to distances \( > 10^{-33} \text{ cm} \) to

**Massive GR** valid up to distances \( > \Lambda_2^{-1} \sim \left[ 10^{-33} m^{-1} (\text{cm}) \right]^{1/2} \text{ cm} \)
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Class of exact solvable potentials

If \( \{ \lambda_i, \ i = 0, \cdots, 3 \} \) are the eigenvalues of \( X \), the potentials

\[
V_n = \sum_{i_1 > i_2 \cdots > i_n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n}
\]

lead to analytically solvable equations

Examples

\[
V_1 = \frac{1}{6|\tilde{g}|} (\epsilon \epsilon \tilde{g} \tilde{g} \tilde{g} g) = \frac{1}{6 \text{Det}(X)} (\tau_1^3 - 3 \tau_2 \tau_1 + 2 \tau_3)
\]

\[
V_2 = \frac{1}{2|\tilde{g}|} (\epsilon \epsilon \tilde{g} \tilde{g} g g) = \text{Det}(X)^{-1} (\tau_1^2 - \tau_2)
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In adapted coordinates \((t, x^i)\), ADM energy measured by an observer with a clock ticking \(t\)

\[
H_{tot} = \int_{t=\text{const}} d^3x \left[ \mathcal{H} N + \mathcal{H}_i N^i \right] + \int_{S^2, r \to \infty} d^2x B
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on shell

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\(S^2\) is 2-sphere bounding space \((t = \text{const})\) at infinity

- The value of \(B\) and then the total energy depends on the detailed asymptotics of \(g_{\mu\nu}\)
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ADM limitations: derivatives of \( h_{ij} \) (extrinsic curvature) must fall-off at least as \( 1/r^2 \) to be well defined

Coordinates must be Cartesian at Infinity No good for our solution!
Large distances: \( D \sim 1/\sqrt{r} \) (for \( \gamma < -1 \)). Too slow

Analogous to the Schwarzschild solution written in Painlevé coordinates: \( dt = dT - f' dr \)

\[
\begin{align*}
    ds^2 &= -J \, dt^2 + J^{-1} \, dr^2 + r^2 \, d\Omega^2 \\
    &= -J \, dT^2 + 2f' \, J \, dT \, dr + dr^2 + 2f' \, J \, dT \, dr + r^2 \, d\Omega^2 \\
    f'^2 &= J^{-2} - 1
\end{align*}
\]

ADM energy is zero in Painleve coordinates !! In reality is not defined in Painleve coordinates. Extrinsic curvature does not have the right fall-off

We need a more general tool: Gravitational energy as a Noether charge
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