F_π and $\Lambda_{\overline{\mathbf{MS}}}$ from Renormalization Group Optimized Perturbation

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1. Introduction, Motivations

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Summary, conclusions

1. Introduction/Motivations

usually considered hopeless to calculate the χSB order parameters from "first principle" (*except* on Lattice):

1. $\langle \bar{q}q \rangle^{1/3}$, F_π,.. $\sim \mathcal{O}(\Lambda_{QCD}) \simeq$ 100–300 MeV
$\to \alpha_S$ large $\to$ invalidates pert. expansion

2. $\langle \bar{q}q \rangle$, F_π,.. perturbative expansions vanish for $m \to 0$ at any pert. order (trivial chiral limit)

3. Other arguments e.g. (infrared) renormalon ambiguities (signature of (factorially) divergent pert. expansion)

Seem to tell that χSB parameters are intrinsically NP..
Aim here: attempt to circumvent at least 1., 2.,

+ try to understand better pert./NP interplay

This talk: concentrate mainly on pion decay constant F_π:

- derive a non-trivial, finite F_π ($F_\pi/\Lambda_{\overline{MS}}$) for $m \to 0$ from a (variationally) modified perturbation ($m \equiv m_u = m_d$)

- thus derive $\Lambda_{\overline{MS}}$ ($n_f = 2$) from F_π experimental value.

NB For $m \to 0$, $\alpha_S(\mu)$ [equivalently $\Lambda_{\overline{MS}} \sim \mu e^{-\frac{1}{2b_0\alpha_S}}(\ldots)$] is the only fundamental QCD parameter.

$\alpha_S(m_Z)$ (also $\alpha_S(m_\tau)$) known with impressive accuracy: PDG World average: $\alpha_S(m_Z) = .1184 \pm .0007$

Still, worth to get $\Lambda_{\overline{MS}}$ from other analyses, specially for $n_f = 2(3)$, not perturbatively extrapolable from high scale (intense activities in Lattice simulations)

2. (Variationally) Optimized Perturbation

$$\mathcal{L}_{QCD}(g, m_q) \to \mathcal{L}_{QCD}(\delta^{\frac{1}{2}} g, m(1-\delta)^a) \quad (\alpha_S \equiv g^2/(4\pi))$$

δ interpolate between $\mathcal{L}_{free}$ and $\mathcal{L}_{int}$
quark mass $m \to$ arbitrary "variational" parameter

- Take any standard (renormalized) pert. series, expand in δ after:

$$m \to m\,(1-\delta)^a; \quad \alpha_S \to \delta\,\alpha_S$$

then take $\delta \to 1$ (recover original massless theory).
$\to$ BUT a m-dependence remains at any finite δ^k-order:
fixed typically by optimisation (PMS):

$$\frac{\partial}{\partial m}(\text{physical quantity}) = 0 \text{ for } m = m_{opt}$$

NB a extra parameter, to be fixed by further physical/technical requirements (see later)

Simpler model's support + properties

•At first order, ressembles a lot to Hartree-Fock (or large N) approximation

•Convergence proof of this procedure for $D = 1\ \lambda\phi^4$ oscillator (cancels large pert. order factorial divergences!)

particular case of 'order-dependent mapping' Seznec Zinn-Justin '79(exponentially fast convergence for ground state energy $E_0 = const.\lambda^{1/3}$; good to % level at 2d δ-order)

In renormalizable QFT, also produces factorial damping at large perturbative orders (JLK, Reynaud '2002)
(delay divergences, but not sufficient for convergence)

•Flexible, Renormalization-compatible, gauge-invariant applications also at finite temperature (phase transitions beyond mean field approx in 2D, 3D GN models, QCD...)

Previous QCD results

NB previous attempts in the past to use this method in QCD gave chiral symmetry breaking order parameters roughly of the right order of magnitude
(dynamical “mass gap”, F_π, $\langle \bar{q} q \rangle$)

(JLK ’96, Arvanitis, Geniet, Neveu, JLK ’96)

- However had very cumbersome way to marry Renormalization Group properties within such modified perturbation. (i.e. not easily generalizable beyond 2-loop)

Here new proposal: a simple, transparent marriage of OPT and RG properties, easily generalizable

3. RG improved OPT

Our main new ingredient (JLK + A. Neveu 1004.4834, PRD 81 125012):
Consider a physical (RG invariant) quantity, e.g. pole mass M: in addition to OPT Eq:

$$\frac{\partial}{\partial\, m} M^{(k)}(m, g, \delta = 1)|_{m \equiv \tilde{m}} \equiv 0 \tag{1}$$

Require (δ-modified!) series to satisfies a standard perturbative RG equation:

$$\mathrm{RG}\left(M^{(k)}(m, g, \delta = 1)\right) = 0$$

with standard RG operator:

$$\text{RG} \equiv \mu \frac{d}{d\,\mu} = \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \gamma_m(g)\, m \frac{\partial}{\partial m}$$

Combining with Eq. (1) implies RG equation takes a reduced form

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right] M^{(k)}(m, g, \delta = 1) = 0$$

and completely fixes $m \equiv \tilde{m}$ and $g \equiv \tilde{g}$ (two constraints for two parameters).

• But $\Lambda_{\overline{\text{MS}}}(g)$ satisfies by def. $\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right] \Lambda_{\overline{\text{MS}}} \equiv 0$ consistently at a given pert. order for $\beta(g)$.

Thus equivalent to:

$$\frac{\partial}{\partial\, m}\left(\frac{M^k(m, g, \delta = 1)}{\Lambda_{\overline{\text{MS}}}}\right) = 0\;; \quad \frac{\partial}{\partial\, g}\left(\frac{M^k(m, g, \delta = 1)}{\Lambda_{\overline{\text{MS}}}}\right) = 0$$

Pre-QCD test on Gross Neveu model

•$D = 2\ O(N)$ GN model shares many properties with $D = 4$ QCD (asymptotic freedom, mass gap,...)

•Mass gap known exactly (for any N):

$$\frac{M^P_{exact}(N)}{\Lambda_{\overline{\mathrm{MS}}}} = \frac{(4e)^{\frac{1}{2N-2}}}{\Gamma[1-\frac{1}{2N-2}]}$$

from exact S matrix + Thermodynamic Bethe Ansatz
(Zamolodchikov's '79 , Forgacs, Niedermayer, Weisz '91)

•large N result ($M^P = \Lambda_{\overline{\mathrm{MS}}}$) exactly recovered at any δ-order
•At δ^2 (2-loop) order, OPT+RG results differ at worst by $\sim 1-2\%$ from exact mass gap for any N
GN also gives useful insight on generic RG+OPT features

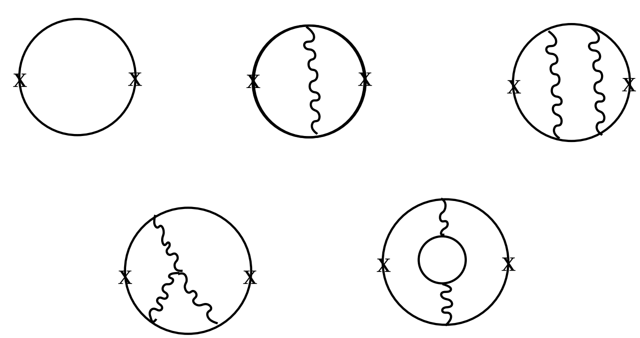
QCD Application; Pion decay constant F_π

Consider $SU(n_f)_L \times SU(n_f)_R \to SU(n_f)_{L+R}$ for massless n_f quarks. (here mostly $n_f = 2$)
Define/calculate pion decay constant F_π from

$$i\langle 0|T A^i_\mu(p) A^j_\nu(0)|0\rangle \equiv \delta^{ij} g_{\mu\nu} F^2_\pi + \mathcal{O}(p_\mu p_\nu)$$

where quark axial current: $A^i_\mu \equiv \bar{q}\gamma_\mu\gamma_5 \frac{\tau_i}{2}\, q$
$F_\pi \neq 0 \to$ Chiral symmetry breaking order parameter

Advantage: Perturbative expression known now to 4 loops
(Avdeev et al ’95); (Chetyrkin et al ’08 ’09)



(Standard) perturbative available information

$$F_\pi^2(pert) = N_c \frac{m^2}{2\pi^2}\left[-L + \frac{\alpha_S}{4\pi}(8L^2 + \frac{4}{3}L + \frac{1}{6})\right.$$
$$\left.+(\frac{\alpha_S}{4\pi})^2[f_{30}L^3 + f_{31}L + f_{32}L + f_{33}] + ..\right] \quad (L \equiv \ln\frac{m}{\mu})$$

+ $O(\alpha_S^3)$ recently available (Maier et al '09, Sturm '09,..)

Note, finite part (after mass + coupling renormalization) not separately RG-inv: (i.e. F_π^2 as defined has its own anomalous dimension)

$\rightarrow$ renormalization by subtraction of the form:
$H(m,\alpha_S) = N_c \frac{m^2}{2\pi^2}(s_0/\alpha_S + s_1 + s_2\alpha_S + ...)$ where s_i fixed requiring RG-inv order by order: $s_0 = \frac{1}{8\pi(\gamma_0 - b_0)}$, $s_1 = ...$

But to fix s_k needs knowing order $k+1$ (the $1/\epsilon$ coefficient)
At $\mathcal{O}(g^2)$ (3-loop) s_3 can be fixed unambiguously from 4-loop

OPT + RG main features

•OPT and RG equations are polynomial in (L, α_S)
At first, one serious drawback: polynomial Eqs of order $k \to$ (too) many solutions, and some complex, at increasing δ-orders
•Solution: requiring RG perturbative asymptotic ($\alpha_S \to 0$) behaviour: $\alpha_S \sim -\frac{1}{2b_0 L} + \cdots$

removes most spurious solutions, which have wrong (perturbative) RG-behaviour!

•After OPT, variational mass m_{opt} is consistently $\mathcal{O}(\Lambda)$ (rather than $m \sim 0$): m_{opt} plays the role of a mass gap, supporting why (modifed) series is more trustable:
$F_\pi^{opt} \sim m_{opt} \times$ pert. series $\sim \Lambda \times$ pert. series

Also, α_S^{opt} is not too large (perturbative value or almost)

F_π OPT+RG estimates

a certain freedom in the basic interpolating Lagrangian:

$$m \to m\,(1-\delta)^a$$

a to be fixed by extra prescription (simplest case $a = 1$).

•For F_π: at arbitrary RG order, asymptotic RG branches only appears for a specific value:

$$m \to m\,(1-\delta)^{\frac{\gamma_0}{2b_0}}$$

Thus fix this a value and follow solutions at successive orders

•However not all RG solutions are real (artefact of polynomial Eq., no physical meaning a priori). Expect 'good' solutions have moderate imaginary parts (see later)

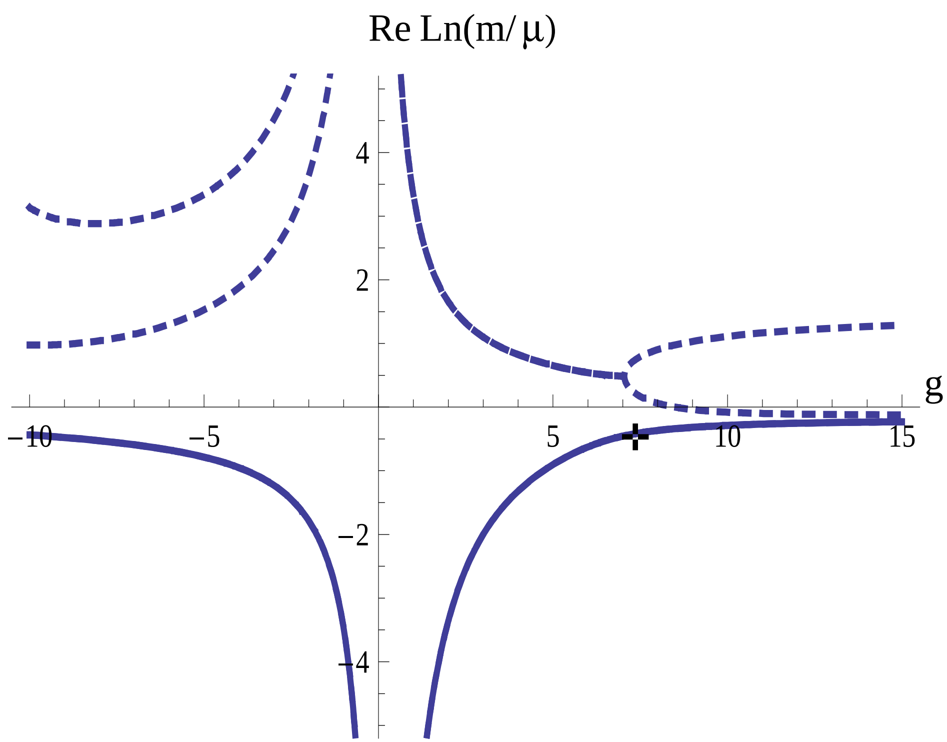


Figure 1: Different branches of the RG solutions $Re[\ln \frac{m}{\mu}(g)]$ to the modified 3rd order (4-loop) perturbative series. ($g = 4\pi\alpha_S$). Unique optimal sol. indicated by a cross

Parenthesis: caution with $\Lambda_{\overline{MS}}$ conventions

Careful with $\Lambda_{\overline{\rm MS}}$ definition for comparison (e.g. with Lattice results). At 3rd order we use

$$\Lambda^{3,\text{Padé}}_{\overline{\rm MS}} \equiv \mu e^{-\frac{1}{2b_0\, g}} \left(\frac{b_0\, g}{1 + (\frac{b_1}{b_0} - \frac{b_2}{b_1})\, g} \right)^{-\frac{b_1}{2b_0^2}} \qquad (g = 4\pi\alpha_S(\mu))$$

(Padé Approximant form, cf some Lattice analysis)
Or alternatively more standard 4-loop perturbative expression

$$\ln \frac{\mu}{\Lambda^4_{\overline{\rm MS}}} = \frac{1}{2b_0\, g} + \frac{b_1}{2b_0^2} \ln g + \cdots + f(b_0, b_1, b_2, b_3) g^2$$

NB gives about 5 % differences on $\Lambda_{\overline{\rm MS}}$ for our typical α_S^{opt} values.

Numerical results ($n_f = 2$)

1st δ-order (2-loop): unique (but complex) solution

$$\frac{F_\pi}{\Lambda^3_{\overline{\text{MS}}}} \sim 0.37 \pm 0.15\, i \quad L \sim -0.45 \pm 0.9\, i, \;\; g \sim 12.7 \pm 1\, i$$

2d order (3-loop): unique solution with right RG behaviour still complex...but smaller imaginary part:

$$\frac{F_\pi}{\Lambda_{\overline{\text{MS}}}} \sim 0.35 \pm 0.03\, i \quad (g \sim 9.15 \pm 0.24\, i\;; \quad L \sim -0.51 \pm 0.69\, i)$$

Note that the value of g_{opt} decreased from 1rst to 2d order, becoming fairly perturbative ($\alpha_S \sim 0.73 \pm 0.02\, i$)
higher orders? complete δ^3 order needs 4-loop results (known) + log terms of 5-loops (unknown) $\rightarrow$ We estimate these from Padé Approximants

Estimate of theoretical uncertainties

•standard: estimates of higher (4-loop) orders (Padés), + truncated RG-equation (only required up to $\mathcal{O}(g^{k+1})$ at pert. order k). $\rightarrow$ Very stable: $\sim$ 1-2% effect on $\Lambda_{\overline{\text{MS}}}$

•In addition, 'intrinsical' uncertainty: complex solutions being artefacts, we may quantify this uncertainty by taking empirically the range spanned by
$Re(F_\pi(g,L)) - F_\pi(Re(g), Re(L))$
increases if $Im[F_\pi]/Re[F_\pi]$ grows.
$\rightarrow$ 1-2% at $\mathcal{O}(\delta^2)$; but 10-13% at $\mathcal{O}(\delta^3)$: Padé effects?

•Subtract effect from explicit chiral symmetry breaking $m_u, m_d \neq 0$: $\frac{F_\pi}{F_0} \sim 1.073 \pm 0.015$ (Lattice FLAG working group 2010)

(alternative: implement explicit sym break. in OPT: under progress)

Combined results with theoretical uncertainties:

$$\frac{F_\pi}{\Lambda^{3,\text{Padé}}_{\overline{\text{MS}}}} \sim 0.29 - 0.34 \rightarrow \Lambda_{\overline{\text{MS}}} \sim 250 - 295 \text{ MeV}$$

To be compare to Lattice results, e.g.:

•'Schrödinger functional scheme' (ALPHA coll. Della Morte et al '05):

$\Lambda_{\overline{\text{MS}}}(n_f = 2) = 245 \pm 16 \pm 16$ MeV

•Wilson fermions (Göckeler et al '05)

$\Lambda_{\overline{\text{MS}}}(n_f = 2) = 261 \pm 17(\text{stat}) \pm 26(\text{syst})$ MeV

•Twisted fermions (+NP power corrections) (Blossier et al '10):

$\Lambda_{\overline{\text{MS}}}(n_f = 2) = 330 \pm 23 \pm 22_{-33}$ MeV

NB those differences seems having to do with
-quark mass effects and different chiral extrapolation
-different calibration (continuum limit) (i.e. better agreement for ratio of physical quantities)

Extrapolation to α_S at high (perturbative) q^2?

Main obstacle: from $n_f = 2$ to $n_f = 3$ i.e. 'crossing' m_s threshold. Deep NP regime a priori, can't use standard perturbative extrapolation.

But, we can calculate similarly $F_\pi/\Lambda_{\overline{\text{MS}}}$ for $n_f = 3$:

mild variation for $n_f = 3$, but $\Lambda_{\overline{\text{MS}}}(n_f = 3) \lesssim \Lambda_{\overline{\text{MS}}}(n_f = 2)$

•Standard perturbative extrapolation (4-loop, with m_c and mb threshold etc) naively gives $\alpha_S(m_Z) \sim .112 - .116$.

However, recall OPT modifies pert. theory: we should use OPT too to extrapolate!

$$\ln \frac{\mu}{\Lambda}\,_{pert} \neq \ln \frac{\mu}{\Lambda}\,_{OPT} \equiv - \ln \frac{m}{\mu}(\alpha_S)\,_{OPT} + \ln \frac{m}{\Lambda_{\overline{\text{MS}}}}\,_{OPT}$$

Preliminary estimate: appears to increase α_S (under progress) But need to better control m_s effects before to conclude...

quark condensate $\langle\bar{q}q\rangle$

$$\langle\bar{q}q\rangle \equiv -i\, lim_{x\to 0} Tr S(x)\ ; \quad S(x) = i\langle 0|T\bar{q}(0)q(x)|0\rangle$$

$\langle\bar{q}q\rangle \neq 0 \to$ Chiral symmetry breaking (CS) order parameter
One considers in fact the (RG invariant) combination

$m(\mu)\langle\bar{q}q\rangle(\mu) \to$ after OPT $(\frac{m}{\Lambda_{\overline{\text{MS}}}})(\frac{\langle\bar{q}q\rangle}{\Lambda^3_{\overline{\text{MS}}}})$

Perturbative expression known to 3 loops

$$m\langle\bar{q}q\rangle(pert) = N_c \frac{m^4}{2\pi^2}\left[-L + \frac{1}{2} + \frac{\alpha_S}{4\pi}(4L^2 - \frac{10}{3}L + \frac{5}{3}) + ...\right]$$

renormalization by subtraction procedure needed similar as F_π one.
(3-loop calculation under progress)

Summary and Outlook

- Variationally optimized perturbation gives a simple procedure to go beyond "large N" in many models, using only perturbative information.

- We proposed a complementary and simple implementation of RG properties
$\rightarrow \mathcal{O}(1-2\%)$ accuracy using ony 2-loop order, for GN model mass gap

- QCD calculation based on 2, 3-loop perturbative information are quite stable,
Estimates of $\Lambda_{\overline{\text{MS}}}$ from F_π compare reasonably well with (some) recent Lattice results.
- Outlook: need to implement explicit chiral sym. breaking in this framework, specially to attack important m_s effects for $n_f = 3$ and extrapolate to perturbative regime.