AdS/CFT and Applications:
Scattering in Planar $\mathcal{N} = 4$ SYM

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“Scattering Amplitudes in Gauge Theories”
Motivation

Why Scattering Amplitudes? ✓

Why $\mathcal{N} = 4$ Super Yang–Mills theory?

- very simple 4D interacting gauge theory model.
- only three parameters ($g_{YM}, \theta, N_c$), no masses, no running: $\beta = 0$.
- for strong coupling can use AdS/CFT duality with string theory.

Why Planar Limit?

- $\mathcal{N} = 4$ SYM becomes simplest gauge theory model!
- integrability to compute observables conveniently.
- finite coupling accessible!

Together: Scattering in Planar $\mathcal{N} = 4$ SYM

- far from trivial functional dependence on particle momenta.
- similar to ordinary QFT (e.g. QCD).
- a lot of progress in recent years . . .
I. Cast of Characters
\( \mathcal{N} = 4 \) Super Yang–Mills Theory

Reminder:
- gauge field \( A_\mu \), 4 fermions \( \Psi \), 6 scalars \( \Phi \).
- gauge group typically \( SU(N_c) \).
- all fields \textit{massless} and adjoint (\( N_c \times N_c \) matrix).
- standard couplings: non-abelian gauge, \( \Psi^2 \Phi, \Phi^4 \)
- coupling constant \( g_{YM} \), topological angle \( \theta \).
- exact super\textit{conformal} symmetry \( \widetilde{PSU}(2, 2|4) \).

Supersymmetry helps:
- protects some quantities, e.g. \( \beta = 0 \), but still model far from trivial!
- \( \mathcal{N} = 4 \) susy relates all fields combine all fields into superfield.
- not care about flavours, helicities: just “scalars”!

Perturbative \( \mathcal{N} = 4 \) SYM through Feynman graphs (\textit{hard}!)
Planar Limit

Planar Limit:

- large-$N_c$ limit: $N_c = \infty$, $g_{YM} = 0$, 
  't Hooft coupling $\lambda = g_{YM}^2 N_c$ remains,
- only planar Feynman graphs, 
  no crossing propagators, 
- drastic combinatorial simplification.

Surface of Feynman graphs becomes 2D string worldsheet:
Strings on $AdS_5 \times S^5$

**AdS/CFT Dual:** Superstrings on curved $AdS_5 \times S^5$ space:

- worldsheet coupling $\lambda$, string coupling: $g_s$,
- weakly coupled for large $\lambda$,
- holographic duality: $\mathcal{N} = 4$ SYM on $\partial AdS_5$,
- symmetry: background isometries $\widetilde{PSU}(2, 2|4)$.

**Planar Limit:**
- no string coupling $g_s = 0$,
  no string splitting or joining.
- worldsheet coupling $\lambda$ remains.
Integrability

Standard QFT approach: **Feynman graphs**
- enormously difficult at higher loops . . .
- . . . but also lower loops and many legs.

Planar $\mathcal{N} = 4$ SYM is **integrable**
- integrability vastly simplifies calculations.
- spectrum of local operators now largely understood.
- can compute observables at **finite coupling** $\lambda$.
- simple **integral equation for cusp dimension** $D_{\text{cusp}}(\lambda)$
- infinite-dimensional **Yangian algebra** $Y(\text{PSU}(2, 2|4))$.
II. Scattering in AdS/CFT
Planar Scattering in Gauge Theory

Consider colour-ordered **planar scattering** (ignore helicities/flavours)

Generic infrared factorisation for $S_n(\lambda, p)$:

$$S_n(0)(p) \exp \left( D_{\text{cusp}}(\lambda) M_n^{(1)}(p) + R_n(\lambda, p) \right)$$

**Required data:**
- tree level $S_n^{(0)}(p)$
- one loop factor $M_n^{(1)}(p)$ (IR-divergent)
- cusp anomalous dimension $D_{\text{cusp}}(\lambda)$
- remainder function $R_n(p, \lambda)$ (finite)

**Intriguing observation** for $n = 4, 5$ legs: $R_n = 0$!
- Computed/confirmed at 4 loops using unitarity.
- Exact result for scattering at finite $\lambda$! Why simple?
- Generalise to $n \geq 6$ legs! Compute exact $R_n$?!
Planar Scattering in String Theory

AdS/CFT provides a **string analog** for planar scattering.

Area of a minimal surface in $AdS_5$ . . . 
. . . ending on a null polygon on $\partial AdS_5$.

- Identify particles with segments:
  \[ p_k = \Delta x_k = x_k - x_{k-1} \]
- on-shell particles $\rightarrow$ null segments:
  \[ p_k^2 = \Delta x_k^2 = 0 \]
- momentum conservation $\rightarrow$ closure:
  \[ \sum_k p_k = \sum_k \Delta x_k = 0 \]

Note:
- Identification uses **T-duality** of $AdS_5 \times S^5$ strings.
- Functional form of exponent $M^{(1)}$ verified in string theory.
Null Polygonal Wilson Loop

AdS/CFT backwards:

- Minimal surfaces correspond to Wilson loops in gauge theory.
- Amplitudes “T-dual” to null polygonal Wilson loops

\[ p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5 \]

\[ x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \]

Weak/weak perturbative duality. **Tested for:**

- all 1-loop amplitudes / Wilson loops
- 2-loop 6-leg amplitude / hexagon Wilson loop
\( \mathcal{N} = 4 \) SYM is superconformal: \( \text{PSU}(2, 2|4) \) symmetry.

- Amplitudes are conformally invariant.*
- Wilson loops are conformally invariant.*

* IR/UV singularities break invariance (in a controllable fashion), see below.

**Two conformal symmetries:**
- different action on amplitudes and Wilson loops
  - ordinary conformal symmetry
  - dual conformal symmetry \( \updownarrow \) T-duality
- together: generate infinite-dimensional . . .
  . . . Yangian algebra \( Y(\text{PSU}(2, 2|4)) \).

Dual conformal symmetry explains simplicity:
- No dual conformal cross ratios for \( n = 4, 5 \).
- Remainder function must be trivial: \( R_n = 0 \).
III. Twistor Representation
Spinor Helicity

Dual picture of scattering leads to novel useful parametrisation.

**Step 1: Spinor Helicity.**

- function of constrained particle momenta \( S(p_k) \)
- particle momenta are on shell \( p_k^2 = 0 \)
- can write momentum in spinor notation as product \( p_k^{\alpha\gamma} = \lambda_k^\alpha \tilde{\lambda}_k^\gamma \).
- function of unconstrained spinor variables \( S(\lambda_k, \tilde{\lambda}_k) \).

Amplitude expressions **simplify**.

Tree amplitude for certain helicity configuration (MHV):

\[
S_n^{\text{MHV}} \sim \frac{1}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}, \quad \langle j, k \rangle := \varepsilon_{\alpha\gamma} \lambda_j^\alpha \lambda_k^\gamma.
\]

Sum of thousands of Feynman trees for large \( n \).
Momentum Twistors

Still have to satisfy momentum conservation constraint: \( \sum_k \lambda_k \tilde{\lambda}_k = 0 \).

Step 2: Momentum Twistors.

- use dual coordinates \( x_k = \sum_{j=1}^{k-1} \Delta x_j = \sum_{j=1}^{k-1} p_j = \sum_{j=1}^{k-1} \lambda_j \tilde{\lambda}_j \).

- define \( \mu_k^\dot{\alpha} := \varepsilon_{\beta\gamma} \lambda_k^\beta x_k^\gamma \dot{\alpha} \) and 4-component twistor \( W_k := (\lambda_k, \mu_k) \).

- amplitude is a function of unconstrained twistor variables \( S(W_k) \).

Many benefits:

- Map between momenta \( p_k \) and twistors \( W_k \) is 1-to-1.
- On-shell and momentum constraints automatically satisfied.
- Twistors transform as (projective) vectors of conformal SU(2, 2).
- Simple expressions for \( S(W_k) \). Recent progress based on twistors.
- Can easily compute twistors \( W_k \) from momenta \( p_k \). Then:

\[
S(p_k) = S(W_k(p_k)).
\]
Twistor Theory

What is the **meaning** of the twistor variables?

In \((3,1)\) Minkowski space:
- single twistor \(W_k\): light ray.
- all \(W_k\)'s: null polygon.
- define shape of Wilson loop.

**Alternative formulation:**
- Chern–Simons theory on \(CP^3\) twistor space.
- dual Wilson loop in \(CP^3\)
- matching results.
IV. Recent Progress
Grassmannian
Polytopes
Polylog Symbols
OPE
Higgs Branch
Representations
Bubble Ansatz
Null Correlators
Progress in Computation
Graßmannian Formula

Twistor space representation $A_n(W)$ ideal for complex analysis:

Poles generated by **Graßmannian integral**

$k$: helicity, $C$: $k \times n$ matrix, $M_j$: $k \times k$ minors of $C$

\[ A_{n,k}(W) = \int \frac{d^{k(n-k)} C \delta^{k(4\mid 4)}(CW)}{M_1 \cdots M_n} \]

**Features:**

- Integral is conformal and Yangian invariant.
- Higher-loop integrands **conveniently constructible**.
- Can isolate remainder function (finite).

**Still:**

- order by order construction
- remaining loop integrals very difficult

Arkani-Hamed, Cachazo, Cheung, Kaplan

Drummond, Ferro

Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka

Arkani-Hamed, Bourjaily, Cachazo, Trnka
Polylogarithms and Symbols

Functional dependence on momenta:
\[
\begin{cases} 
\text{tree: rational} \\
\text{1 loop: } \text{Li}_2, \log^2 \\
\text{n loops: } \text{Li}_{2n} \text{ (generalised)}
\end{cases}
\]

Problems:
- hard to compute: loop integrals.
- hard to handle: partial fractions, polylog identities.

E.g. 2 loops, 6 legs: \( R_6 = \frac{1}{17 + \varepsilon} \) pages of Goncharov polylogs

New idea: Use polylog "symbol" (differential form)

\[
\text{log } x \text{ log } y \mapsto x \otimes y + y \otimes x, \quad \text{Li}_2 x \mapsto - (1 - x) \otimes x, \quad \ldots
\]

Simplify result \( \text{symbol} \mapsto \text{polylog} \mapsto \) (few lines of ordinary polylogs)
Polytopes

Many ways to write amplitudes:
- partial fractions,
- polylog identities,
- kinematical relations.

Some features manifest, others not:
- cyclic symmetry,
- physical poles & cuts,
- limits, symmetries, ... 

Why? What does it mean? Is there a perfect form?

Observation: One-loop amplitudes contain $\text{Li}_2$.
$\text{Li}_2$ also describes volume of simplex in hyperbolic space.

Proposal: Amplitude \(\simeq\) volume of curved space polytope.
Different forms of amplitude \(\simeq\) different triangulations:

- manifest symmetries hard to compute
- cyclic symmetry extra points
- no extra points no cyclic symmetry
- some symmetries some cyclic symmetry
- spurious regions

Hodges 1004.3323
Hodges 0905.1473
Bubble Ansatz

Strong coupling limit: **Minimal surface area.**
- Minimal surface for 4-sided polygon explicitly known.
- **Unclear** how to construct more sides analytically.

**Idea:**
- Not construct minimal surface explicitly.
- Introduce auxiliary complex parameter $z$ (integrability).
- Study $z$-behaviour at boundary and cusps: complex analysis.

Complex analysis problem encodes:
- polygon data (cross ratios)
- minimal surface area.
Apply CFT tools to Wilson loop: **Operator Product Expansion.**

\[
\begin{align*}
\text{Wilson Loop OPE} & = + \int d\tau \dot{x}^\mu \delta x^\nu \frac{F_{\mu\nu}}{x} + \int \int \\
\text{Apply to multiple collinear limit} & \quad \rightarrow \\
\text{Result uses:} & \\
\bullet \text{flux tube between top/bottom WL: spinning string} \\
\bullet \text{cusp dimension,} \\
\bullet \text{excitations of flux tube/spinning string.}
\end{align*}
\]
Null Correlators

Consider further principal objects in CFT:

**Correlation function** of local operators \( \langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle \)

- **tree: rational**
- **1 loop: \( \text{Li}_2, \log^2 \)**
- **\( n \) loops: \( \text{Li}_{2n} \) (generalised)**

Relation to amplitudes and null Wilson loops?

**Proposal:** Null separated local operators. **Triality:**

\[
\begin{align*}
A & \quad \leftrightarrow \quad S \\
\left( A \right)^2 & \quad \leftrightarrow \quad \left( S \right)^2
\end{align*}
\]
Perturbative Symmetry

Can we use conformal/Yangian symmetry to determine amplitudes?

Problem:
- IR/UV regulator breaks symmetry,
- need to understand how symmetry is violated.

Caution: Conformal symmetry already broken at tree level!
Only for singular configurations of external momenta: $p_k \parallel p_{k+1}$.

\[
\delta S = S^* \delta S^{(1)}
\]

Can correct symmetry representation (non-linear in fields).

Loops: Integrate over internal momenta, conformal violation smeared, always apparent.
Massive Regularisation

Alternative regularisation: **Higgs branch** of $\mathcal{N} = 4$ SYM:
- turn on scalar VEV,
- masses to particles,
- IR singularities regularised.

**Benefits:**
- dual conformal symmetry manifest.
- some physics more manifest (whereas DR $\varepsilon$ artificial).

Relations to extra dimensions:
- **holography**: 5th coordinate from $AdS_5$,
- masses from higher dimensional momenta (5th, 6th component).
- cross-dimension relations for loop integrals (4D–6D).
V. Conclusions
Conclusions

Scattering Amplitudes in Planar $\mathcal{N} = 4$ Super Yang–Mills
- have exciting properties, obey interesting relations.
- integrability a key to progress.

More Concretely:
- strong coupling: minimal surface area
- duality to Wilson loops (among others)
- twistor formulation useful
- various novel methods for computing amplitudes.

Outlook:
- Hope to calculate amplitudes exactly.
- Investigate at finite coupling.
- Apply to QCD (quantitatively).

Read More:
- J. Phys. A special issue

Eps 2011, Niklas Beisert
VI. Material
Formula for 2 loops, 6 legs remainder

\[
\begin{align*}
R_{6}^{(2)}(u_1, u_2, u_3) &= \sum_{i=1}^{3} \left( L_4(x_i^+, x_i^-) - \frac{1}{2} \operatorname{Li}_4(1 - 1/u_i) \right) \\
&- \frac{1}{8} \left( \sum_{i=1}^{3} \operatorname{Li}_2(1 - 1/u_i) \right)^2 + \frac{1}{24} J^4 + \frac{\pi^2}{12} J^2 + \frac{\pi^4}{72}.
\end{align*}
\]

\[
\begin{align*}
u_1 &= \frac{s_{12} s_{45}}{s_{123} s_{345}}, & u_2 &= \frac{s_{23} s_{56}}{s_{234} s_{123}}, & u_3 &= \frac{s_{34} s_{61}}{s_{345} s_{234}}, \\
\quad x_i^\pm &= u_i x^\pm, & x^\pm &= \frac{u_1 + u_2 + u_3 - 1 \pm \sqrt{\Delta}}{2u_1u_2u_3}, \\
\Delta &= (u_1 + u_2 + u_3 - 1)^2 - 4u_1u_2u_3, \\
L_4(x^+, x^-) &= \frac{1}{8!!} \log(x^+ x^-)^4 \\
&+ \sum_{m=0}^{3} \frac{(-1)^m}{(2m)!!} \log(x^+ x^-)^m (\ell_{4-m}(x^+) + \ell_{4-m}(x^-)) \\
J &= \sum_{i=1}^{3} (\ell_1(x_i^+) - \ell_1(x_i^-)).
\end{align*}
\]