AdS/CFT and Applications: Scattering in Planar $\mathcal{N} = 4$ SYM

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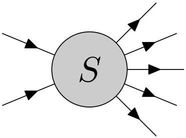
see J. Phys. A Special Issue (∼Sep'2011)
"Scattering Amplitudes in Gauge Theories"

Motivation



Why Scattering Amplitudes? ✓

Why $\mathcal{N}=4$ Super Yang–Mills theory?

- very simple 4D interacting gauge theory model.
- only three parameters (g_{YM}, θ, N_{c}), no masses, no running: $\beta = 0$.
- for strong coupling can use AdS/CFT duality with string theory.

Why Planar Limit?

- $\mathcal{N}=4$ SYM becomes simplest gauge theory model!
- integrability to compute observables conveniently.
- finite coupling accessible!

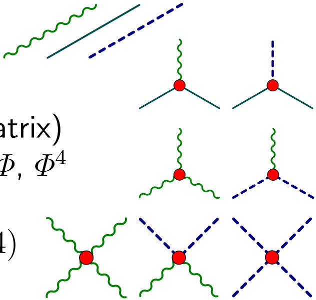
Together: Scattering in Planar $\mathcal{N}=4$ SYM

- far from trivial functional dependence on particle momenta.
- similar to ordinary QFT (e.g. QCD).
- a lot of progress in recent years ...

I. Cast of Characters

$\mathcal{N} = 4$ Super Yang–Mills Theory

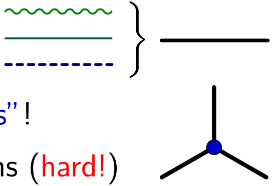
Reminder: [Brink Schwarz Scherk]



- gauge field A_μ, 4 fermions Ψ, 6 scalars Φ.
- gauge group typically $\mathrm{SU}(N_\mathrm{c})$
- all fields massless and adjoint ($N_\mathrm{c} \times N_\mathrm{c}$ matrix)
- standard couplings: non-abelian gauge, $\Psi^2\Phi$, Φ^4
- coupling constant g_YM, topological angle θ
- exact superconformal symmetry $\widetilde{\mathrm{PSU}}(2,2|4)$

Supersymmetry helps:

- protects some quantities, e.g. $\beta = 0$,
  but still model far from trivial!
- $\mathcal{N} = 4$ susy relates all fields
  combine all fields into superfield
  not care about flavours, helicities: just "scalars"!

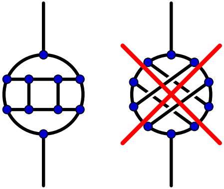


Perturbative $\mathcal{N} = 4$ SYM through Feynman graphs (hard!)

Planar Limit

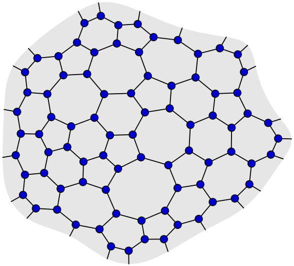
Planar Limit:

['t Hooft Nucl. Phys. B72, 461]



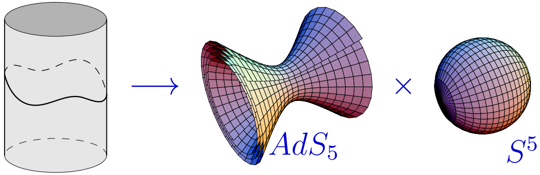
- large-N_c limit: $N_c = \infty$, $g_{YM} = 0$,
  't Hooft coupling $\lambda = g_{YM}^2 N_c$ remains,
- only planar Feynman graphs,
  no crossing propagators,
- drastic combinatorial simplification.

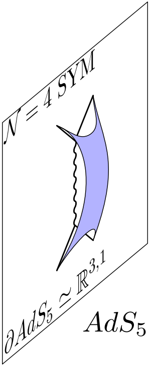
Surface of Feynman graphs becomes 2D string worldsheet:



Strings on $AdS_5 \times S^5$

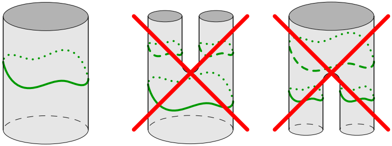
AdS/CFT Dual: Superstrings on curved $AdS_5 \times S^5$ space: [Maldacena hep-th/9711200]





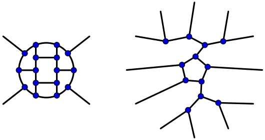
- worldsheet coupling λ, string coupling: g_s,
- weakly coupled for large λ,
- holographic duality: $\mathcal{N} = 4$ SYM on ∂AdS_5,
- symmetry: background isometries $\widetilde{\mathrm{PSU}}(2, 2|4)$.

Planar Limit:



- no string coupling $g_\mathrm{s} = 0$,
  no string splitting or joining.
- worldsheet coupling λ remains.

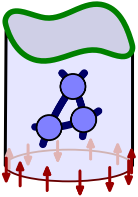
Integrability



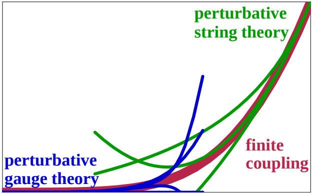
Standard QFT approach: **Feynman graphs**

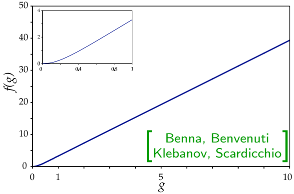
- enormously difficult at higher loops ...
- ... but also lower loops and many legs.

Planar $\mathcal{N} = 4$ SYM is **integrable** see review collection [NB et al. 1012.3982]



- integrability vastly simplifies calculations.
- spectrum of local operators now largely understood.
- can compute observables at finite coupling λ. [NB, Eden Staudacher]
- simple integral equation for cusp dimension $D_{\text{cusp}}(\lambda)$
- infinite-dimensional Yangian algebra $\mathrm{Y}(\mathrm{PSU}(2,2|4))$.

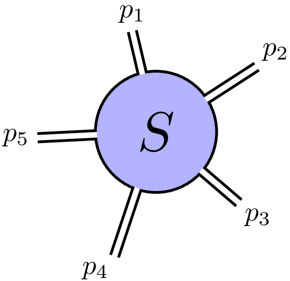




II. Scattering in AdS/CFT

Planar Scattering in Gauge Theory

Consider colour-ordered **planar scattering** (ignore helicities/flavours)



Generic infrared factorisation for $S_n(\lambda, p)$:

$$S_n^{(0)}(p) \exp\big(D_{\text{cusp}}(\lambda) M_n^{(1)}(p) + R_n(\lambda, p)\big)$$

Required data:

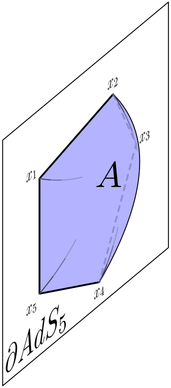
- tree level $S_n^{(0)}(p)$
- one loop factor $M_n^{(1)}(p)$ (IR-divergent)
- cusp anomalous dimension $D_{\text{cusp}}(\lambda)$
- remainder function $R_n(p, \lambda)$ (finite)

Intriguing observation for $n = 4, 5$ legs: $R_n = 0$! [Anastasiou, Bern Dixon, Kosower][Bern Dixon Smirnov] [Bern Dixon Smirnov][Bern, Czakon, Dixon Kosower, Smirnov]

- Computed/confirmed at 4 loops using unitarity.
- Exact result for scattering at finite λ! Why simple?
- Generalise to $n \geq 6$ legs! Compute exact R_n?!

Planar Scattering in String Theory

AdS/CFT provides a **string analog** for planar scattering. [Alday Maldacena]



Area of a minimal surface in AdS_5 ...

... ending on a null polygon on ∂AdS_5.

- Identify particles with segments:

$$p_k = \Delta x_k = x_k - x_{k-1}$$

- on-shell particles → null segments:

$$p_k^2 = \Delta x_k^2 = 0$$

- momentum conservation → closure:

$$\sum_k p_k = \sum_k \Delta x_k = 0$$

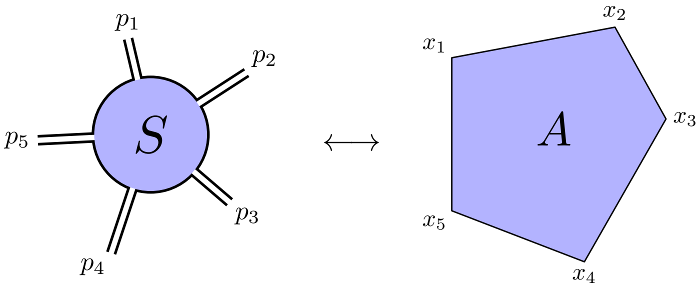
Note:

- Identification uses T-duality of $AdS_5 \times S^5$ strings.
- Functional form of exponent $M^{(1)}$ verified in string theory.

Null Polygonal Wilson Loop

AdS/CFT backwards: [Drummond Korchemsky Sokatchev] [Brandhuber Heslop Travaglini]

- Minimal surfaces correspond to Wilson loops in gauge theory.
- Amplitudes "T-dual" to null polygonal Wilson loops



Weak/weak perturbative duality. **Tested for:**

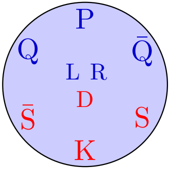
- all 1-loop amplitudes / Wilson loops [Drummond Korchemsky Sokatchev] [Brandhuber Heslop Travaglini]
- 2-loop 6-leg amplitude / hexagon Wilson loop [Bern, Dixon, Kosower Roiban, Spradlin Vergu, Volovich] [Drummond, Henn Korchemsky Sokatchev]

Dual Conformal and Yangian Symmetries

$\mathcal{N}=4$ SYM is superconformal: $\mathrm{PSU}(2,2|4)$ symmetry.

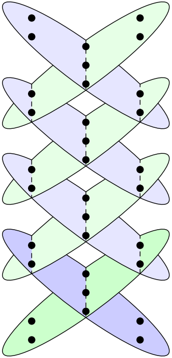
- Amplitudes are conformally invariant.*
- Wilson loops are conformally invariant.*

* IR/UV singularities break invariance (in a controllable fashion), see below.



Two conformal symmetries: [Drummond, Henn Smirnov, Sokatchev][Drummond Korchemsky Sokatchev]

- different action on amplitudes and Wilson loops
- ordinary conformal symmetry ↕ T-duality [Alday Maldacena]
  dual conformal symmetry
- together: generate infinite-dimensional … [NB, Ricci Tseytlin, Wolf]
  … Yangian algebra $\mathrm{Y}(\mathrm{PSU}(2,2|4))$. [Drummond Henn Plefka]



Dual conformal symmetry **explains simplicity**:

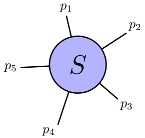
- No dual conformal cross ratios for $n=4,5$.
- Remainder function must be trivial: $R_n=0$.

III. Twistor Representation

Spinor Helicity

Dual picture of scattering leads to novel useful parametrisation.

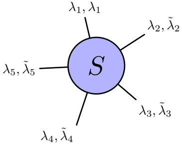
Step 1: Spinor Helicity.



- function of constrained particle momenta $S(p_k)$
- particle momenta are on shell $p_k^2 = 0$
- can write momentum in spinor notation as product

$$p_k^{\alpha\dot{\gamma}} = \lambda_k^\alpha \tilde{\lambda}_k^{\dot{\gamma}}.$$

- function of unconstrained spinor variables $S(\lambda_k, \tilde{\lambda}_k)$.



Amplitude expressions simplify.

Tree amplitude for certain helicity configuration (MHV): [Parke Taylor][Berends Giele]

$$S_n^{\mathrm{MHV}} \simeq \frac{1}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}\,, \qquad \langle j, k\rangle := \varepsilon_{\alpha\gamma}\lambda_j^\alpha \lambda_k^\gamma.$$

Sum of thousands of Feynman trees for large n.

Momentum Twistors

Still have to satisfy momentum conservation constraint: $\sum_k \lambda_k \tilde{\lambda}_k = 0$.

Step 2: Momentum Twistors. [Hodges 0905.1473][Mason Skinner]

- use dual coordinates $x_k = \sum_{j=1}^{k-1} \Delta x_j = \sum_{j=1}^{k-1} p_j = \sum_{j=1}^{k-1} \lambda_j \tilde{\lambda}_j$.
- define $\mu_k^{\dot{\alpha}} := \varepsilon_{\beta\gamma} \lambda_k^{\beta} x_k^{\gamma\dot{\alpha}}$ and 4-component twistor $W_k := (\lambda_k, \mu_k)$.
- amplitude is a function of unconstrained twistor variables $S(W_k)$.

Many benefits:

- Map between momenta p_k and twistors W_k is 1-to-1.
- On-shell and momentum constraints automatically satisfied.
- Twistors transform as (projective) vectors of conformal $\mathrm{SU}(2,2)$.
- Simple expressions for $S(W_k)$. Recent progress based on twistors.
- Can easily compute twistors W_k from momenta p_k. Then:

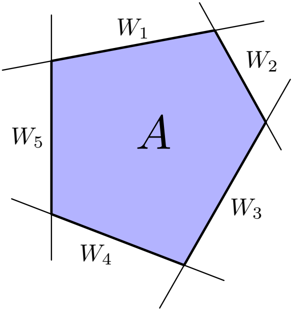
$$S(p_k) = S(W_k(p_k)).$$

Twistor Theory

What is the **meaning** of the twistor variables?

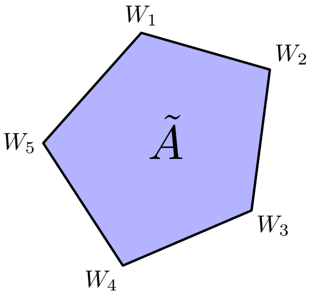
In $(3,1)$ Minkowski space:

- single twistor W_k: light ray.
- all W_k's: null polygon.
- define shape of Wilson loop.



Alternative formulation: [Mason Skinner]

- Chern–Simons theory on CP^3 twistor space.
- dual Wilson loop in CP^3
- matching results.



IV. Recent Progress

Progress in Computation

Graßmannian Formula

Twistor space representation $A_n(W)$ ideal for complex analysis:
Poles generated by **Graßmannian integral** [Arkani-Hamed, Cachazo Cheung, Kaplan]
[k: helicity, C: $k \times n$ matrix, M_j: $k \times k$ minors of C]

$$A_{n,k}(W) = \int \frac{d^{k(n-k)}C \, \delta^{k(4|4)}(CW)}{M_1 \cdots M_n}$$

Features:

- Integral is conformal and Yangian invariant. [Drummond Ferro]
- Higher-loop integrands conveniently constructible. [Arkani-Hamed, Bourjaily Cachazo, Caron-Huot, Trnka]
- Can isolate remainder function (finite). [Arkani-Hamed, Bourjaily Cachazo, Trnka]

Still:

- order by order construction
- remaining loop integrals very difficult

Polylogarithms and Symbols

Functional dependence on momenta: $\begin{cases} \text{tree: rational} \\ \text{1 loop: } \mathrm{Li}_2,\ \log^2 \\ n \text{ loops: } \mathrm{Li}_{2n} \text{ (generalised)} \end{cases}$

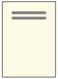
Problems:

- hard to compute: loop integrals.
- hard to handle: partial fractions, polylog identities.

E.g. 2 loops, 6 legs: $R_6 =$  ($17 + \varepsilon$ pages of Goncharov polylogs) [Del Duca Duhr Smirnov]

New idea: Use polylog "symbol" (differential form) [Goncharov, Spradlin Vergu, Volovich]

$$\log x \log y \mapsto x \otimes y + y \otimes x, \quad \mathrm{Li}_2\, x \mapsto -(1-x) \otimes x, \quad \ldots$$

Simplify result  $\xrightarrow{\text{symbol}}$  $\xrightarrow{\text{polylog}}$  (few lines of ordinary polylogs)

Polytopes

Many ways to write amplitudes:

- partial fractions,
- polylog identities,
- kinematical relations.

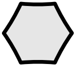
Some features manifest, others not:

- cyclic symmetry,
- physical poles & cuts,
- limits, symmetries, . . .

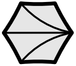
Why? What does it mean? Is there a perfect form?

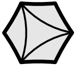
Observation: One-loop amplitudes contain Li_2.
Li_2 also describes volume of simplex in hyperbolic space. [Hodges 1004.3323]

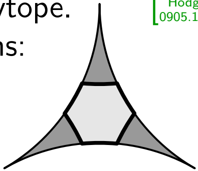
Proposal: Amplitude $\simeq$ volume of curved space polytope. [Hodges 0905.1473]
Different forms of amplitude $\simeq$ different triangulations:


manifest symmetries
hard to compute


cyclic symmetry
extra points


no extra points
no cyclic symmetry


some symmetries
some cyclic symmetry


spurious regions

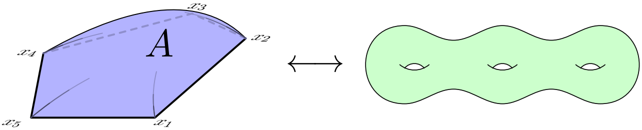
Bubble Ansatz

Strong coupling limit: **Minimal surface area.**

- Minimal surface for 4-sided polygon explicitly known.
- Unclear how to construct more sides analytically.

Idea: [Alday Maldacena][Alday Gaiotto Maldacena][Alday, Maldacena Sever, Vieira]

- Not construct minimal surface explicitly.
- Introduce auxiliary complex parameter z (integrability).
- Study z-behaviour at boundary and cusps: complex analysis.

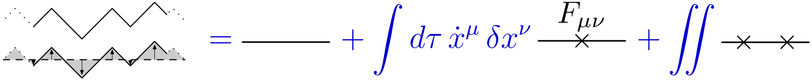


Complex analysis problem encodes:

- polygon data (cross ratios)
- minimal surface area.

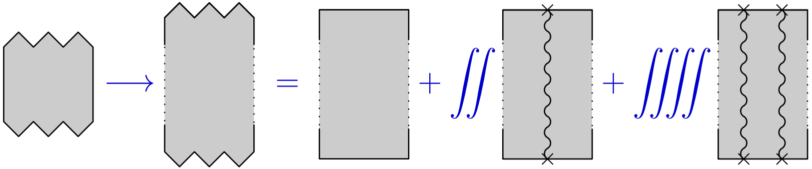
Wilson Loop OPE

Apply CFT tools to Wilson loop: **Operator Product Expansion**.



Apply to multiple collinear limit

[Alday, Gaiotto Maldacena Sever, Vieira]



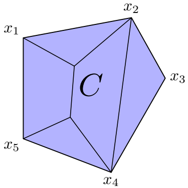
Result uses:

- flux tube between top/bottom WL: spinning string [Gubser Klebanov Polyakov]
- cusp dimension, [NB, Eden Staudacher]
- excitations of flux tube/spinning string. [Basso 1010.5237]

Null Correlators

Consider further principal objects in CFT:

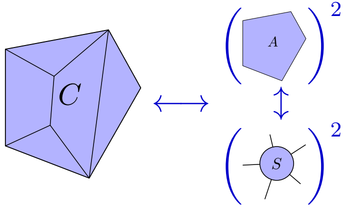
Correlation function of local operators $\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle$



$$\text{dependence} \begin{cases} \text{tree: rational} \\ \text{1 loop: } \mathrm{Li}_2,\ \log^2 \\ n \text{ loops: } \mathrm{Li}_{2n} \text{ (generalised)} \end{cases}$$

Relation to amplitudes and null Wilson loops?

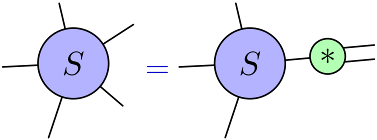
Proposal: Null separated local operators. Triality: [Alday, Eden, Korchemsky Maldacena, Sokatchev] [Eden Korchemsky Sokatchev]

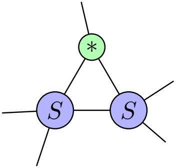


Perturbative Symmetry

Can we use conformal/Yangian symmetry to determine amplitudes?
Problem:
- IR/UV regulator breaks symmetry,
- need to understand how symmetry is violated.

Caution: Conformal symmetry already broken at tree level! [Cachazo Svrcek Witten]
Only for singular configurations of external momenta: $p_k \parallel p_{k+1}$.

δ 

$\delta S^{(1)} =$ 

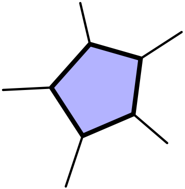
Can correct symmetry representation (non-linear in fields). [Bargheer, NB, Galleas Loebbert, McLoughlin]

Loops: Integrate over internal momenta, [Sever Vieira][NB, Henn McLoughlin, Plefka]
conformal violation smeared, always apparent.

Massive Regularisation

Alternative regularisation: **Higgs branch** of $\mathcal{N} = 4$ SYM: [Alday, Henn Plefka, Schuster]

- turn on scalar VEV,
- masses to particles,
- IR singularities regularised.



Benefits:

- dual conformal symmetry manifest.
- some physics more manifest (whereas DR ε artificial). [Henn, Naculich Schnitzer, Spradlin]

Relations to extra dimensions:

- holography: $5^{\rm th}$ coordinate from AdS_5,
- masses from higher dimensional momenta ($5^{\rm th}$, $6^{\rm th}$ component).
- cross-dimension relations for loop integrals (4D–6D). [Dixon Drummond Henn]

V. Conclusions

Conclusions

Scattering Amplitudes in Planar $\mathcal{N} = 4$ Super Yang–Mills

- have exciting properties, obey interesting relations. 1105.0771
- integrability a key to progress. 1103.3477

1103.1869

More Concretely: 1103.3298

- strong coupling: minimal surface area [Ita]
- duality to Wilson loops (among others) 1012.4493
- twistor formulation useful 1104.3873
- various novel methods for computing amplitudes. 1104.2890

Outlook: 1012.3401

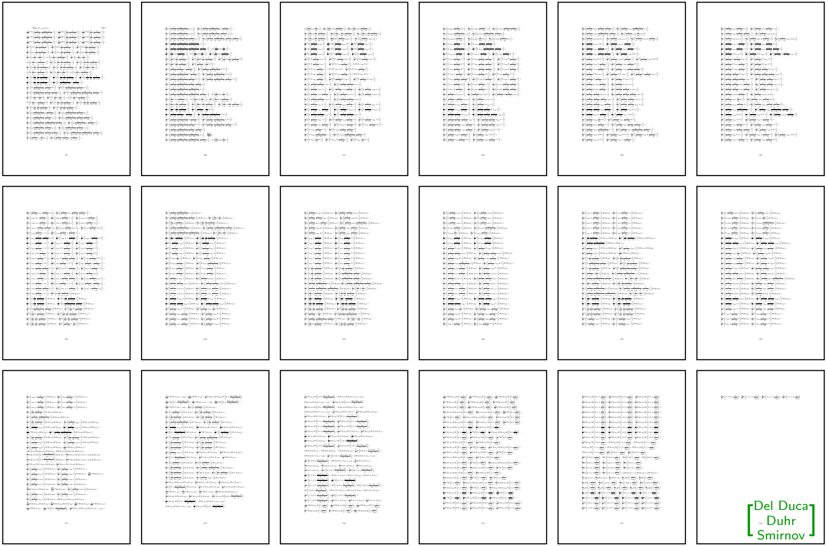
- Hope to calculate amplitudes exactly. 1107.4544
- Investigate at finite coupling. 1103.1016
- Apply to QCD (quantitatively). 1104.0700

Read More:

- J. Phys. A special issue 1104.0816
- Lett. Math. Phys. special issue arxiv:1012.3982

VI. Material

$17 + \varepsilon$ Pages



Few Lines

Formula for 2 loops, 6 legs remainder [Goncharov, Spradlin Vergu, Volovich]

$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\,\mathrm{Li}_4(1-1/u_i)\right)$$
$$-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}.$$

$$u_1 = \frac{s_{12}s_{45}}{s_{123}s_{345}}, \quad u_2 = \frac{s_{23}s_{56}}{s_{234}s_{123}}, \quad u_3 = \frac{s_{34}s_{61}}{s_{345}s_{234}},$$

$$x_i^\pm = u_i x^\pm, \qquad x^\pm = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3},$$
$$\Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3,$$

$$L_4(x^+,x^-) = \frac{1}{8!!}\log(x^+x^-)^4$$
$$+\sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+)+\ell_{4-m}(x^-))$$

$$\ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n\,\mathrm{Li}_n(1/x)\right),$$

$$J = \sum_{i=1}^{3}(\ell_1(x_i^+) - \ell_1(x_i^-)).$$