Determination of the light quark masses from $\eta \to 3\pi$

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Europhysics Conference on High Energy Physics

Grenoble, July 22nd, 2011

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Ph.D. Thesis of S. Lanz, University of Bern, May 12, 2011
Article in preparation

Outline :

1. Introduction and Motivation

2. Dispersive analysis of $\eta \to \pi^+ \pi^- \pi^0$ decays

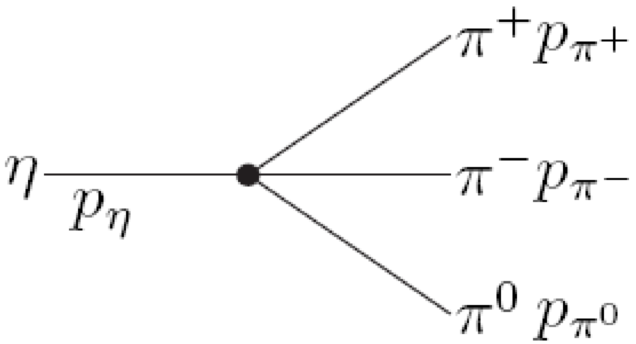
3. Results

4. Conclusion and outlook

1. Introduction and Motivation

1.1 $\eta \to \pi^+ \pi^- \pi^0$ decays: $(m_u - m_d)$ Golden Channel

- η decay: $\eta \to \pi^+ \pi^- \pi^0$



- Decay forbidden by isospin symmetry

  ⇨ $A \sim (m_u - m_d)$ or $A \sim \alpha_{em}$

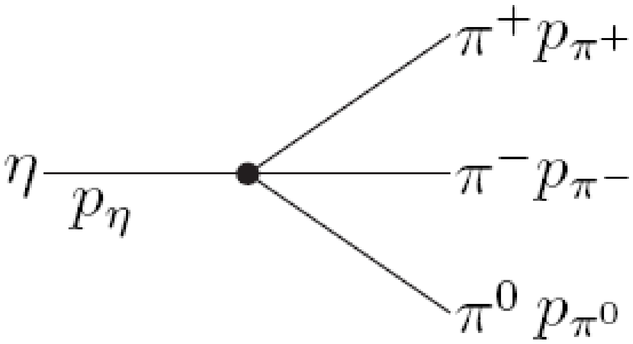
- Electromagnetic effects are small

  Sutherland's theorem'66
  Baur, Kambor & Wyler'95
  Ditsche, Kubis & Meißner'09

- Decay rate measures the size of isospin breaking in the SM

  ⇨ Direct probe of $m_u - m_d$

1.2 $\eta \to \pi^+ \pi^- \pi^0$ decays

- 

$$s = \left(p_{\pi^+} + p_{\pi^-}\right)^2, \quad t = \left(p_{\pi^-} + p_{\pi^0}\right)^2$$

$$u = \left(p_{\pi^0} + p_{\pi^+}\right)^2$$

$$s + t + u = M_\eta^2 + M_{\pi^0}^2 + 2M_{\pi^+}^2 \equiv 3s_0$$

$$\left\langle \pi^+ \pi^- \pi^0{}_{out} \middle| \eta \right\rangle = i\left(2\pi\right)^4 \delta^4\left(p_\eta - p_{\pi^+} - p_{\pi^-} - p_{\pi^0}\right) A(s,t,u)$$

- Lowest order amplitude: Current algebra

Osborn, Wallace'70

$$A(s,t,u) = \frac{B_0\left(m_d - m_u\right)}{3\sqrt{3}F_\pi^2}\left[1 + \frac{3\left(s - s_0\right)}{M_\eta^2 - M_\pi^2} + O(m)\right] + O(e^2 m)$$

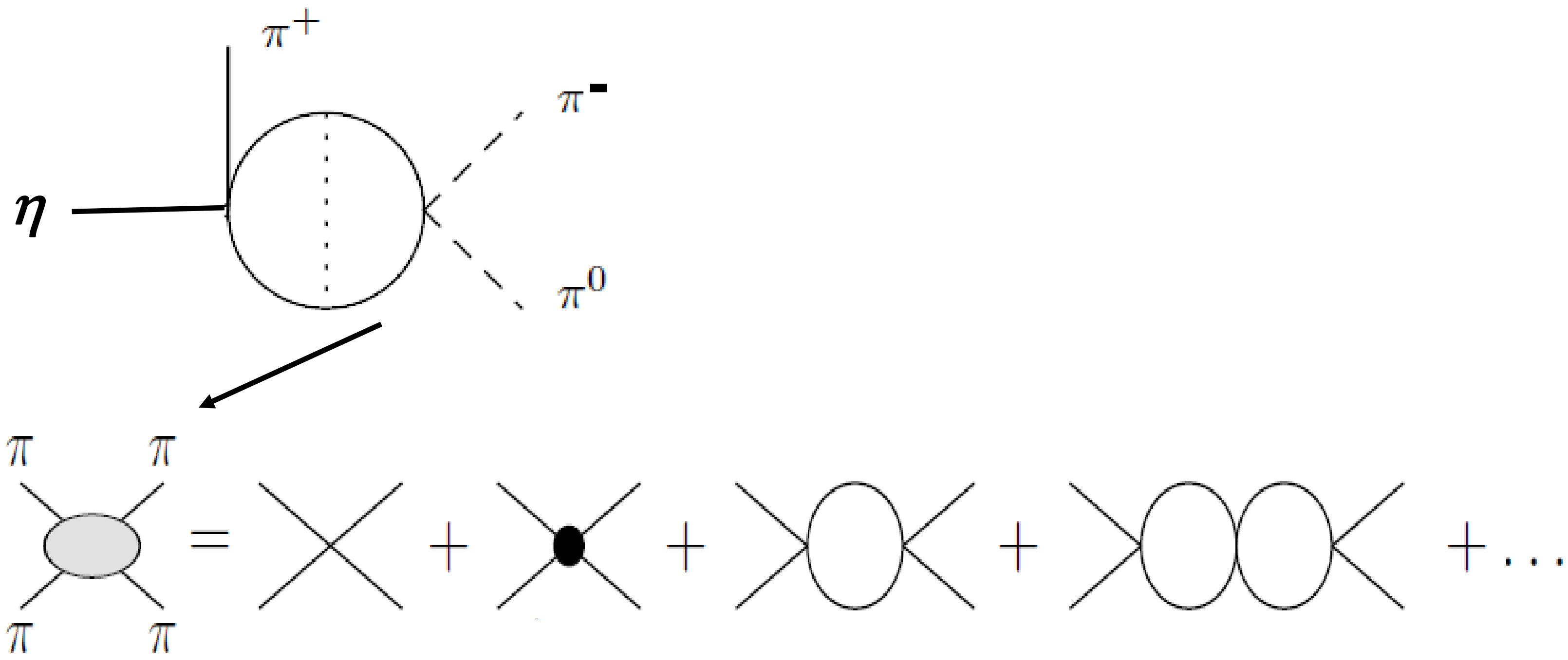
⇨ Prediction: $\Gamma_{\eta \to 3\pi} = 66 \text{ eV}$ and $\Gamma_{exp} = 197 \pm 29 \text{ eV}$ ⇨ *Problem!*

↑ in 1985

1.3 Dispersion relations

- Solution to this problem: *Large final state interactions*

Roiesnel & Truong'81



1.3 Dispersion relations

- Solution to this problem: *Large final state interactions*

Roiesnel & Truong'81

⇒ Chiral Perturbation Theory *Gasser & Leutwyler'85*

$$\Gamma^{\text{one loop}}_{\eta\to 3\pi} = 160 \pm 50 \text{ eV}$$ but $$\Gamma_{\text{exp}} = 295 \pm 20 \text{ eV}$$!

- Higher order corrections
  - ChPT at two loops *Bijnens & Ghorbani'07*
    but many LECs to determine at $\mathcal{O}(p^6)$!
  - Use of *dispersion relations*
    - ➢ analyticity, unitarity and crossing symmetry
    - ➢ Take into account all the rescattering effects

Kambor, Wiesendanger & Wyler'96
Anisovich & Leutwyler'96
Walker'98

1.4 New dispersive analysis

- Dispersive analysis following *Anisovich & Leutwyler* approach with new inputs:
  - New $\pi\pi$ phase shifts available, extracted with a better precision
    Ananthanarayan et al'01, Colangelo et al'01
    Descotes-Genon et al'01
    Kaminsky et al'01, Garcia-Martin et al'09
  - New experimental programs, precise Dalitz plot measurements
    CBall-Brookhaven, KLOE (Frascati)
    TAPS/CBall-MAMI (Mainz), WASA-Celsius (Uppsala), WASA-Cosy (Juelich)

- NB: Other recent analyses
  - Analytic dispersive *Kampf, Knecht, Novotný, Zdráhal '11* ⟵ *see poster*
  - NREFT approach *Schneider, Kubis, Ditsche'11*

2. Dispersive Analysis of $\eta \rightarrow \pi^+ \pi^- \pi^0$ decays

2.1 Strategy

- Instead of determining (m_u - m_d) ⇨ *extraction of Q*

$$Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} \qquad \hat{m} = \frac{m_d + m_u}{2}$$

since $$Q^2 = \frac{M_K^2}{M_\pi^2} \frac{M_K^2 - M_\pi^2}{\left(M_{K^0}^2 - M_{K^+}^2\right)_{QCD}} \left[1 + O(m_q^2, e^2)\right]$$

- $$\Gamma_{\eta\to 3\pi} \propto |A|^2$$

   ➢ $\Gamma_{\eta\to 3\pi}$ measured by *KLOE, MAMI, COSY*

   ➢ $$A(s,t,u) = -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{3\sqrt{3}F_\pi^2} M(s,t,u)$$

   $M(s,t,u)$ computed from dispersive treatment

⇨ *Q*

2.2 Method: Representation of the amplitude

- Decomposition of the amplitude as a function of isospin states

$$M(s,t,u) = M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s)$$

Fuchs, Sazdjian & Stern'93

Anisovich & Leutwyler'96

  - M_I isospin *I* rescattering in two particles
  - Amplitude in terms of S and P waves ⇒ exact up to NNLO ($\mathcal{O}(p^6)$)
  - Main two body rescattering corrections inside M_I

- Functions of only one variable with only right-hand cut of the partial wave ⇒ $disc\left[M_I(s)\right] \equiv disc\left[f_\ell^I(s)\right]$

- Elastic unitarity *Watson's theorem*

$$disc\left[f_\ell^I(s)\right] \propto t_\ell^*(s) f_\ell^I(s)$$

with $t_\ell(s)$ partial wave of elastic $\pi\pi$ scattering

2.3 Dispersion Relations for the $M_I(s)$

- $$M_0(s) = \Omega_0(s)\left(\alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_0^0(s')\,\hat{M}_0(s')}{|\Omega_0(s')|(s'-s-i\varepsilon)}\right)$$

Omnès function

$$\left[\Omega_I(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_\ell^I(s')}{s'(s'-s-i\varepsilon)}\right)\right]$$

Similarly for M_1 and M_2

- Inputs needed for the ππ phase shifts δ_ℓ^I
- $\hat{M}_I(s)$ contain the left-hand cut. They are obtained from angular averages over the $M_I(s)$ ⇨ *Coupled equations*
- Four subtraction constants to be fixed: α_0, β_0, γ_0 and one more in M_1 (β_1)
- Solve dispersion relations numerically by an iterative procedure

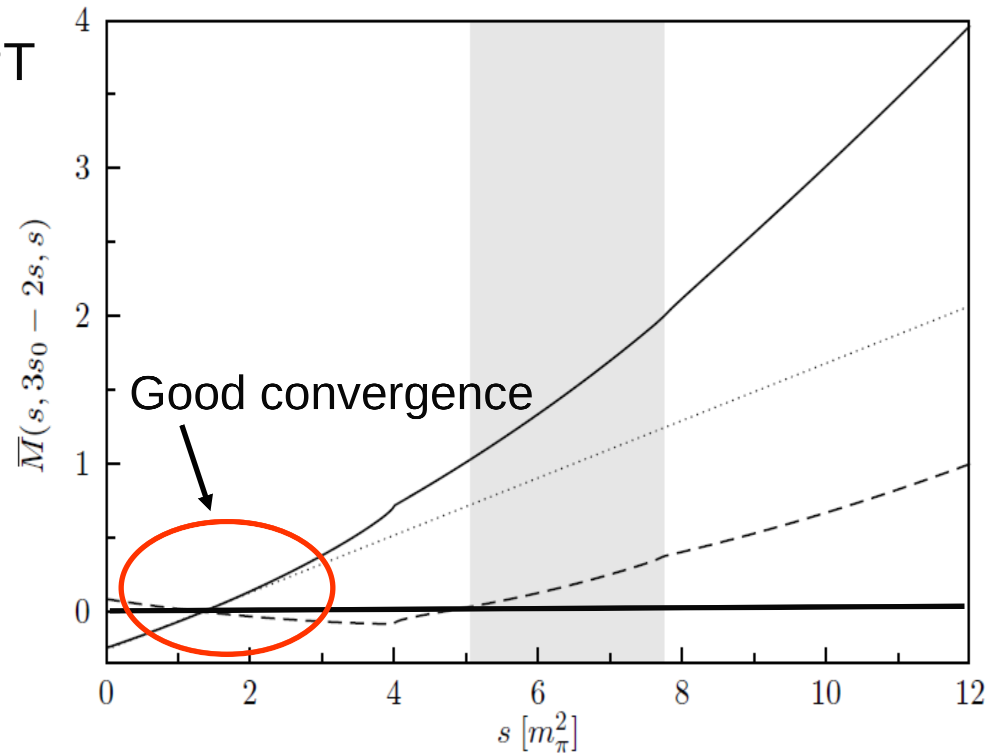
2.4 Subtraction constants

- From a matching to one loop ChPT
  - Sum rules for γ_0 and β_1

$$\gamma_0 \approx 0$$

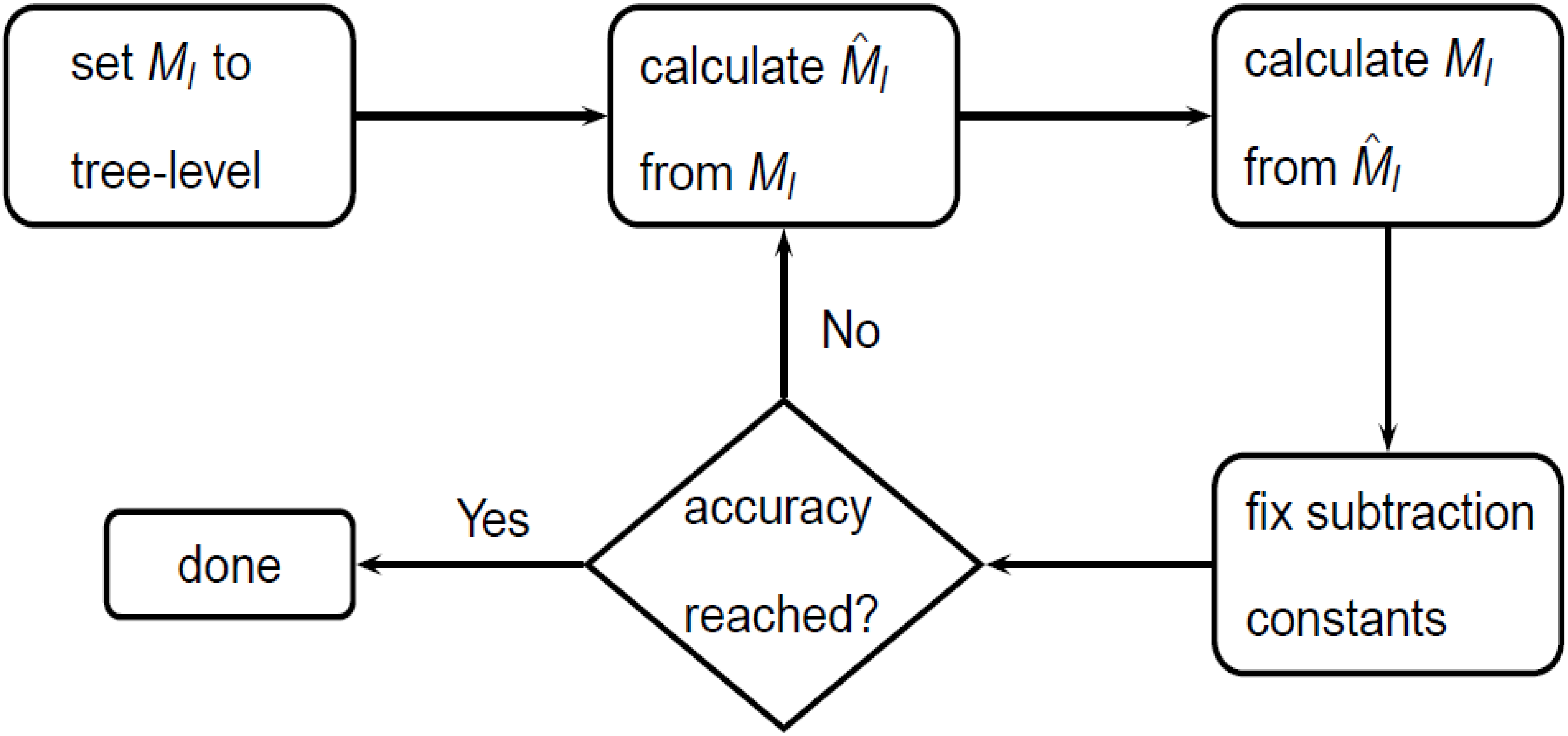
$$\beta_1 \approx -\frac{4L_3 - \frac{1}{64\pi^2}}{F_\pi^2\,(m_\eta^2 - m_\pi^2)} \approx 4.6\ \mathrm{GeV}^{-4}$$

  - Match α_0 and β_0 at Adler zero



- From a fit
  - KLOE data in physical region *Ambrosino et al'08*
  - One loop ChPT in the vicinity of Adler zero

2.5 Iterative Procedure



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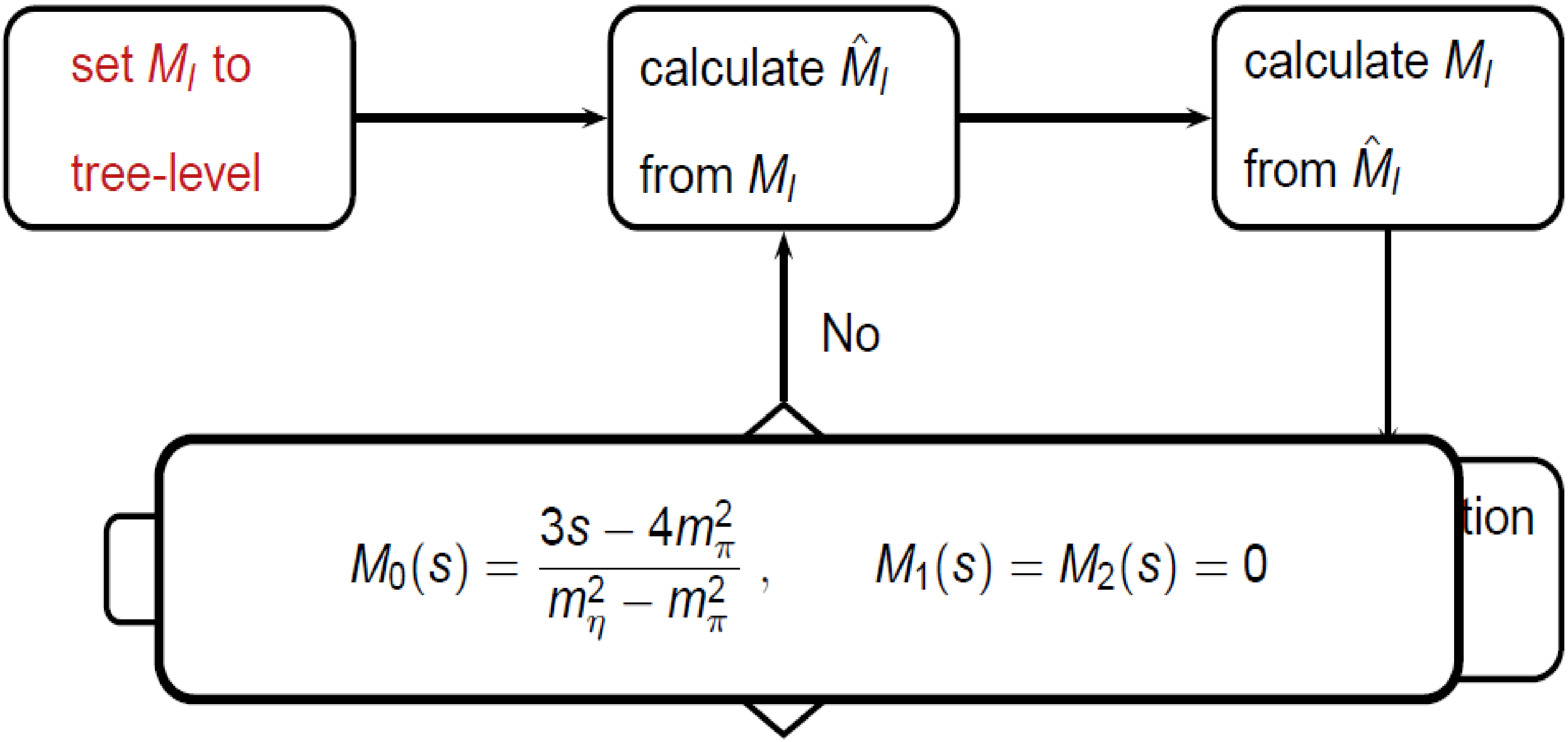
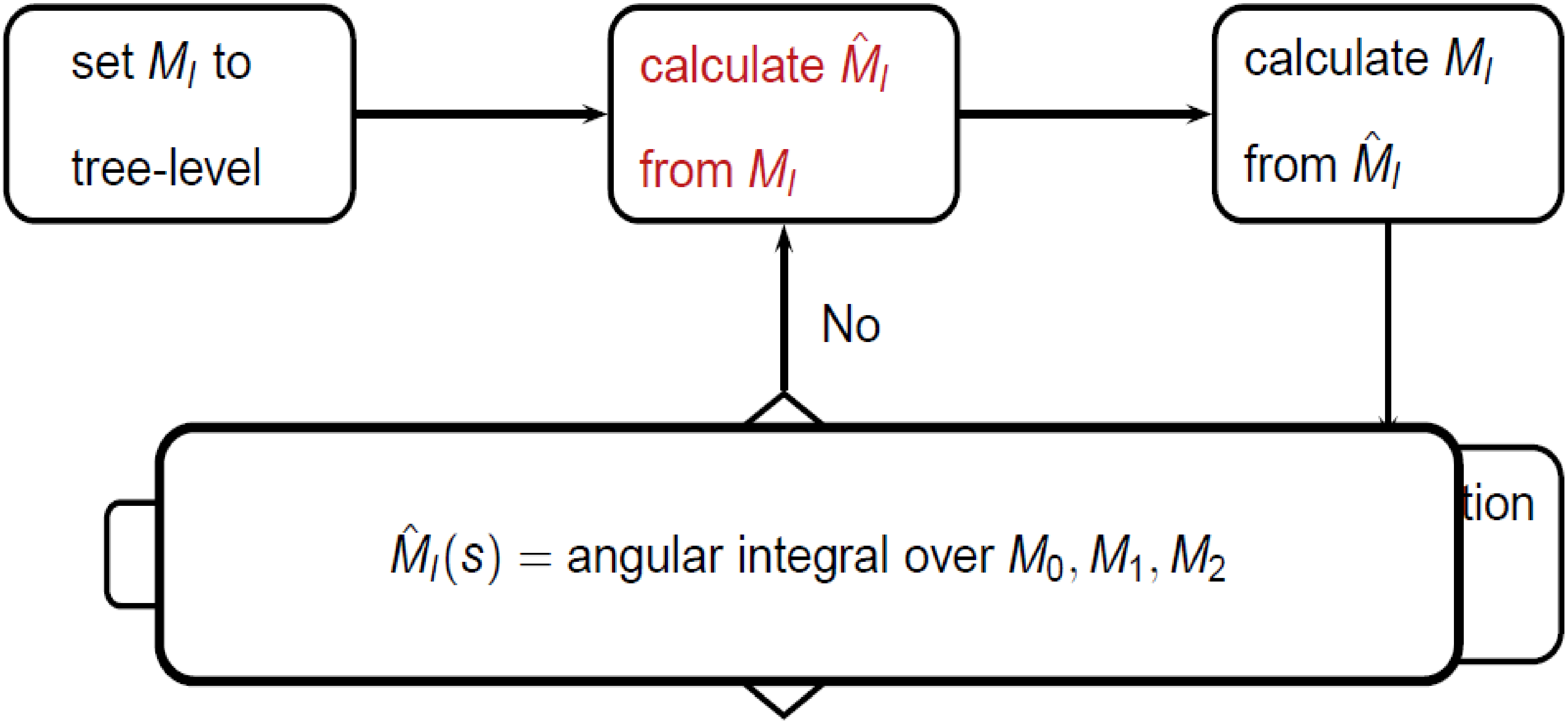


$$M_0(s) = \frac{3s - 4m_\pi^2}{m_\eta^2 - m_\pi^2}\,, \qquad M_1(s) = M_2(s) = 0$$

2.5 Iterative Procedure



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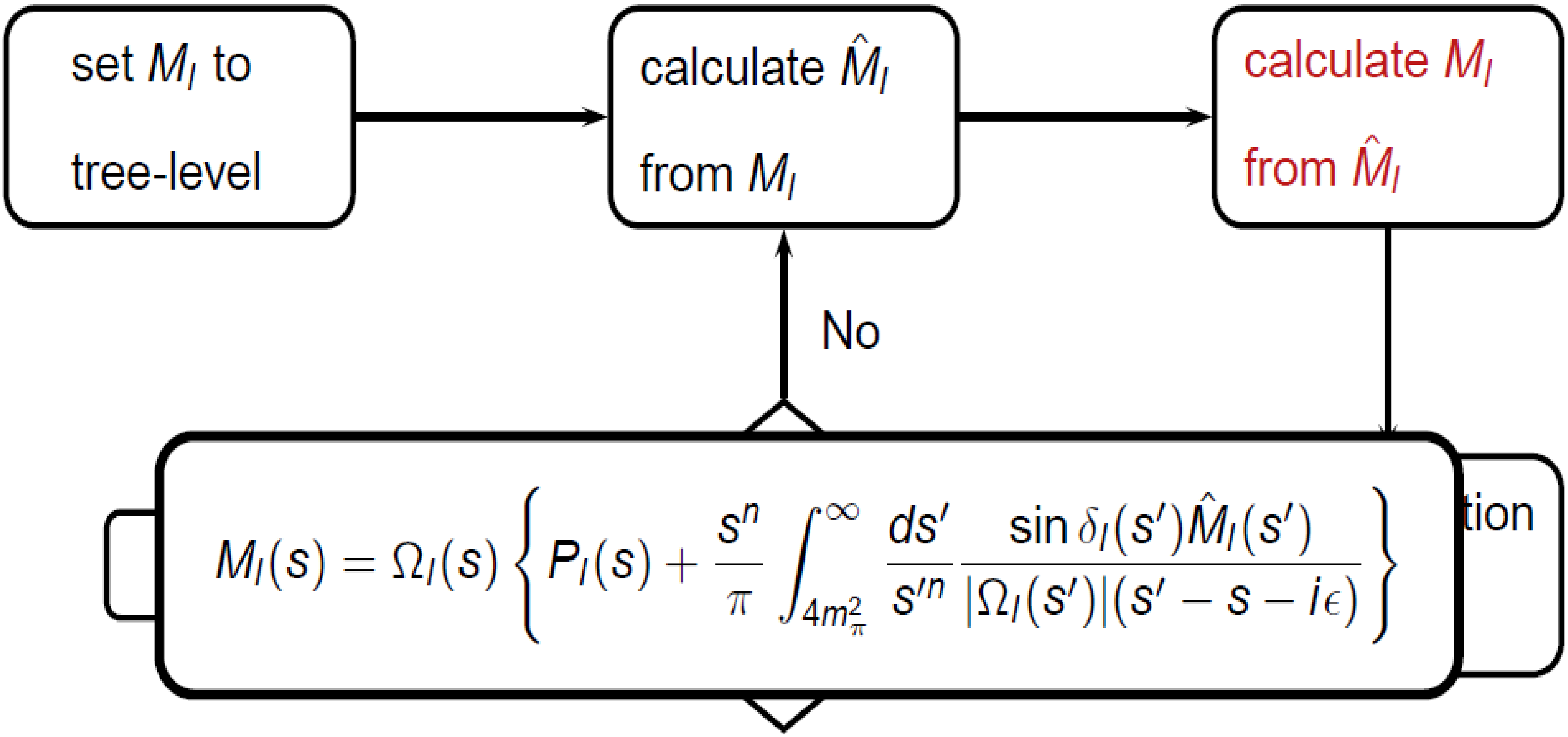

$$M_l(s) = \Omega_l(s)\left\{P_l(s) + \frac{s^n}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^n}\frac{\sin\delta_l(s')\hat{M}_l(s')}{|\Omega_l(s')|(s'-s-i\epsilon)}\right\}$$

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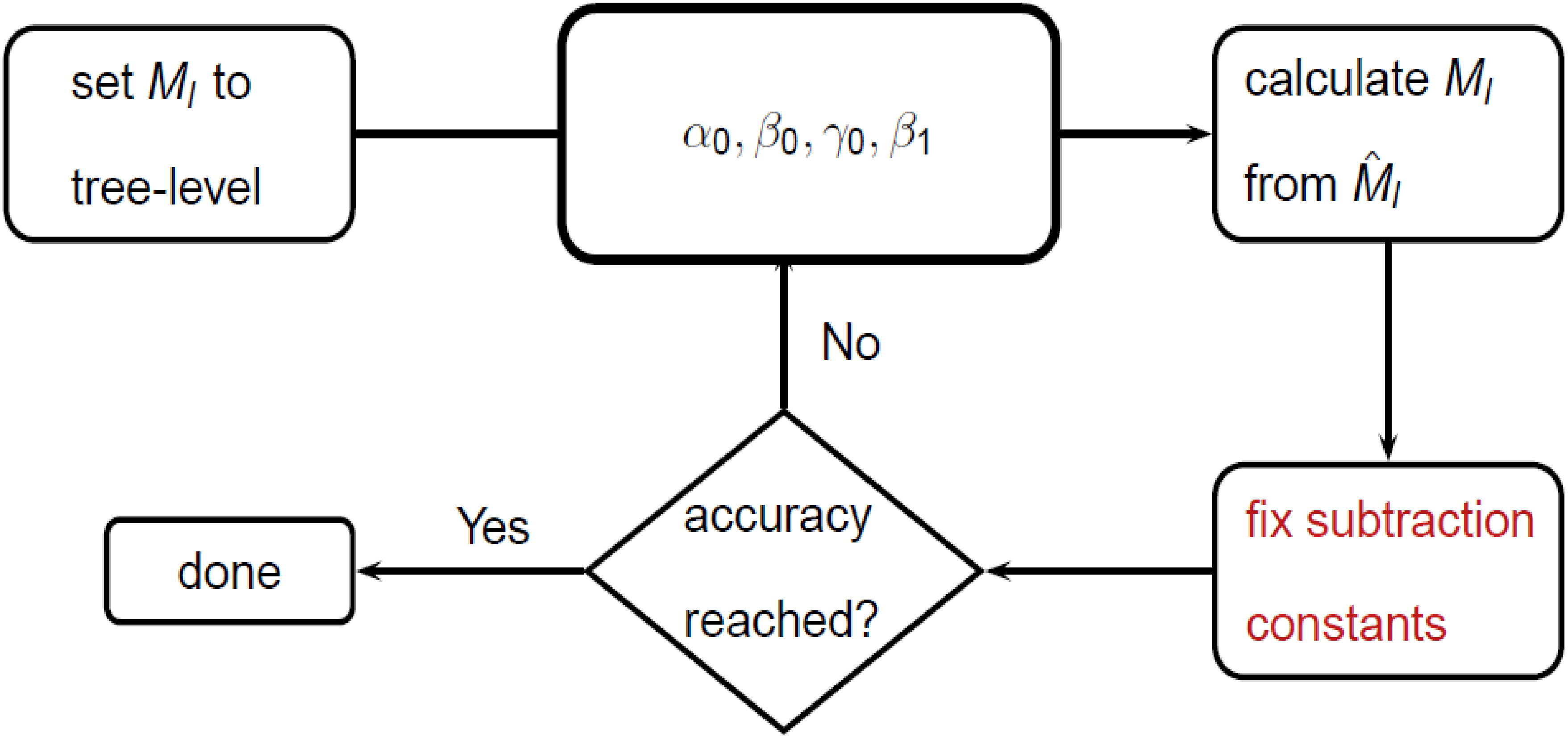

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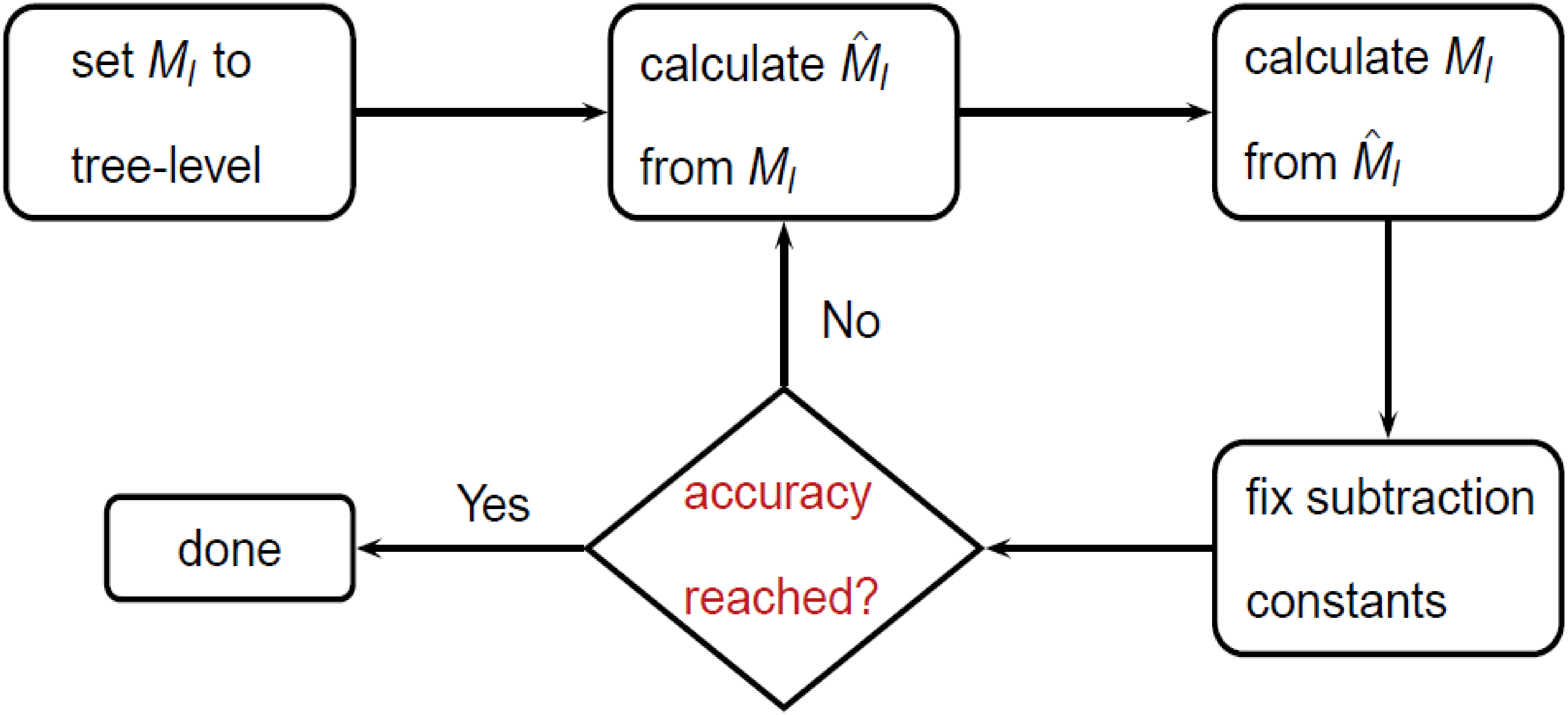

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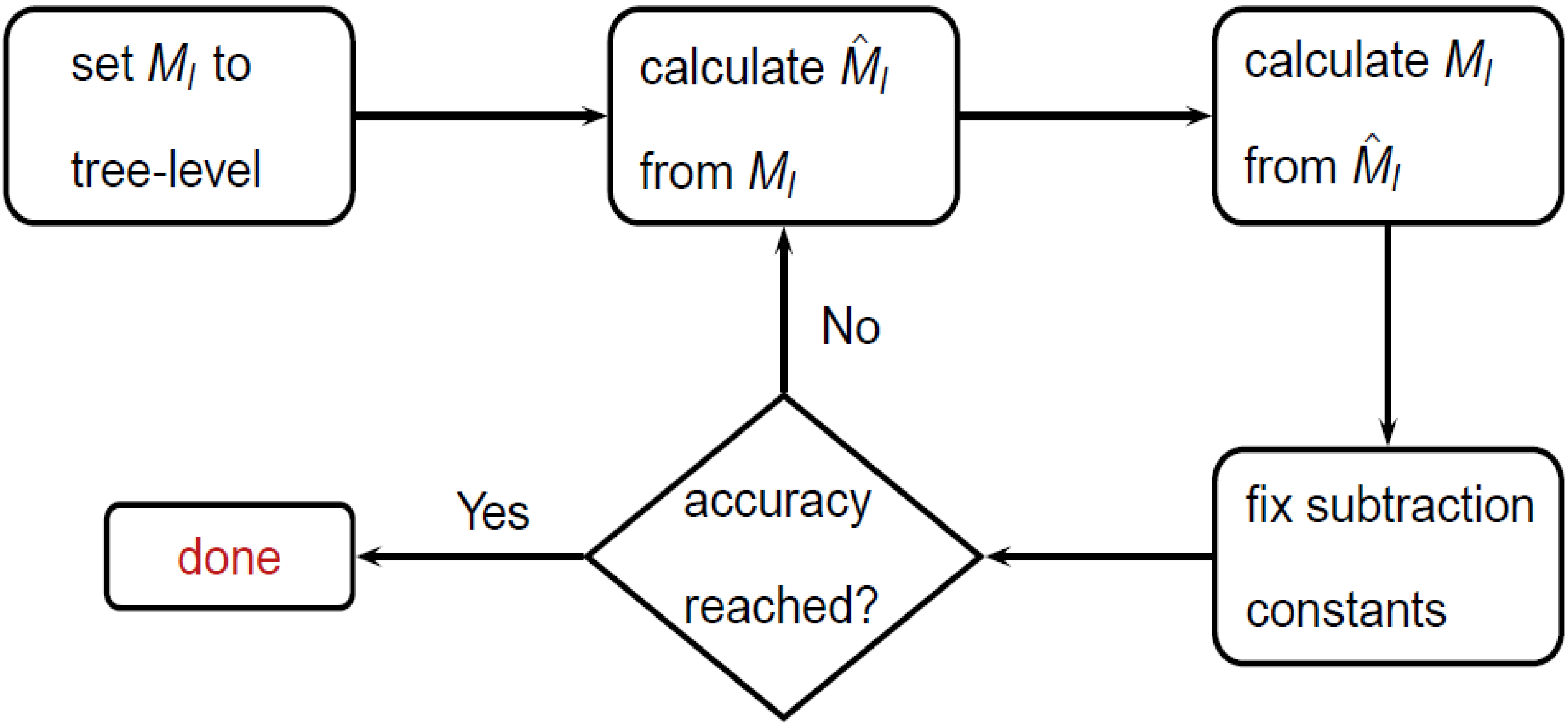
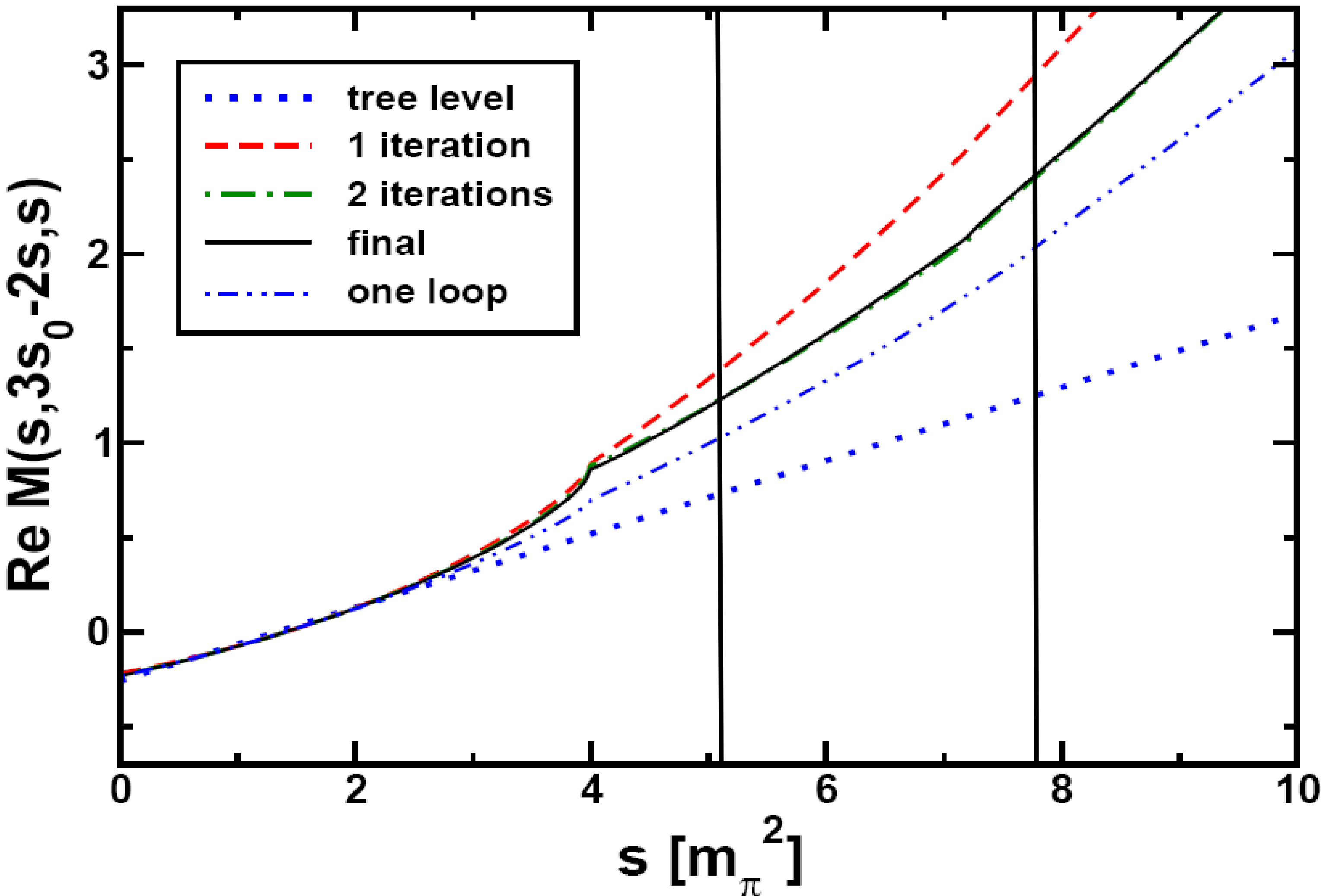


3. Results

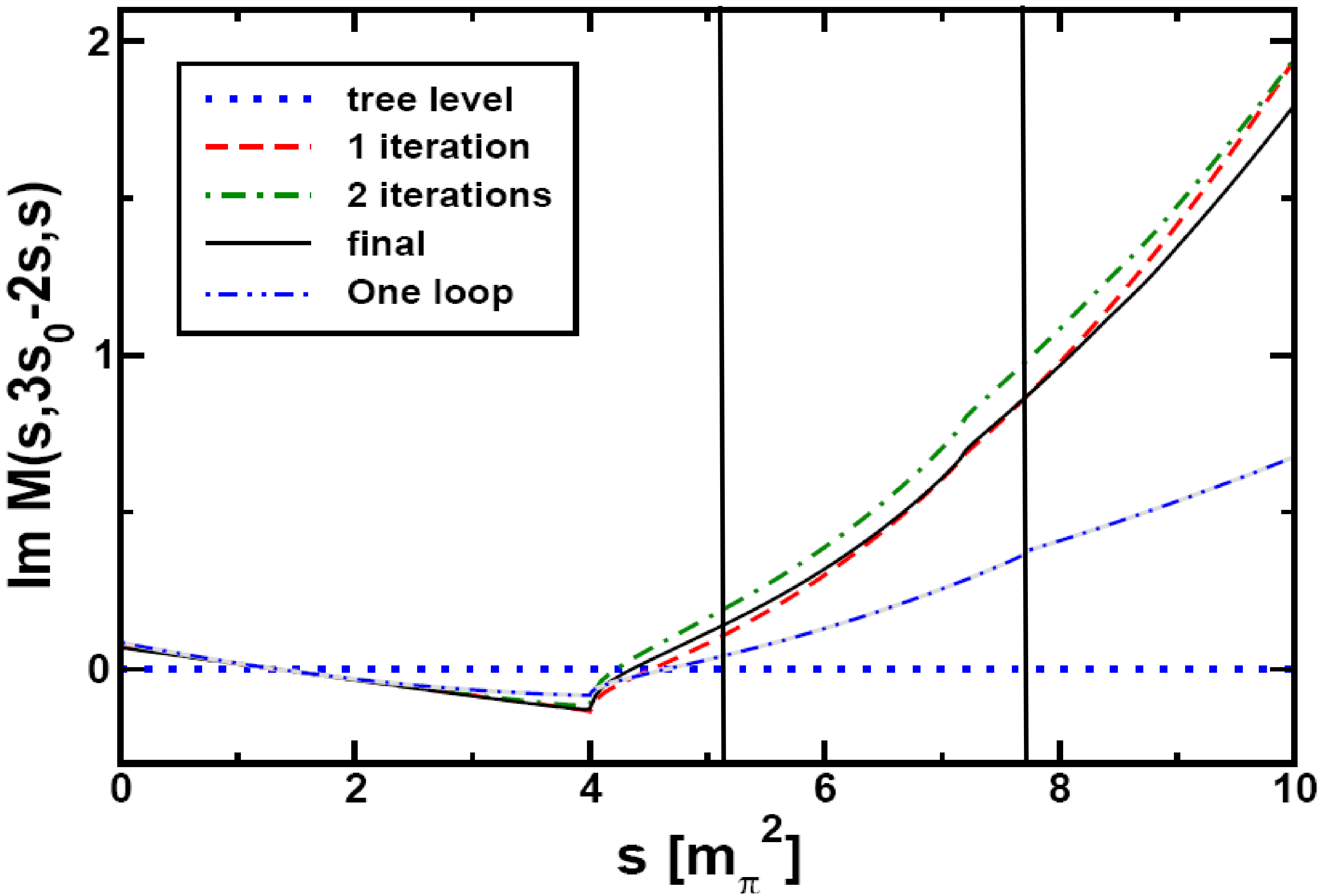
3.1 Result for M(s,t,u) along s=u

- From the matching to one loop ChPT (→ referred as matching in the following)



3.1 Result for M(s,t,u) along s=u

- From the matching to one loop ChPT



3.2 Dalitz plot for $\eta \to \pi^+ \pi^- \pi^0$

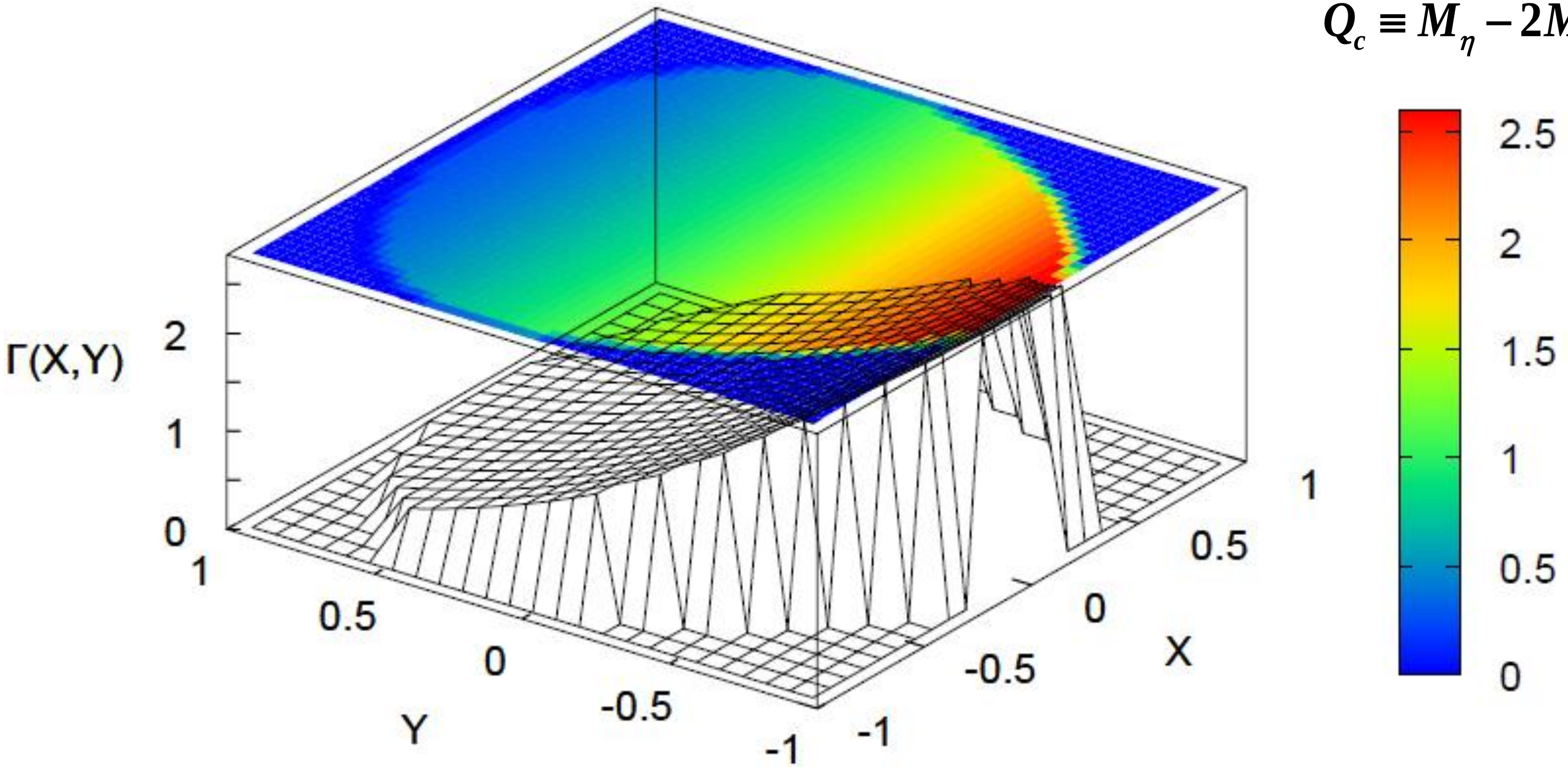
- Amplitude expanded in X and Y around X=Y=0

$$X = \frac{\sqrt{3}}{2M_\eta Q_c}(u-t)$$

$$Y = \frac{3}{2M_\eta Q_c}\left(\left(M_\eta - M_{\pi^0}\right)^2 - s\right) - 1$$

$$Q_c \equiv M_\eta - 2M_{\pi^+} - M_{\pi^0}$$

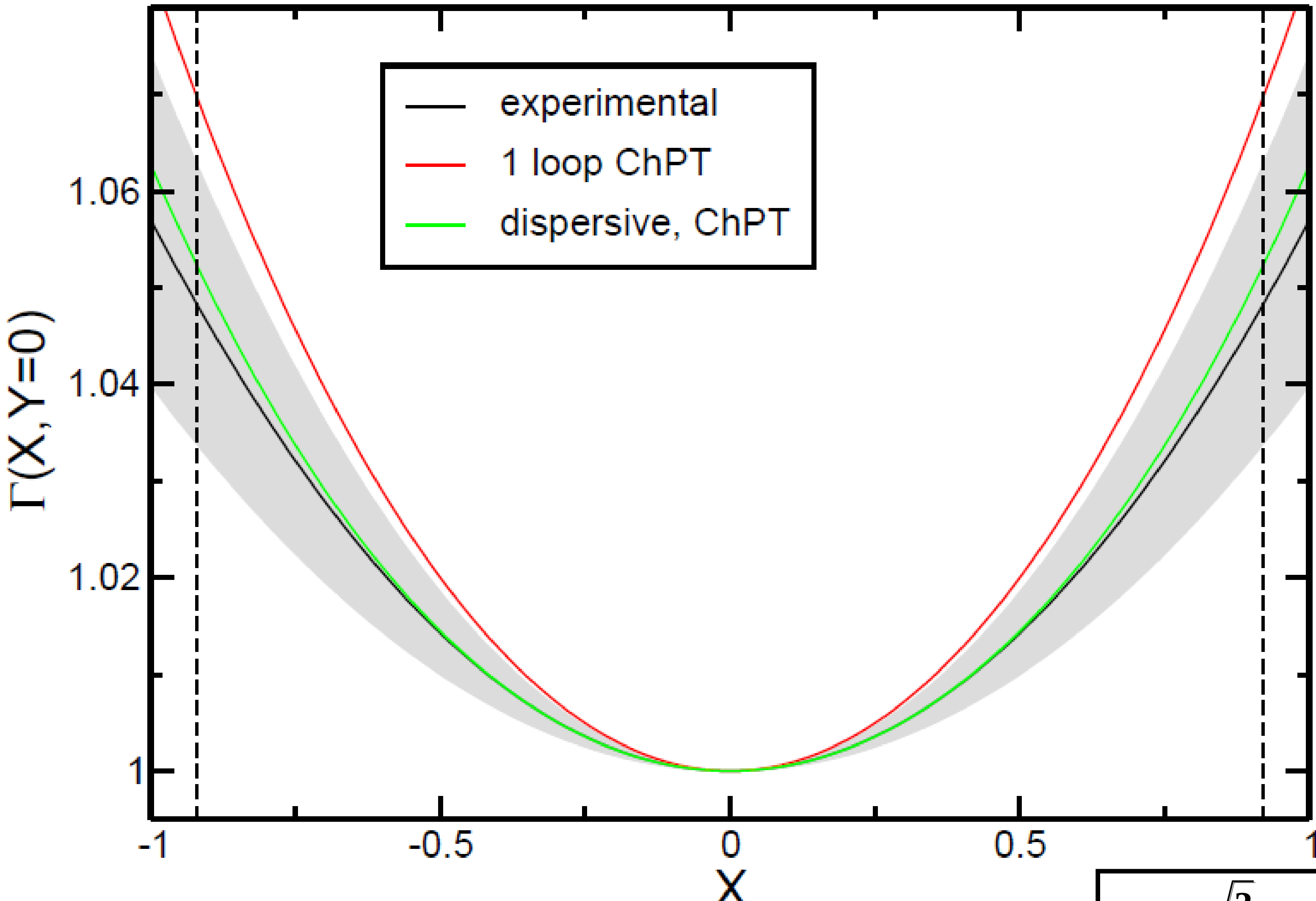
$$\Gamma(X,Y) = N\left(1 + aY + bY^2 + dX^2 + fY^3\right)$$



- Dalitz plot distribution measured by *KLOE*

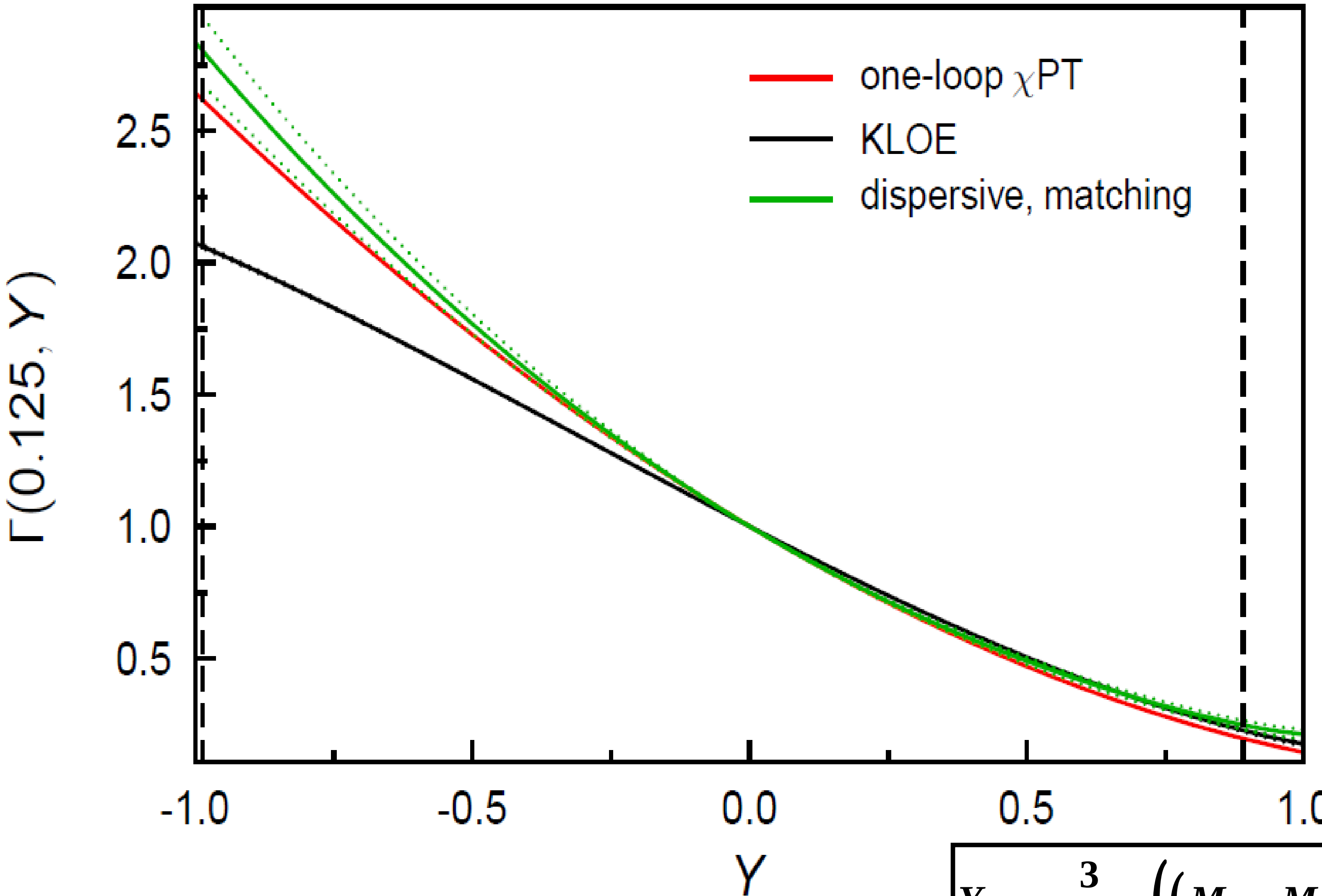
Ambrosino et al'08

3.3 Dalitz plot: Comparison with KLOE



$$X = \frac{\sqrt{3}}{2M_\eta Q_c}(u-t)$$

3.3 Dalitz plot: Comparison with KLOE



$$Y = \frac{3}{2M_\eta Q_c}\left(\left(M_\eta - M_{\pi^0}\right)^2 - s\right) - 1$$

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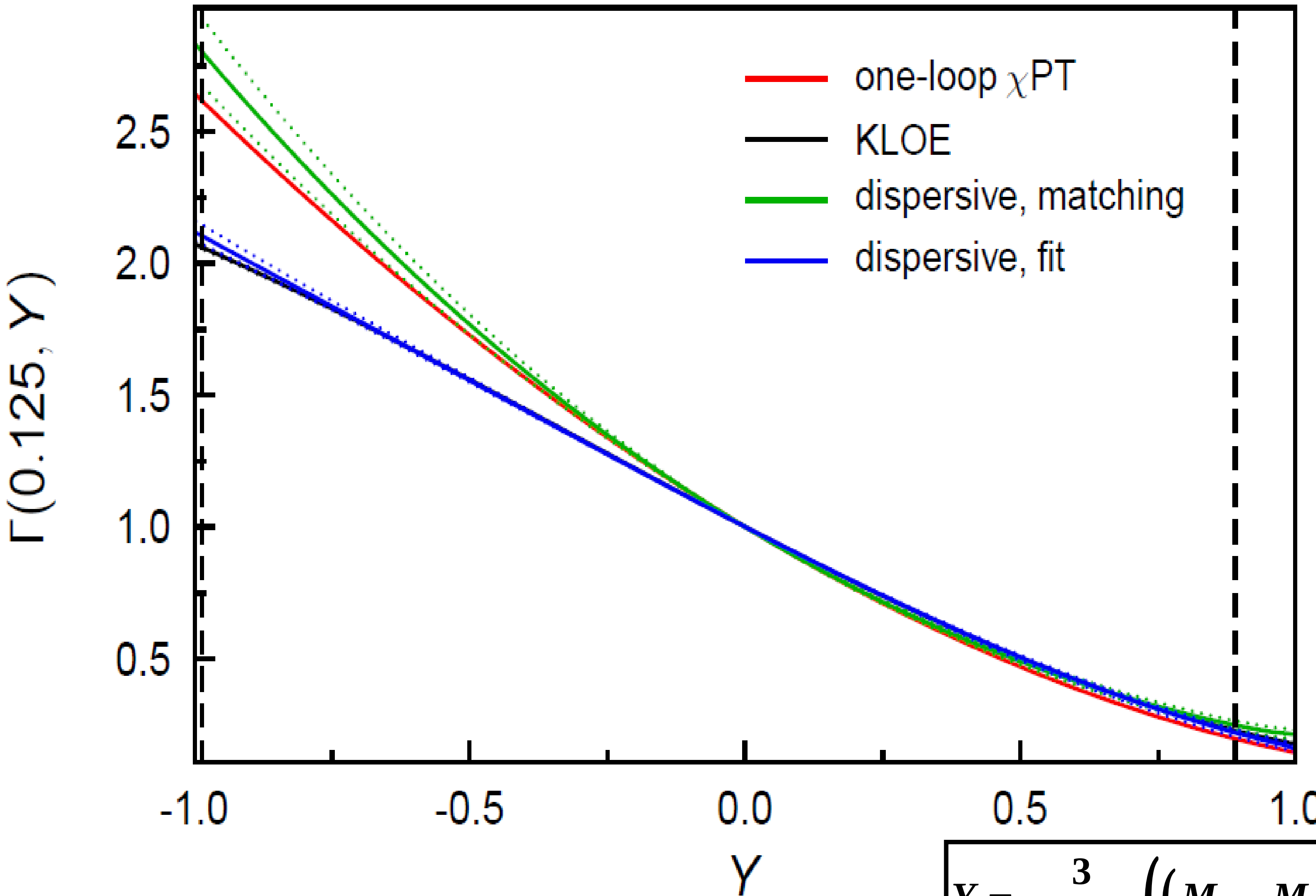
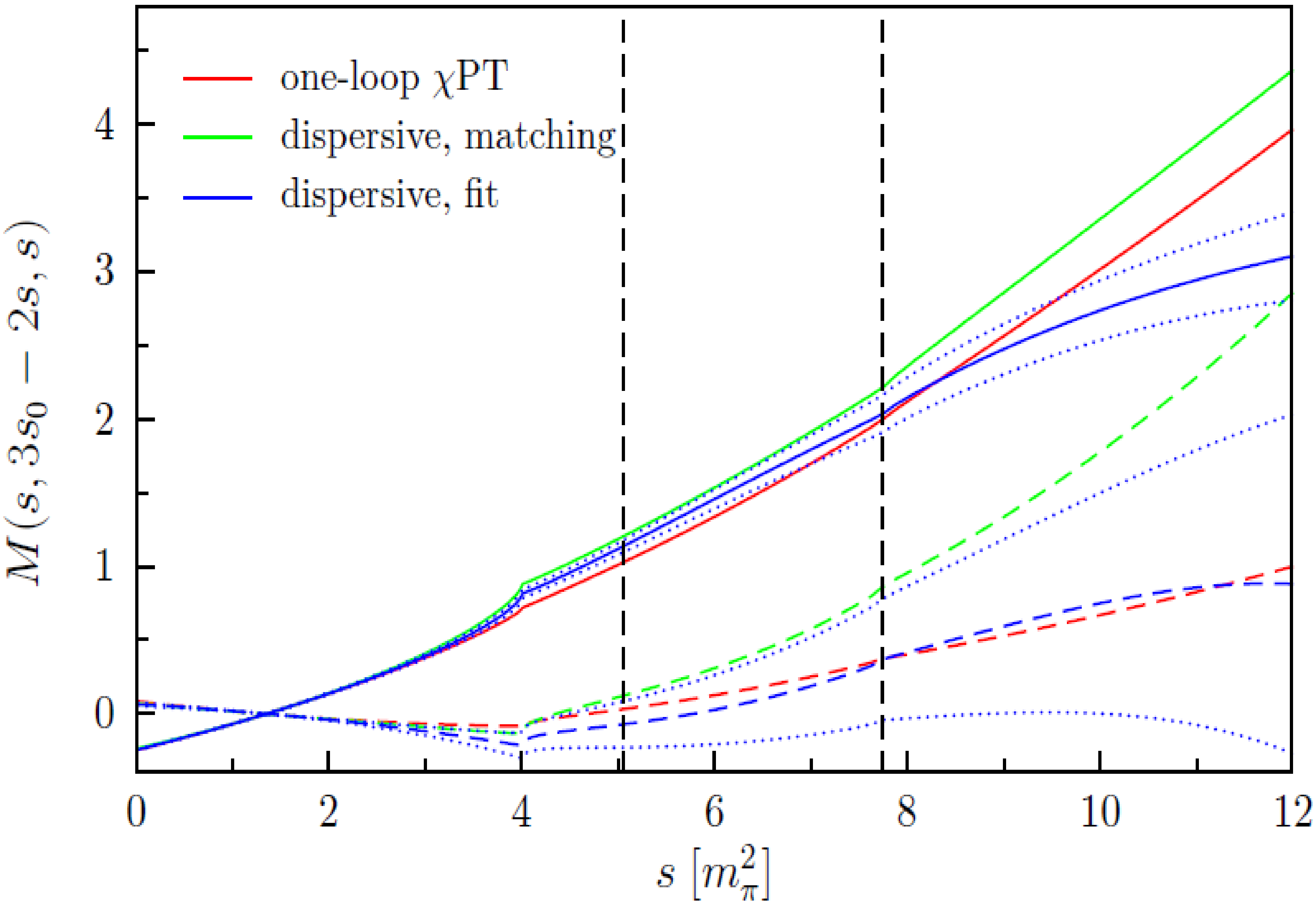
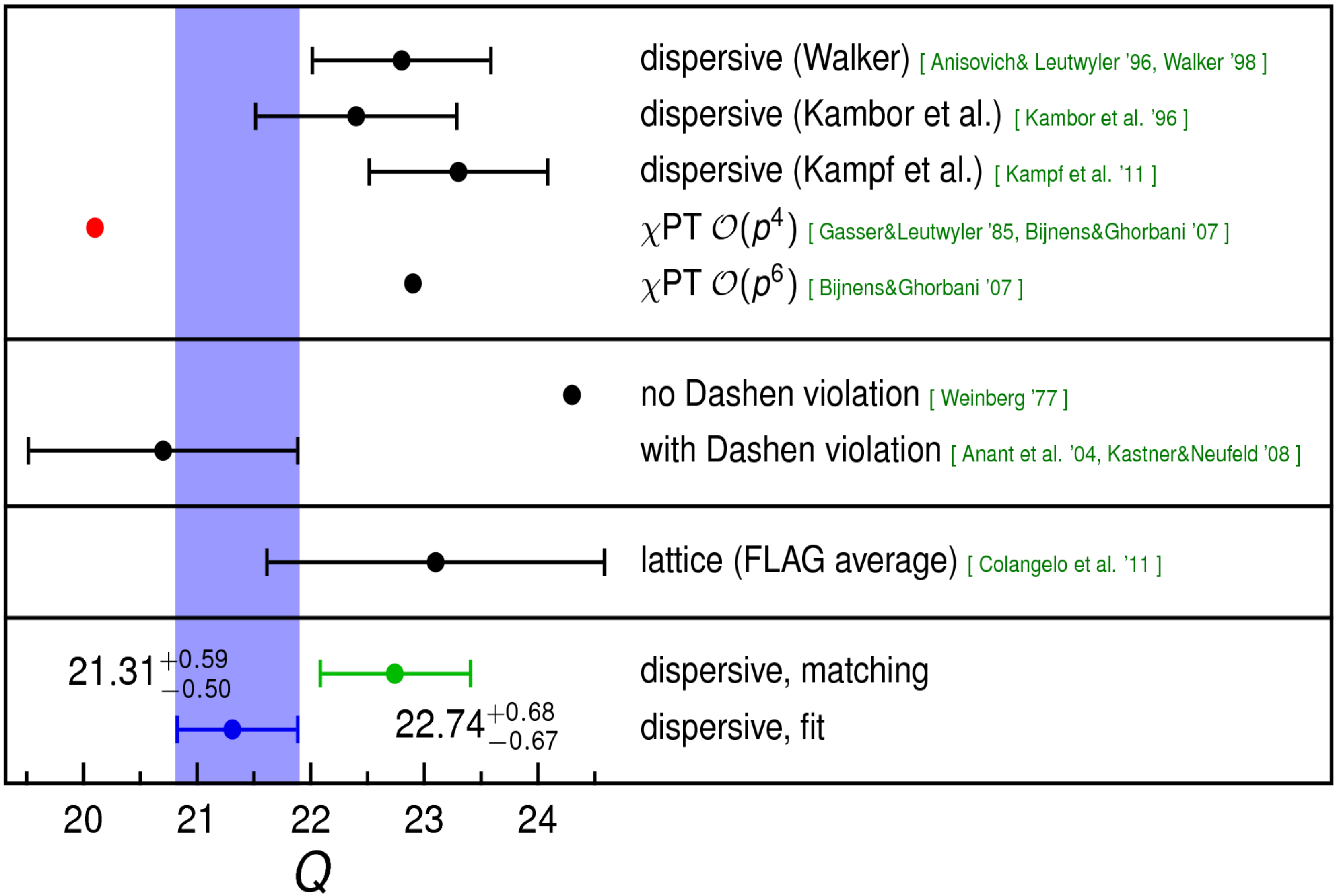


$$Y = \frac{3}{2M_\eta Q_c}\left(\left(M_\eta - M_{\pi^0}\right)^2 - s\right) - 1$$

3.4 Comparison for M(s,t,u) along s=u



3.5 Extraction of Q and comparison with other results



3.6 Results for the neutral mode

- Compute the amplitude for the neutral mode for which there are much more experimental results

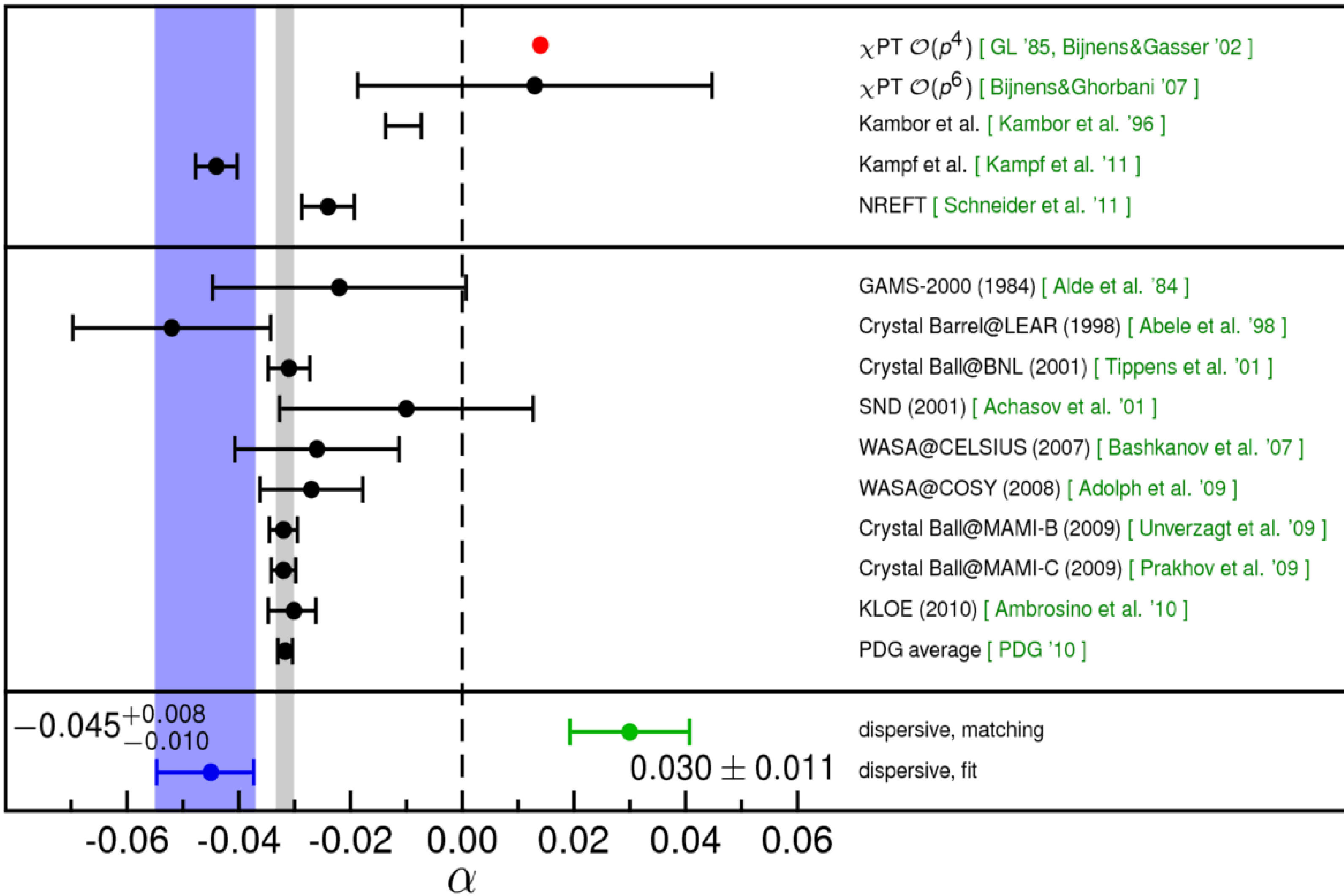
- Amplitude: $\overline{A}(s,t,u) = A(s,t,u) + A(t,u,s) + A(u,s,t)$

- NB: Fit still performed to the charged Dalitz plot distribution

- Dalit plot parametrization: $\Gamma = N\left(1 + 2\alpha Z\right)$ with $Z = X^2 + Y^2$

3.6 Results for the neutral mode

- Extraction of α



4. Conclusion and outlook

4.1 Conclusion

- $\eta \to \pi^+ \pi^- \pi^0$ decays represent a source of information on the quark mass ratio *Q*

- A reliable extraction of *Q* requires having the strong rescattering effects in the final state under control

- This is possible thanks to dispersion relations
  ⇒ need to determine unknown subtraction constants

- Use of experimental measurements of the Dalitz plot distributions to determine the subtraction constants and reduce the uncertainties in the dispersive analysis

4.1 Conclusion

- Analysis presented with subtraction constants from
  - Matching to one loop ChPT ⇒ $Q = 22.74^{+0.68}_{-0.67}$

    Disagreement with the observed Dalitz plot distribution from *KLOE*

  - Fit to the Dalitz plot distribution of the charge mode (*KLOE*) and ChPT ⇒ $Q = 21.31^{+0.59}_{-0.50}$

- Experimental fit removes the discrepancy on the sign of α in the neutral mode but the value of α is only in marginal agreement with the experimental ones

4.2 Outlook

- Try to understand the discrepancy between the two results
  - ➢ use the experimental results on the Dalitz plot distribution from the neutral mode to fix the subtraction constants

  - ➢ More data on $\eta \to \pi^+ \pi^- \pi^0$ in particular on the Dalitz plot distribution needed!

- Matching to NNLO ChPT

  ⇨ Constraints from experiment: possible determination of C_i

- Investigate the differences with the analysis of *Kampf et al.*

- Include electromagnetic corrections in the dispersive analysis

5. Back-up

Light quark masses from Lattice QCD using Q

- Use *Q* and lattice determinations of m_s and $\hat{m}$

  ⇨ *Light quark masses: m_u, m_d*

$$m_u = \hat{m} - \frac{m_s^2 - \hat{m}^2}{4\hat{m}Q^2} \quad \text{and} \quad m_d = \hat{m} + \frac{m_s^2 - \hat{m}^2}{4\hat{m}Q^2}$$

- For instance
  - m_s and $\hat{m}$ from *BMW* $\begin{cases} m_s = 95.5 \pm 1.5 \pm 1.1 \\ \hat{m} = 3.469 \pm 0.048 \pm 0.0047 \end{cases}$ *Durr et al'10*
  - *Q* from the fit: $Q = 21.31^{+0.59}_{-0.50}$

⇨ $m_u = (2.02 \pm 0.14)\ \text{MeV}$ and $m_d = (4.91 \pm 0.11)\ \text{MeV}$

Determination of the light quark masses

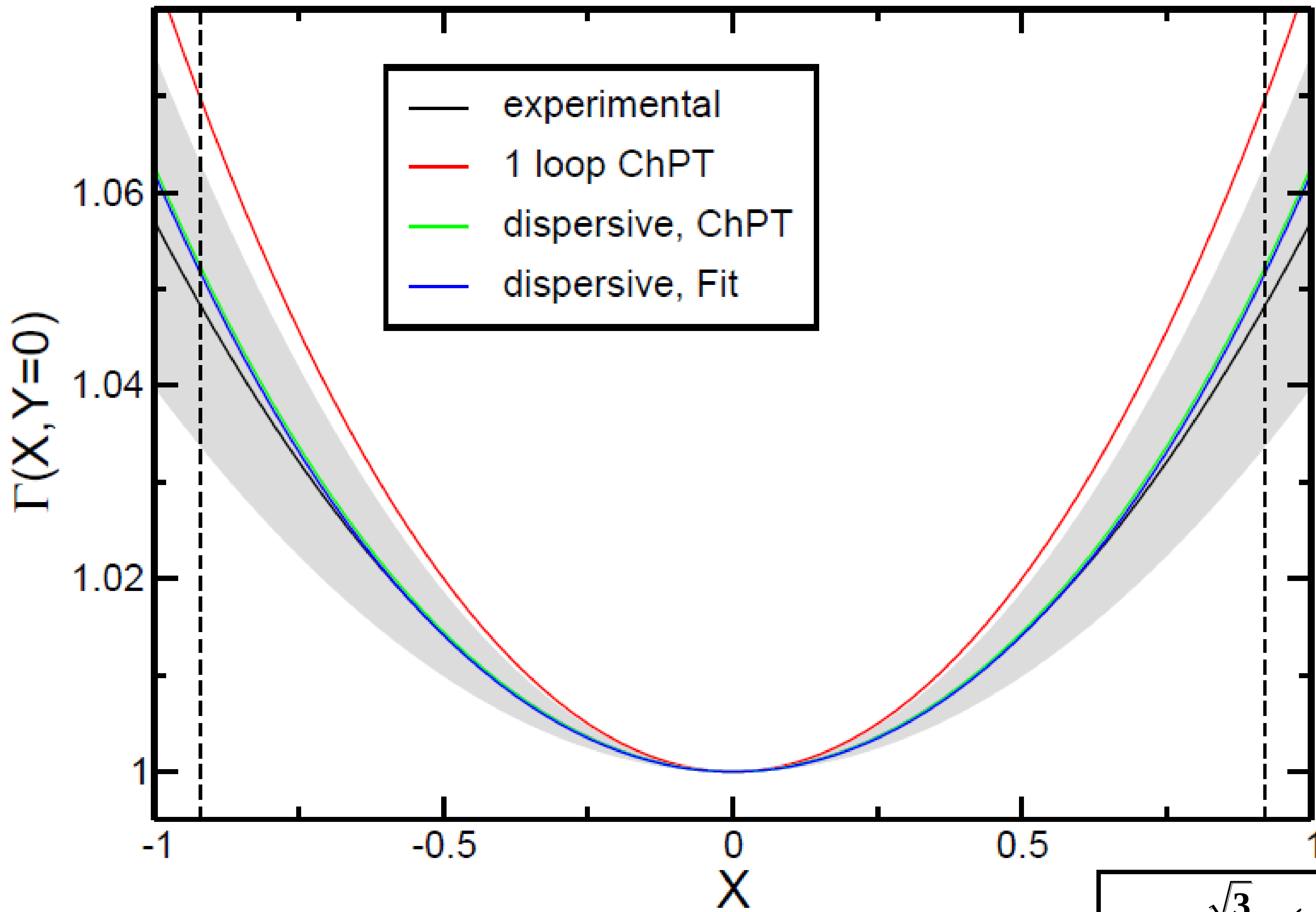
- Fundamental unknowns of the QCD Lagrangian

$$\mathcal{L}_{QCD} = -\frac{1}{4} G_a^{\mu\nu} G_{\mu\nu}^a + \sum_{k=1}^{N_F} \bar{q}_k \left(i\gamma^\mu D_\mu - m_k \right) q_k$$

- High precision physics at low energy as a key of new physics?
  m_d - m_u : small isospin breaking corrections but to be taken into account for high precision physics

- Different approaches:
  - Effective field theory ⇒ *ChPT*
    $\eta \to \pi^+ \pi^- \pi^0$ decays, meson mass splitting
  - Numerical simulations on the lattice
    Hadron spectrum
  - Sum-rules
    Hadronic τ decays

Dalitz plot: Comparison with KLOE



$$X = \frac{\sqrt{3}}{2M_\eta Q_c}(u-t)$$

Extraction of Q

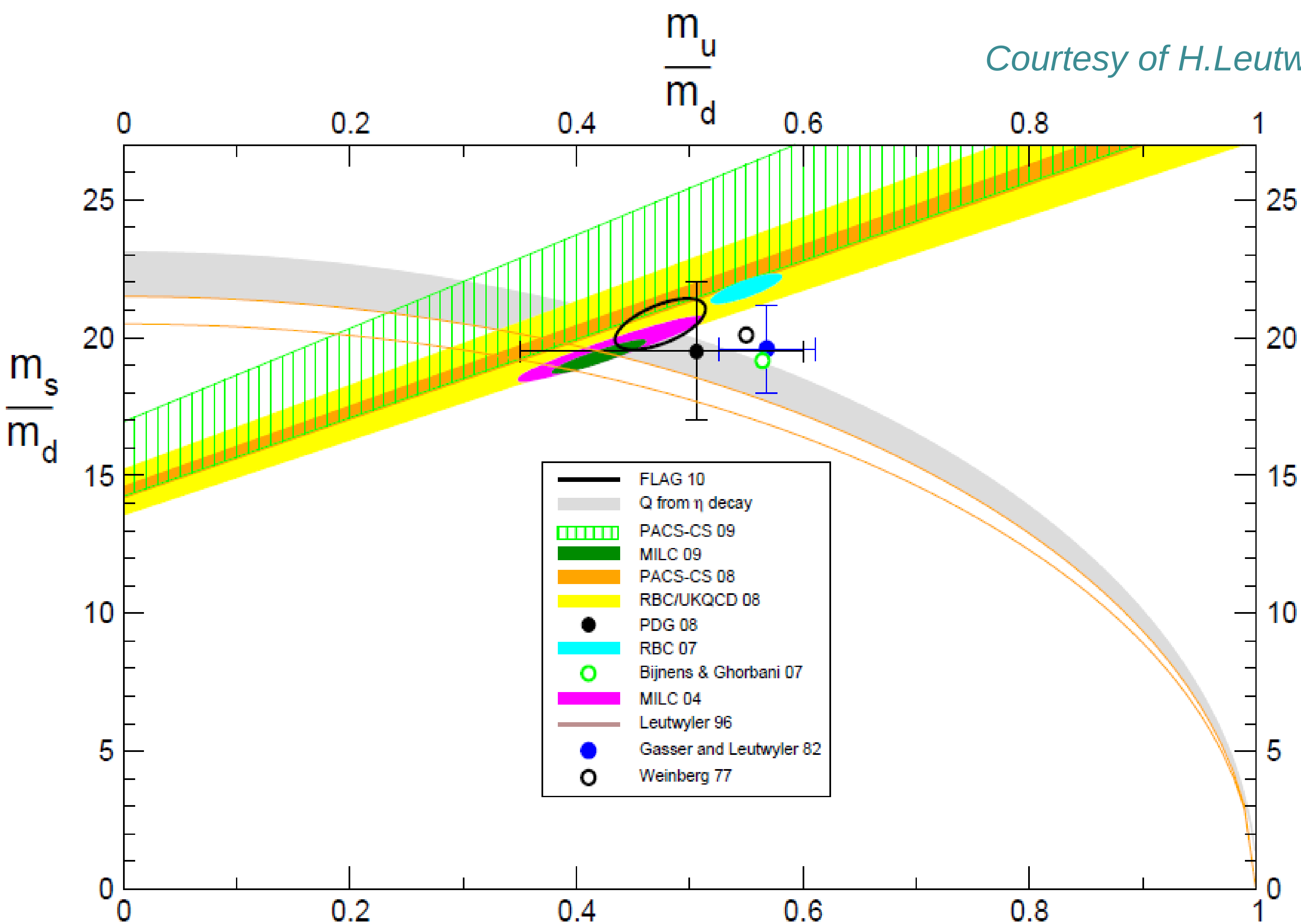
- Error analysis

Matching	$Q(\pi^+\pi^-\pi^0)$
Γ	$\pm\, 0.31$
γ_0	$\pm\, 0.38$
β_1	$\pm\, 0.36$
L_3	$^{+0.025}_{-0.023}$
$\delta_I(s)$	$^{+0.18}_{-0.15}$
inelasticity	$\pm\, 0.2$
cut-off	$\pm\, 0.09$
total uncertainty	$^{+0.68}_{-0.67}$

Fit	$Q(\pi^+\pi^-\pi^0)$
Γ	$\pm\, 0.29$
stat. KLOE	$\pm\, 0.091$
syst. KLOE	$^{+0.45}_{-0.30}$
$\mathcal{N}$ KLOE	$^{+0.030}_{-0.029}$
L_3	$^{+0.21}_{-0.25}$
$\delta_I(s)$	$^{+0.041}_{-0.053}$
W_A	$^{+0.000}_{-0.033}$
total uncertainty	$^{+0.59}_{-0.50}$

Light quark masses

Courtesy of H.Leutwyler



Meson masses

- From LO ChPT without e.m effects:

$$M^2_{\pi^+} = (m_u + m_d)\, B_0 + O(m^2)$$
$$M^2_{K^+} = (m_u + m_s)\, B_0 + O(m^2)$$
$$M^2_{K^0} = (m_d + m_s)\, B_0 + O(m^2)$$

- Electromagnetic effects: *Dashen's theorem*

$$\left(M^2_{K^+} - M^2_{K^0}\right)_{em} - \left(M^2_{\pi^+} - M^2_{\pi^0}\right)_{em} = O\left(e^2 m\right)$$

Dashen'69

⇨ ChPT at leading order + e.m corrections

- $M^2_{\pi^0} = B_0\left(m_u + m_d\right),\ \ M^2_{\pi^+} = B_0\left(m_u + m_d\right) + \Delta_{em}$
- $M^2_{K^0} = B_0\left(m_d + m_s\right),\ \ M^2_{K^+} = B_0\left(m_u + m_s\right) + \Delta_{em}$

2 unknowns B_0 and Δ_{em}

Meson masses

Quark mass ratios *Weinberg'77*

$$\frac{m_u}{m_d} \stackrel{\text{LO}}{=} \frac{M^2_{K^+} - M^2_{K^0} + 2M^2_{\pi^0} - M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}} = 0.56\,,$$

$$\frac{m_s}{m_d} \stackrel{\text{LO}}{=} \frac{M^2_{K^+} + M^2_{K^0} - M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}} = 20.2$$

Q from meson mass splitting

- $Q^2 = \frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{M_{K^0}^2 - M_{K^+}^2}\left[1 + O(m_q^2)\right]$ is only valid for e=0

- Including the electromagnetic corrections, one has

$$Q_D^2 \equiv \frac{(M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2 + M_{\pi^0}^2)(M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2 - M_{\pi^0}^2)}{4M_{\pi^0}^2(M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2 - M_{\pi^0}^2)}$$

⇨ $Q_D = 24.2$

- Corrections to the Dashen's theorem

⇨ The corrections can be large due to $e^2 m_s$ corrections:

$$\left(M_{K^+}^2 - M_{K^0}^2\right)_{\mathrm{em}} - \left(M_{\pi^+}^2 - M_{\pi^0}^2\right)_{\mathrm{em}} = e^2 M_K^2\left(A_1 + A_2 + A_3\right) + O\left(e^2 M_\pi^2\right)$$

Urech'98,
Ananthanarayan & Moussallam'04

Corrections to Dashen's theorem

- Dashen's Theorem

$$\left(M^2_{K^+} - M^2_{K^0}\right)_{em} = \left(M^2_{\pi^+} - M^2_{\pi^0}\right)_{em} \Rightarrow \boxed{\left(M_{K^+} - M_{K^0}\right)_{em} = 1.3 \text{ MeV}}$$

- With higher order corrections
  - Lattice : $\left(M_{K^+} - M_{K^0}\right)_{em} = 1.9 \text{ MeV}, Q = 22.8$ *Ducan et al.'96*
  - ENJL model: $\left(M_{K^+} - M_{K^0}\right)_{em} = 2.3 \text{ MeV}, Q = 22$ *Bijnens & Prades'97*
  - VMD: $\left(M_{K^+} - M_{K^0}\right)_{em} = 2.6 \text{ MeV}, Q = 21.5$ *Donoghue & Perez'97*
  - Sum Rules: $\left(M_{K^+} - M_{K^0}\right)_{em} = 3.2 \text{ MeV}, Q = 20.7$ *Anant & Moussallam'04*

  Update $\Rightarrow$ $Q = 20.7 \pm 1.2$ *Kastner & Neufeld'07*

Lattice QCD

- Compute the quark masses from first principles

  ⇨ $\mathcal{L}_{QCD}$ on the lattice

  - QCD Lagrangian as input
  - Calculate the spectrum of the low-lying states for different quark masses
  - Tune the values of the quark masses such that the QCD spectrum is reproduced
  - Set the scale by adding an external input or extract quark mass ratios

- NB: computation in the isospin limit: $m_u = m_d = \hat{m}$

$$\hat{m} = \frac{m_u + m_d}{2}$$

Light quark masses from Lattice QCD using Q

Q from	$\hat{m}$, m_s from	Q	$\hat{m}$	m_s	m_u	m_d
matching	FLAG	22.74	3.4	95	2.12 ± 0.62	4.68 ± 0.38
matching	RBC/UKQCD	22.74	3.59	96.2	2.35 ± 0.30	4.83 ± 0.17
matching	BMW	22.74	3.469	95.5	2.20 ± 0.13	4.74 ± 0.10
fit	FLAG	21.31	3.4	95	1.94 ± 0.65	4.86 ± 0.39
fit	RBC/UKQCD	21.31	3.59	96.2	2.17 ± 0.31	5.01 ± 0.17
fit	BMW	21.31	3.469	95.5	2.02 ± 0.14	4.91 ± 0.11

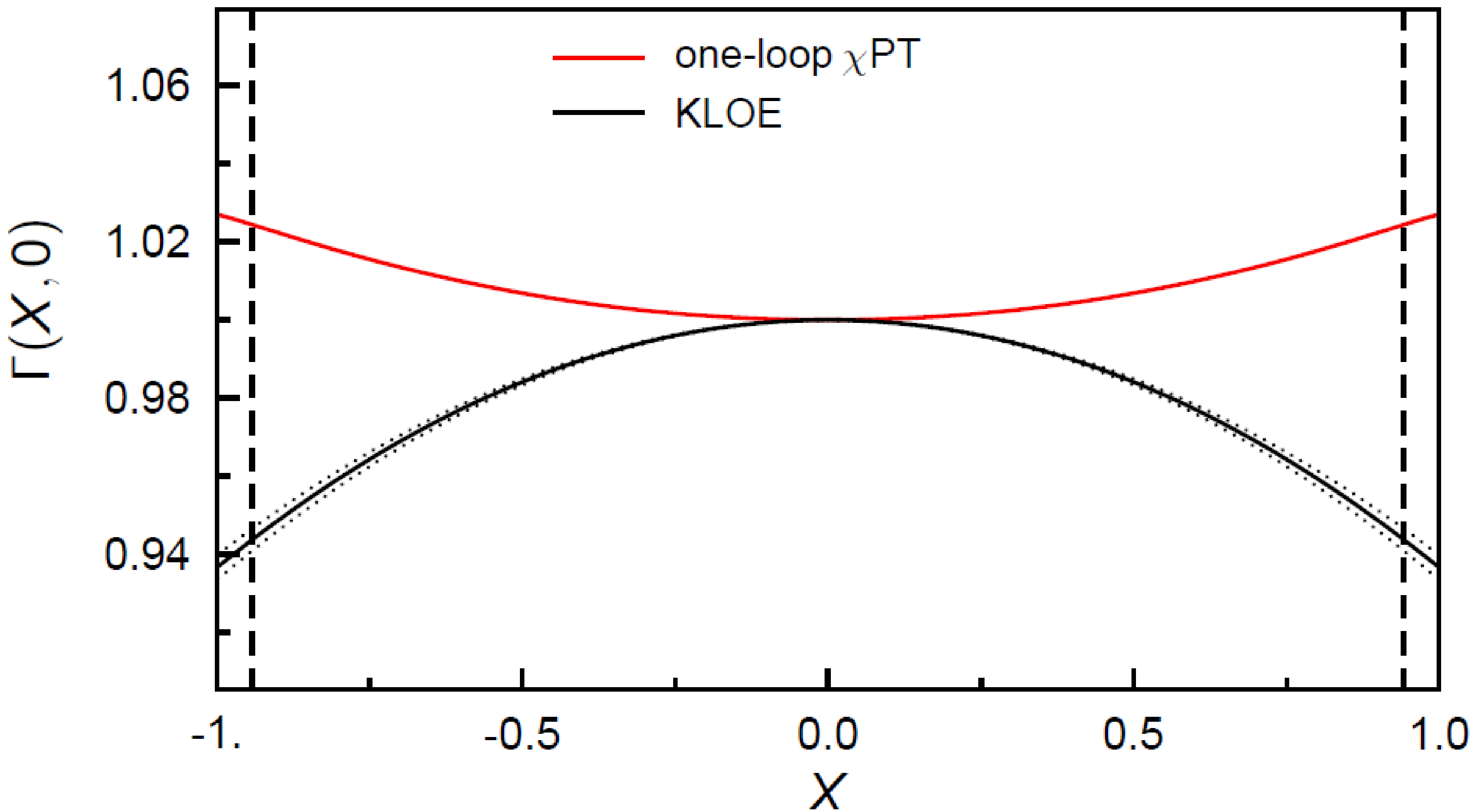
⇨ Main result: m_s and $\hat{m}$ from BMW + Q from fit

$m_u = (2.02 \pm 0.14)$ MeV and $m_d = (4.91 \pm 0.11)$ MeV

m_u = 0 excluded!

Results for the neutral channel

- Amplitude: $\overline{A}(s,t,u) = A(s,t,u) + A(t,u,s) + A(u,s,t)$
- Dalit plot parametrization: $\Gamma = N\left(1 + 2\alpha Z\right)$ with $Z = X^2 + Y^2$



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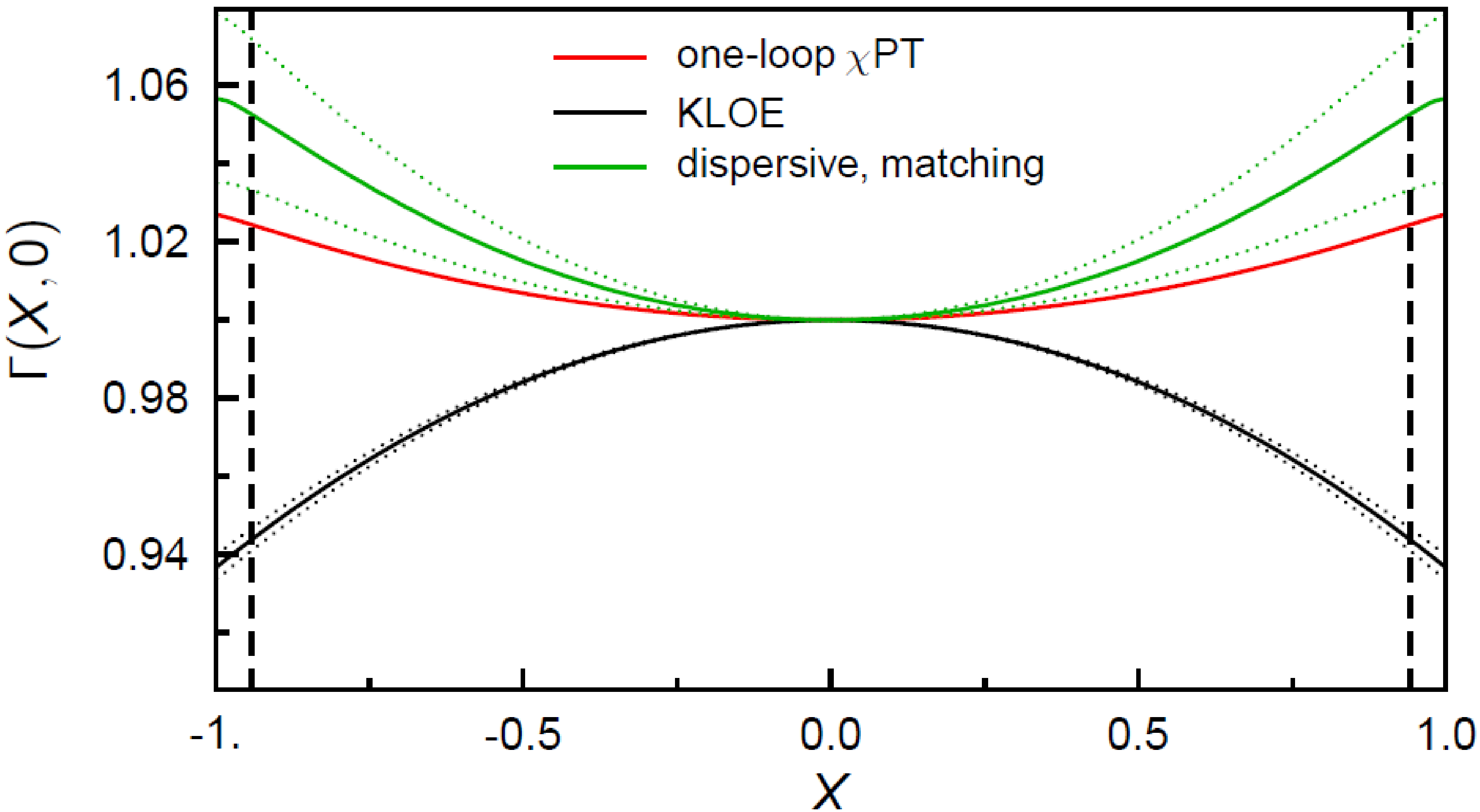

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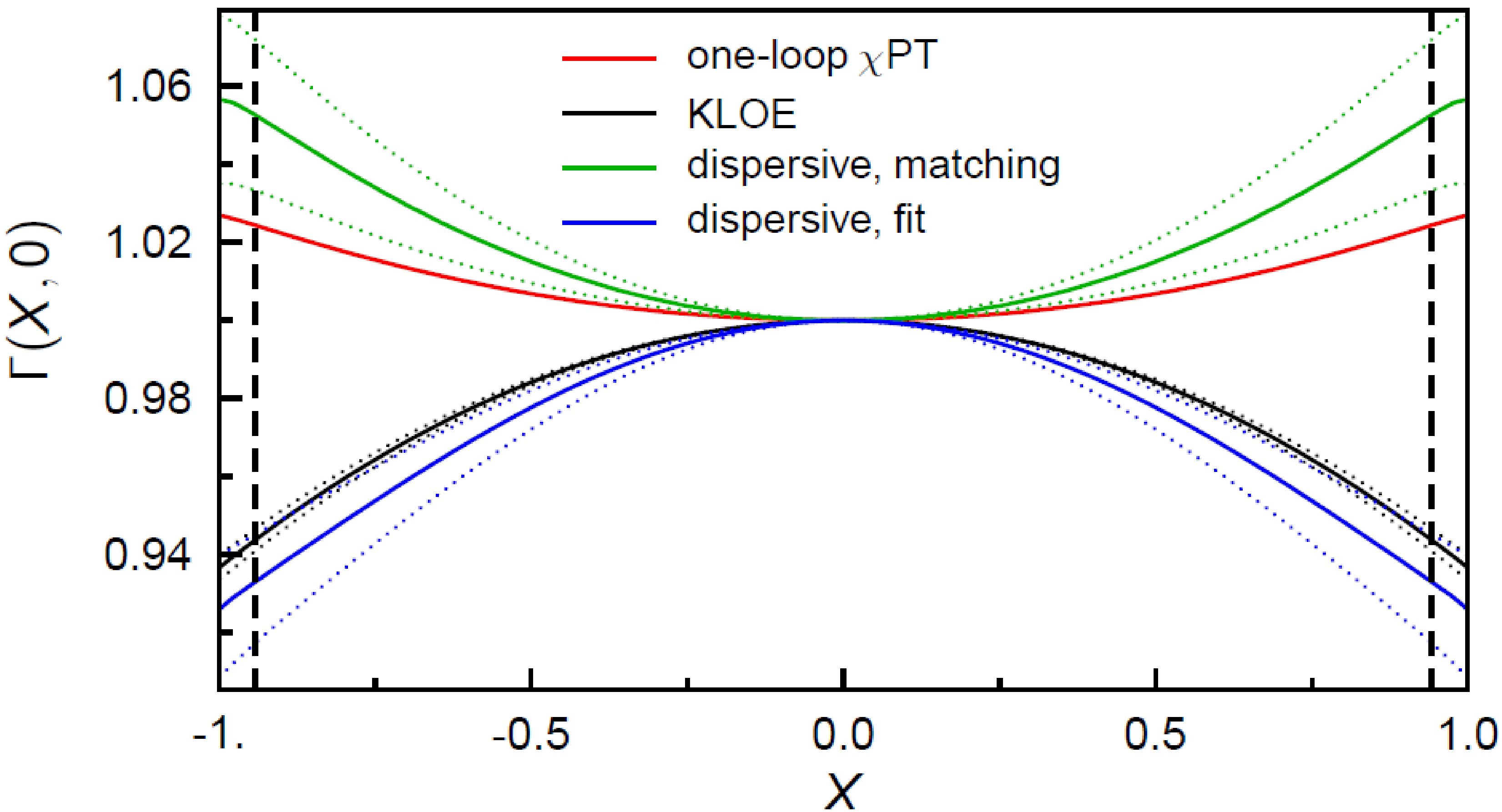


Error analysis Matching

	$Q(\pi^+\pi^-\pi^0)$	$Q(3\pi^0)$	r	α
Γ	$\pm\,0.31$	$\pm\,0.31$	—	—
γ_0	$\pm\,0.38$	$\pm\,0.36$	$\pm\,0.0069$	$\pm\,0.0096$
β_1	$\pm\,0.36$	$\pm\,0.35$	$\pm\,0.0039$	$\pm\,0.0026$
L_3	$^{+0.025}_{-0.023}$	$^{+0.036}_{-0.033}$	$\pm\,0.0026$	$\pm\,0.0009$
$\delta_l(s)$	$^{+0.18}_{-0.15}$	$^{+0.17}_{-0.13}$	$^{+0.0027}_{-0.0032}$	$\pm\,0.0040$
inelasticity	$\pm\,0.2$	$\pm\,0.2$	—	—
cut-off	$\pm\,0.09$	$\pm\,0.09$	$\pm\,0.002$	$\pm\,0.0026$
total uncertainty	$^{+0.68}_{-0.67}$	$^{+0.65}_{-0.64}$	$^{+0.0090}_{-0.0092}$	$\pm\,0.011$

Error analysis Fit

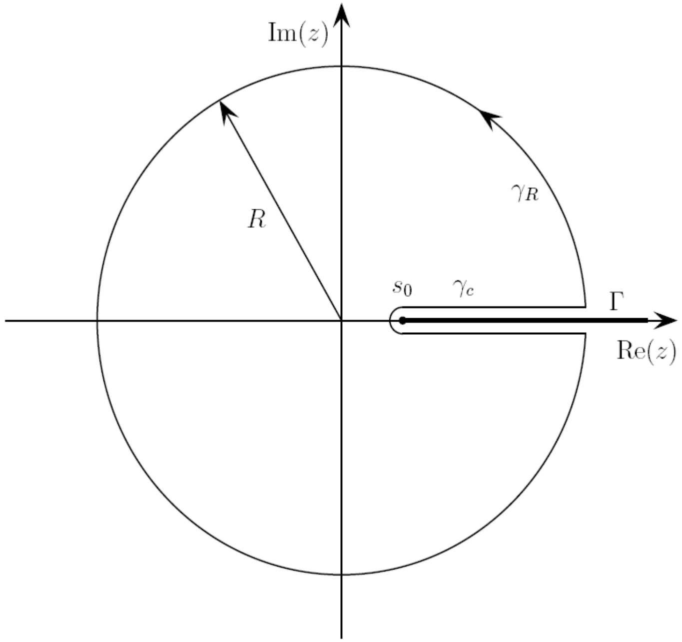
	$Q(\pi^+\pi^-\pi^0)$	$Q(3\pi^0)$	r	α
Γ	$\pm\, 0.29$	$\pm\, 0.29$	—	—
stat. KLOE	$\pm\, 0.091$	$\pm\, 0.086$	$\pm\, 0.0068$	$\pm\, 0.0034$
syst. KLOE	$^{+0.45}_{-0.30}$	$^{+0.42}_{-0.28}$	$^{+0.0078}_{-0.0125}$	$^{+0.0067}_{-0.0094}$
$\mathcal{N}$ KLOE	$^{+0.030}_{-0.029}$	$^{+0.030}_{-0.029}$	$^{+0.0001}_{-0.0001}$	$^{+0.0016}_{-0.0012}$
L_3	$^{+0.21}_{-0.25}$	$^{+0.22}_{-0.26}$	$^{+0.0020}_{-0.0021}$	$^{+0.0018}_{-0.0015}$
$\delta_I(s)$	$^{+0.041}_{-0.053}$	$^{+0.034}_{-0.048}$	$^{+0.0014}_{-0.0018}$	$^{+0.0020}_{-0.0017}$
W_A	$^{+0.000}_{-0.033}$	$^{+0.000}_{-0.032}$	$^{+0.0015}_{-0.0013}$	$^{+0.0013}_{-0.0008}$
total uncertainty	$^{+0.59}_{-0.50}$	$^{+0.56}_{-0.50}$	$^{+0.011}_{-0.015}$	$^{+0.0083}_{-0.0104}$

Method: Representation of the amplitude

- Knowing the discontinuity of M_I ⇨ write a dispersion relation for it

- Cauchy Theorem and Schwarz reflection principle

⇨
$$M_I(s) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \frac{disc\left[M_I(s')\right]}{s'-s-i\varepsilon} ds'$$



M_I can be reconstructed everywhere from the knowledge of $disc\left[M_I(s)\right]$

- If M_I doesn't converge fast enought for $|s| \to \infty$ ⇨ subtract the dispersion relation

$$M_I(s) = P_{n-1}(s) + \frac{s^n}{\pi}\int_{4M_\pi^2}^{\infty} \frac{ds'}{s'^n} \frac{disc\left[M_I(s')\right]}{(s'-s-i\varepsilon)}$$

$P_{n-1}(s)$ polynomial

Hat functions

- Discontinuity of M_I: by definition $disc\left[M_I(s)\right] \equiv disc\left[f_\ell^I(s)\right]$

  ⇨ $f_\ell^I(s) = M_I(s) + \hat{M}_I(s)$

  with $\hat{M}_I(s)$ real on the right-hand cut

- The left-hand cut is contained in $\hat{M}_I(s)$

- Determination of $\hat{M}_I(s)$:
  subtract M_I from the partial wave projection of $M(s,t,u)$

  $$M(s,t,u) = M_0(s) + (s-u)M_1(t) + \ldots$$

- $\hat{M}_I(s)$ singularities in the t and u channels, depend on the other M_I
  Angular averages of the other functions ⇨ Coupled equations

Hat functions

- Ex: $$\hat{M}_0(s) = \frac{2}{3}\langle M_0 \rangle + 2(s - s_0)\langle M_1 \rangle + \frac{20}{9}\langle M_2 \rangle + \frac{2}{3}\kappa(s)\langle z M_1 \rangle$$

where $$\langle z^n M_I \rangle (s) = \frac{1}{2}\int_{-1}^{1} dz\, z^n M_I(t(s,z)),$$

$$z = \cos\theta$$ scattering angle

Non trivial angular averages ⇨ need to deform the integration path to avoid crossing cuts

Anisovich & Anselm'66

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Dispersion Relations for the $M_I(s)$

- Elastic Unitarity

$$\left[\ell = 1 \text{ for } I = 1,\ \ell = 0 \text{ otherwise}\right]$$

$$\Rightarrow\ disc\left[M_I\right] = disc\left[f_\ell^I(s)\right] = \theta\left(s - 4M_\pi^2\right)\left[M_I(s) + \hat{M}_I(s)\right]\sin\delta_\ell^I(s)\, e^{-i\delta_\ell^I(s)}$$

$\pi\pi$ phase shift

δ_ℓ^I phase of the partial wave $f_\ell^I(s)$

⇒ Watson theorem: elastic $\pi\pi$ scattering phase shifts

- Solution: Inhommogeneous Omnès problem

$$M_0(s) = \Omega_0(s)\left(\alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^3}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^3}\frac{\sin\delta_0^0(s')\,\hat{M}_0(s')}{\left|\Omega_0(s')\right|\left(s' - s - i\varepsilon\right)}\right)$$

Omnès function

Similarly for M_1 and M_2

$$\left[\Omega_I(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_\ell^I(s')}{s'(s' - s - i\varepsilon)}\right)\right]$$