EPS Grenoble
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Meaningful characterisation of perturbative theoretical uncertainties

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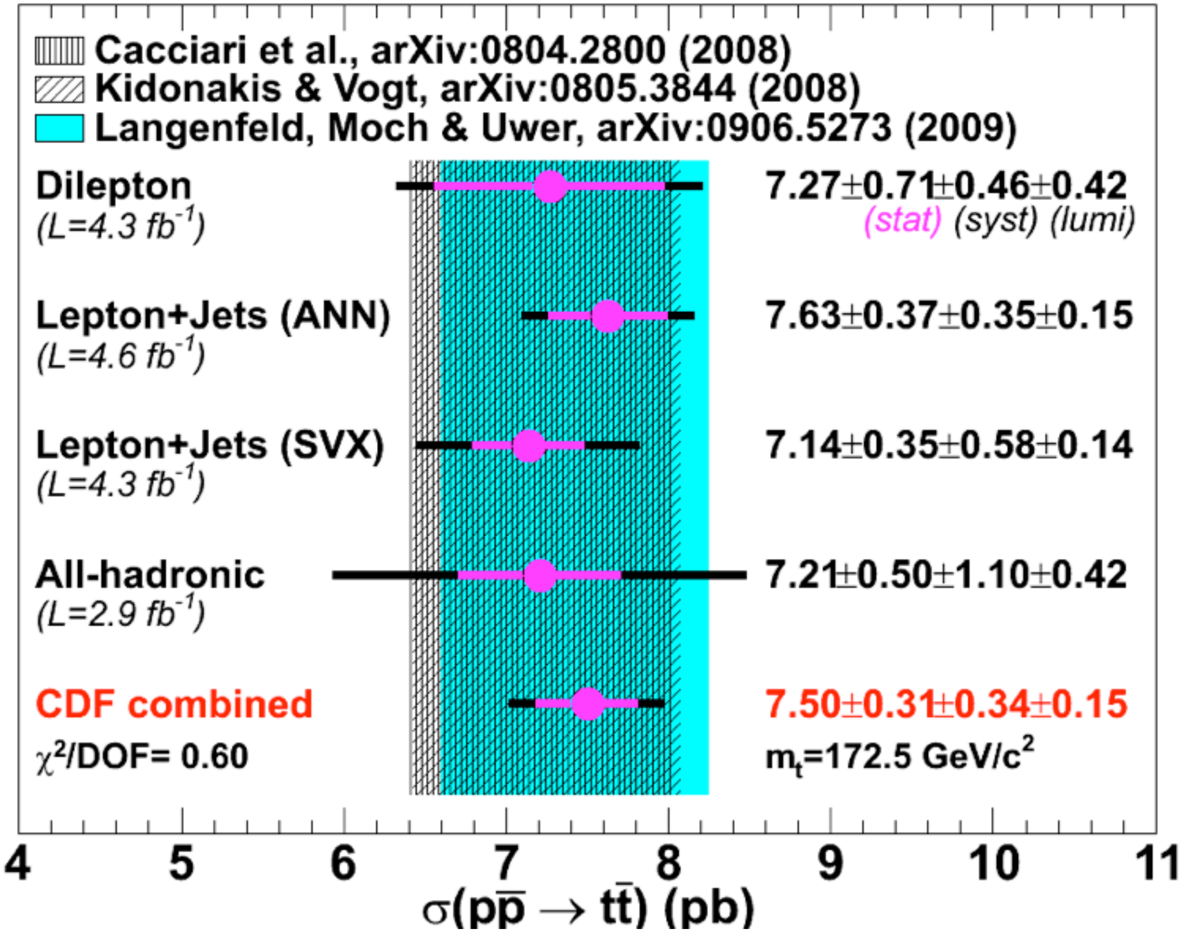
(M. Cacciari and NH, arXiv:1105.5152)

Theory v. experiment

top-antitop total cross section



What goes into the theoretical uncertainty band?

How has it been obtained?

What does it mean?

How **much** are you ready to bet that theory and experiment agree?

Theoretical uncertainties

How confident are we in our way of
estimating theoretical uncertainties?

Would you feel ready to bet (and how much) on the result of a scale variations uncertainty estimate?

If you can't (or don't wish to) answer, then we don't have a proper estimate, and one that can be meaningfully and safely combined with other sources of uncertainty

Understanding scale variations

Theoretical prediction

$$\sigma_{QCD,k} = \sum_{n=1}^{k} c_n(Q,\mu)\alpha_s^n(\mu)$$

Remainder

$$\Delta_k \equiv \sum_{n=k+1}^{\infty} c_n \alpha_s^n$$

Example:

$$\frac{\sigma(e^+e^- \to \text{hadrons}, Q)}{\sigma(e^+e^- \to \mu^+\mu^-, Q)} = R_{EW}(Q)(1+\sigma_{QCD}(Q))$$

Known exactly up to k=3

'Theoretical uncertainty'

$$\sigma_k^{\pm} = \sigma_k \pm \frac{\delta_k}{2}$$ where $$\delta_k \equiv |\sigma_k(Q,\mu=2Q) - \sigma_k(Q,\mu=Q/2)|$$

Why do we say that δ_k represents the theoretical uncertainty?

Understanding scale variations

Approximate δ_k as $\delta_k \simeq \left| \frac{d\sigma_k}{d \ln \mu^2} \right|_{\mu=Q} [\ln(2Q)^2 - \ln(Q/2)^2] \simeq 3k\beta_0 \alpha_s^{k+1} |c_k|$

This is the last known coefficient, multiplied by a further power of α_s

The true uncertainty, $O(\Delta_k)$, starts at α_s^{k+1} c_{k+1}

Logically equating $|\Delta_k|$ and δ_k means assuming that $|c_k| \approx |c_{k+1}|$

Bayesian model

Even if they can somehow estimate Δ_k, the bands given by scale variations have no statistical meaning: what is our **degree of belief** that they contain the right result?

We want to construct a credibility distribution for Δ_k from which to calculate the degree of belief of a given interval: $f(\Delta_k | c_0, \ldots c_k)$

For this, we use explicitly the implicit assumption that allows the scale variations method to work, i.e. that $|c_k| \approx |c_{k+1}|$

Bayesian model

We suppose that all coefficients of the series are bounded by an (unkown) maximum value

$$f(c_n|\bar{c}) = \frac{1}{2\bar{c}} \begin{cases} 1 & \text{if } |c_n| \leq \bar{c} \\ 0 & \text{if } |c_n| > \bar{c} \end{cases}$$

whose orders of magnitude are a priori equally probable

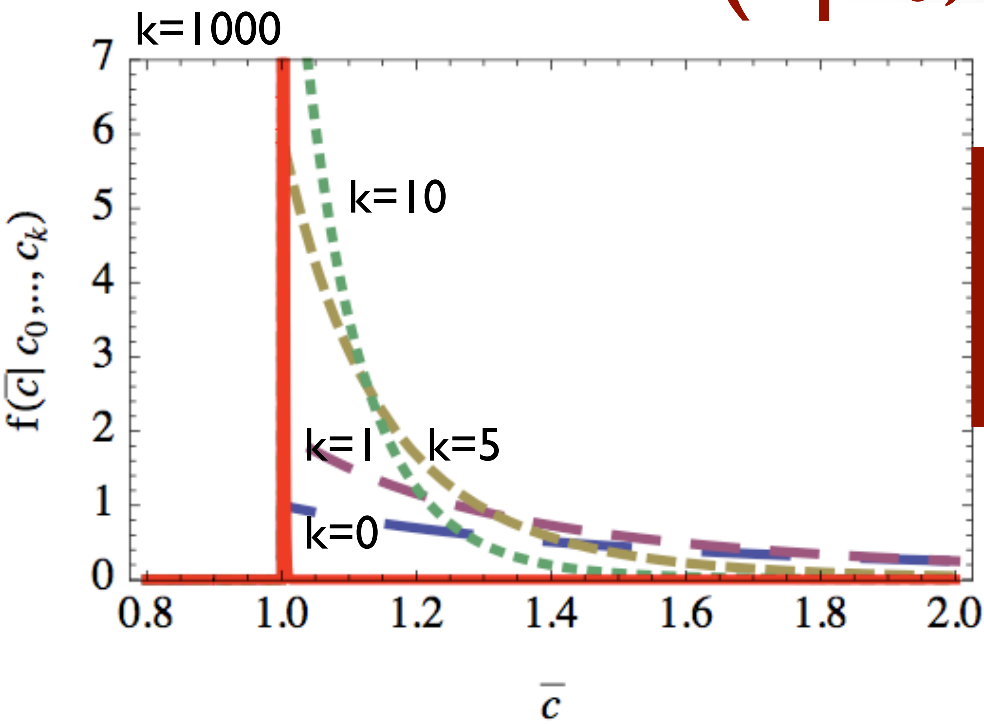
$$f_\epsilon(\ln \bar{c}) = \frac{1}{2|\ln \epsilon|} \chi_{|\ln \bar{c}| \leq |\ln \epsilon|}$$

and that they are independent with the exception of this common bound

$$f(\{c_i, i \in I\}|\bar{c}) = \prod_{i \in I} f(c_i|\bar{c})$$

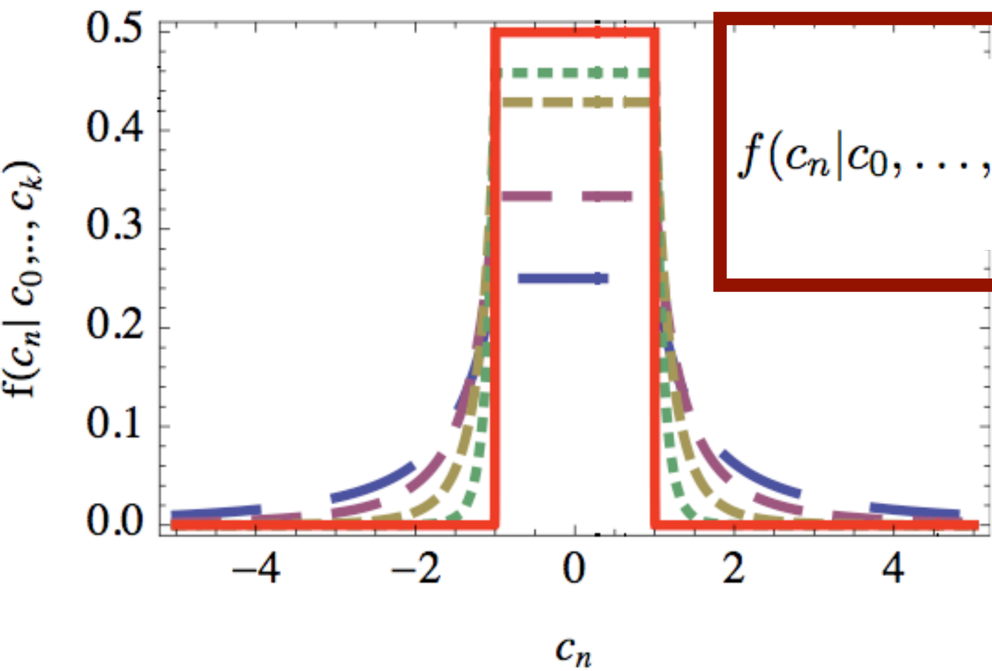
The results will be found in the $\varepsilon \rightarrow 0$ limit

$f(\bar{c}|c_0,\ldots,c_k)$ and $f(c_n|c_0,\ldots,c_k)$



$$f(\bar{c}|c_0,\ldots,c_k) = (k+1)\,\frac{\bar{c}_{(k)}^{k+1}}{\bar{c}^{k+2}}\,\chi_{\bar{c}\geq\bar{c}_{(k)}}$$

$$\bar{c}_{(k)} = \max(|c_0|,\ldots,|c_k|)$$



$$f(c_n|c_0,\ldots,c_k) = \left(\frac{k+1}{k+2}\right)\frac{1}{2\bar{c}_{(k)}}\begin{cases} 1 & \text{if } |c_n|\leq\bar{c}_{(k)} \\ \frac{1}{(|c_n|/\bar{c}_{(k)})^{k+2}} & \text{if } |c_n|>\bar{c}_{(k)} \end{cases}$$

Knowing more perturbative coefficents improves our estimates of $\bar{c}$ and of the unknown coefficients

$f(\Delta_k|c_0,\ldots,c_k)$

$$f(\Delta_k|c_0,\ldots,c_k) = \int \left[\delta(\Delta_k - \sum_{n=k+1}^{\infty} \alpha_s^n c_n)\right] f(c_{k+1}, c_{k+2}, \ldots |c_0,\ldots,c_k)\; \mathrm{d}c_{k+1}\mathrm{d}c_{k+2}\ldots$$

Making the approximation $\Delta_k \simeq \alpha_s^{k+1} c_{k+1}$
i.e. assuming that the coupling is reasonably small, one finds

$$f(\Delta_k|c_0,\ldots,c_k) \simeq \left(\frac{k+1}{k+2}\right) \frac{1}{2\alpha_s^{k+1}\bar{c}_{(k)}} \begin{cases} 1 & \text{if } |\Delta_k| \le \alpha_s^{k+1}\bar{c}_{(k)} \\ \frac{1}{(|\Delta_k|/(\alpha_s^{k+1}\bar{c}_{(k)}))^{k+2}} & \text{if } |\Delta_k| > \alpha_s^{k+1}\bar{c}_{(k)} \end{cases}$$

This is, in a sense, our 'main' result

From the credibility distribution of Δ_k one can calculate by integration the **degree of belief of any given interval**, or a ***p*%-credible interval**

DoB of the conventional interval

Setting n_c **as the number of known (i.e. calculated) coefficients**, the degree of belief of the interval given by scale variations can be found to be

$$\mathbb{C}(\Delta_k \in [-\frac{\delta_k}{2}, \frac{\delta_k}{2}] | c_l, \ldots, c_k) = \begin{cases} 1 - \frac{1}{n_c+1}\left[\frac{2}{3k\beta_0}\frac{\bar{c}_{(k)}}{|c_k|}\right]^{n_c} & \text{if } \frac{\delta_k}{2} \geq \alpha_s^{k+1}\bar{c}_{(k)} \Leftrightarrow |c_k| \geq \frac{2}{3k\beta_0}\bar{c}_{(k)} \\ \frac{n_c}{n_c+1}\frac{3k\beta_0}{2}\frac{|c_k|}{\bar{c}_{(k)}} & \text{if } \frac{\delta_k}{2} < \alpha_s^{k+1}\bar{c}_{(k)} \Leftrightarrow |c_k| < \frac{2}{3k\beta_0}\bar{c}_{(k)} \end{cases}$$

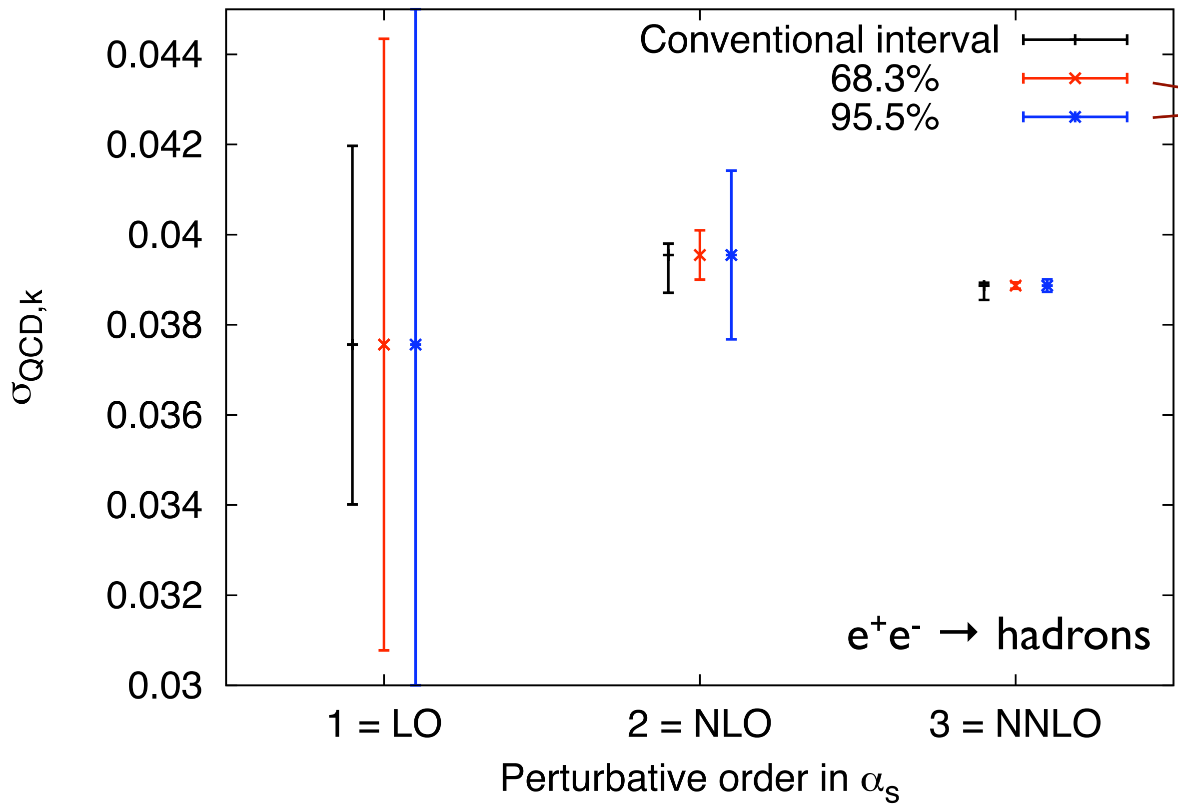
For $e^+e^- \rightarrow$ hadrons at 90 GeV, this gives

Perturbative order	Degree of belief
k=1 (LO)	46%
k=2 (NLO)	90%
k=3 (NNLO)	98.8%

p%-credible intervals

Independently from scale variations, one can calculate the **interval [$\sigma_k - d_k^{(p)}$, $\sigma_k + d_k^{(p)}$] that has a p% degree of belief**

$$d_k^{(p)} = \begin{cases} \alpha_s^{k+1} \max\{|c_l|,\ldots,|c_k|\} \frac{n_c+1}{n_c} p\% & \text{if } p\% \leq \frac{n_c}{n_c+1} \\ \alpha_s^{k+1} \max\{|c_l|,\ldots,|c_k|\} \left[(n_c+1)(1-p\%)\right]^{-1/n_c} & \text{if } p\% > \frac{n_c}{n_c+1} \end{cases}$$



p%-credible intervals in Bayesian model

Conclusions

- The Bayesian model allows one to calculate the degree of belief of given intervals of the remainder of a perturbative series
- The aim is **not** to 'add knowledge' to the perturbative calculation, but rather to **formalise** one way of estimating its uncertainty
- The priors we have used are no more arbitrary than the conventional method of scale variations. The added value is that the resulting intervals have a proper interpretation in terms of degree of belief, in principle allowing a 'coherent bet'
- Outlook: more extensive exploration of priors, extension to hadronic processes, comparisons with known perturbative calculations, efficient numerical implementation