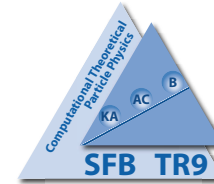


Singlet Contributions to the Vector Correlator



Johann H. Kühn

in collaboration with

P. Baikov, K. Chetyrkin, and J. Rittinger

**based on P. Baikov, K. Chetyrkin, and J. K.,
Nucl.Phys.Proc.Suppl.205-206:237-241,2010;**

**P. Baikov, K. Chetyrkin, J. K. and J. Rittinger, in
preparation**

Outline

- status of vector (VV) correlator in massless QCD: singlet versus nonsinglet
- tool-box
- singlet contribution $\mathcal{O}(\alpha_s^4)$ for a generic gauge group
- QED β -function in five loops
- phenomenological implications for $\sigma_{tot}(e^+e^- \rightarrow hadrons)$
- Gross-Llewellyn Smith sum rule in $\mathcal{O}(\alpha_s^4)$ and test of the constraints on the singlet part of the Adler function coming from the Crewther relation
- Conclusions

“gold plated” (Bjorken, 1979) QCD observables:

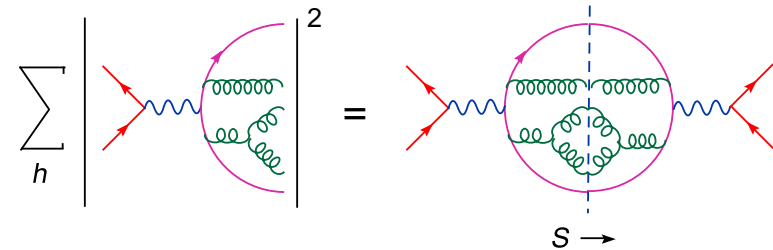
$$R_Z = \Gamma(Z_0 \rightarrow \text{hadrons}) / \sigma(Z_0 \rightarrow \mu^+ \mu^-)$$

$$R_\tau = \Gamma(\tau \rightarrow \text{hadrons} + \nu_\tau) / \Gamma(\tau \rightarrow l + \bar{\nu}_l + \nu_\tau)$$

$$R(s) = \sigma_{tot}(e^+ e^- \rightarrow \text{hadrons}) / \sigma(e^+ e^- \rightarrow \mu^+ \mu^-)$$

(via unitarity) $R(s) \approx \Im \Pi(s - i\delta)$

$$\Pi(Q^2) \approx \int e^{iqx} \langle 0 | T[j_\mu^v(x) j_\mu^v(0)] | 0 \rangle dx$$



$$R(s) \leftrightarrow D(Q) \quad \Leftarrow \text{Adler function} \equiv Q^2 \frac{d}{dQ^2} \Pi(q^2) = Q^2 \int \frac{R(s)}{(s + Q^2)^2} ds$$

$$R(s) = 1 + \sum_{i \geq 1} r_i a_s(s)^i, \quad D = 1 + \sum_{i \geq 1} d_i a_s(Q)^i, \quad (a_s \equiv \alpha_s / \pi, \mu = Q, Q^2 \equiv -q^2)$$

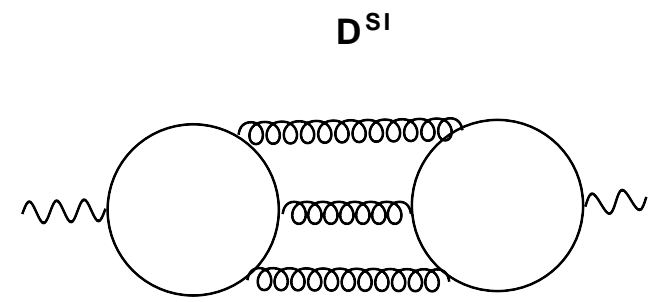
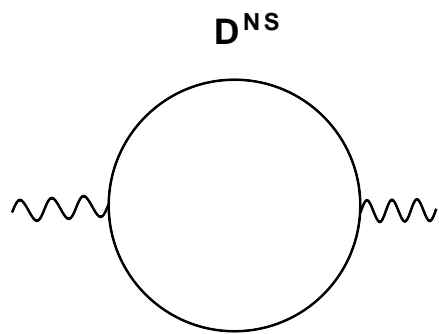
- status of theory (in the massless limit) •

$$R^{NS} = 3 \sum_i Q_i^2 \left(1 + \frac{\alpha_s}{\pi} + \# \left(\frac{\alpha_s}{\pi} \right)^2 + \# \left(\frac{\alpha_s}{\pi} \right)^3 + \# \left(\frac{\alpha_s}{\pi} \right)^4 + \dots \right)$$

parton model	QED Källen+ Sabry 1955	Chetyrkin, Kataev, Tkachov; Dine, Sapirstein; Celmaster 1979	Gorishny, Kataev, Larin; Surguladze, Samuel 1991 Chetyrkin /gen. gauge/ 1996	Baikov, Chetyrkin, Kühn 2008; Baikov, Chetyrkin, Kühn 2010 (Feynman Gauge only)
--------------	---------------------------------	---	---	--



$$R^{SI} = \left(\sum_i Q_i \right)^2 \left(\# \left(\frac{\alpha_s}{\pi} \right)^3 + \boxed{?? \left(\frac{\alpha_s}{\pi} \right)^4} + \dots \right)$$



Recall non-singlet results (PLR 101 (2008) 012002):

$$R^{NS} = 1 + a_s + (1.9857 - 0.1152 n_f) a_s^2 \\ (-6.63694 - 1.20013 n_f - 0.00518 n_f^2) a_s^3 \\ + (-156.61 + 18.77 n_f - 0.7974 n_f^2 + 0.0215 n_f^3) a_s^4$$

Impact on α_s from Z -decays:

$$\mathcal{O}(\alpha_s^3) : \alpha_s(M_Z)^{NNLO} = 0.1185 \pm 0.0026^{\text{exp}} \pm 0.002^{\text{th}}$$

Including the α_s^4 term leads to an increase of $\delta\alpha_s(M_Z) = 0.0005$ and to *four-fold* decrease of the theory error!

$$\mathcal{O}(\alpha_s^4) : \alpha_s(M_Z)^{NNNLO} = 0.1190 \pm 0.0026^{\text{exp}} \pm 0.0005^{\text{th}}$$

Impact on α_s from τ -decays (FO and CI): α_s decreased by $\delta\alpha_s(M_Z) = 0.0016$

$$\delta\alpha_s(M_Z)^{NNNLO} = 0.1202 \pm 0.0019^{\text{exp}}$$

Massless Correlators: Technicalities

Π related to the corresponding absorptive part $R(s)$ through


$$R^{jj}(s) \approx \Im \Pi^{jj}(s - i\delta)$$

RG equation

$$\begin{aligned} \Pi^{jj} &= Z^{jj} + \Pi^{\text{Bare}}(-Q^2, \alpha_s^{\text{Bare}}) \\ \left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(a_s) \frac{\partial}{\partial a_s} \right) \Pi &= \gamma^{jj}(a_s) \end{aligned}$$

extremely useful for determining the absorptive part of Π^{jj}

For Π at 5 loop

$$\frac{\partial}{\partial \log(\mu^2)} \Pi = \gamma^{jj}(a_s) - \left(\beta(a_s) \frac{\partial}{\partial a_s} \right) \Pi$$


anom.dim. at $\mathcal{O}(\alpha_s^4)$
5 loop integrals

4-loop integrals at $\mathcal{O}(\alpha_s^3)$ only
contribute
due to the factor of
 $\beta(a_s) \frac{\partial}{\partial a_s} = \mathcal{O}(a_s)$

- to find Log-dependent part of Π at 5-loops one needs 5-loop anomalous dimension γ^{jj} and 4-loop Π (BUT! including its constant part)
- 5-loop anom.dim. reducible to 4-loop p-integrals

Tool-box for massless correlators at α_s^4 :

- IRR / Vladimirov, (78)/ + IR R^* -operation /Chetyrkin, Smirnov (1984)/
+ resolved combinatorics /Chetyrkin, (1997)/
- reduction to Masters: “direct and automatic” construction of CF’s through $1/D$ expansion within the Baikov’s representation for Feynman integrals (Phys. Lett. B385 (1996) 403; B474 (2000) 385; Nucl.Phys.Proc.Suppl.116:378-381,2003)
- computing: MPI-based (PARFORM) as well as thread-based (TFORM) versions of FORM
Vermaseren, Retey, Fliegner, Tentyukov, ...(2000 – ...)

Results

Singlet contribution to the Adler function (Last missing term!)

$$D^{SI}(Q^2) = d_R \left(\sum_{i=3}^{\infty} d_i^{SI} a_s^i(Q^2) \right)$$

$$d_3^{SI} = \frac{d^{abc} d^{abc}}{d_R} \left(\frac{11}{192} - \frac{1}{8} \zeta_3 \right), \quad d_4^{SI} = \frac{d^{abc} d^{abc}}{d_R} (C_F d_{4,1}^{SI} + C_A d_{4,2}^{SI} + T n_f d_{4,3}^{SI})$$

$$d_{4,1}^{SI} = -\frac{13}{64} - \frac{\zeta_3}{4} + \frac{5\zeta_5}{8}, \quad d_{4,3}^{SI} = \frac{-149}{576} - \frac{13}{32} \zeta_3 + \frac{5}{16} \zeta_5 + \frac{1}{8} \zeta_3^2$$

$$d_{4,2}^{SI} = -\frac{3893}{4608} + \frac{169}{128} \zeta_3 - \frac{45}{64} \zeta_5 - \frac{11}{32} \zeta_3^2$$

Phenomenological implications for $\sigma_{tot}(e^+e^- \rightarrow hadrons)$

Numerically:

$$\begin{aligned}
 R(s) = & 3 \sum_f Q_f^2 \left\{ 1 + a_s + a_s^2 (1.986 - 0.1153n_f) \right. \\
 & + a_s^3 \left(-6.637 - 1.200n_f - 0.00518n_f^2 \right) \left. \right\} \\
 & - \left(\sum_f Q_f \right)^2 \left(1.2395 a_s^3 + \underline{(-17.8277 + 0.57489n_f) a_s^4} \right)
 \end{aligned}$$

for $n_f=5$

$$\frac{11}{3} \left[1 + a_s + a_s^2 1.409 - 12.767 a_s^3 - 79.98 a_s^4 \right] + \frac{1}{9} \left[-1.240 a_s^3 - 14.95 a_s^4 \right]$$

Extra suppression factor $\frac{3}{99} \approx 0.03!$

QED β -function in five loops

By a proper change of color factors we arrive at the **full** Adler function of QED in five loops \implies the QED β -function;

for a QED with one charged fermion we get ($A \equiv \frac{e^2}{16\pi^2}$)

$$\beta^{QED} = \frac{4}{3} A + 4 A^2 - \frac{62}{9} A^3 - A^4 \left(\frac{5570}{243} + \frac{832}{9} \zeta_3 \right)$$

Gorishny, Kataev,
Larin, Surguladze, 1991

$$-A^5 \left(\frac{195067}{486} + \frac{800}{3} \zeta_3 + \frac{416}{3} \zeta_4 - \frac{6880}{3} \zeta_5 \right)$$

Numerically ($A = \frac{\alpha}{4\pi} \approx 5.81 \cdot 10^{-4}$)

$$\beta^{QED} = \frac{4}{3} A (1 + 3 A - 5.1667 A^3 - 100.534 A^4 + 1129.51 A^5)$$

To check reduction to masters, two more calculations in $\mathcal{O}(\alpha_s^4)$, all for *general gauge group!*:

1. Perturbative factor $C^{Bjp}(a_s)$ in Bjorken sum rule:

$$\int_0^1 [g_1^{ep}(x, Q^2) - g_1^{en}(x, Q^2)] dx = \frac{1}{6} \left| \frac{g_A}{g_V} \right| C^{Bjp}(a_s)$$

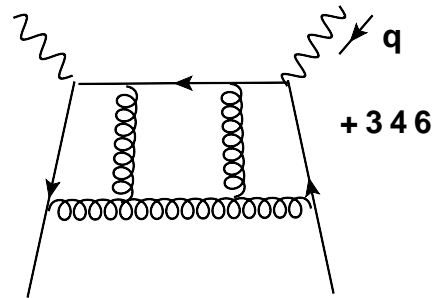
2. Perturbative factor $C_{GLS}(a_s)$ in Gross-Llewellyn Smith sum rule:

$$\frac{1}{2} \int_0^1 F_3^{\nu p + \bar{\nu} p}(x, Q^2) dx = 3 C_{GLS}(a_s)$$

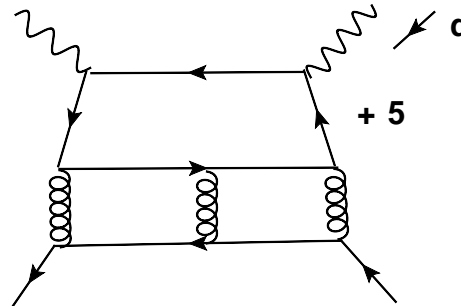
Both sum rules are unambiguous QCD predictions /modulo higher twists!/ confrontable with data

Typical diagrams at α_s^3 (computed in early nineties /[Larin & Vermaseren](#)/),

Bjp and GLS
(non-singlet)



GLS only
(singlet)



(1)

$$\begin{aligned}
 C_{GLS}^{NS} \equiv C_{BJp} &= 1 - a_s + a_s^2 [-4.583 + 0.3333 n_f] \\
 &+ a_s^3 [-41.44 + 7.607 n_f - 0.1775 n_f^2] \\
 &+ a_s^4 [-479.4 + 123.4 n_f - 7.697 n_f^2 + 0.1037 n_f^3]
 \end{aligned}$$

$$C_{GLS}^{SI} = 0.4132 n_f a_s^3 + a_s^4 n_f (5.80157 - 0.233185 n_f)$$

Note: $C_{GLS}^{SI} \ll C_{GLS}^{NS}$ as expected ($\overline{\text{MS}}$ -scheme):

(Generalized) Crewther relation for D^{NS}

$$C^{Bjp}(a_s) D^{NS}(a_s) = 1 + \frac{\beta(a_s)}{a_s} \left[K^{NS} = K_1 a_s + K_2 a_s^2 + K_3 a_s^3 + \dots \right] (\star)$$

with $\frac{\beta(a_s)}{a_s} \equiv -\beta_0 a_s + \dots$, $\beta_0 = \frac{11}{12} C_A - \frac{T_f n_f}{3}$

(\star) implies **6** constraints on 12 color structures

$$C_F^4, C_F^3 C_A, C_F^2 C_A^2, C_F C_A^3, C_F^3 T_F n_f, C_F^2 C_A T_F n_f, \\ C_F C_A^2 T_F n_f, C_F^2 T_F^2 n_f^2, C_F C_A T_F^2 n_f^2, C_F T_F^3 n_f^3, d_F^{abcd} d_A^{abcd}, n_f d_F^{abcd} d_F^{abcd}$$

appearing at $\mathcal{O}(\alpha_s^4)$ in the difference

$$D^{NS} - 1/C^{Bjp}$$

All 6 constraints are met identically! (which means $6 \cdot 7 = 42$ separate constraints on coefficients of $\zeta_3, \zeta_3^2, \dots$)

Crewther relation between $D = D^{NS} + D^{SI}$ and C_{GLS}

$$\left(D^{NS} + d_3^{SI} a_s^3 + d_4^{SI} a_s^4 \right) \left(C_{GLS}^{NS} + c_3^{SI} a_s^3 + c_4^{SI} a_s^4 \right) =$$

$$1 + \frac{\beta(\alpha_s)}{\alpha_s} \left[K^{NS} + a_s^3 K_3^{SI} n_f \frac{d^{abc} d^{abc}}{d_R} \right] \quad \star$$

with $\frac{\beta(\alpha_s)}{\alpha_s} \equiv -\beta_0 a_s + \dots$, $\beta_0 = \frac{11}{12} C_A - \frac{T_f}{3}$

$$d_3^{SI} = n_f \frac{d^{abc} d^{abc}}{d_R} d_{3,1}^{SI}, \quad d_4^{SI} = n_f \frac{d^{abc} d^{abc}}{d_R} (C_F d_{4,1}^{SI} + C_A d_{4,2}^{SI} + T_F d_{4,3}^{SI})$$

$$c_3^{SI} = n_f \frac{d^{abc} d^{abc}}{d_R} c_{3,1}^{SI}, \quad c_4^{SI} = n_f \frac{d^{abc} d^{abc}}{d_R} (C_F c_{4,1}^{SI} + C_A c_{4,2}^{SI} + T_F c_{4,3}^{SI})$$

rhs of \star depends on only 1 unknown parameter, K_3^{SI} , thus 3-1 = 2 constraints on three coefficients in d_4^{SI}

Obvious solution of these constraints reads:

$$d_{4,1}^{SI} = -\frac{3}{2}c_{3,1}^{SI} - c_{4,1}^{SI} = -\frac{13}{64} - \frac{\zeta_3}{4} + \frac{5\zeta_5}{8}$$

$$d_{4,2}^{SI} = -c_{4,2}^{SI} + \frac{11}{12}K_{3,1}^{SI} \quad d_{4,3}^{SI} = -c_{4,3}^{SI} + \frac{1}{3}K_{3,1}^{SI}$$

All 2 constraints are met identically! (which means $2*7=14$ separate constraints on coefficients of front of $\zeta_3, \zeta_3^2, \zeta_4, \zeta_4\zeta_3, \zeta_5, \zeta_7, \frac{n}{m}$)

CONCLUSIONS

- Singlet and non-singlet parts of the Adler function and the perturbative factors $C(a_s)$ of Bjorken and GLS sum rule have been both analytically evaluated for generic gauge group at $\mathcal{O}(\alpha_s^4)$
- The generalized Crewther relation puts 42 constraints on the non-singlet result and 14 constraints on the difference $d_4^{SI} - C_4^{CLS}$ which **are all** fulfilled!
- Numerically, the singlet contribution to the VV-correlator is tiny