

# Study of $A \leq 6$ helium clusters using soft-core potentials

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# Outline

## METHOD

- Non-Symmetrized HH
  - ▶ Motivation
  - ▶ Jacobi and Hyperspherical Coordinates
  - ▶ HH properties - Raynal-Revai and Kil'dyushov coefficients
  - ▶ Hamiltonian (non)construction and diagonalization

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## APPLICATIONS

- Volkov Potential
  - ▶ Permutation symmetry
  - ▶ Symmetry breaking

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## APPLICATIONS

- Volkov Potential
  - ▶ Permutation symmetry
  - ▶ Symmetry breaking
- $A \leq 6$  helium clusters
  - ▶ Soft-core Potential description
  - ▶ Three-body force
  - ▶ Efimov physics

Method

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  - 😊 No need of symmetrization procedure
  - 😊 Simpler matrix-element calculations
  - 😊 Simpler permutation-breaking calculations
  - ☹ Bigger basis set

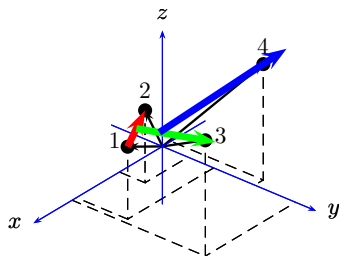
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  - ☹ Bigger basis set
- Method to avoid Hamiltonian construction
  - 😊 Hamiltonian as  $\sum \Pi$ (Sparse Matrices)
  - 😊 Iterative Diagonalization (ex. Lanczos)
  - 😊 Only action on a vector needed

# Jacobi's coordinates - $A \rightarrow N = A - 1$

Kinetic Energy

$$T = -\frac{\hbar^2}{2M} \nabla_{\mathbf{X}}^2 - \frac{\hbar^2}{m} \sum_{i=1}^N \nabla_{\mathbf{x}_i}^2$$



Center of Mass

$$\vec{X} = \frac{1}{M} \sum_{i=1}^A m_i \vec{r}_i, \quad M = \sum_{i=1}^A m_i$$

Jacobi's coordinates

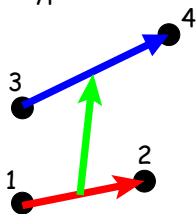
$$\vec{x}_3 = \vec{r}_2 - \vec{r}_1$$

$$\vec{x}_2 = \sqrt{\frac{4}{3}} \left( \vec{r}_3 - \frac{\vec{r}_1 + \vec{r}_2}{2} \right)$$

$$\vec{x}_1 = \sqrt{\frac{3}{2}} \left( \vec{r}_4 - \frac{\vec{r}_1 + \vec{r}_2 + \vec{r}_3}{3} \right)$$

# Different choices for Jacobi's coordinates

H-type

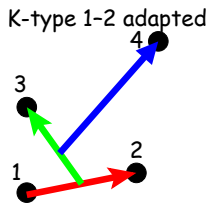


$$\vec{x}_3 = \vec{r}_2 - \vec{r}_1$$

$$\vec{x}_2 = \frac{\vec{r}_4 + \vec{r}_3}{\sqrt{2}} - \frac{\vec{r}_2 + \vec{r}_1}{\sqrt{2}}$$

$$\vec{x}_1 = \vec{r}_4 - \vec{r}_3$$

# Different choices for Jacobi's coordinates



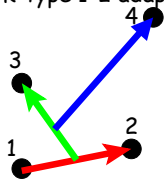
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K-type 1-2 adapted

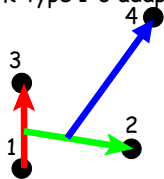


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K-type 1-3 adapted



$$\vec{x}_3 = \vec{r}_3 - \vec{r}_1$$

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# Hyperspherical Coordinates

Hyperradius

$$\rho = \left( x_1^2 + x_2^2 + x_3^2 \right)^{1/2}$$





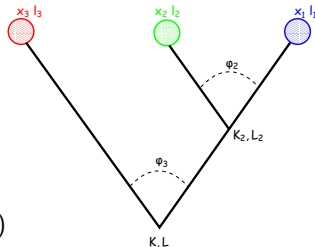
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Hyperangles

$$\Omega_3 = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \varphi_2, \varphi_3)$$



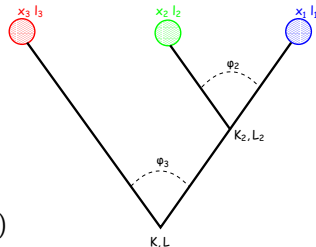
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$$x_3 = \rho \cos \varphi_3$$

$$x_2 = \rho \sin \varphi_3 \cos \varphi_2$$

$$x_1 = \rho \sin \varphi_3 \sin \varphi_2$$

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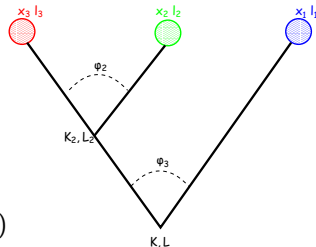
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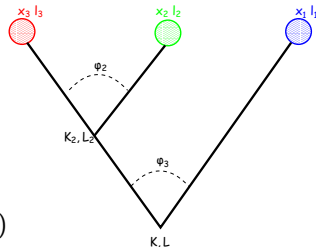
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Grand-Angular Momentum  $\Lambda_N^2$

$$\Delta = \sum_{i=1}^N \nabla_{\mathbf{x}_i}^2 = \left( \frac{\partial^2}{\partial \rho^2} + \frac{3N-1}{\rho} \frac{\partial}{\partial \rho} + \frac{\Lambda_N^2(\Omega_N)}{\rho^2} \right)$$

# Hyperspherical Harmonics

## Defining Equation

$$\left( \Delta_N^2(\Omega_N) + K(K + 3N - 2) \right) \mathcal{Y}_{[K]}(\Omega_N) = 0$$

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$$\mathcal{Y}_{[K]}^{LM}(\Omega_N) \text{ depends on } \begin{cases} \text{Jacobi} & \mathcal{Y}_{[K]}(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ | & / & & \\ & & \backslash & \\ & & & 2 \end{smallmatrix}) \neq \mathcal{Y}_{[K]}(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ | & & / & \\ & \backslash & & \\ & & & 2 \end{smallmatrix}) \\ \text{Hyperspherical} & \mathcal{Y}_{[K]}(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ | & / & / & \\ & & \backslash & \\ & & & 2 \end{smallmatrix}) \neq \mathcal{Y}_{[K]}(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ | & / & \backslash & \\ & & / & \\ & & & 2 \end{smallmatrix}) \end{cases}$$

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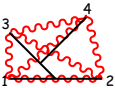
## Raynal-Revai

$$\mathcal{Y}_{[K]}(\begin{smallmatrix} 3 & 4 \\ | & \diagdown \\ 1 & 2 \end{smallmatrix}) = \sum_{K'} \mathcal{A}_{[K],[K']}^{(23)} \mathcal{Y}_{[K']}(\begin{smallmatrix} 3 & 4 \\ \diagdown & | \\ 1 & 2 \end{smallmatrix})$$

## Kil'dyushov T-coefficients

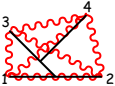
$$\mathcal{Y}_{[K]}(\begin{smallmatrix} \dots & \dots \\ \diagdown & \diagdown \\ \dots & \dots \end{smallmatrix}) = \sum_{K'} \mathcal{T}_{[K],[K']} \mathcal{Y}_{[K']}(\begin{smallmatrix} \dots & \dots \\ \diagdown & \diagup \\ \dots & \dots \end{smallmatrix})$$

# Potential Energy

$$V_{[K],[K']} = \langle \mathcal{Y}_{[K]}(\begin{array}{c} 3 \quad 4 \\ \diagdown \quad / \\ 1 \quad 2 \end{array}) \mid \sum_{i < j}^A V(r_{ij}) \mid \mathcal{Y}_{[K']}(\begin{array}{c} 3 \quad 4 \\ \diagdown \quad / \\ 1 \quad 2 \end{array}) \rangle =$$


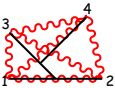


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A diagram showing a square lattice with four sites labeled 1, 2, 3, and 4. The sites are arranged in a square: 1 at the bottom-left, 2 at the bottom-right, 3 at the top-left, and 4 at the top-right. Wavy red lines represent interactions between adjacent sites: 1-2, 2-4, 4-3, and 3-1. A diagonal line also connects sites 1 and 3.

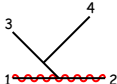
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- ☹ Huge Dense Matrix!!!

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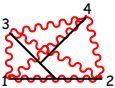
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**BUT !!!**

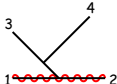
$$V(r_{12}) =$$


# Potential Energy

$$V_{[K],[K']} = \langle \mathcal{Y}_{[K]}(\text{diagram}) | \sum_{i < j}^A V(r_{ij}) | \mathcal{Y}_{[K']}(\text{diagram}) \rangle =$$


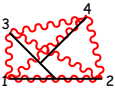
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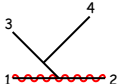
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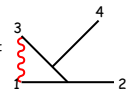
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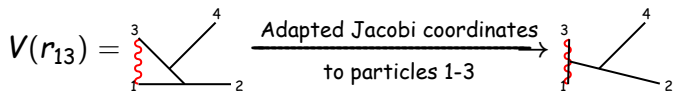
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- ☹ Huge but **Sparse Matrix!!!**
- ☹ Sparse is Good!!!

# "Adapted" Jacobi coordinates

$$V(r_{13}) =$$


## "Adapted" Jacobi coordinates



# "Adapted" Jacobi coordinates

$$V(r_{13}) = \begin{array}{c} 3 \\ \text{wavy red line} \\ 1 \end{array} \begin{array}{c} 4 \\ \diagup \\ \diagdown \\ 2 \end{array} \xrightarrow[\text{to particles 1-3}]{\text{Adapted Jacobi coordinates}} \begin{array}{c} 3 \\ \text{wavy red line} \\ 1 \end{array} \begin{array}{c} 4 \\ \diagup \\ \diagdown \\ 2 \end{array} = \begin{array}{c} 3 \\ \diagup \\ \diagdown \\ 2 \\ \text{wavy red line} \\ 1 \end{array}$$

The diagram illustrates the transformation of a potential function  $V(r_{13})$  from a coordinate system where the potential is between particles 1 and 3 to a coordinate system where it is between particles 1 and 2. The first diagram shows particle 1 at the bottom left, particle 3 above it, and particle 2 to its right. Particle 4 is above particle 2. A red wavy line connects particles 1 and 3. An arrow labeled "Adapted Jacobi coordinates" and "to particles 1-3" points to the second diagram. In the second diagram, particle 1 is at the bottom left, particle 3 above it, and particle 2 to its right. Particle 4 is above particle 2. A red wavy line connects particles 1 and 2. The third diagram is identical to the second, showing the potential between particles 1 and 2.

# "Adapted" Jacobi coordinates

$$V(r_{13}) = \begin{array}{c} \text{3} \\ \text{1} \text{---} \text{2} \end{array} \xrightarrow[\text{to particles 1-3}]{\text{Adapted Jacobi coordinates}} \begin{array}{c} \text{3} \\ \text{1} \text{---} \text{2} \end{array} = \begin{array}{c} \text{3} \\ \text{1} \text{---} \text{2} \end{array}$$

## Rotation (Transposition) Matrix

$$A_{[K],[K']}^{(23)} = \int d\left(\begin{array}{c} \text{3} \\ \text{1} \text{---} \text{2} \end{array}\right) \mathcal{Y}_{[K]}^* \left(\begin{array}{c} \text{3} \\ \text{1} \text{---} \text{2} \end{array}\right) \mathcal{Y}_{[K']} \left(\begin{array}{c} \text{3} \\ \text{1} \text{---} \text{2} \end{array}\right)$$



# "Adapted" Jacobi coordinates

$$V(r_{13}) = \begin{array}{c} \text{3} \\ \text{1} \text{---} \text{2} \end{array} \begin{array}{c} \text{4} \\ \text{---} \end{array} \xrightarrow[\text{to particles 1-3}]{\text{Adapted Jacobi coordinates}} \begin{array}{c} \text{3} \\ \text{1} \text{---} \text{2} \end{array} \begin{array}{c} \text{4} \\ \text{---} \end{array} = \begin{array}{c} \text{3} \\ \text{1} \text{---} \text{2} \end{array} \begin{array}{c} \text{4} \\ \text{---} \end{array}$$

The diagram shows a transformation of a potential function  $V(r_{13})$ . On the left, a red wavy line connects particle 1 and particle 3, while particle 2 is positioned below particle 1. An arrow labeled "Adapted Jacobi coordinates" and "to particles 1-3" points to the right. On the right, the wavy line now connects particle 1 and particle 2, while particle 3 is positioned above particle 1. This is followed by an equals sign and a final diagram where the wavy line connects particle 1 and particle 2, and particle 3 is positioned above particle 1.

## Rotation (Transposition) Matrix

$$A_{[K],[K']}^{(23)} = \int d\left(\begin{array}{c} \text{3} \\ \text{1} \text{---} \text{2} \end{array} \begin{array}{c} \text{4} \\ \text{---} \end{array}\right) \mathcal{Y}_{[K]}^* \left(\begin{array}{c} \text{3} \\ \text{1} \text{---} \text{2} \end{array} \begin{array}{c} \text{4} \\ \text{---} \end{array}\right) \mathcal{Y}_{[K']} \left(\begin{array}{c} \text{3} \\ \text{1} \text{---} \text{2} \end{array} \begin{array}{c} \text{4} \\ \text{---} \end{array}\right) = \left( \begin{array}{c} \text{3} \\ \text{1} \text{---} \text{2} \end{array} \begin{array}{c} \text{4} \\ \text{---} \end{array} \right)$$

The equation defines the rotation matrix  $A_{[K],[K']}^{(23)}$  as an integral over the coordinates of a three-particle system (particles 1, 2, 3) and a fourth particle (particle 4). The coordinates are represented by a diagram where particle 1 is at the bottom left, particle 2 is at the bottom right, and particle 3 is above particle 1. Particle 4 is to the right of the 1-2 line. The integral involves the product of the complex conjugate of a spherical harmonic  $\mathcal{Y}_{[K]}^*$  and a spherical harmonic  $\mathcal{Y}_{[K']}$ , both evaluated at the same coordinate diagram. The result is a matrix, represented by a red grid of dots, with a coordinate diagram as its index.

# "Adapted" Jacobi coordinates

$$V(r_{13}) = \begin{array}{c} 3 \\ \text{wavy} \\ 1 \end{array} \begin{array}{c} 4 \\ \diagup \\ \diagdown \\ 2 \end{array} \xrightarrow[\text{to particles 1-3}]{\text{Adapted Jacobi coordinates}} \begin{array}{c} 3 \\ \text{wavy} \\ 1 \end{array} \begin{array}{c} 4 \\ \diagup \\ \diagdown \\ 2 \end{array} = \begin{array}{c} 3 \\ \diagup \\ \diagdown \\ 1 \end{array} \begin{array}{c} 4 \\ \text{wavy} \\ 2 \end{array}$$

## Rotation (Transposition) Matrix

$$A_{[K],[K']}^{(23)} = \int d(\begin{array}{c} 3 \\ \diagup \\ \diagdown \\ 1 \end{array} \begin{array}{c} 4 \\ \diagup \\ \diagdown \\ 2 \end{array}) \mathcal{Y}_{[K]}^* (\begin{array}{c} 3 \\ \diagup \\ \diagdown \\ 1 \end{array} \begin{array}{c} 4 \\ \diagup \\ \diagdown \\ 2 \end{array}) \mathcal{Y}_{[K']} (\begin{array}{c} 3 \\ \diagup \\ \diagdown \\ 1 \end{array} \begin{array}{c} 4 \\ \diagup \\ \diagdown \\ 2 \end{array}) = \left( \begin{array}{c} \text{matrix} \end{array} \right)$$

## $V_{13}$ as Product of Sparse Matrices

$$\begin{array}{c} 3 \\ \text{wavy} \\ 1 \end{array} \begin{array}{c} 4 \\ \diagup \\ \diagdown \\ 2 \end{array} = A^{(23)} \cdot \begin{array}{c} 3 \\ \text{wavy} \\ 1 \end{array} \begin{array}{c} 4 \\ \diagup \\ \diagdown \\ 2 \end{array} \cdot A^{(23)}$$

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Rotation (Transposition) Matrix

$$A_{[K],[K']}^{(23)} = \int d(\begin{array}{c} 3 \\ \diagup \\ \diagdown \\ 1 \end{array} \begin{array}{c} 4 \\ \diagup \\ \diagdown \\ 2 \end{array}) \mathcal{Y}_{[K]}^* (\begin{array}{c} 3 \\ \diagup \\ \diagdown \\ 1 \end{array} \begin{array}{c} 4 \\ \diagup \\ \diagdown \\ 2 \end{array}) \mathcal{Y}_{[K']} (\begin{array}{c} 3 \\ \diagup \\ \diagdown \\ 1 \end{array} \begin{array}{c} 4 \\ \diagup \\ \diagdown \\ 2 \end{array}) = \begin{pmatrix} \text{red matrix} \end{pmatrix}$$

$V_{13}$  as Product of Sparse Matrices

$$\begin{array}{c} 3 \\ \text{wavy} \\ 1 \end{array} \begin{array}{c} 4 \\ \diagup \\ \diagdown \\ 2 \end{array} = A^{(23)} \cdot \begin{array}{c} 3 \\ \text{wavy} \\ 1 \end{array} \begin{array}{c} 4 \\ \diagup \\ \diagdown \\ 2 \end{array} \cdot A^{(23)}$$

Always possible! Use Jacobi-adjacent transpositions

$$\begin{array}{c} 3 \\ \text{wavy} \\ 1 \end{array} \begin{array}{c} 4 \\ \diagup \\ \diagdown \\ 2 \end{array} = A^{(23)} \cdot A^{(12)} \cdot A^{(23)} \cdot \begin{array}{c} 3 \\ \text{wavy} \\ 1 \end{array} \begin{array}{c} 4 \\ \diagup \\ \diagdown \\ 2 \end{array} \cdot A^{(23)} \cdot A^{(12)} \cdot A^{(23)}$$

# Three-body force

## Hyper-central three-body force

$$V^{(3)} = \sum_{i < j < k} W(\rho_{ijk}) \quad \text{with} \quad \rho_{ijk}^2 = \frac{2}{3}(r_{ij}^2 + r_{jk}^2 + r_{ik}^2)$$

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$\rho_{123}^2 = x_N^2 + x_{N-1}^2$

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## Adapted Jacobi coordinates

$$\begin{array}{c} 3 \\ \diagdown \\ 1 \text{---} 2 \\ \diagup \\ 4 \end{array} = A^{(12)} \cdot \begin{array}{c} 3 \\ \diagdown \\ 1 \text{---} 2 \\ \diagup \\ 4 \end{array} \cdot A^{(12)}$$

$$W(\rho_{124}) = A^{(12)} \cdot \mathcal{T} \cdot \widetilde{W}(\rho_{123}) \cdot \mathcal{T}^\dagger \cdot A^{(12)}$$

# How to use Hyperspherical Harmonics



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
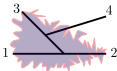
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
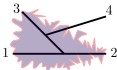
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
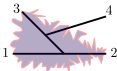
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$$H = \sum \left( \prod \text{Sparse Matrices} \right)$$

Only action on a vector (Iterative Diagonalization)

$$\vec{v}_{\text{out}} = H \cdot \vec{v}_{\text{in}}$$

# Applications

# Volkov's Potential

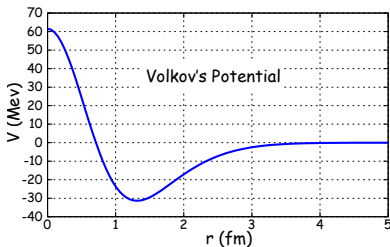
- Mass parameter

$$\hbar^2/m = 41.47 \text{ Mev fm}^2$$

- Potential

$$V(r) = E_1 e^{-r^2/R_1^2} + E_2 e^{-r^2/R_2^2}$$

- $E_1 = 144.86 \text{ Mev}$ ,  $R_1 = 0.82 \text{ fm}$ ,  $E_2 = -83.34 \text{ Mev}$ ,  $R_2 = 1.6 \text{ fm}$



# Volkov's Potential

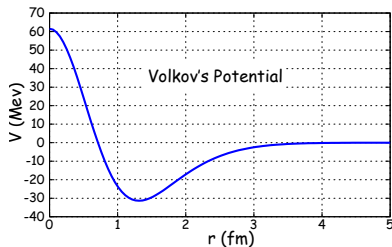
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- S-wave potential - only acts when  $l_{ij} = 0$



# Spectrum & Symmetries

Permutation of the  $A$  particles is a symmetry

$$[H, S_A] = 0$$

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-122.78 MeV  $[6] =$ 

--	--	--	--	--	--

 (1 level)



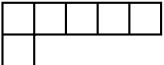
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-70.28 MeV [51] =  (6 levels)

-73.49 MeV [6] =  (1 level)

-122.78 MeV [6] =  (1 level)

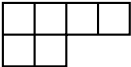
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- For  $A = 6$  with Volkov potential ( $L^\pi = 0^+$ ,  $K = 22$ )

-66.49 MeV  $[4\ 2] =$   (9 levels)

-70.28 MeV  $[5\ 1] =$   (6 levels)

-73.49 MeV  $[6] =$   (1 level)

-122.78 MeV  $[6] =$   (1 level)

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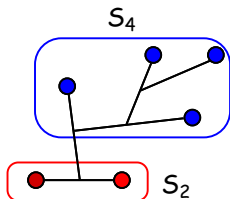
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Coulomb Interaction



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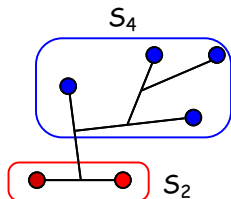
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$$S_6 \rightarrow S_2 \otimes S_4$$

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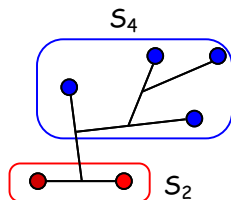
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$$S_6 \rightarrow S_2 \otimes S_4$$

A diagram showing the decomposition of the  $S_6$  irreducible representation into the tensor product of  $S_2$  and  $S_4$  irreps. The  $S_6$  irrep is represented by a Young diagram with three rows: the first row has three boxes, the second row has two boxes, and the third row has one box. This is followed by an arrow pointing to four terms, each representing a tensor product of  $S_2$  and  $S_4$  irreps, with their respective multiplicities in parentheses:

- $\square \otimes \square \quad (2)$
- $\square \otimes \square \quad (3)$
- $\square \otimes \square \quad (1)$
- $\square \otimes \square \quad (3)$

# Spectrum & Symmetries

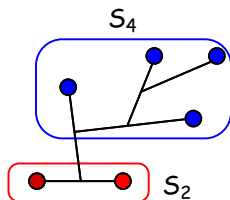
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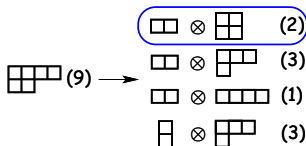
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# All-wave Volkov for $A = 6, L^\pi = 0^+$

$K_{max}$	$N_{HH}$	$E_0$ (MeV) [6]	$E_1$ (MeV) [6]	$E_2$ (MeV) [5 1]	$E_3$ (MeV) [4 2]
2	15	117.205	64.701	62.513	61.142
4	120	118.861	69.450	64.277	62.015
6	680	120.345	70.544	66.268	63.377
8	3045	121.738	71.443	67.280	64.437
10	11427	122.317	71.923	68.371	65.354
12	37310	122.597	72.477	69.029	65.886
14	108810	122.711	72.822	69.531	66.201
16	288990	122.752	73.101	69.842	66.360
18	709410	122.768	73.284	70.051	66.437
20	1628328	122.774	73.407	70.189	66.474
22	3527160	122.776	73.485	70.283	66.491
SVM*					66.25

\* K. Varga and Y. Suzuki, Phys. Rev. C **52**, 2885 (1995)

# All-wave Volkov - Summary

0.546 MeV [2] 0<sup>+</sup>

$A = 2$

0.599 MeV [3] 0<sup>+</sup>

8.465 MeV [3] 0<sup>+</sup>

$A = 3$

8.562 MeV [4] 0<sup>+</sup>

10.406 MeV [3 1] 1<sup>-</sup>

30.418 MeV [4] 0<sup>+</sup>

$A = 4$

28.72 MeV [4 1] 0<sup>+</sup>

31.72 MeV [5] 0<sup>+</sup>

43.03 MeV [4 1] 1<sup>-</sup>

68.28 MeV [5] 0<sup>+</sup>

$A = 5$

66.49 MeV [4 2] 0<sup>+</sup>

70.28 MeV [5 1] 0<sup>+</sup>

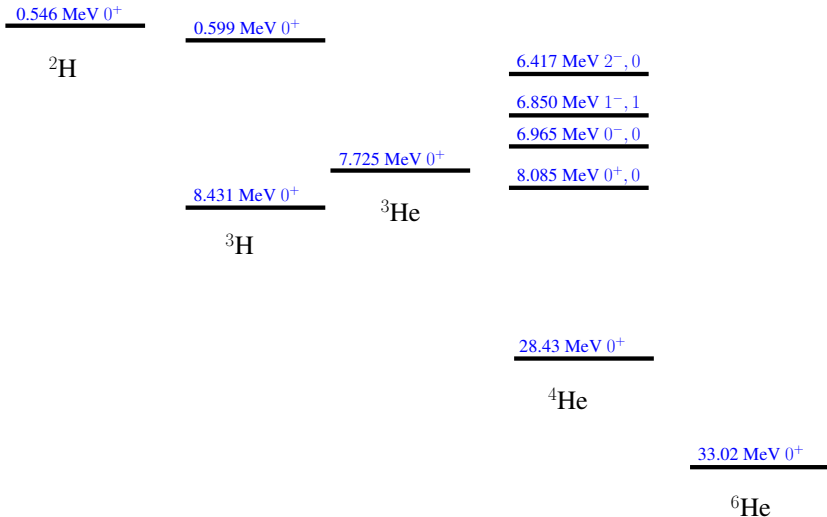
73.49 MeV [6] 0<sup>+</sup>

122.78 MeV [6] 0<sup>+</sup>

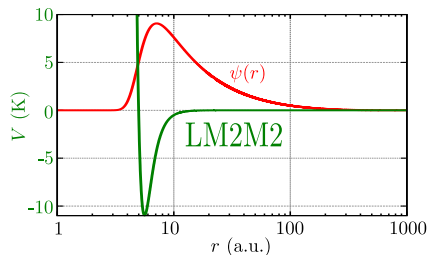
$A = 6$



# S-wave Volkov - "Physics"



# Helium Potential



- Helium-Helium interaction

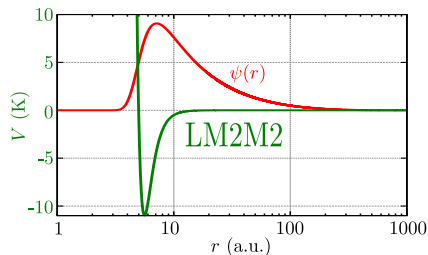
$$\ell_{\text{vdW}} \approx 10 \text{ a.u.}$$

$$r_0 \approx 14 \text{ a.u.}$$

$$a_0 \approx 190 \text{ a.u.}$$

$$E_2 \approx -1.30 \text{ mK}$$

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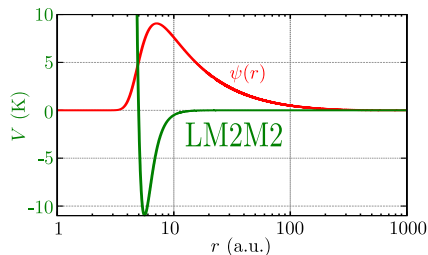
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- Efimov physics

$$r_0/a_0 \approx (\hbar^2/2ma_0^2 - E_2)/E_2$$

$$E_3^{(0)} \simeq -126 \text{ mK} \quad \text{and} \quad E_3^{(1)} \simeq -2.3 \text{ mK}$$

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- Strong short-range repulsion

- ▶ Difficult to treat with orthogonal basis
- ▶ Difficult to have converged excited states

## Soft Two-Body Gaussian Potential

- Effective low-energy gaussian soft potential

$$V(r) = V_0 e^{-r^2/R^2}$$

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- ▶ Fix  $V_0$  to reproduce one low-energy LM2M2 datum
- ▶ Use the cut-off  $R$  to reproduce a second datum

$$V_0 = -1.227 \text{ K}$$
$$R = 10.03 \text{ a.u.}$$

$\Rightarrow$

	Soft-Gaussian	LM2M2
$a_0$ (a.u.)	189.95	189.05
$r_0$ (a.u.)	13.85	13.84
$E_2$ (mK)	-1.296	-1.302

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		$r_0$ (a.u.)	13.85	13.84
		$E_2$ (mK)	-1.296	-1.302

- Problem in the three-body sector

	Soft-Gaussian	LM2M2
$E_3^{(0)}$ (mK)	-150.4	-126.4
$E_3^{(1)}$ (mK)	-2.467	-2.265

# Soft Hyper-Central Three-Body Potential

- Effective low-energy three-body-soft potential

$$W(\rho_{ijk}) = W_0 e^{-2\rho_{ijk}^2/\rho_0^2}$$

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potential	$E_{3b}^{(0)}$ (mK)	$E_{3b}^{(1)}$ (mK)
LM2M2	-126.4	-2.265
gaussian	-150.4	-2.467
$(W_0 \text{ [K]}, \rho_0 \text{ [a.u.]})$		
(306.9, 4)	-126.4	-2.283
(18.314, 6)	-126.4	-2.287
(4.0114, 8)	-126.4	-2.289
(1.4742, 10)	-126.4	-2.292
(0.721, 12)	-126.4	-2.295
(0.422, 14)	-126.4	-2.299
(0.279, 16)	-126.4	-2.302

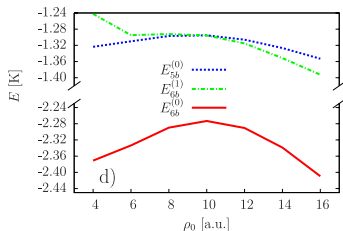
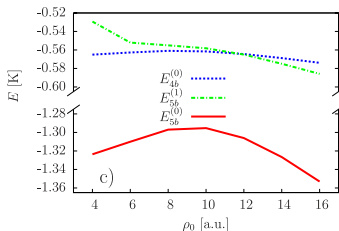
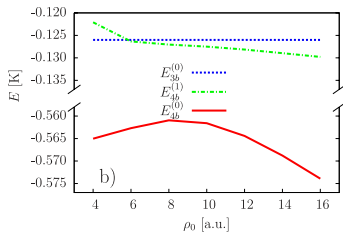
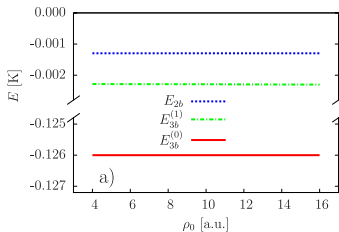
# Helium Clusters with Soft Potential

- Soft Potential sum of two- and three-body terms

$K$	$E_{4b}^{(0)}$ [mK]	$E_{4b}^{(1)}$ [mK]	$E_{5b}^{(0)}$ [mK]	$E_{5b}^{(1)}$ [mK]	$E_{6b}^{(0)}$ [mK]	$E_{6b}^{(1)}$ [mK]
0	538.93	4.557	1288.1	365.1	2293.8	1109.9
2	538.93	4.557	1288.1	365.1	2293.8	1109.9
4	561.69	40.29	1319.6	460.4	2331.8	1237.3
6	566.68	67.47	1324.4	497.6	2336.6	1273.0
8	568.21	84.22	1326.1	527.0	2338.4	1307.7
10	568.58	96.04	1326.5	542.7	2338.7	1323.1
12	568.73	105.30	1326.6	554.0	2338.8	1334.4
14	568.77	111.17	1326.6	561.0	2338.9	1340.9
16	568.78	115.58	1326.6	565.9	2338.9	1345.3
18	568.79	118.78	1326.6	569.3	2338.9	1348.2
20	568.79	121.20	1326.6	571.8	2338.9	1350.2
22	568.79	122.98	1326.6	573.6	2338.9	1351.6
24	568.79	124.38	1326.6	574.9		
26	568.79	125.47				
28	568.79	126.33				
30	568.79	127.02				
32	568.79	127.57				
34	568.79	128.02				
36	568.79	128.40				
38	568.79	128.70				
40	568.79	128.96				
Lewerenz, JCP <b>106</b> , 4596 (1997)	558.4		1302.2		2319.4	
Blume&Greene, JCP <b>112</b> , 8053 (2000)	559.7	132.6	1309.3	597.1	2329.4	1346.7

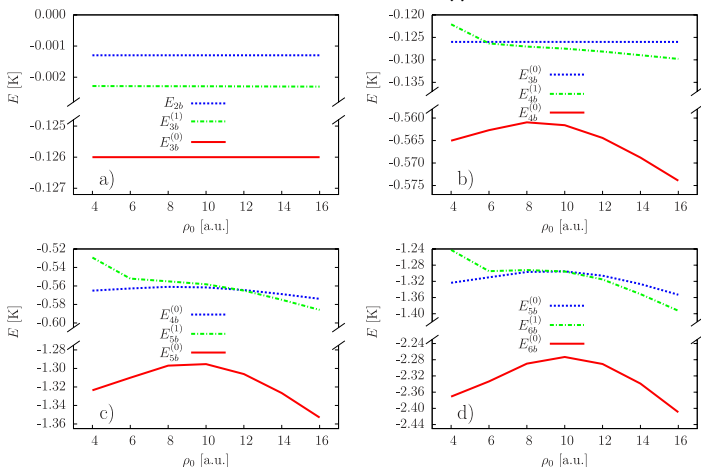
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- The range  $\rho_0$  is not independent ...

$$\rho_0^2/2 \geq R^2 \quad \Rightarrow \quad \rho_0 \gtrsim 14 \text{ a.u.}$$

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- Universal ratios

$\rho_0$ [a.u.]	$E_{4b}^{(0)}/E_{3b}^{(0)}$	$E_{4b}^{(1)}/E_{3b}^{(0)}$	$E_{5b}^{(0)}/E_{3b}^{(0)}$	$E_{5b}^{(1)}/E_{4b}^{(0)}$	$E_{6b}^{(0)}/E_{3b}^{(0)}$	$E_{6b}^{(1)}/E_{5b}^{(0)}$
12	4.47	1.01	10.33	1.001	18.12	1.005
14	4.50	1.02	10.50	1.011	18.50	1.018
16	4.54	1.03	10.70	1.021	19.06	1.029

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MG, A. Kievsky, and M. Viviani, arXiv:1106.3853, accepted PRA