Scales and Universality in Three-Body Systems

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What's Universality? → Independence of the potential

Two-body scattering length >> range of the potential

❖Some consequences

Two-body sector — Ex: Two identical bosons interacting in s-wave $E_2 \approx \dfrac{\hbar^2}{ma^2}$

Helium-4 dimer
$$\begin{cases} a \approx 100\ \text{Å} \\ r_{eff} \approx 50\ \text{Å} \\ E_2 = 1.1\ \text{mK} \end{cases} \qquad \frac{\hbar^2}{m} = \frac{48.12}{m}\ \text{K Å}^2$$

Three-body sector — Ex: Three identical bosons interacting in s-wave

$|E_2^{(1)}|$ —Decrease→ $|E_2^{(2)}| < |E_2^{(1)}|$ —Decrease→

$E_2 = 0$ → Appearance of an effective potential $\propto \frac{1}{\rho^2}$ Infinite three-body bound states

Efimov states

discovered by Vitaly Efimov in 1970

"Evidence of Efimov quantum states in an ultracold gas of cesium atoms" !

T. Kraemer et al. Nature **440**, 315 (2006)

СЛАБОСВЯЗАННЫЕ СОСТОЯНИЯ ТРЕХ РЕЗОНАНСНО ВЗАИМОДЕЙСТВУЮЩИХ ЧАСТИЦ

В. И. ЕФИМОВ

ФИЗИКО-ТЕХНИЧЕСКИЙ ИНСТИТУТ им. А. Ф. ИОФФЕ АКАДЕМИИ НАУК СССР

(Поступила в редакцию 16 февраля 1970 г.)

Показано, что при достаточной резонансности парных сил у трех тождественных частиц возникает семейство связанных состояний малой энергии. Квантовые числа всех состояний одинаковы: для бесспиновых бозонов 0^+, для нуклонов $1/2^+$, $T = 1/2$. Размер состояний больше радиуса парных сил. Наиболее благоприятные условия появления семейства уровней имеют место для трех бесспиновых нейтральных бозонов; менее благоприятные — для заряженных частиц и частиц со спином и изоспином. Обсуждается возможность существования таких уровней в системе трех α-частиц (в ядре C^{12}) и трех нуклонов (H^3).

SOVIET JOURNAL OF NUCLEAR PHYSICS VOLUME 12, NUMBER 5 MAY, 1971

WEAKLY-BOUND STATES OF THREE RESONANTLY-INTERACTING PARTICLES

V. N. EFIMOV

A. F. Ioffe Physico-technical Institute, USSR Academy of Sciences

Submitted February 16, 1970

Yad. Fiz. 12, 1080–1091 (November, 1970)

It is shown that if the pair forces of three identical particles are sufficiently resonant, a family of bound states of low energy is produced. The quantum numbers of all the states are the same: for spinless bosons 0^+ and for nucleons $1/2^+$, $T = 1/2$. The dimension of the states is larger than the radius of the pair forces. The most favorable conditions for the appearance of a family of levels occur for three spinless neutral bosons; the conditions are less favorable for charged particles and particles with spin and isospin. The possibility of existence of such levels in a system of three particles (in the C^{12} nucleus) and of three nucleons (H^3) is considered.

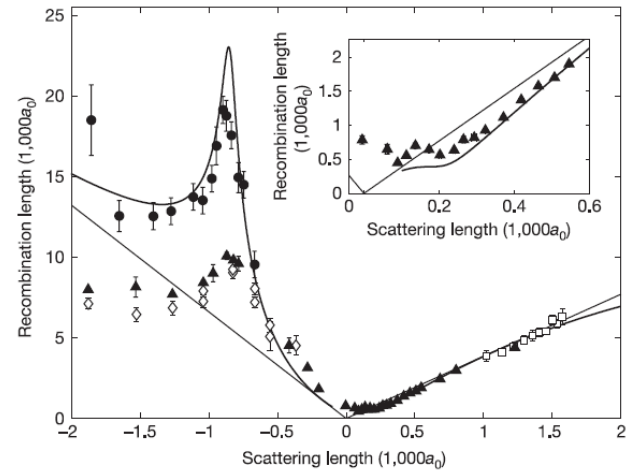


Figure 2 | Observation of the Efimov resonance in measurements of three-body recombination. The recombination length $\rho_3 \propto L_3^{1/4}$ is plotted as a function of the scattering length a. The dots and the filled triangles show the experimental data from set-up A for initial temperatures around 10 nK and 200 nK, respectively. The open diamonds are from set-up B at temperatures of 250 nK. The open squares are previous data[20] at initial temperatures between 250 and 450 nK. The solid curve represents the analytic model from effective field theory[7] with $a_- = -850a_0$, $a_+ = 1{,}060a_0$, and $\eta_- = \eta_+ = 0.06$. The straight lines result from setting the $\sin^2$ and $\cos^2$-terms in the analytic theory to 1, which gives a lower recombination limit for $a < 0$ and an upper limit for $a > 0$. The inset shows an expanded view for small positive scattering lengths with a minimum for $C(a) \propto (\rho_3/a)^4$ near $210a_0$. The displayed error bars refer to statistical uncertainties only. Uncertainties in the determination of the atomic number densities may lead to additional calibration errors for ρ_3 of up to 20%.

Efimov states

- Energy ratio between two consecutive states 515.03
- rms hyperradius ratio between two consecutive states 22.7

Describing universal systems

2: scattering length (*a*) / two-body energy

3: two-body energy + Three-body scale

Thomas collapse in 1935

$r_0 \to 0$
$V_0 \to \infty$
$E_2 \to$ fixed

$E_3^{deepest} \to \infty$

Scaling function

$$O(E, E_2, E_3) = (E_3)^\eta F\left(\sqrt{\frac{E}{E_3}}, \sqrt{\frac{E_2}{E_3}}\right)$$

Three-body bound state equation with zero-range interaction with momentum cutoff

Skorniakov and Ter-Martirosian equation (1956)

$$\chi(\vec{y}) = \frac{-\pi^2}{\pm\sqrt{\varepsilon_2} - \sqrt{\varepsilon_3 + \frac{3}{4}y^2}} \int d^3x \, \frac{\theta\left(1-\left|\vec{x}\right|\right)}{\varepsilon_3 + y^2 + x^2 + \vec{y}\cdot\vec{x}} \, \chi\left(\vec{x}\right)$$

momenta	energies
$\vec{p} = \Lambda\vec{x}$	$E_2 = \Lambda^2\varepsilon_2$
$\vec{q} = \Lambda\vec{y}$	$E_3 = \Lambda^2\varepsilon_3$

$\Lambda = r_0^{-1}$

$\varepsilon_2 \longrightarrow 0$

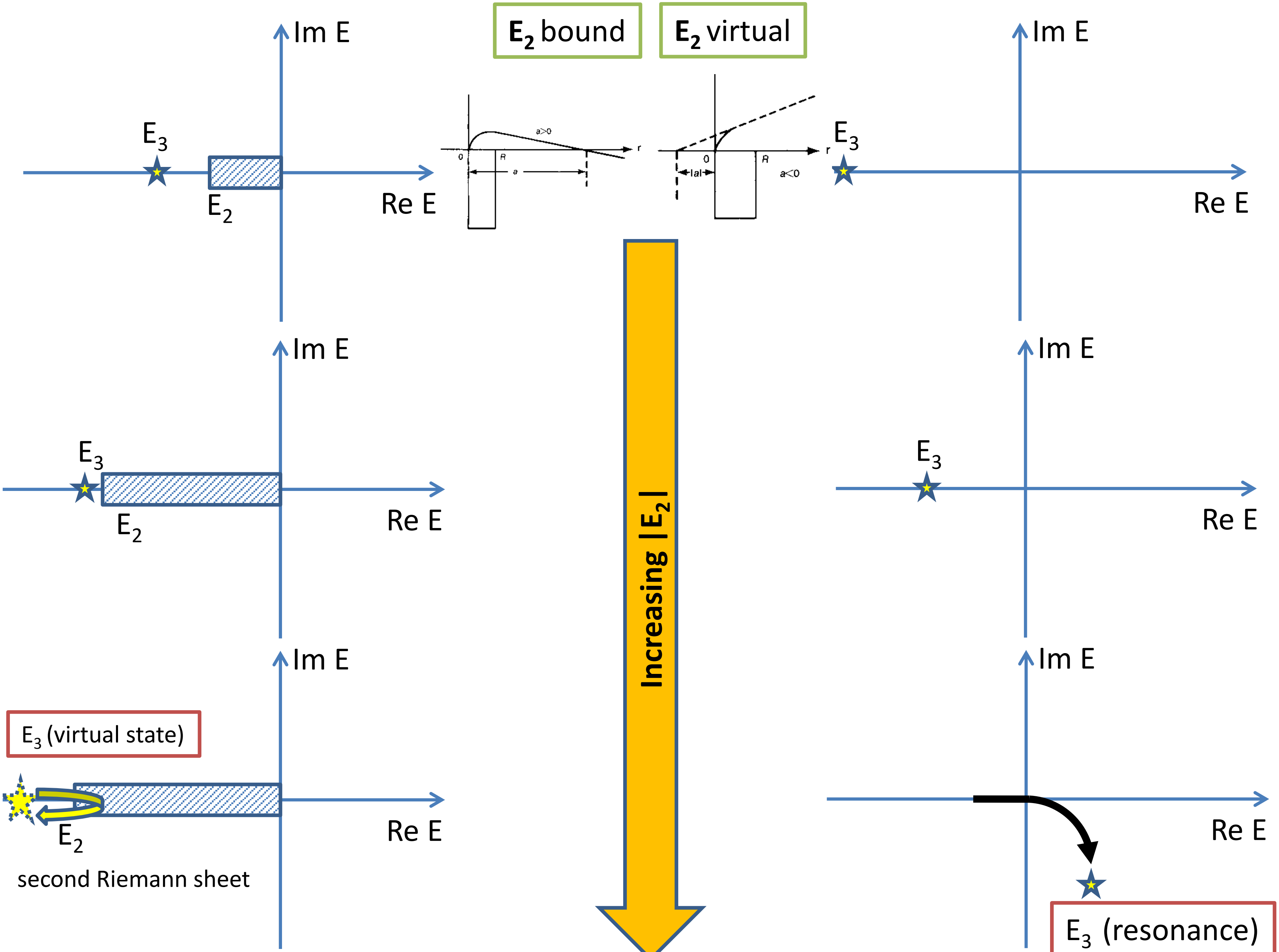
$\varepsilon_3 = \varepsilon_3^{(N)}$ **(N = 0, 1, 2, ...)** **Efimov states**

1) E_2 tends to zero with Λ fixed – Efimov effect

2) Λ tends to infinity with E_2 fixed – Thomas collapse

S.K. Adhikari, A. Delfino, T. Frederico, I.D. Goldman and L. Tomio, Phys. Rev. A 37, 3666 (1988)

If E2 ≠ 0: what happens to the Efimov states after they disappear?



E2 bound
E2 virtual
Im E
Re E
E3
E2
a>0
a<0
0
R
a
|a|
r
Increasing |E2|
E3 (virtual state)
second Riemann sheet
E3 (resonance)

Subtracted T-matrix Equation

S.K. Adhikari, T. Frederico and I.D. Goldman, Phys. Rev. Lett. 74, 487 (1995)

$$T(E) = V + VG_0(E)T(E)$$

$$T(-\mu^2) = (1 + T(-\mu^2)G_0(-\mu^2))V$$

$$V = (1 + T(-\mu^2)G_0(-\mu^2))^{-1}T(-\mu^2)$$

$$\boxed{T_R(E) = T_R(-\mu^2) + T_R(-\mu^2)\left(G_0^{(+)}(E) - G_0(-\mu^2)\right)T_R(E)}$$

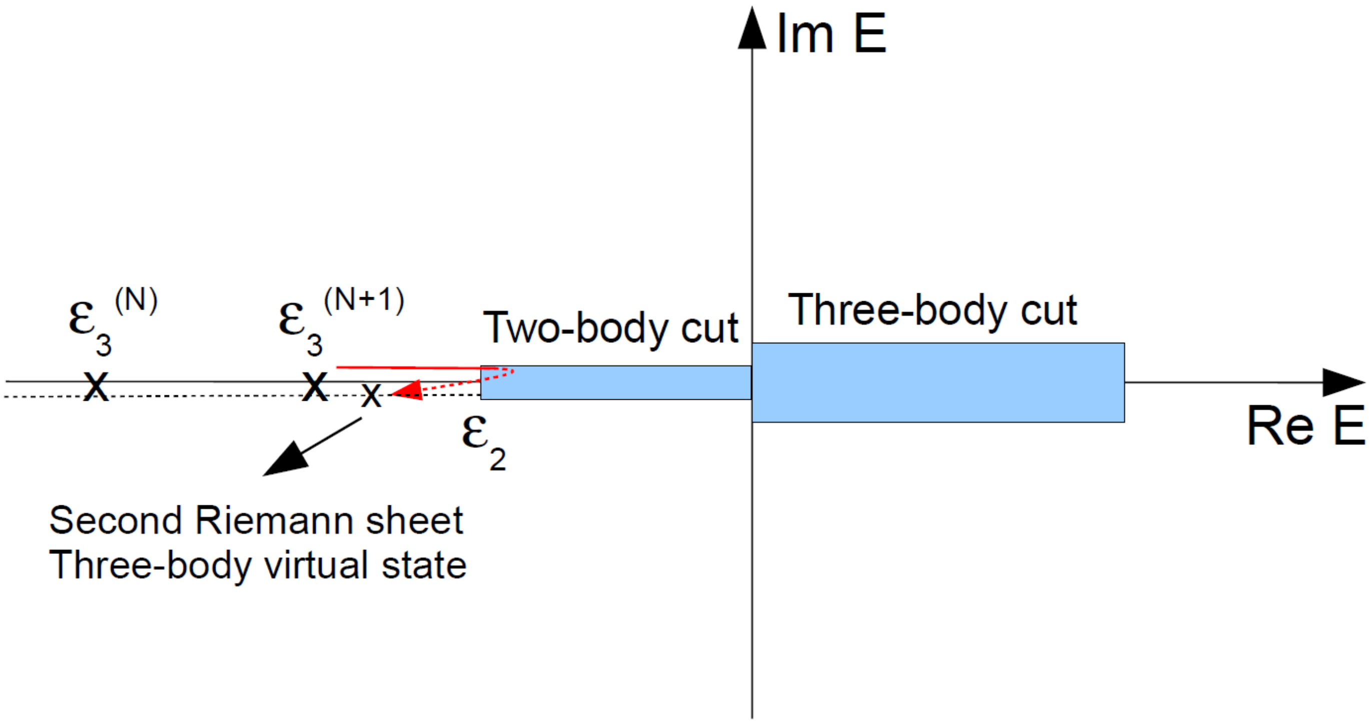
Three-body bound state equation for zero-range interaction with subtraction

M.T. Yamashita, T. Frederico, A. Delfino and L. Tomio, Phys. Rev. A 66, 052702 (2002)

$$f(y) = \frac{-2/\pi}{\pm\sqrt{\epsilon_2} - \sqrt{\epsilon_{3L} + \frac{3}{4}y^2}} \int_0^\infty dx x^2 \int_1^{-1} dz \left[\frac{1}{\epsilon_{3L} + y^2 + x^2 + xyz} - \frac{1}{1 + y^2 + x^2 + xyz}\right] f(x)$$

Virtual states – extension to the second Riemann sheet

M.T. Yamashita, T. Frederico, A. Delfino and L. Tomio, Phys. Rev. A 66, 052702 (2002)



Defining $h(\eta) \equiv \left(\epsilon_2 - \epsilon_{3L} - \frac{3}{4}y^2\right) f(\eta)$ we can write the bound state equation as

$$h(y) = -\frac{2}{\pi}\left(\sqrt{\epsilon_2} + \sqrt{\epsilon_{3B} + \frac{3}{4}y^2}\right)\int_0^\infty dx x^2 \int_1^{-1} dz \left[\frac{1}{\epsilon_{3B} + y^2 + x^2 + xyz} - \frac{1}{1 + y^2 + x^2 + xyz}\right]\frac{h(x)}{\epsilon_2 - \epsilon_{3B} - \frac{3}{4}x^2 + i\delta}$$

Then we can write the cut explicitly

$$\begin{aligned}
h_V(y) &= -\frac{2}{\pi}\left(\sqrt{\epsilon_2}+\sqrt{\epsilon_{3V}+\frac{3}{4}y^2}\right)\int_0^\infty dx x^2 \int_1^{-1} dz \left[\frac{1}{\epsilon_{3V}+y^2+x^2+xyz}\right. \\
&- \left.\frac{1}{1+y^2+x^2+xyz}\right]\left[\frac{1}{\epsilon_2-\epsilon_{3V}-\frac{3}{4}x^2+i\delta}-\frac{1}{\epsilon_2-\epsilon_{3V}-\frac{3}{4}x^2-i\delta}\right] h_V(x) \\
&- \frac{2}{\pi}\left(\sqrt{\epsilon_2}+\sqrt{\epsilon_{3V}+\frac{3}{4}y^2}\right)\int_0^\infty dx x^2 \int_1^{-1} dz \left[\frac{1}{\epsilon_{3V}+y^2+x^2+xyz}\right. \\
&- \left.\frac{1}{1+y^2+x^2+xyz}\right]\frac{h_V(x)}{\epsilon_2-\epsilon_{3V}-\frac{3}{4}x^2-i\delta}
\end{aligned}$$

After integration and defining $\epsilon_2 - \epsilon_{3V} \equiv \frac{3}{4}\kappa^2$

$$\kappa \equiv -i\sqrt{\tfrac{4}{3}(\epsilon_{3V}-\epsilon_2)} \equiv -i\kappa_V$$

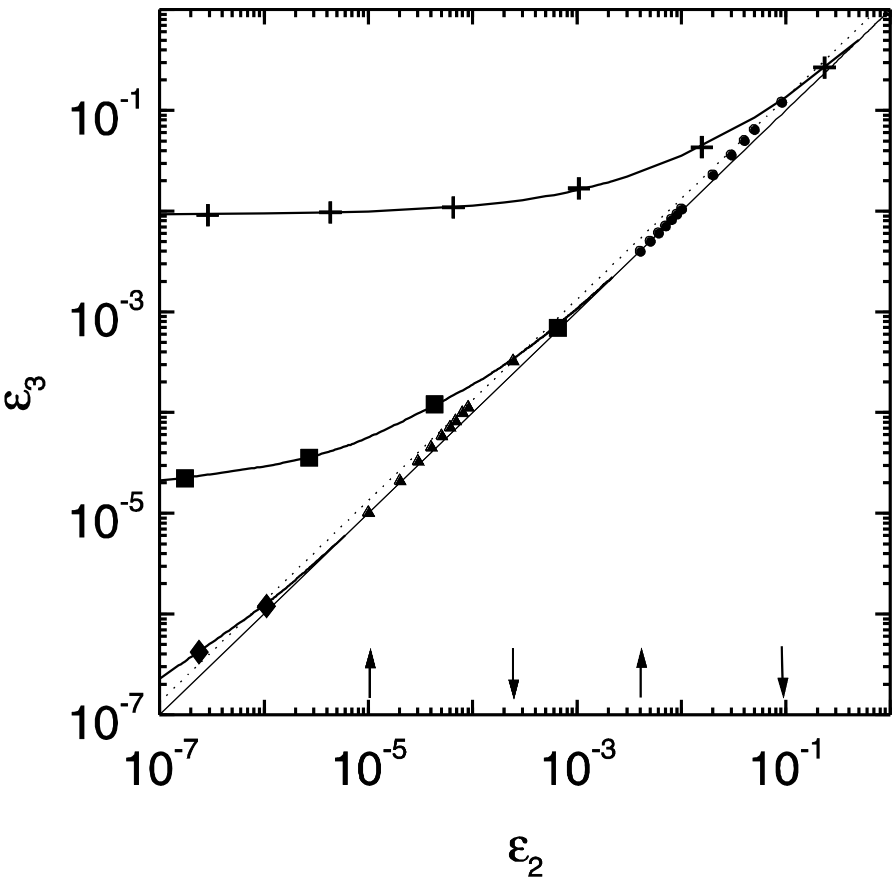
We have finally

$$h_V(y) = \frac{8}{3}\kappa_V\left(\sqrt{\epsilon_2}+\sqrt{\epsilon_{3V}+\frac{3}{4}y^2}\right) \times \int_1^{-1} dz\left[\frac{1}{\epsilon_{3V}+y^2-\kappa_V^2-i\kappa_V yz}-\frac{1}{1+y^2-\kappa_V^2-i\kappa_V yz}\right] h_V(-i\kappa_V)$$

$$- \frac{2}{\pi}\left(\sqrt{\epsilon_2}+\sqrt{\epsilon_{3V}+\frac{3}{4}y^2}\right)\int_0^\infty dx x^2\int_1^{-1} dz\left[\frac{1}{\epsilon_{3V}+y^2+x^2+xyz} - \frac{1}{1+y^2+x^2+xyz}\right]\frac{h_V(x)}{\epsilon_2-\epsilon_{3V}-\frac{3}{4}x^2}.$$

ϵ_{3V} should be outside the cut $\frac{4}{3}\epsilon_2 \le \epsilon_{3cut} \le 4\epsilon_2$ thus $\epsilon_2 \le \epsilon_{3V} \le \frac{4}{3}\epsilon_2$

Efimov states – Bound and virtual states



Lines – Bound states

crosses – ground

squares – first excited

diamonds – second excited

Symbols – Virtual states

circles - refers to the first excited state

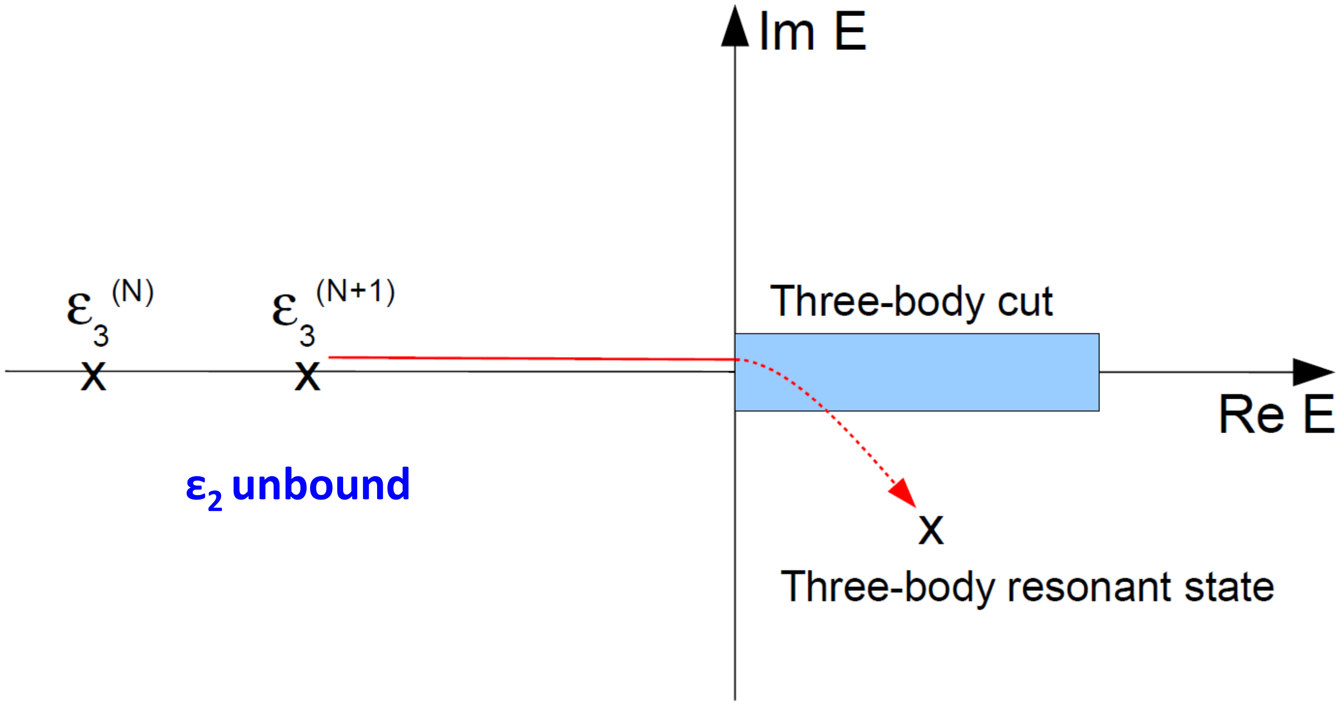
triangles – refers to the second excited state

Appearance of the virtual state (dotted line) $\varepsilon_3 = \frac{4}{3}\varepsilon_2$

The virtual state turns into an excited state (solid line) $\varepsilon_3 = \varepsilon_2$

<u>Resonances</u>

F. Bringas, M.T. Yamashita and T. Frederico Phys. Rev. A 69, 040702(R) (2004)



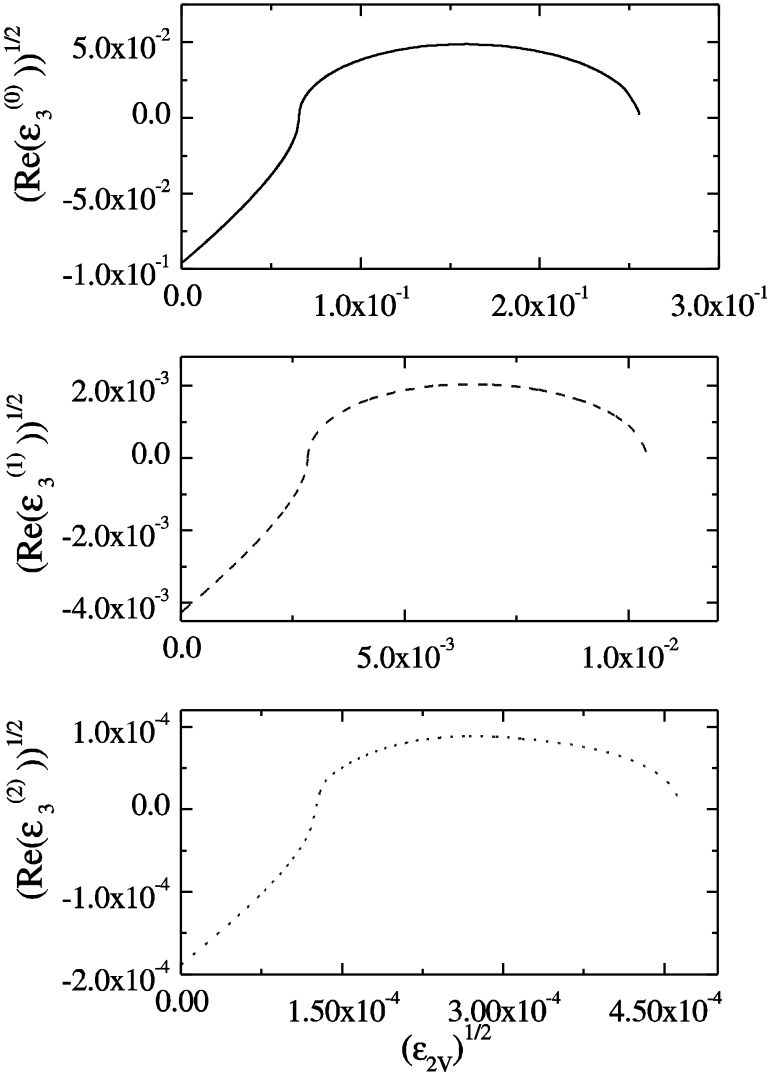
$$f(y) = \frac{-2/\pi}{-\sqrt{\epsilon_2} - \sqrt{\epsilon_{3R} + \frac{3}{4}y^2}} \int_0^\infty dx x^2 \int_1^{-1} dz \left[\frac{1}{\epsilon_{3R} + y^2 + x^2 + xyz} - \frac{1}{1 + y^2 + x^2 + xyz} \right] f(x)$$

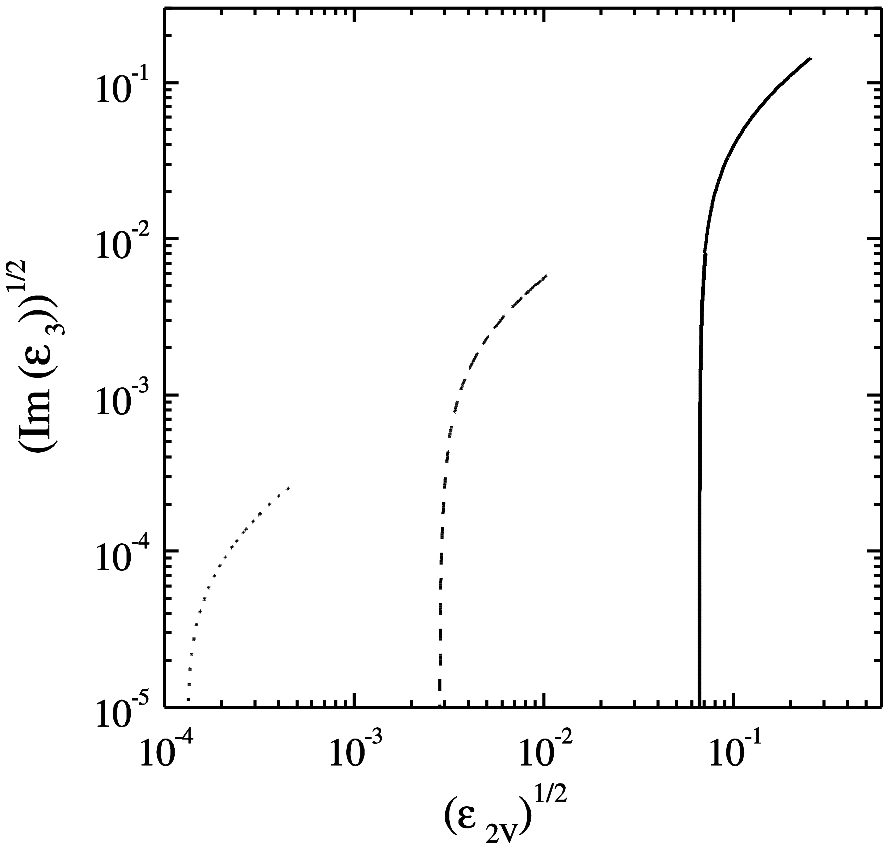
ϵ_{3R} is complex ⟵ ⟶

$$x \to xe^{-i\theta}$$

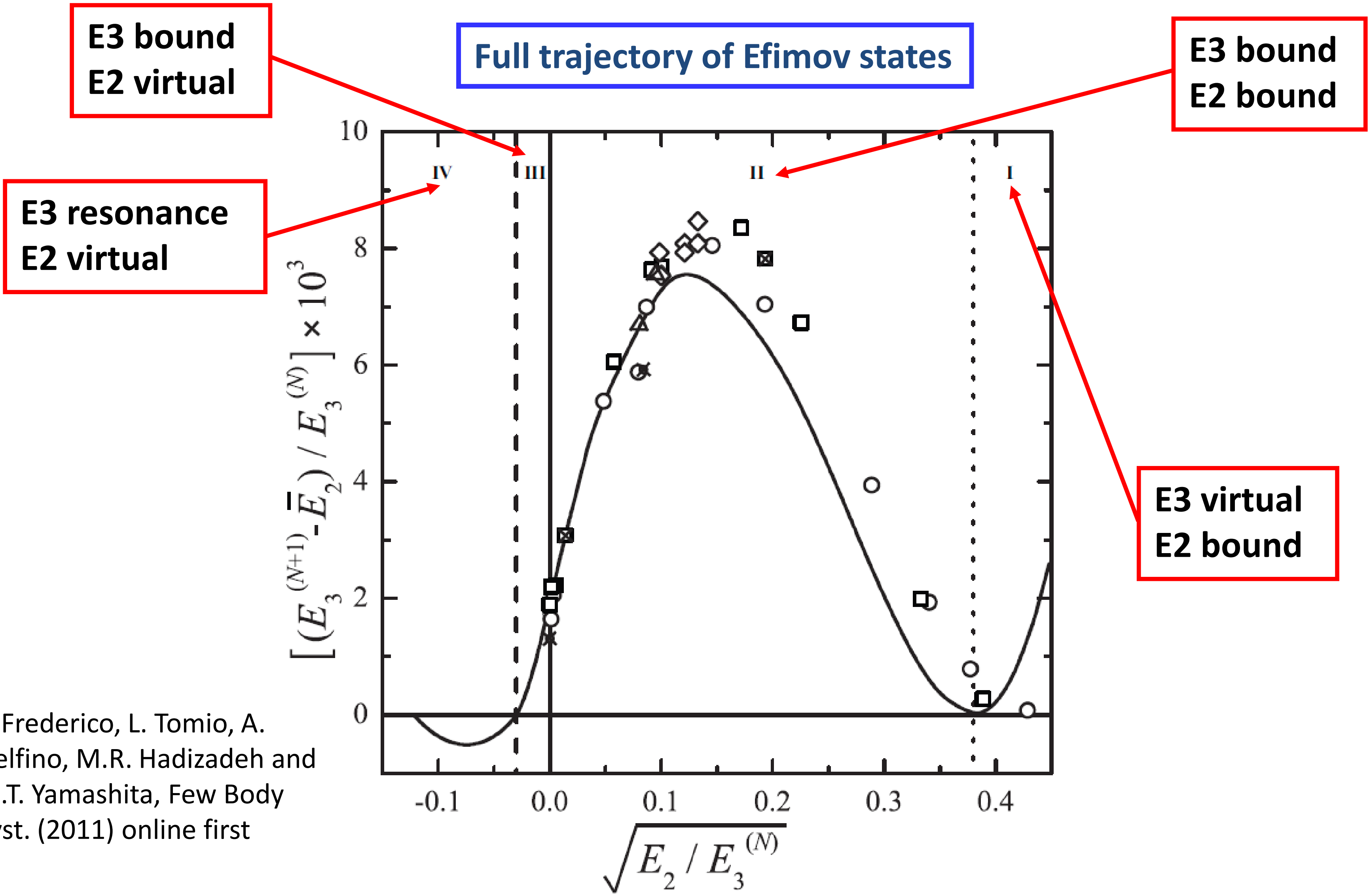
$$y \to ye^{-i\theta}$$

Efimov states - Resonances





ε_2 virtual



T. Frederico, L. Tomio, A. Delfino, M.R. Hadizadeh and M.T. Yamashita, Few Body Syst. (2011) online first

□ *s* wave (*N*=0) ○ *s*+*d* waves (*N*=0) ✖ *s* wave (*N*=1) Th. Cornelius, W. Glöckle. *J. Chem. Phys.* **85**, 1 (1996).

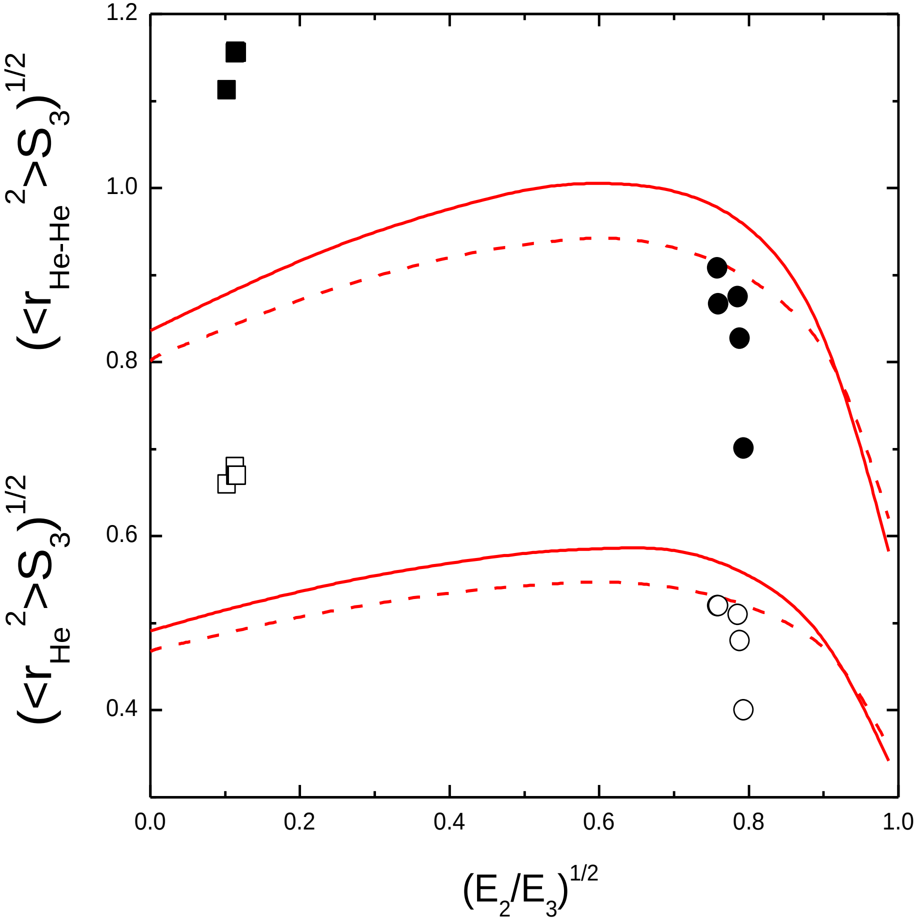
△ S. Huber. *Phys. Rev.* **A31**, 3981 (1985).

⊠ B. D. Esry, C. D. Lin, C. H. Greene. Phys. Rev. A 54, 394 (1996).

◇ E. A. Kolganova, A. K. Motovilov e S. A. Sofianos. *Phys. Rev.* **A56**, R1686 (1997).

Weakly-bound molecules – Helium trimer

M.T. Yamashita, R.S. Marques de Carvalho, L. Tomio and T. Frederico, Phys. Rev. A 68, 012506 (2003)



$$\sqrt{\left\langle r^2_{He-He} \right\rangle S_3} = R_{He-He}\left(\sqrt{\frac{E_2}{E_3}}\right)$$

$$S_3 = E_3 - E_2$$

Ground

First excited

Symbols from
P. Barletta and A. Kievsky
Phys. Rev. A **64**, 042514 (2001)

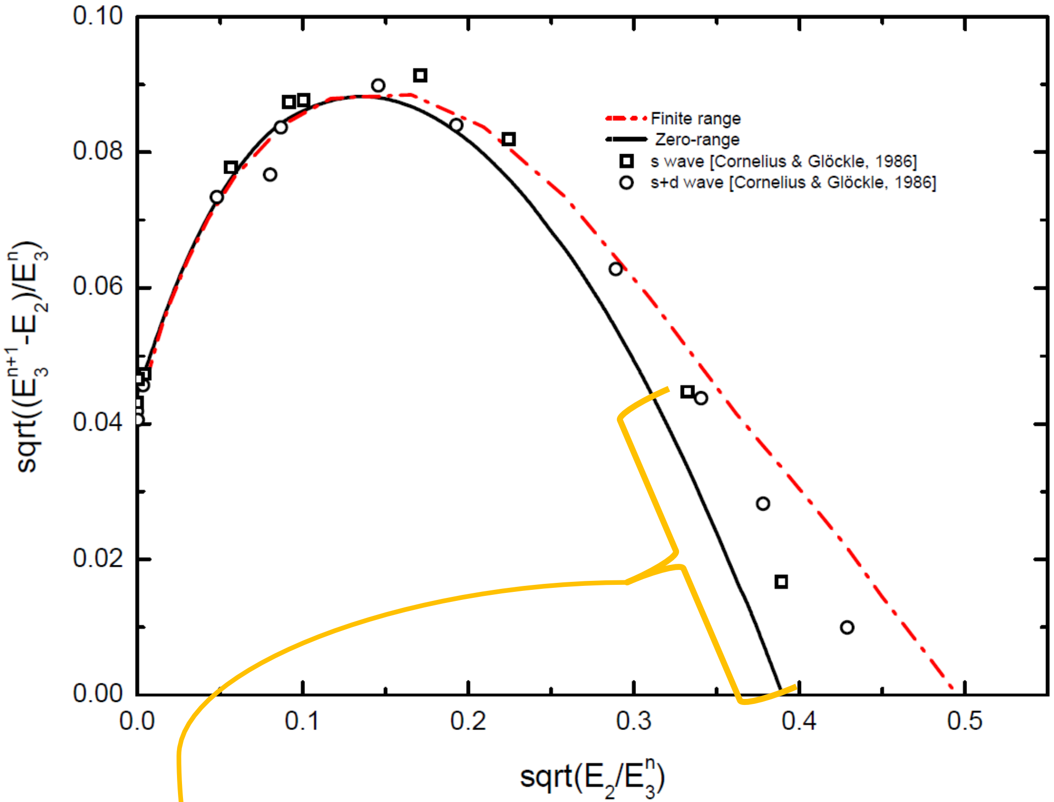
squares - Ground state
circles - First excited state

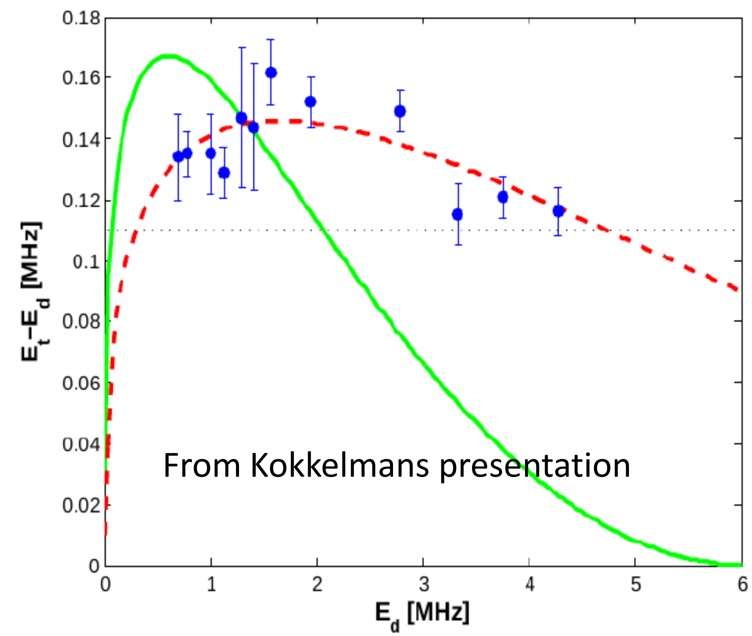
Potentials: HFDB, LM2M2, TTY, SAPT1, SAPT2

Range correction for bound states

D. S. Ventura, M.T. Yamashita, L. Tomio and T. Frederico, in preparation

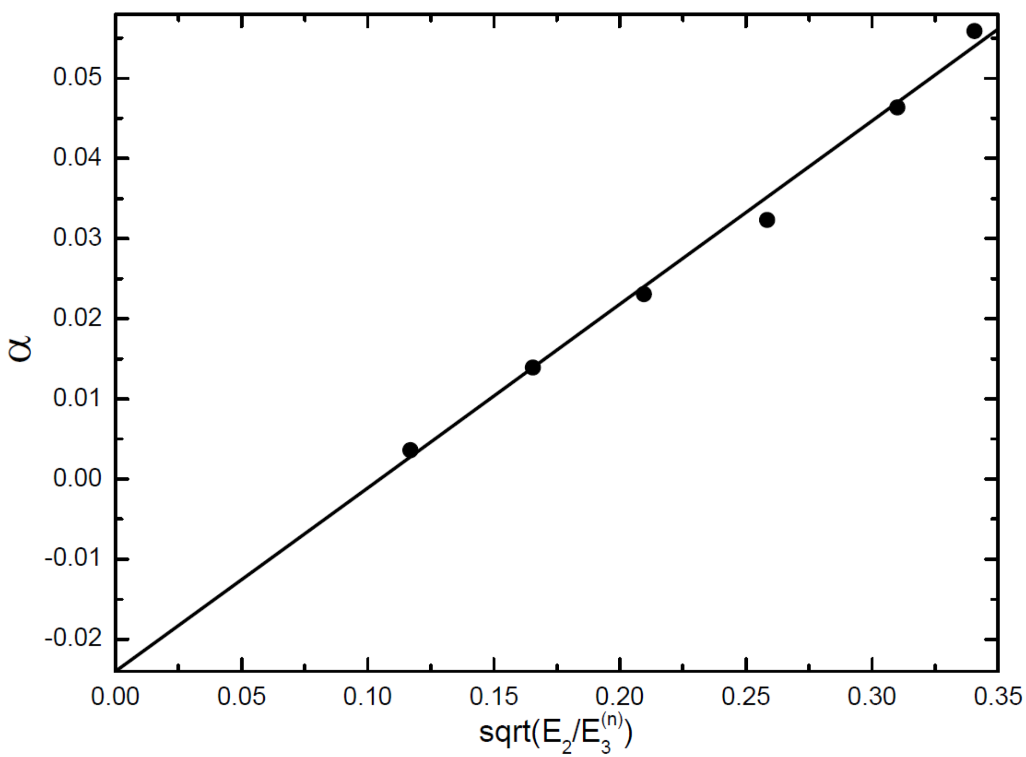
$$\sqrt{\frac{E_3^{(N+1)}-E_2}{E_3^{(N)}}}=\mathcal{F}\left(\sqrt{\frac{E_2}{E_3^{(N)}}}\right) \qquad \mathcal{F}\left(\sqrt{\frac{E_2}{E_3^{(N)}}};\frac{r_0}{a}\right)=\mathcal{F}\left(\sqrt{\frac{E_2}{E_3^{(N)}}};0\right)+\left.\frac{\partial\mathcal{F}}{\partial\frac{r_0}{a}}\right|_{\frac{r_0}{a}=0}\frac{r_0}{a}+\mathcal{O}\left(\left(\frac{r_0}{a}\right)^2\right)$$





$$\mathcal{F}\left(\sqrt{\frac{E_2}{E_3^{(n)}}};0\right)\cong 0.523\left(0.38-\sqrt{\frac{E_2}{E_3^{(n)}}}\right)$$

$$\alpha \equiv \frac{\partial \mathcal{F}}{\partial \frac{r_0}{a}} = \frac{\mathcal{F}\left(\sqrt{\frac{E_2}{E_3^{(N)}}}; \frac{r_0}{a}\right) - \mathcal{F}\left(\sqrt{\frac{E_2}{E_3^{(N)}}}; 0\right)}{\frac{r_0}{a}}$$



$$\frac{\partial \mathcal{F}}{\partial \frac{r_0}{a}} = -0.024 + 0.229\sqrt{\frac{E_2}{E_3^{(n)}}}$$

Point where an excited three-body state becomes virtual/bound

$$\left.\sqrt{\frac{E_2}{E_3^{(N)}}}\right|_{\mathrm{cut}} = 0.38 + 0.12\left(\frac{r_0}{a}\right) + \mathcal{O}\left(\left(\frac{r_0}{a}\right)^2\right)$$

PHYSICAL REVIEW C VOLUME 6, NUMBER 5 NOVEMBER 1972

Parametrization of the Three-Body D Function. II*

Sadhan K. Adhikari and R. D. Amado
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104
(Received 9 June 1972)

We have also studied the question of what happens to the Efimov states as they disappear into the two-body scattering threshold for coupling strengths above the critical values. We find they become virtual states and not resonances. Their

PHYSICAL REVIEW C VOLUME 19, NUMBER 3 MARCH 1979

Virtual state of the three nucleon system

B. A. Girard and M. G. Fuda
Department of Physics and Astronomy, State University of New York at Buffalo, Amherst, New York 14260
(Received 10 October 1978)

The existence of a virtual state of the three nucleon system is established on the basis of three different analyses. Values for its pole position and residue in the doublet, s-wave, n-d elastic scattering amplitude, are obtained from a fit to the experimental data, from partial wave dispersion relations, and from an exact three-particle, separable potential calculation. The calculations indicate that these parameters are determined mainly by the one-nucleon exchange mechanism and the doublet scattering length a_2. For $a_2 = 0.65$ fm our best calculation gives an energy of 0.482 MeV below the elastic threshold, on the second Riemann sheet, and a residue parameter $C_v^2 = 0.0504$, where C_v^2 is defined in analogy to the triton asymptotic normalization parameter.

PHYSICAL REVIEW C VOLUME 26, NUMBER 1 JULY 1982

Method for resonances and virtual states: Efimov virtual states

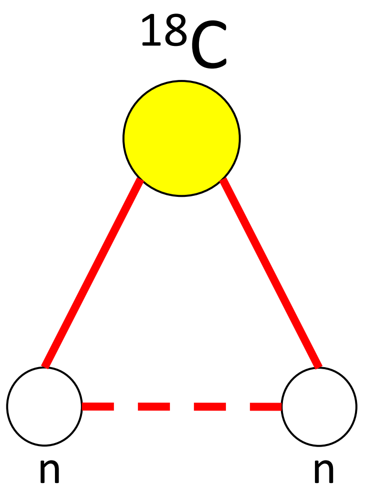
Sadhan K. Adhikari, António C. Fonseca,* and Lauro Tomio
Departamento de Física, Universidade Federal de Pernambuco, 50.000 Recife, PE, Brazil
(Received 1 December 1981)

A simple method is proposed for the calculation of the position and residue of low-energy resonances and virtual states located in the unphysical sheet associated with the lowest two body scattering threshold. Instead of analytically continuing the scattering equation for the t matrix into the unphysical sheet we propose an approximate solution of the scattering equation with known analytic properties and continue it analytically into the unphysical sheet. The poles of the analytically continued solution in the unphysical sheet correspond to virtual states and resonances. The present method is applied to study the $^1S_0$ virtual state of the two nucleon system, the Efimov virtual states in the three-boson Amado model, and the spin doublet virtual state of the three-nucleon system.

The transition bound-virtual does not depend on the particles mass ratio

M.T. Yamashita, T. Frederico and L. Tomio, Phys. Lett. B 660, 339 (2008); Phys. Rev. Lett. 99, 269201 (2007)

Example: ^{20}C (3.5 MeV)



n-^{18}C: 160 (110) keV

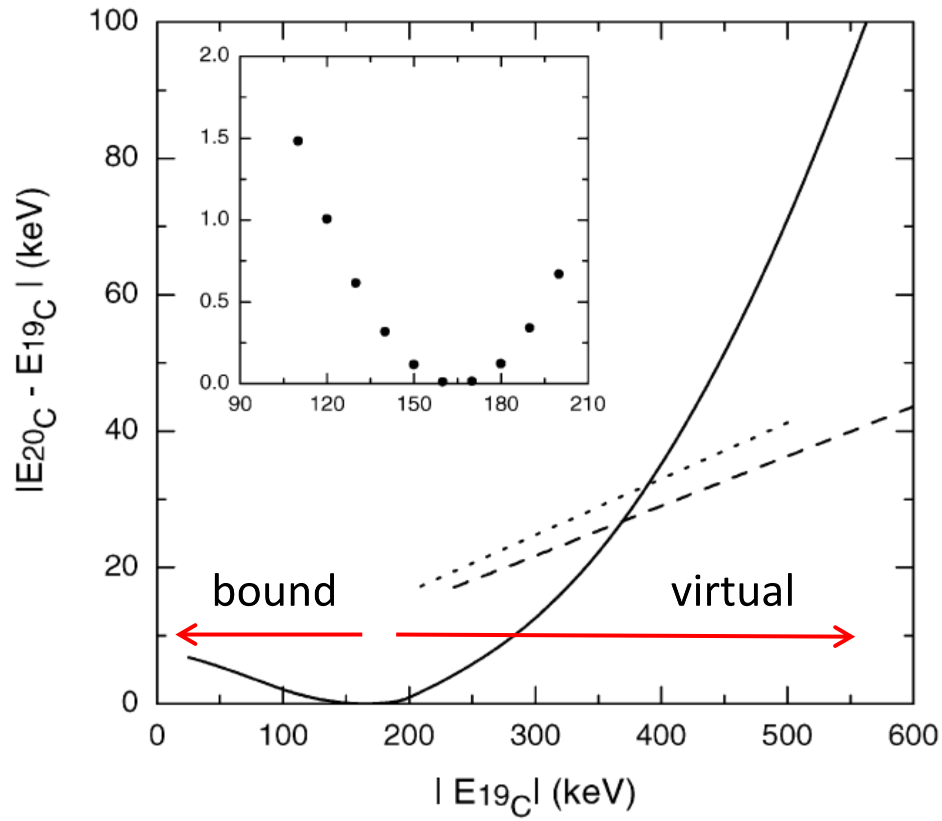
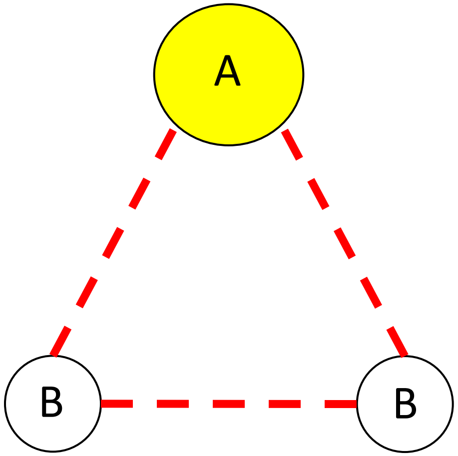


Fig. 2. Three-body n–n–^{18}C results for the first excited state, with respect to the threshold ($|E_{^{20}C} - E_{^{19}C}|$) for varying ^{19}C binding energies. Three-body bound (virtual) states occur when $|E_{^{19}C}|$ is approximately smaller (larger) than 170 keV. s-wave results (solid line) are also presented in the inset (with dots). Results for the p- and d-waves, divided by a factor 10, are shown with dashed and dotted lines, respectively.

Root mean square radii

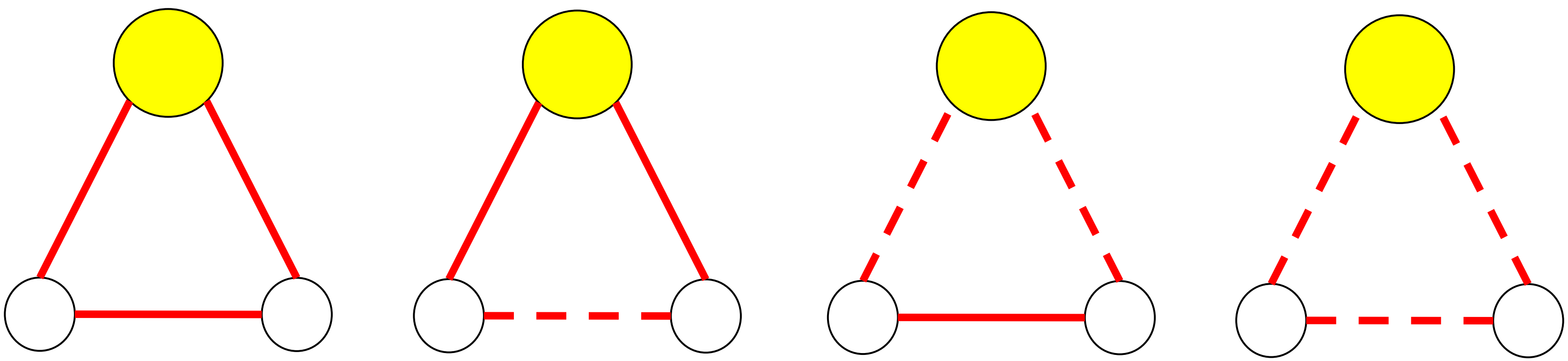


Scaling function for the radii

$$\sqrt{\left\langle r_{A\gamma}^{2} \right\rangle |E_3|} = R_{A\gamma}\left(\pm\sqrt{\frac{E_{AB}}{E_3}}, \pm\sqrt{\frac{E_{BB}}{E_3}}; M \right)$$

γ = *A* or *B*

+ two-body bound state
- two-body virtual state

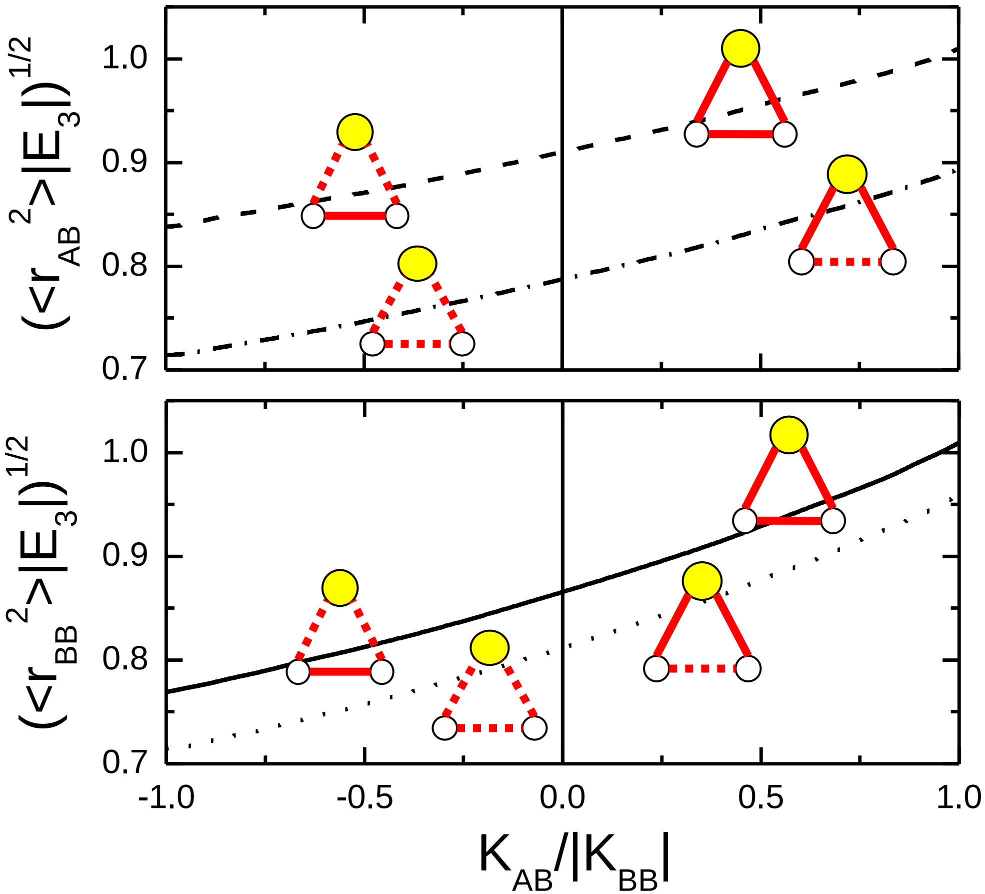


bound state

virtual state

Root mean square radii

- - - - BB bound
- · - · BB virtual



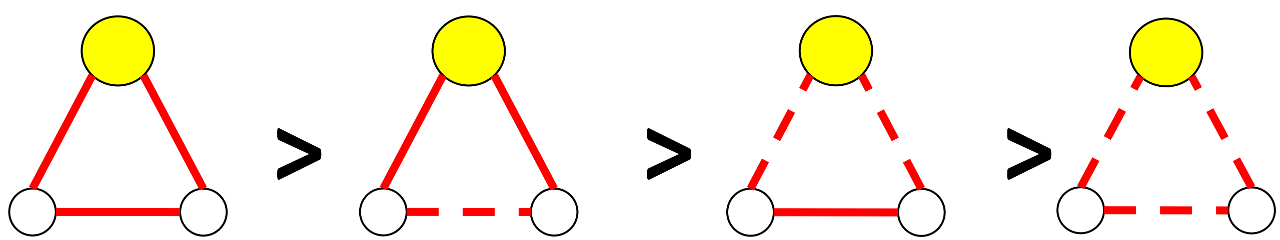
$$K_{AB}^2 = \frac{E_{AB}}{E_3} \to K_{AB} = \pm\sqrt{\frac{E_{AB}}{E_3}}$$

$$K_{BB}^2 = \frac{E_{BB}}{E_3} \to K_{BB} = \pm\sqrt{\frac{E_{BB}}{E_3}}$$

$$K_{BB}^2 = 0.1$$

$$M = 1$$

——— BB bound
·········· BB virtual



M.T. Yamashita, L. Tomio and T. Frederico, Nucl. Phys. A 735, 40 (2004)

Summary

- If at least one two-body subsystem is bound: Efimov state $\longrightarrow$ virtual

- All two-body subsystems are virtual (borromean case): Efimov state $\longrightarrow$ resonance

- Range correction for the point where an excited Efimov state disappears

$$\left.\sqrt{\frac{E_2}{E_3^{(N)}}}\right|_{\rm cut} = 0.38 + 0.12\left(\frac{r_0}{a}\right) + \mathcal{O}\left(\left(\frac{r_0}{a}\right)^2\right)$$

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Thank you!