Exotic atoms in two dimensions

available at *http://www.ipnl.in2p3.fr/perso/richard/SemConf/Talks.html*

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References

- J. of Phys. A: Math. Theor., **44**, 275302 (2011).
- Int. J. Mod. Phys. B, **21**, 3765-3781 (2007).

Table of contents

Introduction

Exotic atoms: Plateaux and rearrangement

- History: Coulomb + short-range, Zel'dovich, Shapiro, etc.

$$-1/r + \lambda\, V(r)$$

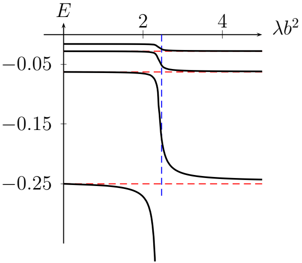
- Where $V(r)$ is very strong but with very short-range
- Energy shift δE small but non perturbative,
- Rearrangement if $V(r)$ attractive, when $\lambda \to$ coupling threshold for binding,
- Pattern explained by the Trueman–Deser formula

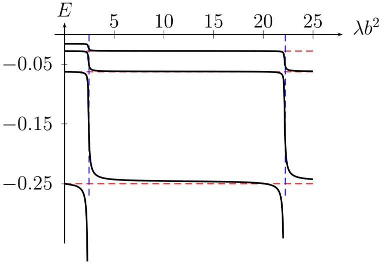
$$\delta E \propto a|\phi(o)|^2$$

- Holds beyond the case of a Coulomb interaction

Exotic atoms in 3D

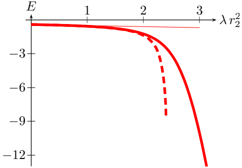
Coulomb + Square well

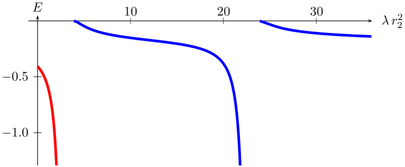




Results 3D

Sum of two square wells





Theory 3D

- Pattern recovered with a point interaction
- See Albeverio et al., Combescure et al.
- But only one “nuclear” bound state in this model

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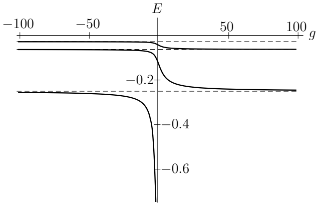
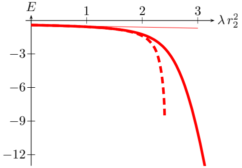
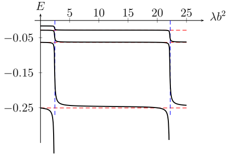


Theory 3D





Trueman-Deser

$$\delta E \propto a|\phi(o)|^2$$

- improves ordinary perturbation
- works for small a,
- indicates the trend for large a,
- works again when a becomes small (but positive)
- many corrections
  - Coulomb-corrected scattering length
  - Effective range terms
  - P-wave analogs (Partensky + Ericson)
  - etc.

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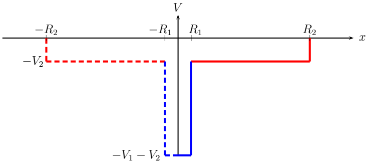
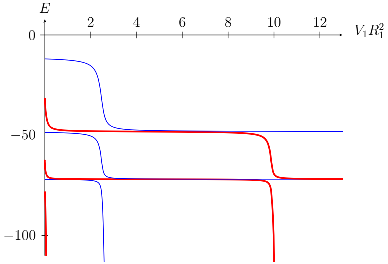
Exotic atoms in 1D

Symmetric case





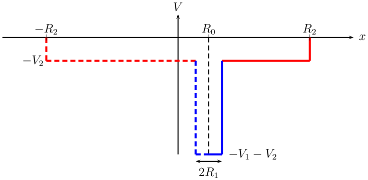
- Odd sector: analog to 3D
- Even sector: evolves to the odd one.
- Trueman–Deser formula now

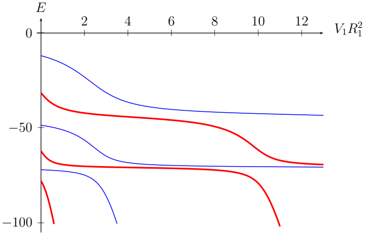
$$\delta E \propto |\phi(0)|^2 a$$

- as $\lambda = V2 - V1 \nearrow$, the ground-state drops immediately

Exotic atoms in 1D

Asymmetric case



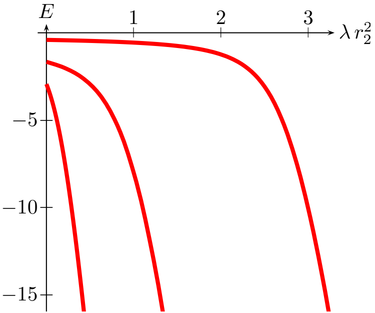


- as $\lambda = V2 - V1 \nearrow$, the ground-state drops immediately
- The spectrum evolves from one square well
- to two square wells
- and then can get rearranged

Exotic atoms in 2D

Comparison 1D–2D–3D

$$V = V_{LR} + \lambda\, V_{SR}$$

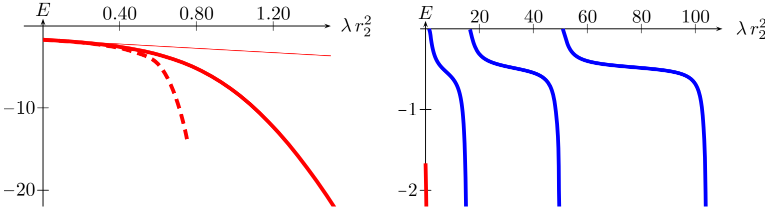


$d = 2$ last dimension for which a potential with $\int V(r)\mathrm{d}^d\vec{r}$, however weak, binds

Exotic atoms in 2D

Patterns in 2D

$$V = V_{LR} + \lambda\, V_{SR}$$



Left: Trueman–Deser, Exact, Simple perturbation

Exotic atoms in 2D
Trueman–Deser in 2D

3D

$$\delta E_n \propto |\phi_n(0)|^2\, a + \cdots$$

from, e.g., a Fermi effective contact interaction. n dependence.
a defined as zero of the $E = 0$ asymptotic wave function

$$k \cot\delta(k) = -\frac{1}{a} + \frac{1}{2} r_0^2\, k^2 + \cdots$$

1D

$$\delta E \propto \frac{|\phi(0)|^2}{a} + \cdots$$

2D

$$\phi(r) = \frac{1}{\sqrt{2\,\pi}} \frac{u(r)}{\sqrt{r}}\ , \qquad -u''(r) + V(r)\, u(r) = E\, u(r)\ ,$$

$$u_{\rm as} = \sqrt{r}\, \ln r/a\ , \ \text{at}\ E = 0$$

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Exotic atoms in 2D

Trueman–Deser in 2D

2D

$$\phi(r) = \frac{1}{\sqrt{2\,\pi}}\frac{u(r)}{\sqrt{r}}\ , \qquad -u''(r) + V(r)\,u(r) = E\,u(r)\ ,$$

$$u_{\rm as} = \sqrt{r}\,\ln r/a\ ,\ \text{at}\ E = 0$$

$$\cot\delta(k) = \frac{2}{\pi}\,[\ln(a\,k/2) + \gamma] + \frac{1}{2} r_0\,k^2 + \cdots$$

where r_0 is given by an integral similar the the 3D formula. For weak binding, this gives

$$E \propto a^{-2} \propto \exp(-A/g^2)\ ,$$

as discussed in other contributions.

Exotic atoms in 2D
Derivation of the Trueman formula

External potential alone

$$-u'' - \frac{u}{4\,r^2} + V_1\,u - E\,u = 0\,, \quad E = -k^2\,, \tag{1}$$

$u = h(E,r) \propto \sqrt{r}\,K_0(k\,r)$ at $r \to \infty$, normalized $\int_0^{+\infty} h(E,r)^2\,\mathrm{d}r = 1$
At small r, if E_0 = unperturbed energy and E = modified energy

$$h(E,r) = B(E)\,\sqrt{r}\,\ln r + A(E)\,\sqrt{r} + \cdots\,,$$
$$h(E_0,r) = A_0\,\sqrt{r} + \cdots \quad A_0 = A(E_0) > 0\,, \quad B(E_0) = 0\,.$$

Combining (1) for E and E_0 gives

$$A_0\,B(E) = \underbrace{(E - E_0)}_{\delta E} \int_0^{+\infty} h(E,r)\,h(E_0,r)\,\mathrm{d}r = \delta E\left[1 + \mathcal{O}(\delta E)^2\right]$$

Exotic atoms in 2D

Derivation of the Trueman formula

$$\delta E = A_0\, B(E)$$

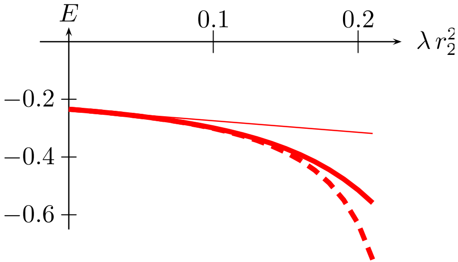
Now, if $A(E) = A_0 + \tilde{A}_0\, \delta E + \cdots$,
and if $h(E, r) = B(E)\sqrt{r}\ln r + A(E)\sqrt{r} + \cdots$ is compared to $h(E, r)$ with $\sqrt{r}\,[\ln r - \ln a]$ as provided by the asymptotic in a short-range interaction, one gets

$$\frac{B}{1} = \frac{A_0 + \tilde{A}_0\, \delta E}{-\ln a}\,,$$

which when combined with $B \simeq \delta E / A_0$ gives

$$\boxed{\delta E \simeq -\frac{A_0^2}{\ln a + A_0\, \tilde{A}_0}\,.}$$

Example



Exponential well supplemented by another exponential of shorter range. Thick line: exact, dashed line: SL formula, thin line: perturbation theory. We use here
$V = -g_1 \exp(-r/r1) - \lambda \exp(-r/r_2)$ with $r_1 = 2$, $g_1 = 1$ and
$r_2 = 0.02$.

Summary

- $V(r) = V_{\text{ext}} + V_{\text{int}}$
- $$\delta E = \begin{cases} A_0^2/a & (d=1) \\ A_0^2/\ln(a/R) & (d=2) \\ A_0^2\, a & (d=3) \end{cases}$$
- where in V_{ext} the reduced wave function is $u(r) = A_0\{1, \sqrt{r}, r, \ldots\}$
- i.e., $A_0 \propto \phi(0)$
- a scattering length in V_{int}
- factorisation int–ext for $d = 1$ and $d = 3$,
- $d = 2$ no factorisation, but R given by the external potential
- link to the Fermi potential and thus to the statistical mechanics of boson systems to be clarified.

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