UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Stockholm University

S. A. Rakityansky and N. Elander

ANALYTIC STRUCTURE AND POWER-SERIES EXPANSION OF THE JOST MATRIX

Taylor-type power-series expansion in scattering theory:

$$k^{2\ell+1}\cot\delta_\ell(k) = \sum_{n=0}^{\infty} c_{\ell n} k^{2n}$$

short-range potential

Effective-range expansion

$\ell = 0$

$$k\cot\delta_0(k) = -\frac{1}{a} + \frac{1}{2}r_0 k^2 - P r_0^3 k^4 + Q r_0^5 k^6 + \cdots$$

Scattering length (a) — **Effective radius** (r_0)

Limitations :

- valid near ***k=0*** (low energies)
- single-channel problem
- three-dimensional problems

Present work → generalization →

- **Expansion near any complex *E***
- ***N*-channel problem**
- **2D-problems**

Single-channel problem

$$V(r) \underset{r\to\infty}{\longrightarrow} 0$$ exponentially

$$\left[\partial_r^2 + k^2 - \frac{\ell(\ell+1)}{r^2}\right] u_\ell(k,r) = V(r) u_\ell(k,r)$$

$r \to \infty$ ⇒

$$\left[\partial_r^2 + k^2 - \frac{\ell(\ell+1)}{r^2}\right] u_\ell(k,r) \simeq 0$$

Independent solutions →

$$j_\ell(kr), \quad y_\ell(kr), \quad h_\ell^{(+)}(kr), \quad h_\ell^{(-)}(kr)$$

General solution →

$$u_\ell(k,r) \underset{r\to\infty}{\longrightarrow} C_1 h_\ell^{(-)}(kr) + C_2 h_\ell^{(+)}(kr)$$

Incoming and outgoing spherical waves

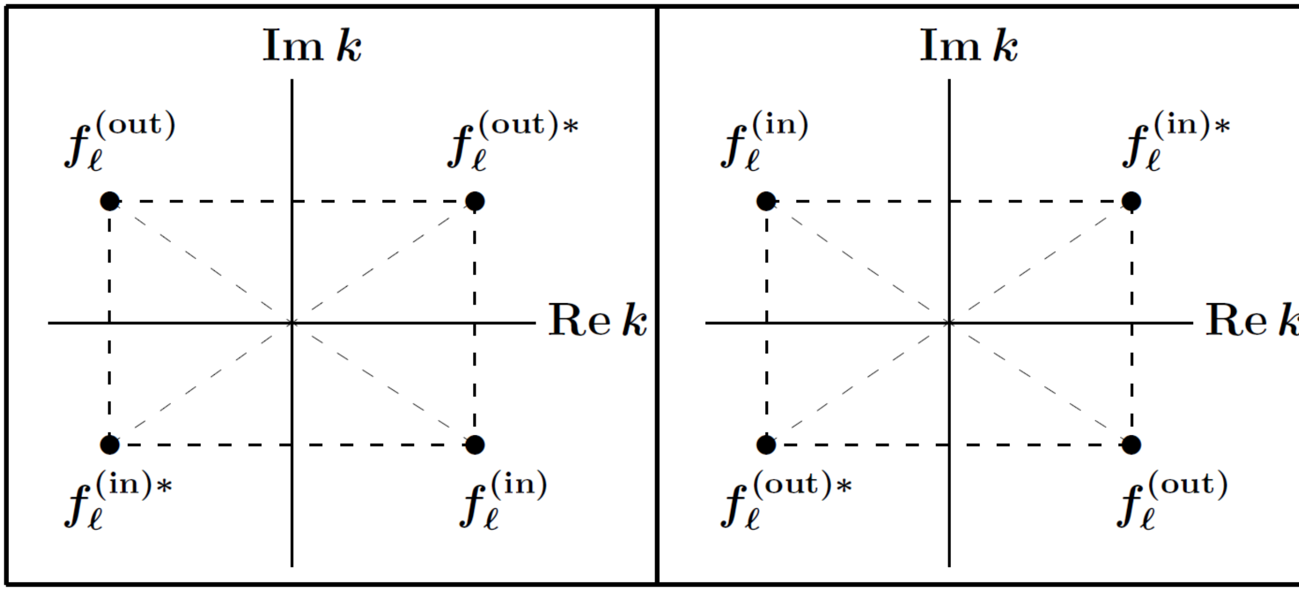
$$h_\ell^{(-)}(kr) \underset{|kr|\to\infty}{\longrightarrow} +i \exp\left(-ikr + i\frac{\ell\pi}{2}\right)$$

$$h_\ell^{(+)}(kr) \underset{|kr|\to\infty}{\longrightarrow} -i \exp\left(+ikr - i\frac{\ell\pi}{2}\right)$$

$$u_\ell(k,r) \underset{r\to\infty}{\longrightarrow} f_\ell^{(\mathrm{in})}(k)h_\ell^{(-)}(kr) + f_\ell^{(\mathrm{out})}(k)h_\ell^{(+)}(kr)$$

Jost functions

Symmetry properties



Scattering

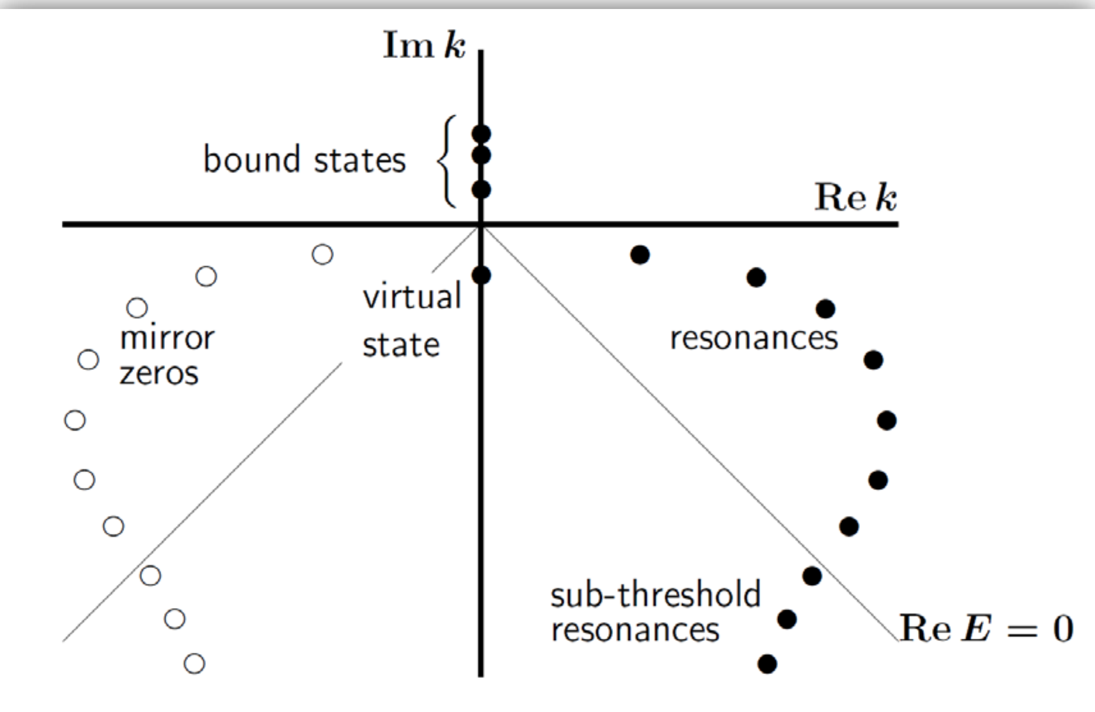
$$s_\ell(k) = \frac{f_\ell^{(\mathrm{out})}(k)}{f_\ell^{(\mathrm{in})}(k)}$$

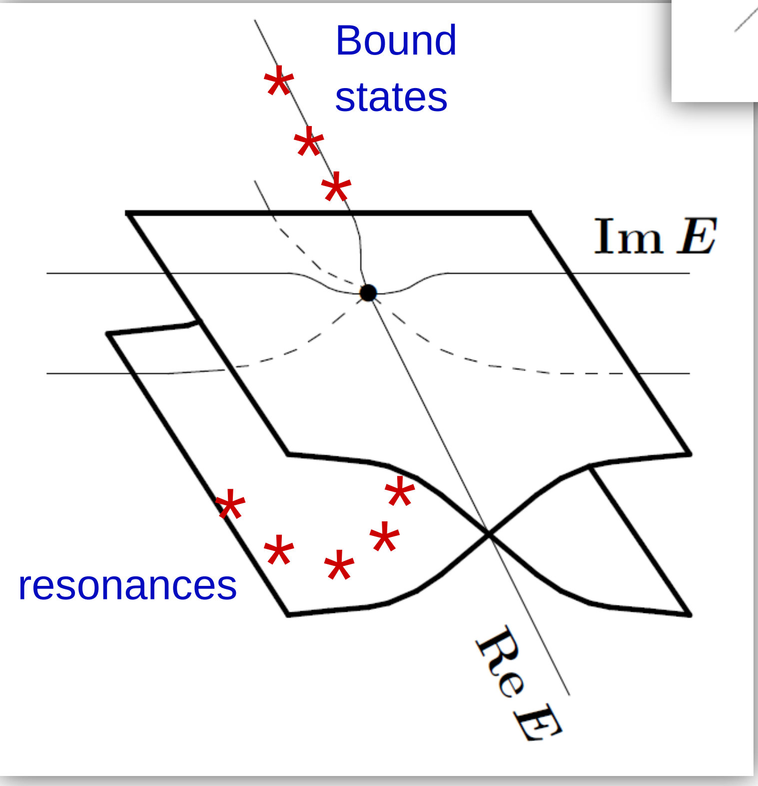
Spectral points

$$f_\ell^{(\mathrm{in})}(k_n) = 0$$

Spectral points

$$f_\ell^{(\mathrm{in})}(k_n) = 0$$





$$f_\ell^{(\mathrm{in/out})}(k) \rightarrow f_\ell^{(\mathrm{in/out})}(E)$$

$$k = \pm\sqrt{\frac{2mE}{\hbar^2}}$$

How can we calculate the Jost functions for a given potential ?

$$u_\ell(E,r) \underset{r\to\infty}{\longrightarrow} f_\ell^{(\mathrm{in})}(E) h_\ell^{(-)}(kr) + f_\ell^{(\mathrm{out})}(E) h_\ell^{(+)}(kr)$$

$$u_\ell(E,r) = F_\ell^{(\mathrm{in})}(E,r) h_\ell^{(-)}(kr) + F_\ell^{(\mathrm{out})}(E,r) h_\ell^{(+)}(kr)$$

$$V(r>R) \equiv 0 \quad \Rightarrow \quad F_\ell^{(\mathrm{in/out})}(E,R) = f_\ell^{(\mathrm{in/out})}(E)$$

$$\lim_{r\to\infty} F_\ell^{(\mathrm{in/out})}(E,r) = f_\ell^{(\mathrm{in/out})}(E)$$

Schrödinger equation

$$\partial_r F_\ell^{(\mathrm{in})} = -\frac{1}{2ik} h_\ell^{(+)} V \left[F_\ell^{(\mathrm{in})} h_\ell^{(-)} + F_\ell^{(\mathrm{out})} h_\ell^{(+)} \right]$$

$$\partial_r F_\ell^{(\mathrm{out})} = \frac{1}{2ik} h_\ell^{(-)} V \left[F_\ell^{(\mathrm{in})} h_\ell^{(-)} + F_\ell^{(\mathrm{out})} h_\ell^{(+)} \right]$$

Boundary conditions

$$F_\ell^{(\mathrm{in})}(E,0) = 1/2$$

$$F_\ell^{(\mathrm{out})}(E,0) = 1/2$$

Factorization of the branching point

$f_\ell^{(\mathrm{in})}(E)$ and $f_\ell^{(\mathrm{out})}(E)$ are NOT single valued functions of ***E***

because they involve **ODD POWERS** of $k = \pm\sqrt{\dfrac{2mE}{\hbar^2}}$

$$\begin{aligned} A_\ell &= F_\ell^{(\mathrm{in})} + F_\ell^{(\mathrm{out})} \\ B_\ell &= i\left[F_\ell^{(\mathrm{in})} - F_\ell^{(\mathrm{out})}\right] \end{aligned} \quad \longleftrightarrow \quad \begin{aligned} F_\ell^{(\mathrm{in})} &= (A_\ell - iB_\ell)/2 \\ F_\ell^{(\mathrm{out})} &= (A_\ell + iB_\ell)/2 \end{aligned}$$

$$\begin{aligned} \partial_r A_\ell &= -\frac{1}{k} y_\ell V \left[A_\ell j_\ell - B_\ell y_\ell\right] \\ \partial_r B_\ell &= -\frac{1}{k} j_\ell V \left[A_\ell j_\ell - B_\ell y_\ell\right] \end{aligned}$$

$$\begin{aligned} A_\ell(E,0) &= 1 \\ B_\ell(E,0) &= 0 \end{aligned}$$

Absolutely convergent power-series expansions for the Riccati-Bessel and Riccati-Neumann functions

$$j_\ell(kr) = \left(\frac{kr}{2}\right)^{\ell+1} \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{\pi}}{\Gamma(\ell + 3/2 + n) n!} \left(\frac{kr}{2}\right)^{2n} = k^{\ell+1} \tilde{j}_\ell(E, r)$$

Possible odd powers of ***k***

$$y_\ell(kr) = \left(\frac{2}{kr}\right)^{\ell} \sum_{n=0}^{\infty} \frac{(-1)^{n+\ell+1}}{\Gamma(-\ell + 1/2 + n) n!} \left(\frac{kr}{2}\right)^{2n} = k^{-\ell} \tilde{y}_\ell(E, r)$$

$$j_\ell(kr) = k^{\ell+1}\tilde{j}_\ell(E,r)$$

$$y_\ell(kr) = k^{-\ell}\tilde{y}_\ell(E,r)$$

Absolutely convergent series

Holomorphic functions

$$\tilde{j}_\ell(E,r) = \left(\frac{r}{2}\right)^{\ell+1} \sum_{n=0}^{\infty} \frac{(-1)^n\sqrt{\pi}}{\Gamma(\ell+3/2+n)n!}\left(\frac{kr}{2}\right)^{2n}$$

$$\tilde{y}_\ell(E,r) = \left(\frac{2}{r}\right)^{\ell} \sum_{n=0}^{\infty} \frac{(-1)^{n+\ell+1}}{\Gamma(-\ell+1/2+n)n!}\left(\frac{kr}{2}\right)^{2n}$$

$$\tilde{A}(E,r) = A_\ell(E,r)$$

$$\tilde{B}(E,r) = k^{-(2\ell+1)}B_\ell(E,r)$$

$$\partial_r \tilde{A}_\ell = -\tilde{y}_\ell V\left[\tilde{A}_\ell\tilde{j}_\ell - \tilde{B}_\ell\tilde{y}_\ell\right]$$

$$\partial_r \tilde{B}_\ell = -\tilde{j}_\ell V\left[\tilde{A}_\ell\tilde{j}_\ell - \tilde{B}_\ell\tilde{y}_\ell\right]$$

Boundary conditions

$$\tilde{A}_\ell(E,0) = 1$$

$$\tilde{B}_\ell(E,0) = 0$$

Poincare theorem ⟹ $\tilde{A}_\ell(E,r),\ \tilde{B}_\ell(E,r)$ holomorphic functions of ***E***

$$f_\ell^{(\mathrm{in})}(E) = \lim_{r\to\infty} \frac{1}{2}\left[\tilde{A}_\ell(E,r) - ik^{2\ell+1}\tilde{B}_\ell(E,r)\right]$$

$$f_\ell^{(\mathrm{out})}(E) = \lim_{r\to\infty} \frac{1}{2}\left[\tilde{A}_\ell(E,r) + ik^{2\ell+1}\tilde{B}_\ell(E,r)\right]$$

$$\tilde{a}_\ell(E) = \lim_{r\to\infty} \tilde{A}_\ell(E,r)$$

$$\tilde{b}_\ell(E) = \lim_{r\to\infty} \tilde{B}_\ell(E,r)$$

$$f_\ell^{(\mathrm{in})}(E) = \frac{1}{2}\left[\tilde{a}_\ell(E) - ik^{2\ell+1}\tilde{b}_\ell(E)\right]$$

$$f_\ell^{(\mathrm{out})}(E) = \frac{1}{2}\left[\tilde{a}_\ell(E) + ik^{2\ell+1}\tilde{b}_\ell(E)\right]$$

Single-valued, analytic functions

$$\tilde{a}_\ell(E) = \sum_{n=0}^{\infty} \alpha_n (E-E_0)^n$$

$$\tilde{b}_\ell(E) = \sum_{n=0}^{\infty} \beta_n (E-E_0)^n$$

$$\tilde{A}_\ell(E,r) = \sum_{n=0}^{\infty} \mathcal{A}_n(r)(E-E_0)^n$$

$$\tilde{B}_\ell(E,r) = \sum_{n=0}^{\infty} \mathcal{B}_n(r)(E-E_0)^n$$

$$\tilde{j}_\ell(E,r) = \sum_{n=0}^{\infty} \mathcal{J}_n(r)(E-E_0)^n$$

$$\tilde{y}_\ell(E,r) = \sum_{n=0}^{\infty} \mathcal{Y}_n(r)(E-E_0)^n$$

At any fixed ***r***, these functions are also analytic

$$\partial_r \tilde{A}_\ell = -\tilde{y}_\ell V \left[\tilde{A}_\ell \tilde{j}_\ell - \tilde{B}_\ell \tilde{y}_\ell\right]$$

$$\partial_r \tilde{B}_\ell = -\tilde{j}_\ell V \left[\tilde{A}_\ell \tilde{j}_\ell - \tilde{B}_\ell \tilde{y}_\ell\right]$$

$$\partial_r \mathcal{A}_n = -\sum_{i+j+k=n} \mathcal{Y}_i V (\mathcal{A}_j \mathcal{J}_k - \mathcal{B}_j \mathcal{Y}_k)$$

$$\partial_r \mathcal{B}_n = -\sum_{i+j+k=n} \mathcal{J}_i V (\mathcal{A}_j \mathcal{J}_k - \mathcal{B}_j \mathcal{Y}_k)$$

Boundary conditions at ***r = 0***

$$\mathcal{A}_n(0) = \delta_{n0}$$

$$\mathcal{B}_n(0) = 0$$

$$\partial_r \mathcal{A}_n = - \sum_{i+j+k=n} \mathcal{Y}_i V (\mathcal{A}_j \mathcal{J}_k - \mathcal{B}_j \mathcal{Y}_k)$$

$$\partial_r \mathcal{B}_n = - \sum_{i+j+k=n} \mathcal{J}_i V (\mathcal{A}_j \mathcal{J}_k - \mathcal{B}_j \mathcal{Y}_k)$$

$$i + j + k = n$$

$$\partial_r \mathcal{A}_0 = -\mathcal{Y}_0 V (\mathcal{A}_0 \mathcal{J}_0 - \mathcal{B}_0 \mathcal{Y}_0)$$

$$\partial_r \mathcal{B}_0 = -\mathcal{J}_0 V (\mathcal{A}_0 \mathcal{J}_0 - \mathcal{B}_0 \mathcal{Y}_0)$$

$$\partial_r \mathcal{A}_1 = -\mathcal{Y}_1 V (\mathcal{A}_0 \mathcal{J}_0 - \mathcal{B}_0 \mathcal{Y}_0) - \mathcal{Y}_0 V (\mathcal{A}_1 \mathcal{J}_0 - \mathcal{B}_1 \mathcal{Y}_0) - \mathcal{Y}_0 V (\mathcal{A}_0 \mathcal{J}_1 - \mathcal{B}_0 \mathcal{Y}_1)$$

$$\partial_r \mathcal{B}_1 = -\mathcal{J}_1 V (\mathcal{A}_0 \mathcal{J}_0 - \mathcal{B}_0 \mathcal{Y}_0) - \mathcal{J}_0 V (\mathcal{A}_1 \mathcal{J}_0 - \mathcal{B}_1 \mathcal{Y}_0) - \mathcal{J}_0 V (\mathcal{A}_0 \mathcal{J}_1 - \mathcal{B}_0 \mathcal{Y}_1)$$

$$R \to \infty \qquad \alpha_n = \mathcal{A}_n(R) \qquad \beta_n = \mathcal{B}_n(R)$$

Choice of the Riemann sheet

$$k = \pm \sqrt{\frac{2mE}{\hbar^2}}$$

$$f_\ell^{(\mathrm{in})}(E) \approx \frac{1}{2} \sum_{n=0}^{M} \left[\alpha_n - i k^{2\ell+1} \beta_n\right] (E - E_0)^n$$

$$f_\ell^{(\mathrm{out})}(E) \approx \frac{1}{2} \sum_{n=0}^{M} \left[\alpha_n + i k^{2\ell+1} \beta_n\right] (E - E_0)^n$$

Standard effective-range expansion

$$E_0 = 0$$

$$f_\ell^{(\mathrm{in})} \approx \frac{1}{2}\sum_{n=0}^{M}\left(\alpha_n - ik^{2\ell+1}\beta_n\right)E^n$$

$$f_\ell^{(\mathrm{out})} \approx \frac{1}{2}\sum_{n=0}^{M}\left(\alpha_n + ik^{2\ell+1}\beta_n\right)E^n$$

$$f_\ell^{(\mathrm{in})} = e^{-i\delta_\ell}$$

$$f_\ell^{(\mathrm{out})} = e^{+i\delta_\ell}$$

$$f_\ell^{(\mathrm{out})} + f_\ell^{(\mathrm{in})} = 2\cos\delta_\ell$$

$$f_\ell^{(\mathrm{out})} - f_\ell^{(\mathrm{in})} = 2i\sin\delta_\ell$$

$$\cot\delta_\ell = \frac{\alpha_0 + \alpha_1 E + \alpha_2 E^2 + \cdots}{k^{2\ell+1}(\beta_0 + \beta_1 E + \beta_2 E^2 + \cdots)}$$

$$k^{2\ell+1}\cot\delta_\ell = \frac{\alpha_0}{\beta_0} + \left(\frac{\alpha_1}{\beta_0} - \frac{\alpha_0\beta_1}{\beta_0^2}\right)E + \cdots$$

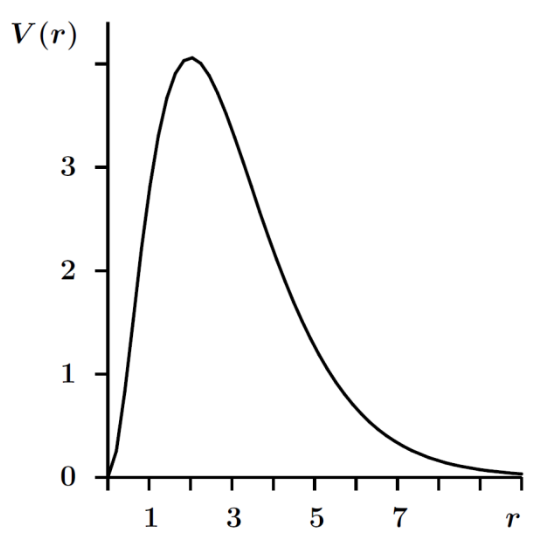
S-wave

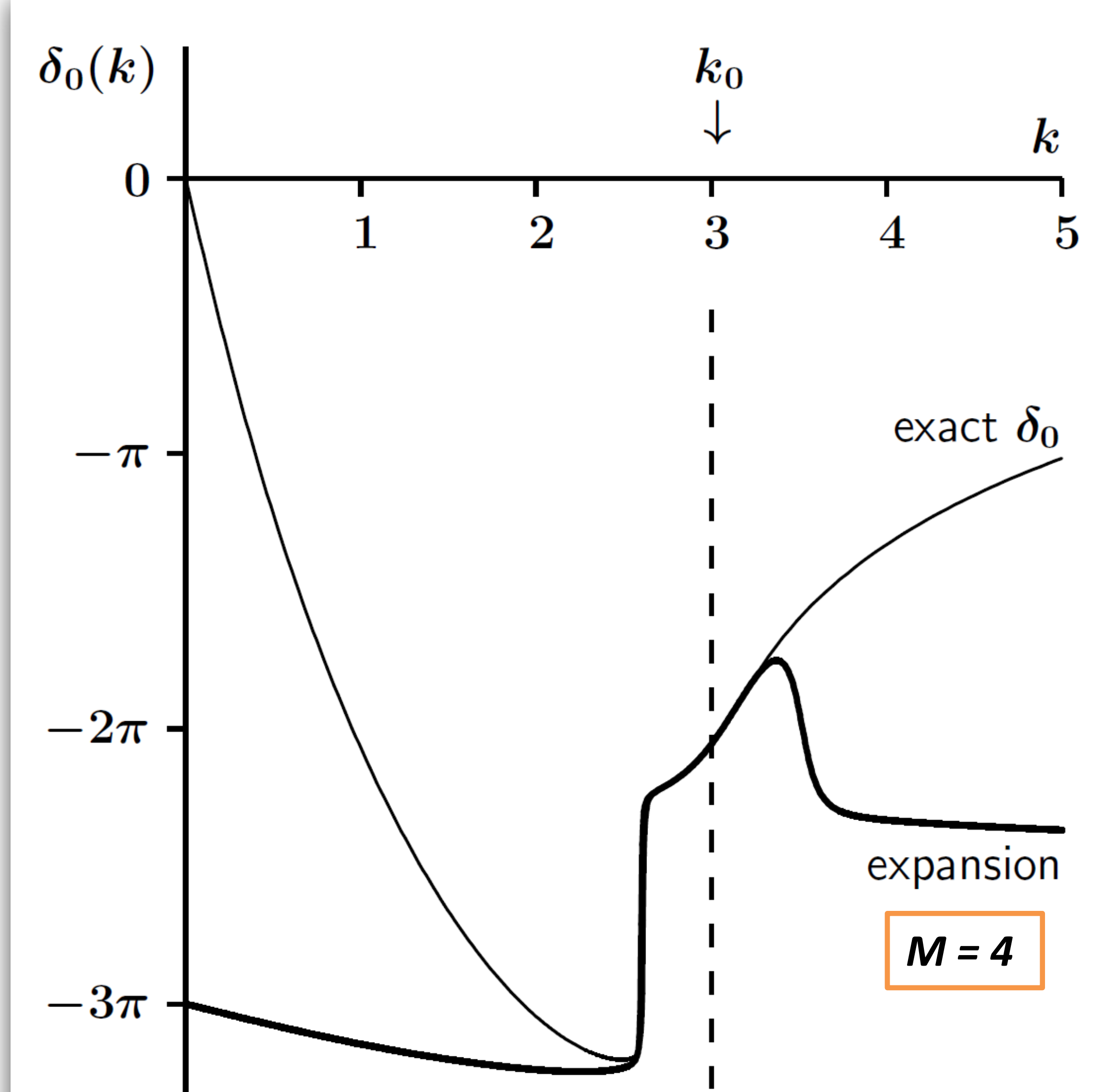
$$k\cot\delta_0(k) = -\frac{1}{a} + \frac{1}{2}r_0k^2 - Pr_0^3k^4 + Qr_0^5k^6 + \cdots$$

Numerical example

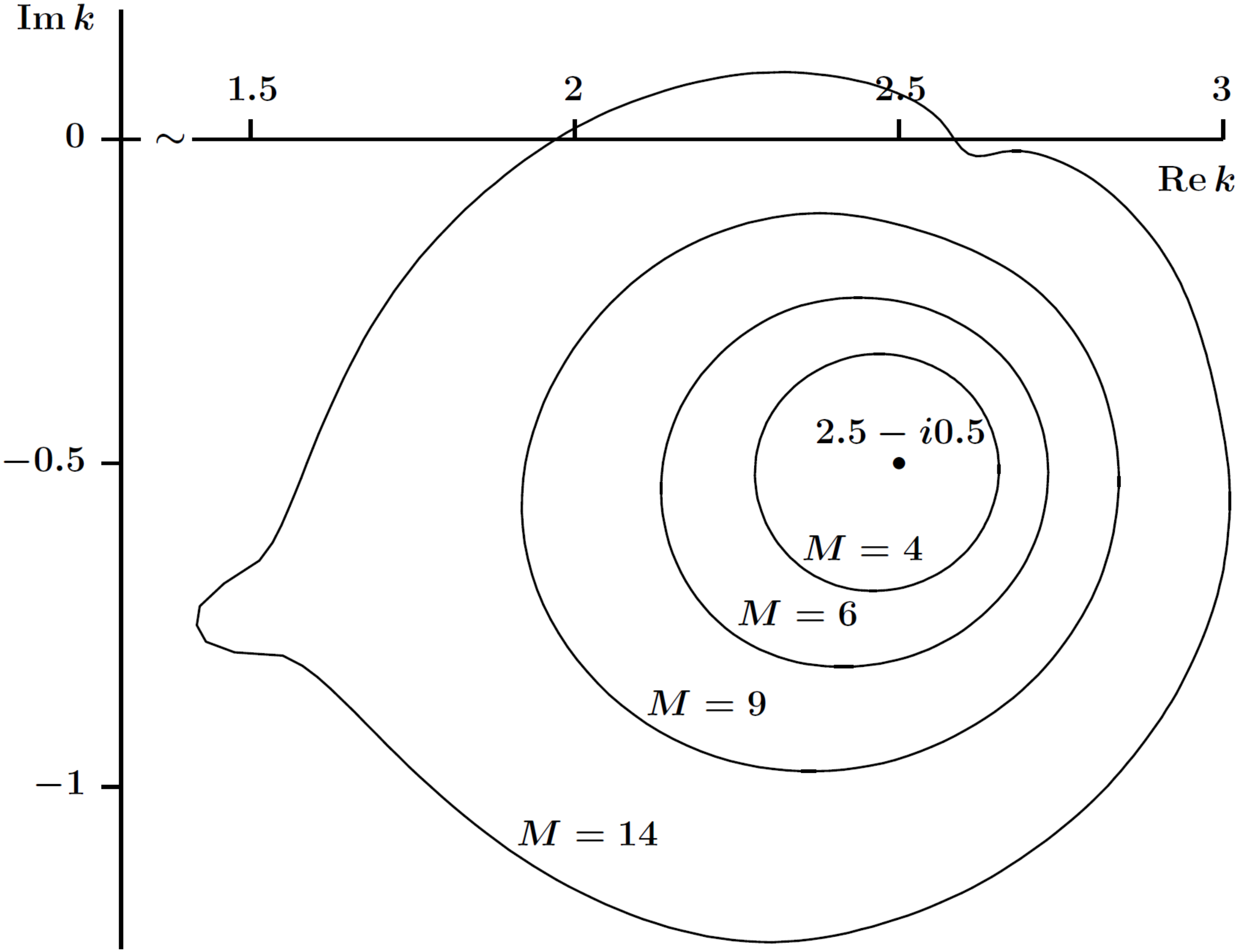
$$V(r) = 7.5r^2 \exp(-r)$$

$\hbar = m = 1$

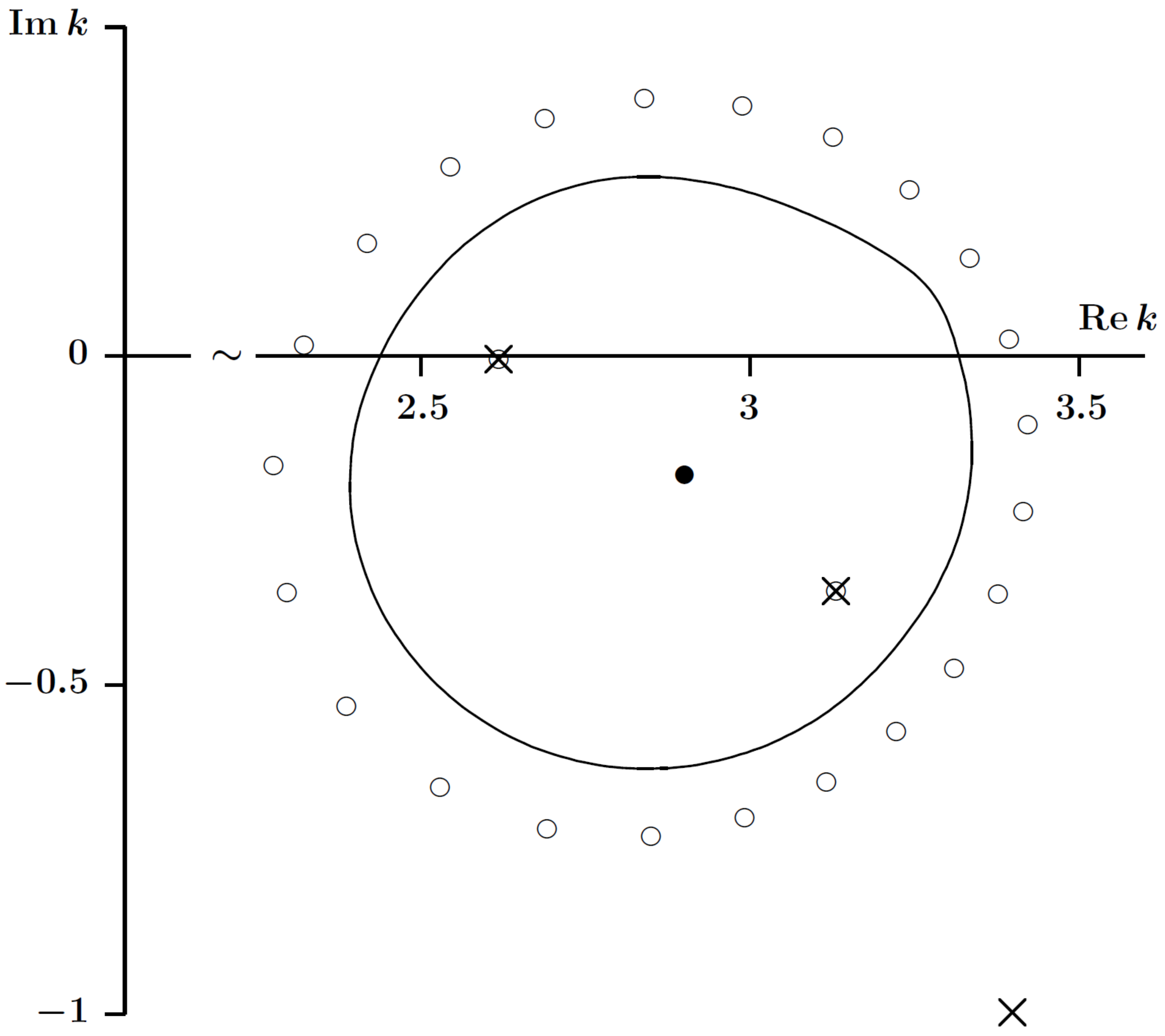




Domain of 1% accuracy



25 terms in the expansion



Multi-channel problem

$$\left[\partial_r^2 + k_n^2 - \frac{\ell_n(\ell_n+1)}{r^2}\right] u_n(E,r) = \sum_{n'=1}^{N} V_{nn'}(r) u_{n'}(E,r)$$

2N linearly independent solutions; ***N*** of them are regular at ***r = 0***

fundamental matrix of regular solutions (the basis)

$$\Phi(E,r) = \begin{pmatrix} \phi_{11}(E,r) & \phi_{12}(E,r) & \cdots & \phi_{1N}(E,r) \\ \phi_{21}(E,r) & \phi_{22}(E,r) & \cdots & \phi_{2N}(E,r) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{N1}(E,r) & \phi_{N2}(E,r) & \cdots & \phi_{NN}(E,r) \end{pmatrix}$$

Physical solution

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} = C_1 \begin{pmatrix} \phi_{11} \\ \phi_{21} \\ \vdots \\ \phi_{N1} \end{pmatrix} + C_2 \begin{pmatrix} \phi_{12} \\ \phi_{22} \\ \vdots \\ \phi_{N2} \end{pmatrix} + \cdots + C_N \begin{pmatrix} \phi_{1N} \\ \phi_{2N} \\ \vdots \\ \phi_{NN} \end{pmatrix}$$

Regular at ***r = 0***

C_n are chosen to give certain asymptotics $r \longrightarrow \infty$ (bound, resonant, scattering)

$r \to \infty$

$$W^{(\text{in})} = \begin{pmatrix} h^{(-)}_{\ell_1}(k_1 r) & 0 & \cdots & 0 \\ 0 & h^{(-)}_{\ell_2}(k_2 r) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & h^{(-)}_{\ell_N}(k_N r) \end{pmatrix}$$

Multi-channel spherical waves

$$W^{(\text{out})} = \begin{pmatrix} h^{(+)}_{\ell_1}(k_1 r) & 0 & \cdots & 0 \\ 0 & h^{(+)}_{\ell_2}(k_2 r) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & h^{(+)}_{\ell_N}(k_N r) \end{pmatrix}$$

$$\Phi(E,r) \underset{r\to\infty}{\longrightarrow} W^{(\text{in})}(E,r) f^{(\text{in})}(E) + W^{(\text{out})}(E,r) f^{(\text{out})}(E)$$

Scattering

$$S(E) = f^{(\text{out})}(E) \left[f^{(\text{in})}(E) \right]^{-1}$$

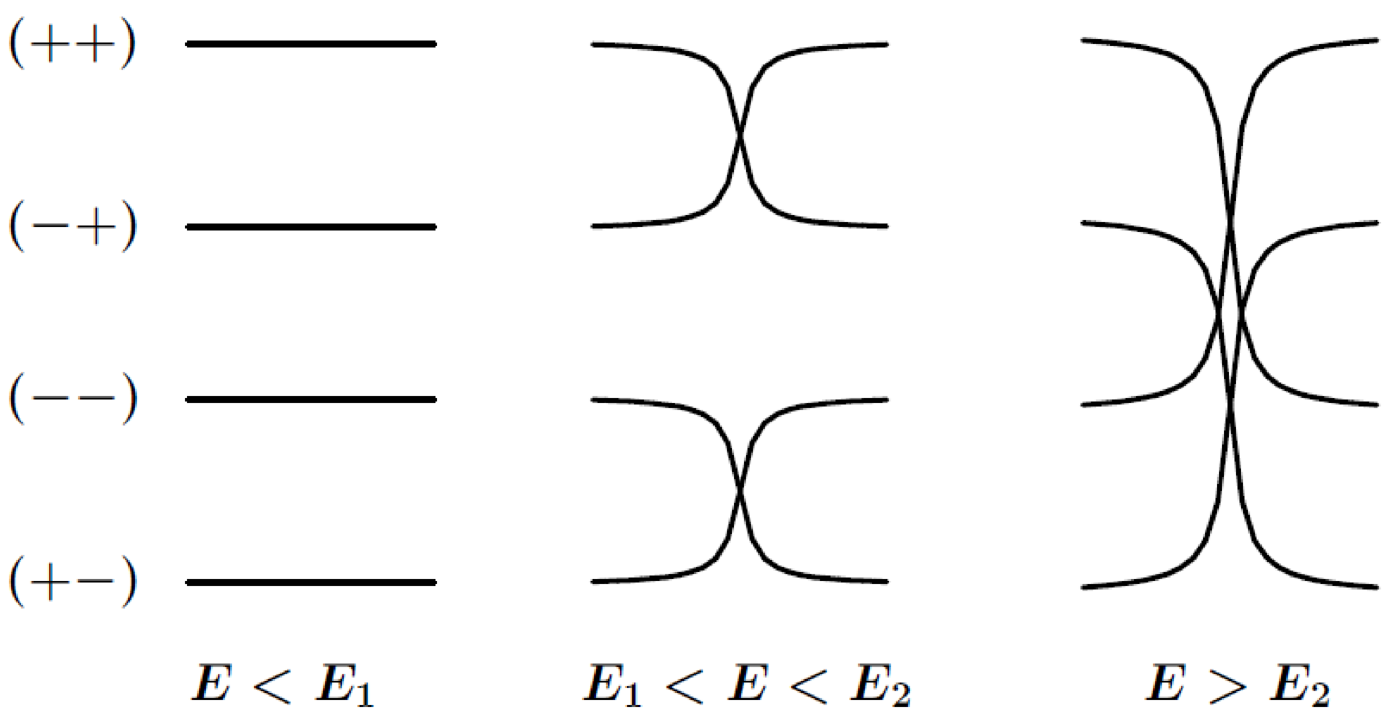
Spectral points

$$\det f^{(\text{in})}(\mathcal{E}_n) = 0$$

Riemann surface

$$k_n = \pm\sqrt{\frac{2\mu_n}{\hbar^2}(E - E_n)}\ , \qquad n = 1, 2$$

Schematically shown interconnections of the layers of the Riemann surface for a two-channel problem at three different energy intervals. The layers correspond to different combinations of the signs (indicated in brackets) of $\mathrm{Im}\, k_1$ and $\mathrm{Im}\, k_2$



In the present work, we construct the Jost matrices in such a way that in their matrix elements the dependences on odd powers of all channel momenta are factorized analytically

$$f_{mn}^{(\mathrm{in})}(E) = \frac{k_n^{\ell_n+1}}{2k_m^{\ell_m+1}}\tilde{a}_{mn}(E) - i\frac{k_m^{\ell_m}k_n^{\ell_n+1}}{2}\tilde{b}_{mn}(E)$$

$$f_{mn}^{(\mathrm{out})}(E) = \frac{k_n^{\ell_n+1}}{2k_m^{\ell_m+1}}\tilde{a}_{mn}(E) + i\frac{k_m^{\ell_m}k_n^{\ell_n+1}}{2}\tilde{b}_{mn}(E)$$

Single-valued, analytic functions of *E*

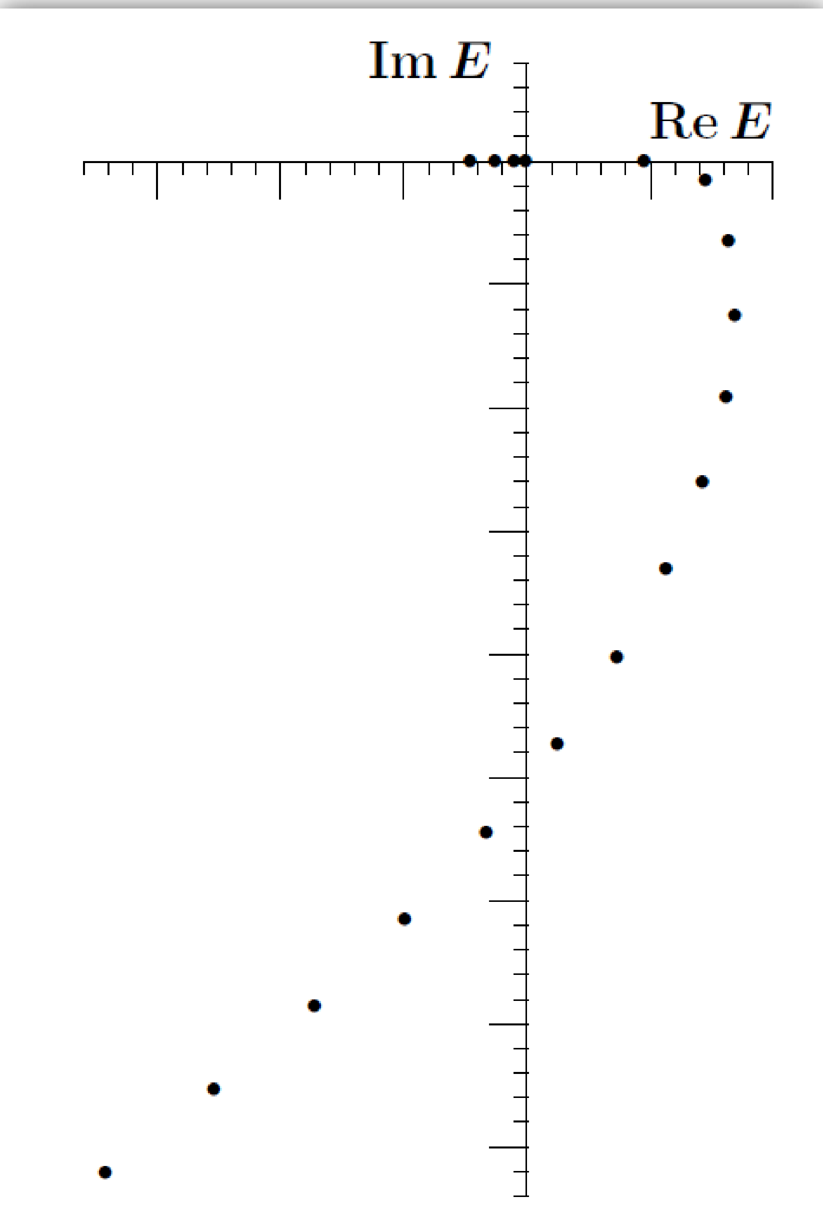
$$\tilde{a}(E) = \sum_{n=0}^{\infty}\alpha_n(E-E_0)^n$$

$$\tilde{b}(E) = \sum_{n=0}^{\infty}\beta_n(E-E_0)^n$$

Matrices $\tilde{a}(E)$ and $\tilde{b}(E)$ are asymptotic values of solutions of a system of differential equations

Example

Two-channel model



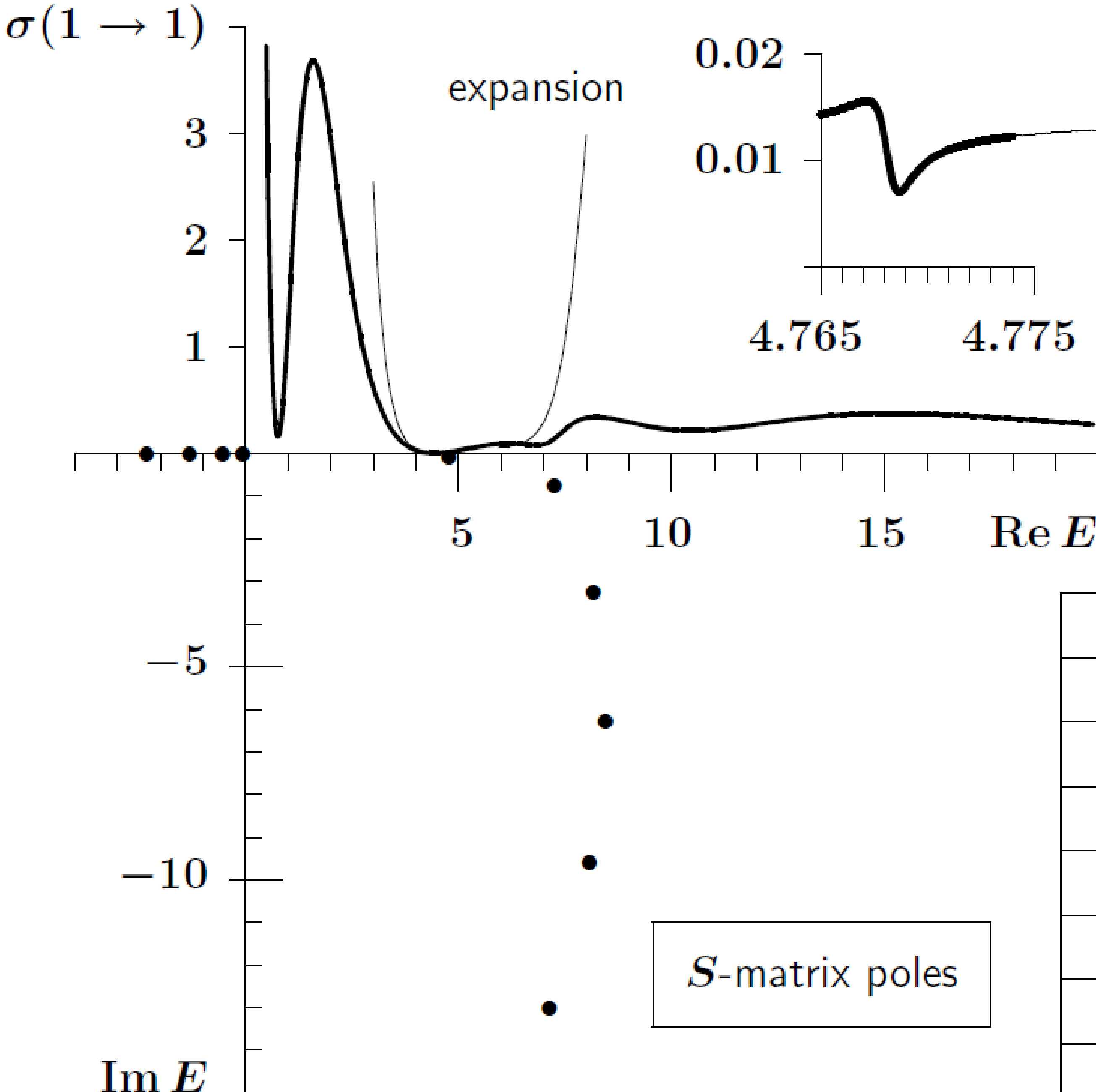
$$V(r) = \begin{pmatrix} -1.0 & -7.5 \\ -7.5 & 7.5 \end{pmatrix} r^2 e^{-r}$$

$$\mu_1 = \mu_2 = \hbar c = 1$$

$$E_1 = 0 \text{ and } E_2 = 0.1$$

$$\ell_1 = \ell_2 = 0.$$

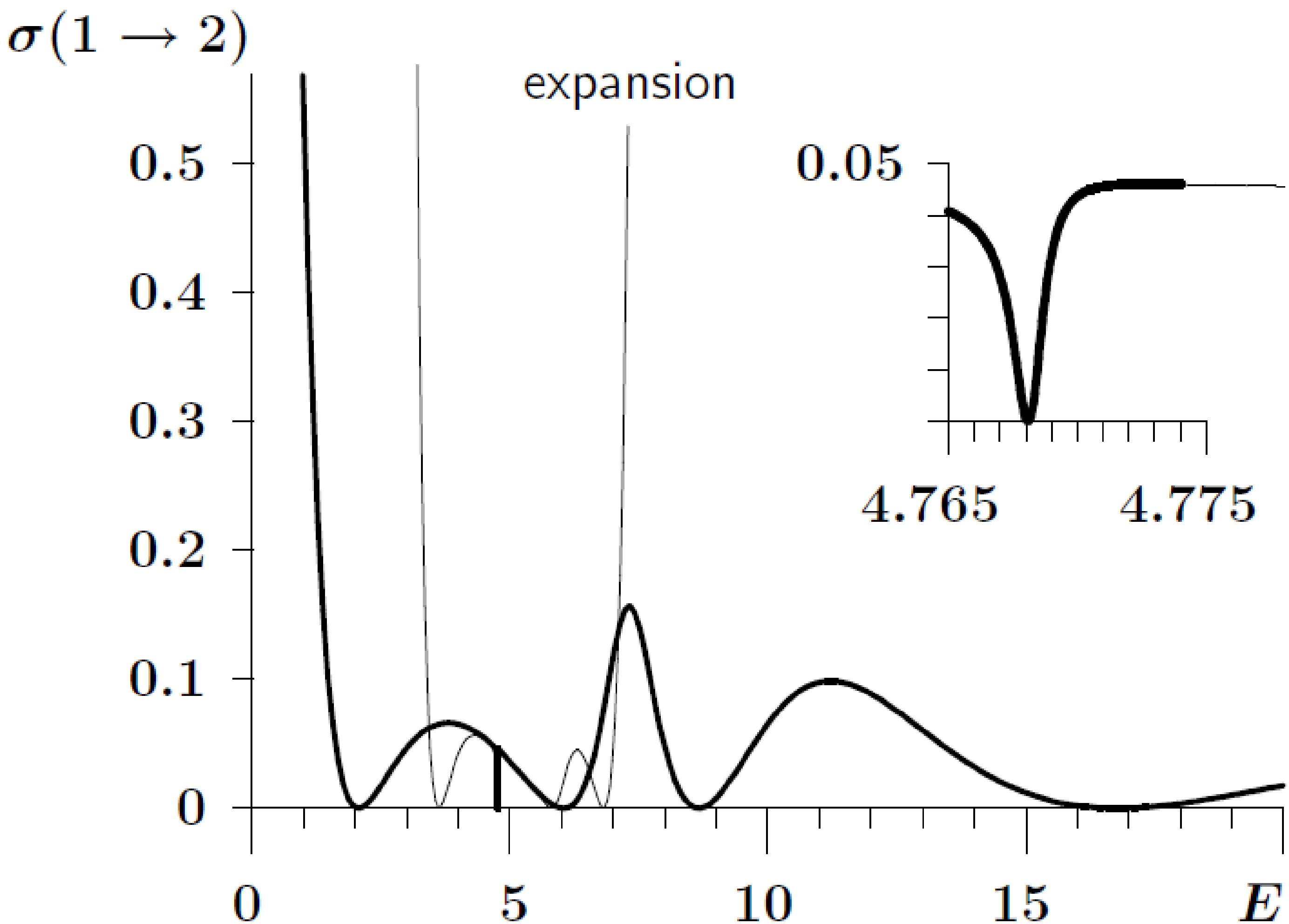
E_r	Γ
-2.314391	0
-1.310208	0
-0.537428	0
-0.065258	0
4.768197	0.001420
7.241200	1.511912
8.171217	6.508332



$$E_0 = 5 + i0$$

$$M = 5$$

E_r	Γ
-2.314391	0
-1.310208	0
-0.537428	0
-0.065258	0
4.768197	0.001420
7.241200	1.511912
8.171217	6.508332



$$E_0 = 5 + i0$$

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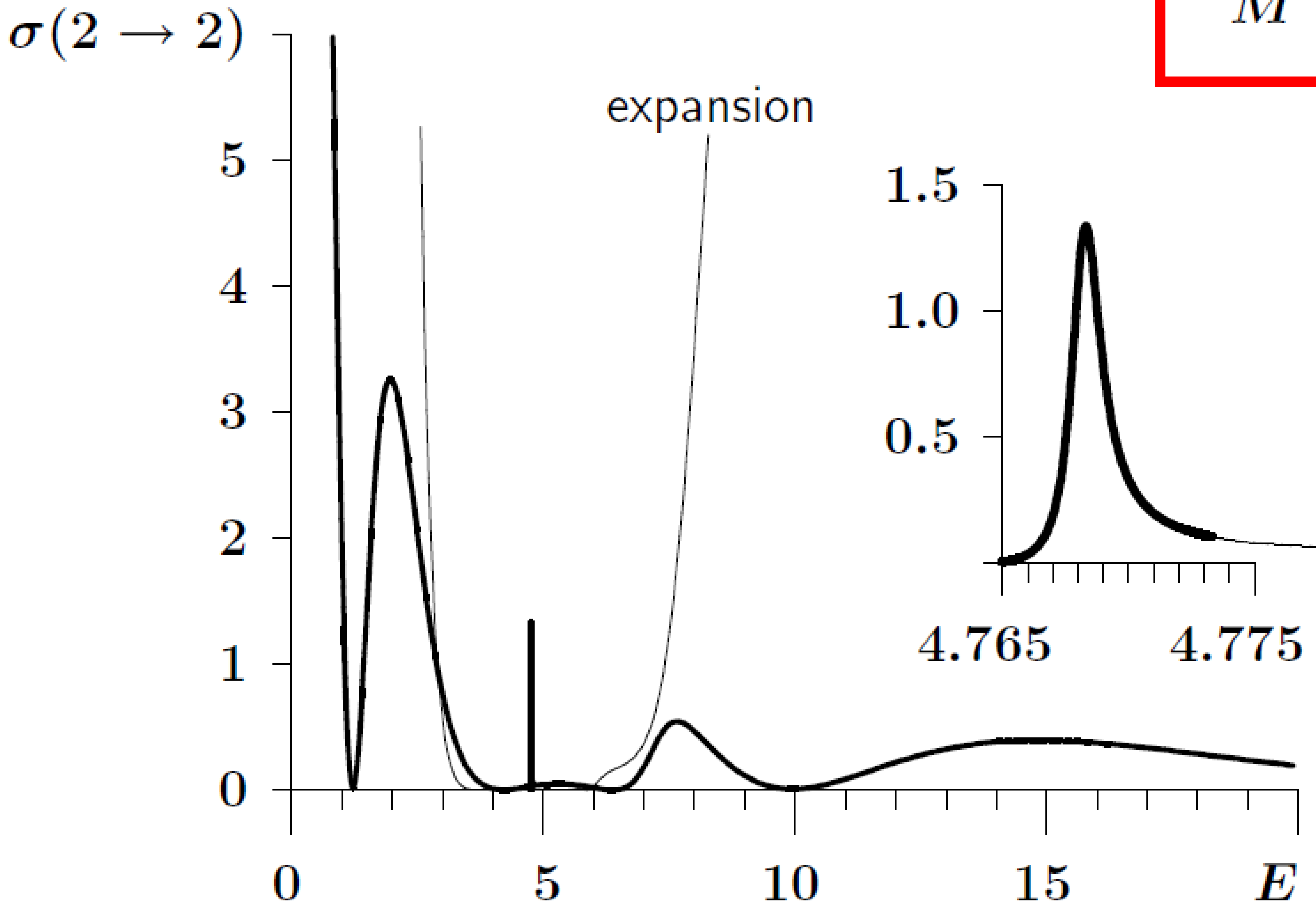
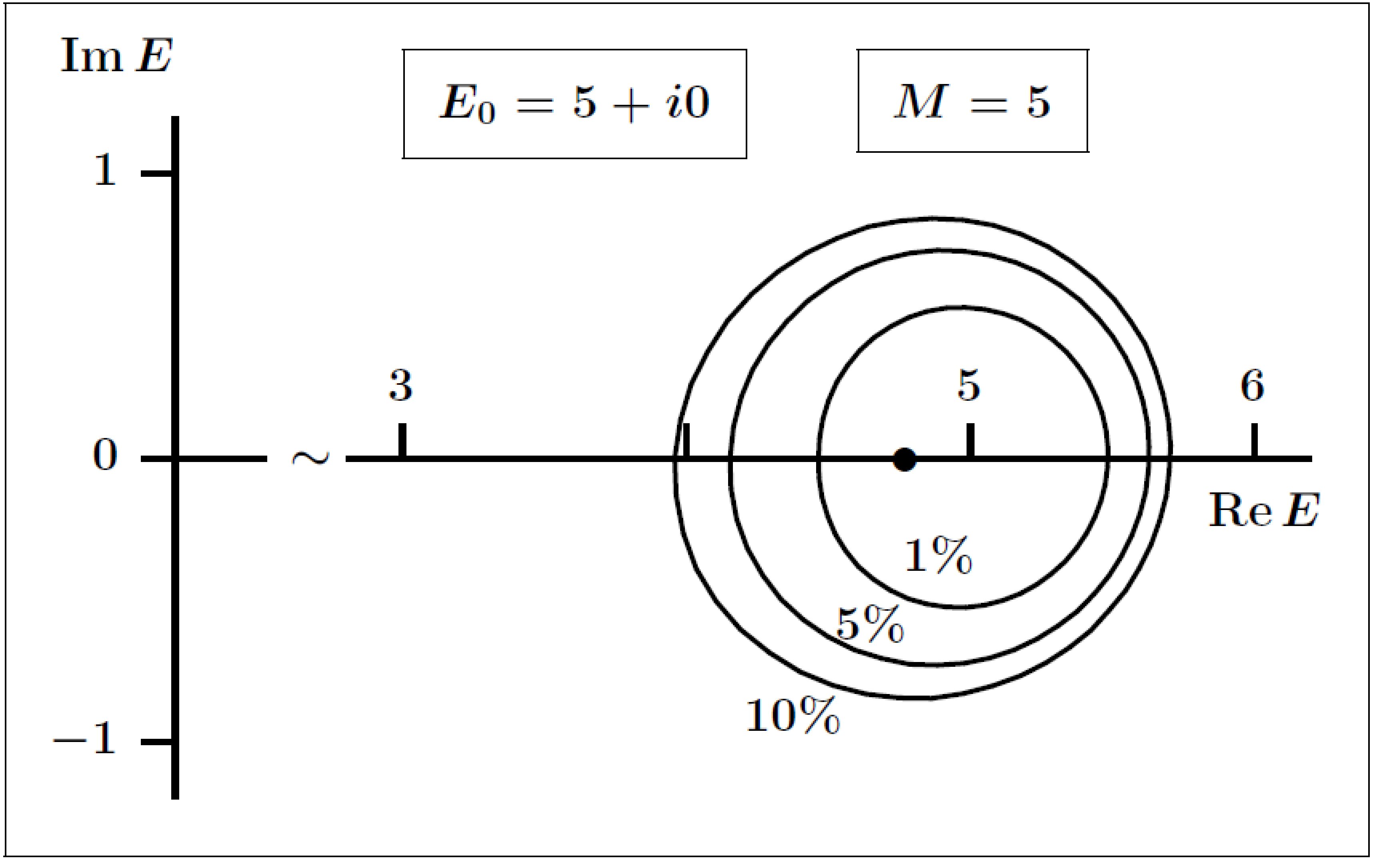
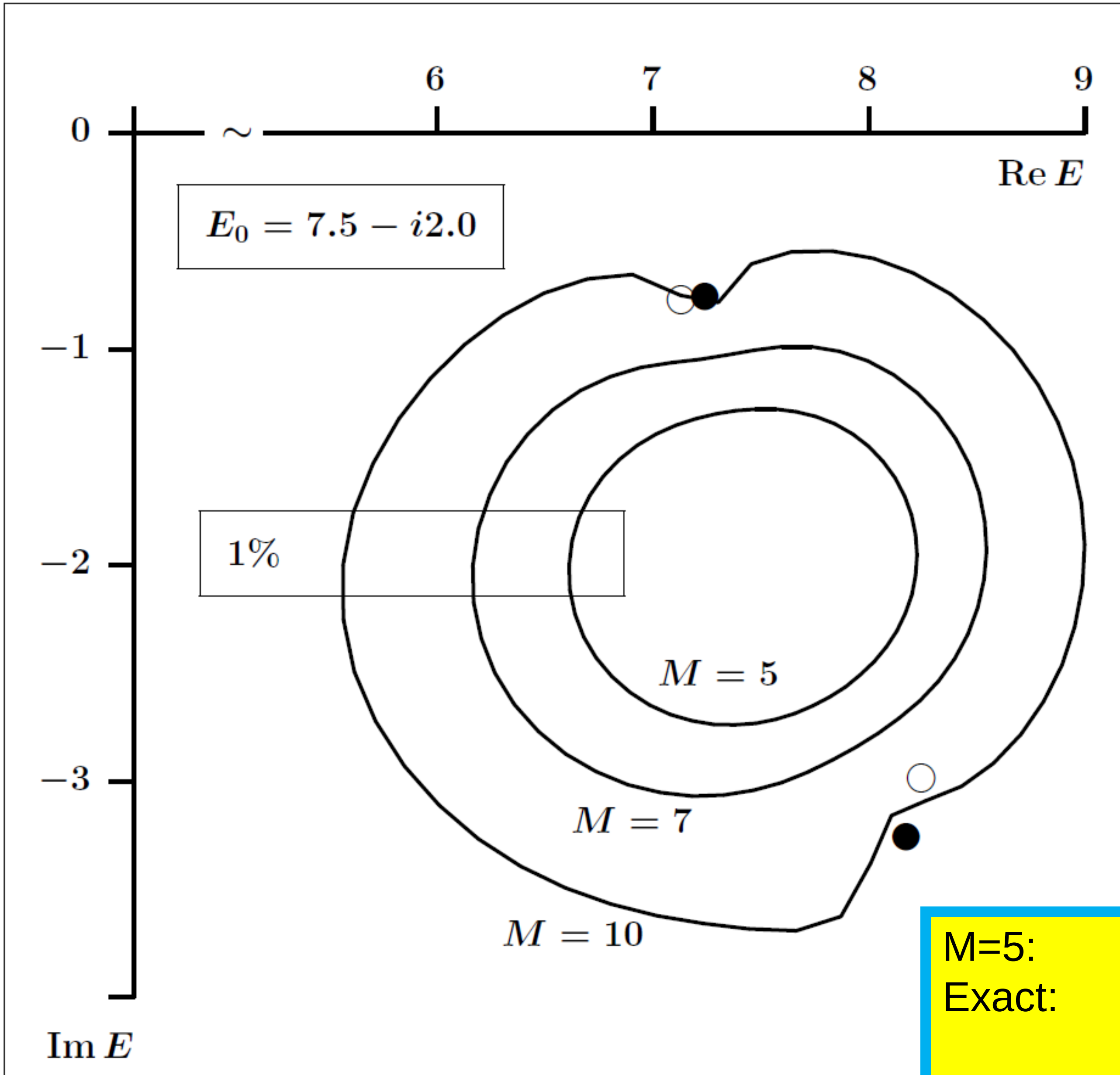




First resonance

Expansion (M=5): E = **4.768178−*i*0.000686**
Exact value: E = **4.768197−*i*0.000710**

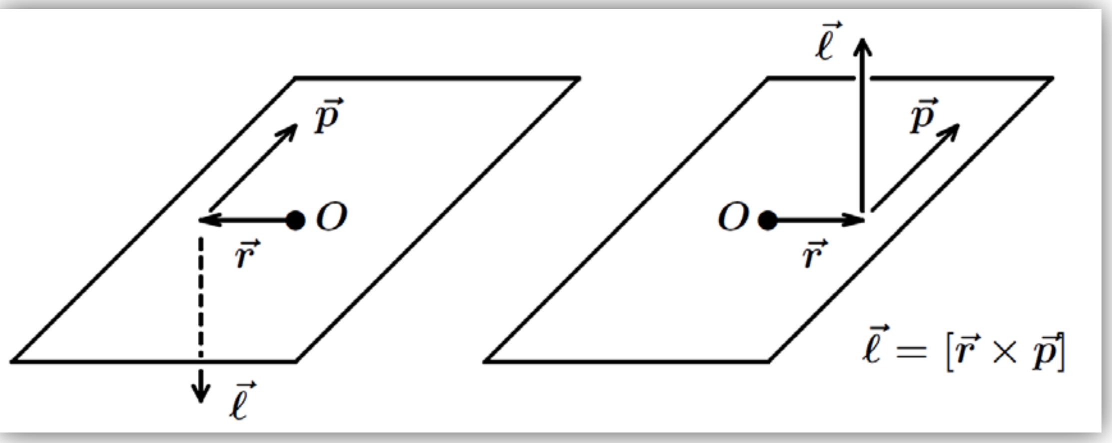


Two resonances

M=5: E = **7.241200 −i0.755956**
Exact: E = **7.131204− i0.768670**

M=5: E **= 8.241795 − i2.982867**
Exact: E = **8.171217 − i3.254166**

Two-dimensional problem



$$\lambda = \ell - \frac{1}{2}$$

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{\lambda(\lambda+1)}{r^2} - V(r)\right] u_\ell(E,r) = 0$$

$$u_\ell(E,r) \underset{r\to\infty}{\longrightarrow} f_\ell^{(\mathrm{in})}(E) h_{\ell-1/2}^{(-)}(kr) + f_\ell^{(\mathrm{out})}(E) h_{\ell-1/2}^{(+)}(kr)$$

$$h_\lambda^{(\pm)}(z) = j_\lambda(z) \pm i y_\lambda(z)$$

For half-integer λ, the Riccati-Neumann function has logarithmic branching point

$$y_\lambda(kr) = k^{-\lambda}\sum_{n=0}^{\infty} k^{2n} g_n^{(\lambda)}(r) + h(k) j_\lambda(kr)$$

Half-integer λ

$$g_n^{(\lambda)}(r) = \begin{cases} -\dfrac{(\lambda-n-1/2)!}{\sqrt{\pi}n!}\left(\dfrac{r}{2}\right)^{2n-\lambda} & , \quad 0 \leqslant n \leqslant \lambda - \frac{1}{2} \\ \dfrac{2}{\pi}\ln\left(\dfrac{r}{R}\right) f^{(\lambda)}_{n-\lambda-\frac{1}{2}}(r) - & \\ -\dfrac{(-1)^{n-\lambda-\frac{1}{2}}\left[\psi(n+1)+\psi(n-\lambda+\frac{1}{2})\right]}{\sqrt{\pi}n!(n-\lambda-\frac{1}{2})!}\left(\dfrac{r}{2}\right)^{2n-\lambda} & , \quad \lambda + \frac{1}{2} \leqslant n < \infty \end{cases}$$

$$h(k) = \frac{2}{\pi}\ln\left(\frac{kR}{2}\right)$$

$$f_\ell^{(\mathrm{in})}(E) = \frac{1}{2}\left\{\tilde{a}_\ell(E) + k^{2\lambda+1}[h(k) - i]\tilde{b}_\ell(E)\right\}$$

$$f_\ell^{(\mathrm{out})}(E) = \frac{1}{2}\left\{\tilde{a}_\ell(E) + k^{2\lambda+1}[h(k) + i]\tilde{b}_\ell(E)\right\}$$

$$\tilde{a}_\ell(E) = \sum_{n=0}^{\infty} \alpha_n^{(\ell)}(E_0)(E - E_0)^n$$

$$\tilde{b}_\ell(E) = \sum_{n=0}^{\infty} \beta_n^{(\ell)}(E_0)(E - E_0)^n$$

Single-valued, analytic functions of *E*

$$f_\ell^{(\mathrm{in})}(E) \approx \frac{1}{2}\sum_{n=0}^{N}\left\{\alpha_n^{(\ell)}(E_0) + k^{2\lambda+1}[h(k) - i]\beta_n^{(\ell)}(E_0)\right\}(E - E_0)^n$$

$$f_\ell^{(\mathrm{out})}(E) \approx \frac{1}{2}\sum_{n=0}^{N}\left\{\alpha_n^{(\ell)}(E_0) + k^{2\lambda+1}[h(k) + i]\beta_n^{(\ell)}(E_0)\right\}(E - E_0)^n$$

$$f^{(\mathrm{in})}_{mn}(E) = \frac{k_n^{\ell_n+1}}{2k_m^{\ell_m+1}}\tilde{a}_{mn}(E) - i\frac{k_m^{\ell_m}k_n^{\ell_n+1}}{2}\tilde{b}_{mn}(E)$$

$$f^{(\mathrm{out})}_{mn}(E) = \frac{k_n^{\ell_n+1}}{2k_m^{\ell_m+1}}\tilde{a}_{mn}(E) + i\frac{k_m^{\ell_m}k_n^{\ell_n+1}}{2}\tilde{b}_{mn}(E)$$

$$\tilde{a}(E) = \sum_{n=0}^{\infty}\alpha_n(E-E_0)^n$$

$$\tilde{b}(E) = \sum_{n=0}^{\infty}\beta_n(E-E_0)^n$$

Possible applications

Semi-analytic expression for the S-matrix

Locating resonances

Moving a resonance to a given point E_0

Fitting experimental cross-section and thus extracting the resonance parameters from experimental data

SUMMARY

- odd powers of the channel momenta in the Jost matrices are factorized
- for the remaining energy dependent factors, a system of differential equations is obtained
- these energy dependent functions are expanded in power series
- the expansion coefficients are determined by a system of differential equations