Universality in Four-Boson Systems

Tobias Frederico

Instituto Tecnológico de Aeronáutica, DCTA, 12228-900, São José dos Campos, SP, Brazil
tobias@ita.br

Sixth International and Interdisciplinary Workshop on
the Dynamics of Critically Stable Quantum Few-Body Systems,
Erice, Italy, 10 Oct. 2011

Outline

1. **Scalings in Few-Body Systems**
   - Motivation: Four-body problem
   - Weakly-bound FB systems, Efimov effect & Thomas collapse
   - Scaling plots - 3B
   - Four-boson systems and Efimov effect
2. **Formalism**
   - Solving few-body bound state problems
   - 4B Bound State in Faddeev-Yakubovsky Scheme
3. **Model results - The tetramer spectrum**
   - Tetramer Binding Energies
   - Tetramer scaling function
   - Yakubovsky Components
   - 4B Wave Function
   - Momentum Distribution Functions
4. **Conclusion**
5. **Further Details**

Main Goal

- The Four-body Problem - Scaling and Universality

Motivation

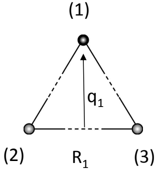
- Short-range interactions & large quantum systems: dominance of the classically forbidden region;
- Universality & model independence;
- Zero-range interaction: modeling the tail of the wave function & few parameter description - How Many?
- Correlation between few-body observables & Limit-cycles;
- Interwoven 4-body and 3-body Limit-cycles;
- Reality: Cold atom physics close to a Feshbach resonance!

Why are weakly bound state problems interesting?

- Identify and understand universal properties of large few-body quantum systems expressed as correlations between observables, the common physics of the classically forbidden region of short-ranged interactions;
- Model independence, i.e., the details of the interactions between particles are not relevant apart few scales or parameters, such that one can consider a zero-range interaction [In a three-body (3B) system, once one 3B observable is given, in addition to a two-body parameter (scatt. length), the other observables are found correlated with the first one.]
- *"One goal is to be able to engineer the interaction between atoms to achieve a quantum system in which multiple-body interactions dominate the physical behavior"* [G. Modugno, Science 326 (2009) 1640]. How to tune few-body parameters or scales?

Illustration: 3-boson wave function & contact interaction

Weakly bound system wave function & contact interaction



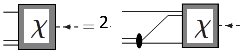
Three-boson wave function:

$$(E - H_0)\psi = 0$$

$$\psi = \int d^3q_1 \frac{\exp\{i[E-(3/4)q_1^2]^{1/2}R_1\}}{R_1} e^{i\mathbf{q}_1\cdot\mathbf{r}_1}\chi(\mathbf{q}_1)$$

$$+ \ (1 \rightarrow 2) \ + \ (1 \rightarrow 3)$$

- Skorniakov and Ter-Martirosian equations (1956)
- Danilov, Sov. Phys. JETP 13 (1961) 349
- scatt. length + 3B short-range parameter
- Thomas collapse & Efimov effect



Three-body Efimov effect

- Efimov effect: If two bosons interact in such a way that a two-body bound state is exactly on the verge of being formed, then in a three-boson system one should observe an infinite number of bound states. This phenomenon appears in a three-dimensional formalism for the three-body systems, and does not exist in one or two dimensions.
- This effect, predicted by Vitaly Efimov in 1970 [Phys. Lett. B 33 (1970) 563; Sov. J. Nucl. Phys. 12 (1971) 589], have been recently verified in ultracold atom laboratories, with the increasing number of three-body bound-state levels, as the two-body scattering length goes to infinity.
- The observation of this effect, first reported by Kraemer et al. [Nature 440 (2006) 315], was confirmed by several other atomic experimental groups, which are looking for the properties of such states [Zaccanti et al., Nature Phys. 5 (2009) 586; Ferlaino et al, PRL 102 (2009) 140401; Pollack et al., Science 326 (2009) 1683; etc.]

Why are weakly bound state problems interesting?

Efimov Physics (1970): Nuclear Physics

Vitaly Efimov

- an infinite sequence of weakly bound 3-body states as $a \to \pm\infty$
- Unitary limit ($a \to \pm\infty$): $E_3^{n+1}/E_3^n \approx 1/22.7^2$ $\Longrightarrow$ discrete scaling with scaling factor 22.7
- For finite a: discrete scaling is exact when range$\to 0$

Vol 440|16 March 2006|doi:10.1038/nature04626 nature

LETTERS

Evidence for Efimov quantum states in an ultracold gas of caesium atoms

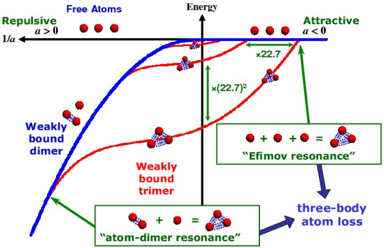
T. Kraemer[1], M. Mark[1], P. Waldburger[1], J. G. Danzl[1], C. Chin[1,2], B. Engeser[1], A. D. Lange[1], K. Pilch[1], A. Jaakkola[1], H.-C. Nägerl[1] & R. Grimm[1,3]

nature physics LETTERS

PUBLISHED ONLINE: 22 FEBRUARY 2009 | DOI: 10.1038/NPHYS1203

Observation of an Efimov-like trimer resonance in ultracold atom-dimer scattering

S. Knoop[1]*, F. Ferlaino[1], M. Mark[1], M. Berninger[1], H. Schöbel[1], H.-C. Nägerl[1] and R. Grimm[1,2]



Thomas collapse

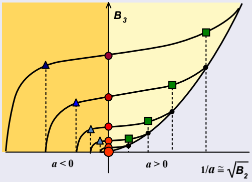
Efimov effect is the counterpart of the Thomas collapse [Phys. Rev. 47 (1935) 903] of the three-body ground-state energy, when the range of the two-body interaction r_0 goes to zero. Both effects can be described by the same dimensionless non-relativistic three-body equation [Adhikari et al. Phys. Rev. A 37 (1988) 3666.]
The Thomas collapse was crucial in determining the range of the nuclear forces, as pointed out by Bethe and Bacher [Rev. Mod. Phys. 8 (1936) 82].

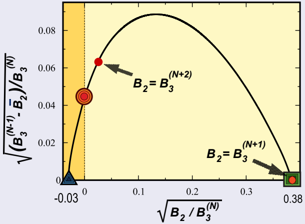
$$\frac{|a|}{r_0} \longrightarrow \infty$$

$r_0 \to 0$ (Thomas collapse), $|a| \to \infty$ (Efimov effect)

Scaling plots

- The trimer energies can be presented in a scaling plot where a limit-cycle can be easily identified in terms of the relevant physical scales (scatt. length and one three-boson energy) - renormalization (More in Marcelo Yamashita Talk);
- Scheme proposed in 1999 [Frederico et al., Phys. Rev. A 60, R9 (1999) “Scaling limit of weakly bound triatomic states”], which was appropriate for revealing the **three-body scaling**, with the corresponding **Efimov cycles**.





- Extension to 4-boson system shows the effect of a new scale!

The addition of one more particle to the quantum three-body system has long challenged the Efimov picture:

R. D. Amado and F. C. Greenwood, *There is no Efimov effect for four or more particles*, *Phys. Rev. D* **7**, 2517 (1973).

H. Kröger and R. Perne, *Efimov effect in the four-body case*, *Phys. Rev. C* **22**, 21 (1980).

S. K. Adhikari and A. C. Fonseca, *Four-body Efimov effect in a Born-Oppenheimer model*, *Phys. Rev. D* **24**, 416 (1981).

H. W. L. Naus and J. A. Tjon, *The Efimov effect in a four-body system*, *Few-Body Systems* **2**, 121 (1987).

Correlation in four-nucleon systems: Tjon line

Notes:

- Nonexistence of a proper four-body scale relies on a strong suppression of the short-range physics, beyond that already accounted for in the three-boson system.
- Amado and Greenwood showed that there is no infrared divergence (estimating the trace of the four-body kernel in momentum space) which led them to conclude against the existence of Efimov effect in the case of four or more particles. However, the momentum integrals should also implicitly have an ultraviolet cutoff (the four-body one) to regulate them.
- Conclusions drawn within the nuclear physics context are obviously limited, in view of the dominance of the two-body potential and repulsion at short-range: the four-body scale is suppressed!

Dominance of 2-body forces in the nuclear physics context leads to the Tjon-line [Phys. Lett. B **56**, 217 (1975)]: ^{4}He and the triton binding energies are strongly correlated with a fixed slope.

Correlation in Four-nucleon systems: Tjon line

- Tjon was not convinced on the non-existence of a proper four-body scale, in a more general case, as shown by his work with Naus [Few-Body Syst. **2** (1987) 121].
- In our recent study on a general four-boson problem, within a renormalized zero-ranged model, we verify that it is not enough only two parameters (which determines trimer properties) to describe the the four-boson system.
- As we move the four-body scale in relation to the three-body one, a new Tjon lines exist. Therefore, we can have a **family of Tjon lines** with slopes depending on the new four-boson scale.
- We performed a number of calculations of tetramer properties within a zero-range model, to show how the dependence on the new short-range parameter is evidenced through their structure in momentum space. Our starting point was the exact unitary limit (infinite scattering length).

Recent studies on four-boson systems - Theory

- Platter, Hammer, Meissner, *Phys. Rev.* **A 70**, 52101 (2004); H.-W. Hammer and L. Platter, *Eur. Phys. J. A* **32**, 113 (2007);
- Yamashita, et al., *Europhys. Lett.* **75**, 555 (2006) - *4-boson scale*;
- Lazauskas and Carbonell, *Phys. Rev. A* **73**, 062717 (2006).
- Thøgersen, Fedorov, Jensen, *Europhys. Lett.* 83, 30012 (2008);
- von Stecher, D'Incao, Greene, *Nature Physics* **5**, 417 (2009); von Stecher, *J. Phys. B: At. Mol. Opt. Phys.* **43**, 101002 (2010);
- Wang and Esry, *Phys. Rev. Lett.* **102**, 133201 (2009).
- Deltuva, arXiv:1009.1295v1 [physics.atm-clus] and PRA82 (2010);
- Hadizadeh, Yamashita, Tomio, Delfino, Frederico, *Phys. Rev. Lett.* **107**, 135304 (2011); **- 4-boson scale & limit cycle -**

Recent studies on four-boson system - Experiments

- F. Ferlaino *et al.*, *Phys. Rev. Lett.* **102**, 140401 (2009).
- M. Zaccanti *et al.*, *Nature Phys.* **5**, 586 (2009).
- S. Pollack *et al.*, *Science* **326**, 1683 (2009), and www.sciencemag.org/cgi/content/full/1182840/DC1 for Supporting Online Material.

The recombination rates measured by Zaccanti et al. with ^{39}K, and by Pollack et al. with ^{7}Li, suggest a change of the three-body parameter when crossing the Feshbach resonance.
The same effect was also seen in an experiment of atom-dimer loss in an ultracold trapped gas of a mixture with three hyperfine states of ^{6}Li performed by Nakajima et al.

- Nakajima et al. [Phys. Rev. Lett. **105** (2010) 023201; **106** (2011) 143201].

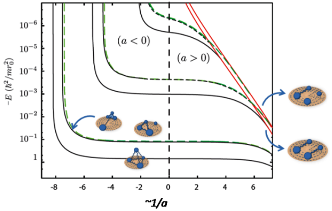
How many scales in four-boson systems with short-range interactions?

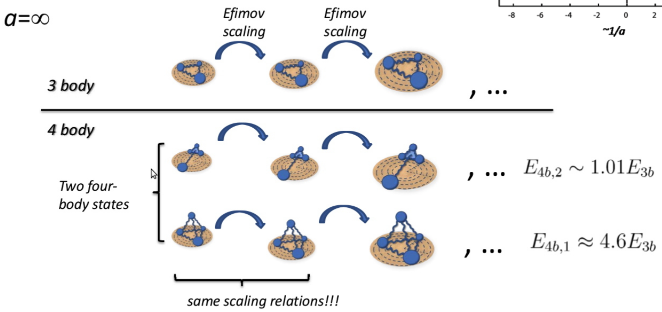
nature physics — LETTERS

PUBLISHED ONLINE: 26 APRIL 2009 | DOI: 10.1038/NPHYS1253

Signatures of universal four-body phenomena and their relation to the Efimov effect

J. von Stecher, J. P. D'Incao and Chris H. Greene*





- *Hammer et al., EPJA32 (2007)*
- *Deltuva, PRA82 (2010)*

No four-body parameter is needed!

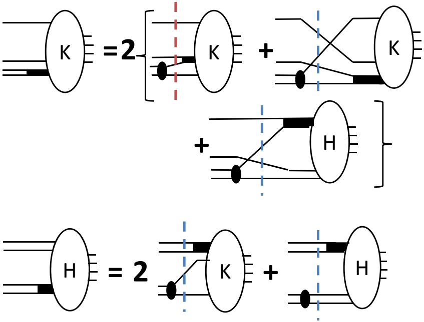
Outline

Numerical approaches

- GFMC (Wiringa et al. PRC62, 014001 (00)),
- NCSM (Navratil et al. PRC62, 054311 (00)),
- CRCGV (Hiyama et al. PRL85, 270 (00)),
- SV (Usukura et al. PRB59, 5652 (99)),
- HH (Viviani et al. PRC 71, 024006 (05)) and EIHH (Barnea et al. PRC67, 054003 (03))
- FY (Yamashita et al EPL75, 555 (06)& Hadizadeh et al. PRC83, 054004 (11))

Subtracted FY coupled equations - regularization -

2-boson scatt. amplitude: $\tau(\epsilon) = \left[2\pi^2(\frac{1}{a} - \sqrt{-\epsilon})\right]^{-1}$



$G_0^{(N)} = \frac{1}{E-H_0} - \frac{1}{-\mu_N^2-H_0}$ with μ_3 (RED) (3B collapse) and μ_4 (BLUE)

Outline

Our results for the tetramer spectrum

Tetramer ground and excited state binding energies for $B_2 = 0$

μ_4/μ_3	$B_4^{(0)}/B_3$	$B_4^{(1)}/B_3 - 1$	$B_4^{(2)}/B_3 - 1$	$B_4^{(3)}/B_3 - 1$
1	3.10			
1.6	**4.70**	7.1×10^{-4}		
5	12.5	0.531		
10	24.6	1.44		
21	63.5	**3.62**	3.2×10^{-4}	
40	184	7.65	0.203	
$\Rightarrow$ 70	5.20×10^{2}	12.9	0.629 $\Leftarrow$	
100	1.04×10^{3}	20.5	1.17	
200	4.06×10^{3}	50.8	2.86	
240				≈ 0
300	9.11×10^{3}	102	4.53	
400	1.62×10^{4}	153	6.28	

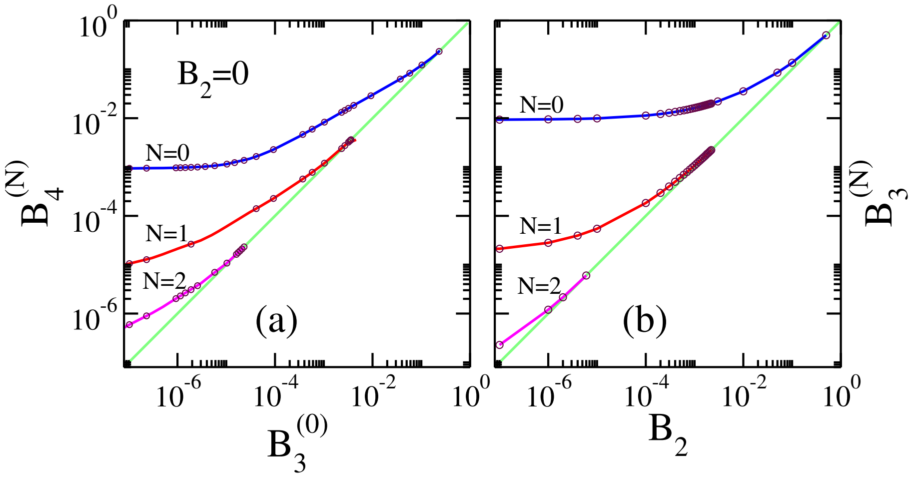
Tetramer ground and excited state binding energies for $B_2 = 0.02\ B_3$

μ_4/μ_3	$B_4^{(0)}/B_3$	$B_4^{(1)}/B_3 - 1$
1	2.66	
1.76	4.24	9.8×10^{-4}
5	10.0	0.421
20	45.9	2.77
40	139	6.10
80	506	13.0
200	2.86×10^{3}	39.5
300	6.00×10^{3}	69.3
400	9.81×10^{3}	104

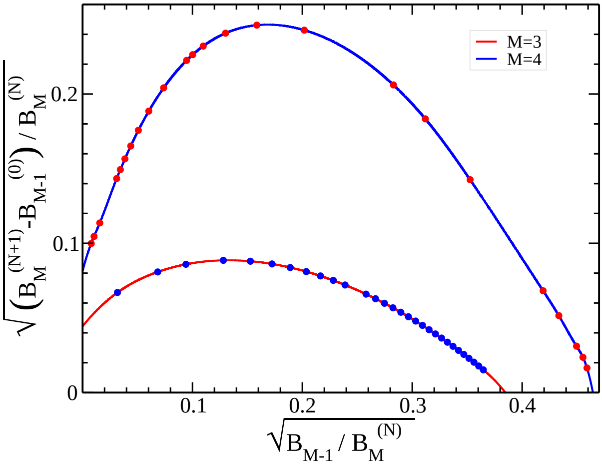
Tetramer ground and excited state binding energies for $B_2^{virtual} = 0.02\ B_3$

μ_4/μ_3	$B_4^{(0)}/B_3$	$B_4^{(1)}/B_3 - 1$
1	3.62	
1.7	5.91	0.014
5	15.4	0.658
20	74.8	4.18
40	236	9.46
80	873	20.6
200	5.02×10^3	64.5
300	1.06×10^4	115
400	1.73×10^4	174

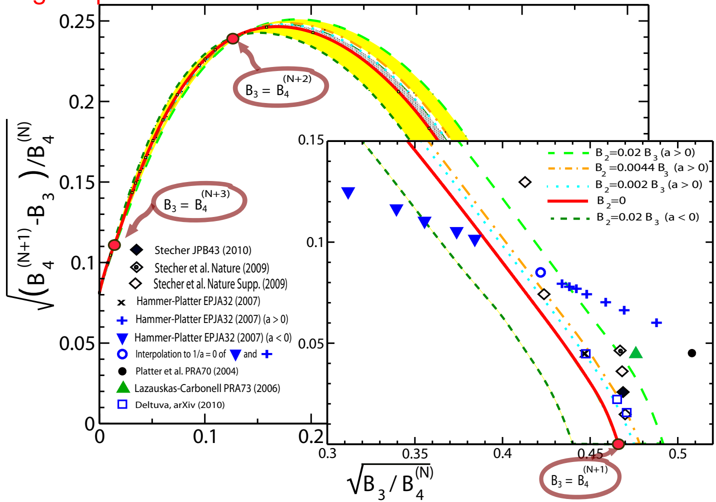
Three- and Four-boson Binding Energies
4B at unitary limit $a = \pm\infty$



Three- and Four-boson scaling plots
4B at unitary limit $a = \pm\infty$



"Scaling Properties of Universal Tetramers"

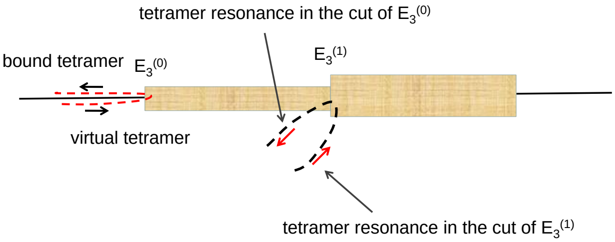


- For $\sqrt{B_3/B_4} > 1/22.7 = 0.044$ at most three tetramers fit between two consecutive Efimov trimers -

Hadizadeh et al., PRL 107, 135304 (2011).

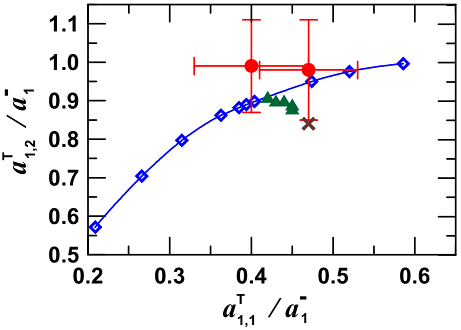
"Trajectory of Universal Tetramers"

Pattern of 4-boson energies trajectory by increasing the four-body scale in respect to the three-body one:



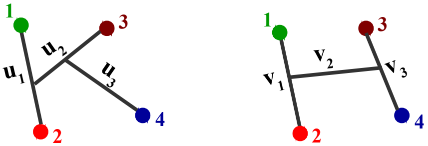
“4-atom resonances”

Positions of four-atom recombination peaks ($a < 0$) where two successive tetramers become unbound (blue-solid line with boxes). For comparison, we show results from calculations given in Stecher et al. (Nat. Phys.’09) (green-triangles) and from experiments reported in Pollack et al. (Science’09) (red-bullets with error bars) and Ferlaino et al (PRL’09)(brown $\times$). (Our first point from left corresponds to $B_4 \simeq 64\ B_3$ at the unitary limit.)

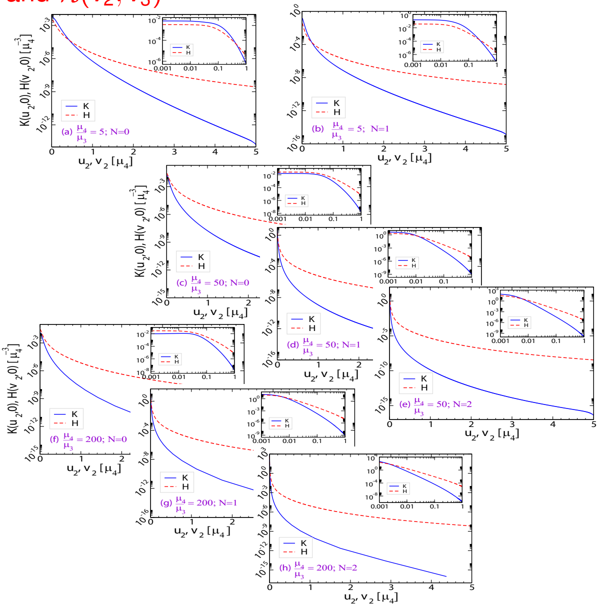


Jacobi momenta

- $\mathcal{K}(u_2, u_3)$ and $\mathcal{H}(v_2, v_3)$;
- $\langle \mathcal{K}|u_2|\mathcal{K}\rangle \;<\; \langle \mathcal{K}|u_3|\mathcal{K}\rangle$;
- $\langle \mathcal{H}|v_3|\mathcal{H}\rangle \;<\; \langle \mathcal{H}|v_2|\mathcal{H}\rangle$;

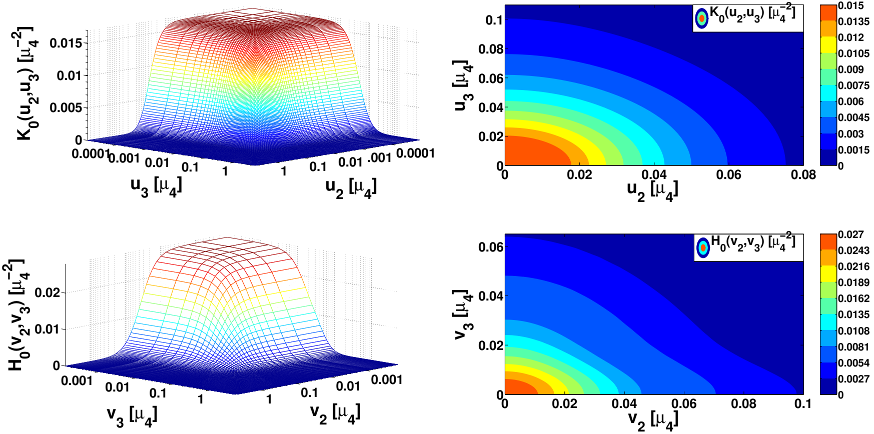


$\mathcal{K}(u_2, u_3)$ and $\mathcal{H}(v_2, v_3)$



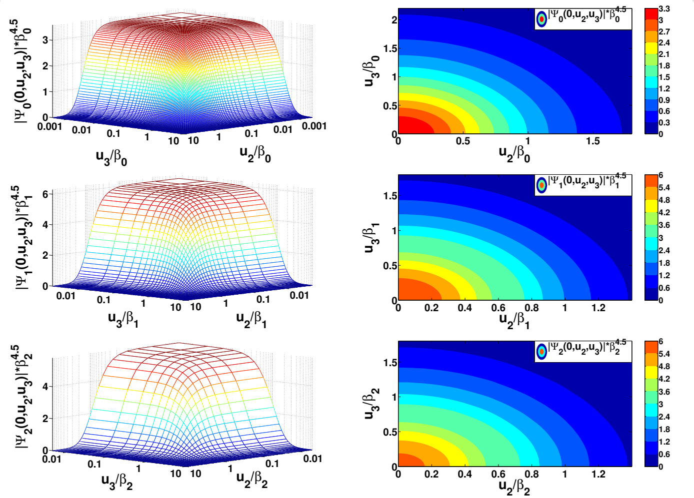
Example

$\mathcal{K}(u_2, u_3)$ and $\mathcal{H}(v_2, v_3)$ for $\frac{\mu_4}{\mu_3} = 50$; ground tetramer:



Example

$\Psi(0, u_2, u_3)$ for $\frac{\mu_4}{\mu_3} = 50$: where $\beta_N = \sqrt{E_4^{(N)}}$, $N = 0, 1, 2$

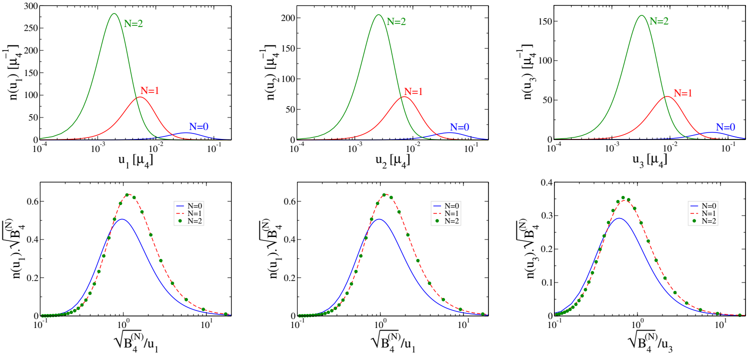


Momentum Distribution Functions

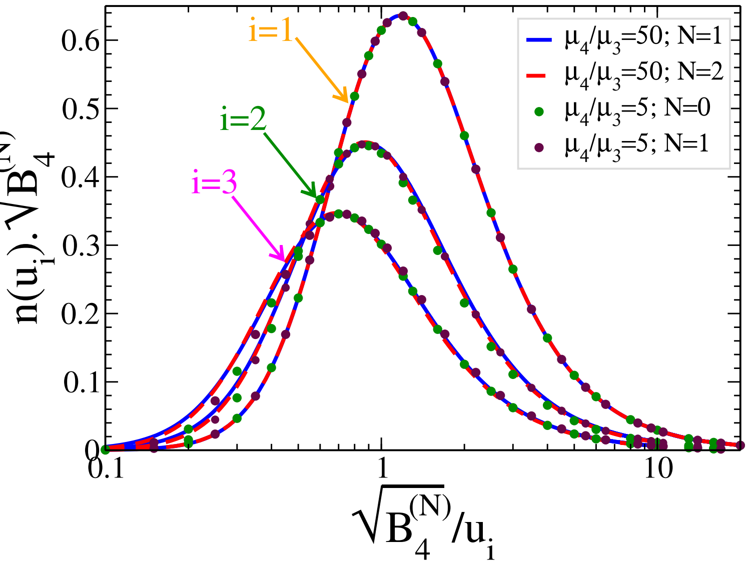
$$n(u_i) = u_i^2 \int du_j\, u_j^2 \int du_k\, u_k^2\, \Psi^2(u_i, u_j, u_k); \quad (i,j,k) \equiv (1,2,3) \quad (1)$$

Example

$n(u_1)$, $n(u_2)$ and $n(u_3)$ for $\frac{\mu_4}{\mu_3} = 50$:



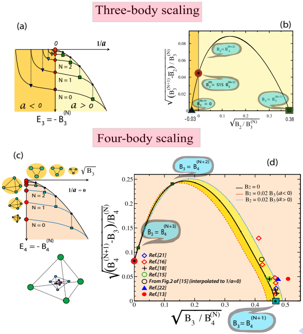
Universal momentum distribution functions at unitary ($B_2 = 0$)

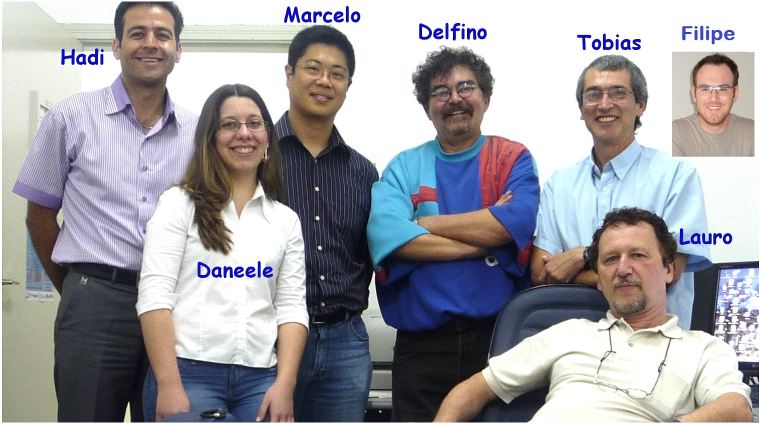


Outline

Conclusion

- Four-body scale moving two consecutive tetramer states below a given trimer;
- Model independence of the limit cycle - comparison with other models;
- No more than 3 tetramers between consecutive Efimov trimers for $a = \pm\infty$;
- Interwoven 3B and 4B limit cycles?
- More bosons?

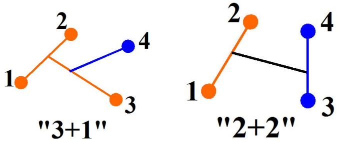




Thanks for your attention! [1]

[1] Work supported by the Brazilian agencies FAPESP and CNPq

Outline



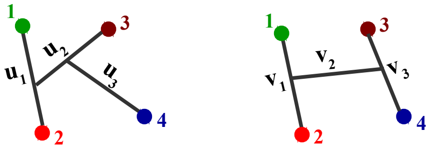
$$
\begin{aligned}
|K_{12,3}^{4}\rangle &= G_0 t_{12} P\left[\left(1+P_{34}\right)|K_{12,3}^{4}\rangle + |H_{12,34}\rangle\right] \\
|H_{12,34}\rangle &= G_0 t_{12} \tilde{P}\left[\left(1+P_{34}\right)|K_{12,3}^{4}\rangle + |H_{12,34}\rangle\right]
\end{aligned} \tag{2}
$$

$$
|\Psi\rangle = \left(1+P+P_{34}P+\tilde{P}\right)\left[\left(1+P_{34}\right)|K_{12,3}^{4}\rangle + |H_{12,34}\rangle\right] \tag{3}
$$

$$
P = P_{12}P_{23} + P_{13}P_{23} \quad \tilde{P} = P_{13}P_{24} \tag{4}
$$

Kamada et al., NPA548 (1992)

Definition of 4B basis states



$$\begin{cases} \mathbf{u}_1 = \frac{1}{2}\left(\mathbf{k}_1 - \mathbf{k}_2\right) \\ \mathbf{u}_2 = \frac{2}{3}\left(\mathbf{k}_3 - \left(\mathbf{k}_1 + \mathbf{k}_2\right)\right) \\ \mathbf{u}_3 = \frac{3}{4}\left(\mathbf{k}_4 - \frac{1}{3}\left(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3\right)\right) \end{cases} \tag{5}$$

$$\begin{cases} \mathbf{v}_1 = \frac{1}{2}\left(\mathbf{k}_1 - \mathbf{k}_2\right) \\ \mathbf{v}_2 = \frac{1}{2}\left(\mathbf{k}_1 + \mathbf{k}_2\right) - \frac{1}{2}\left(\mathbf{k}_3 + \mathbf{k}_4\right) \\ \mathbf{v}_3 = \frac{1}{2}\left(\mathbf{k}_3 - \mathbf{k}_4\right) \end{cases} \tag{6}$$

Subtracted FY coupled equations II - regularization

$$|\,\mathcal{K}_{ij,k}^{\;l}\,\rangle = 2\,\tau(\epsilon_{ij,k}^{\;l})\left[\mathcal{G}^{(3)}_{ij;ik}\,|\,\mathcal{K}_{ik,j}^{\;l}\,\rangle + \mathcal{G}^{(4)}_{ij;ik}\left(|\,\mathcal{K}_{ik,l}^{\;j}\,\rangle + |\,\mathcal{H}_{ik,jl}\,\rangle\right)\right],$$

$$|\,\mathcal{H}_{ij,kl}\,\rangle = \tau(\epsilon_{ij,kl})\,\mathcal{G}^{(4)}_{ij;kl}\left[2\,|\,\mathcal{K}_{kl,i}^{\;j}\,\rangle + |\,\mathcal{H}_{kl,ij}\,\rangle\right].$$

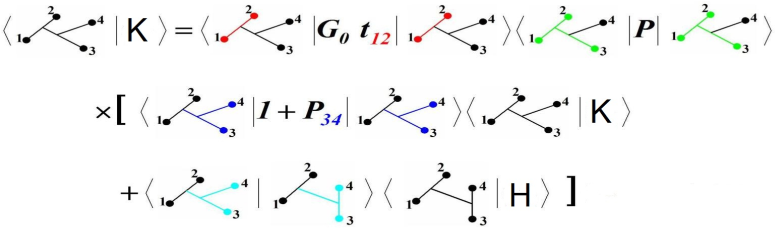
Projected Green function operators for $N = 3$ (RED) or 4 (BLUE):

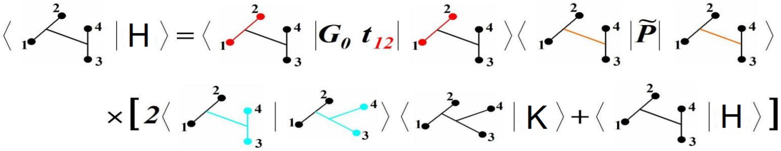
$$\mathcal{G}^{(N)}_{ij;ik} = \langle\chi_{ij}|\frac{1}{E-H_0} - \frac{1}{-\mu_N^2 - H_0}|\chi_{ik}\rangle$$

with μ_3 (avoids the Thomas-collapse) and μ_4 3- and 4- body regularization parameters, respectively.
2-boson scatt. amplitude:

$$t_{ij}(\epsilon) = |\,\chi_{ij}\,\rangle\,\tau_{ij}(\epsilon)\,\langle\,\chi_{ij}\,|, \quad \tau_{ij}^{-1}(\epsilon) = 2\pi^2\left(\frac{1}{a} - \sqrt{-\epsilon}\right), \quad \langle\,\mathbf{p}_{ij}\,|\,\chi_{ij}\,\rangle = 1,$$

Explicit representation of Yakubovsky equations





Hadizadeh et al., FBS40 (2007)

Zero-range interaction

$$V(\mathbf{r}) = (2\pi)^3 \,\lambda\, \delta(\mathbf{r})$$

$$\langle \mathbf{p}|V|\mathbf{p}'\rangle = \lambda\langle \mathbf{p}|\chi\rangle\langle\chi|\mathbf{p}'\rangle; \quad \langle \mathbf{p}|\chi\rangle = \int d^3r\, e^{i\mathbf{p}.\mathbf{r}}\,\delta(\mathbf{r}) = 1 \tag{7}$$

$$\tau(\epsilon) = \left[\lambda^{-1} - \int d^3p\,\frac{1}{\epsilon - p^2}\right]^{-1} \tag{8}$$

$$\lambda^{-1} = \int d^3p\,\frac{1}{B_2 - p^2} \tag{9}$$

$$\tau(\epsilon) = \frac{1}{2\pi^2}\left(\sqrt{-B_2} - \sqrt{-\epsilon}\right)^{-1} = \frac{1}{2\pi^2}\left(\frac{1}{a} - \sqrt{-\epsilon}\right)^{-1} \tag{10}$$

Amorim et al., PRC46 (1992)

Subtracted Yakubovsky integral equations

$$
\begin{aligned}
\mathcal{K}(u_2, u_3) &= 4\pi\, \tau(\epsilon) \int du_2' u_2'^2 \int dx \\
&\times \Big[G_0^{(3)}\Big(\Pi(u_2', u_2), u_2', u_3\Big)\, \mathcal{K}\Big(u_2', u_3\Big) \\
&+ \frac{1}{2} \int dx'\, G_0^{(4)}\Big(\Pi(u_2', u_2), \Pi_2(u_2', u_3, x'), \Pi_3(u_2', u_3, x')\Big)\, \mathcal{K}\Big(\Pi_2(u_2', u_3, x'), \Pi_3(u_2', u_3, x')\Big) \\
&+ \frac{1}{2} \int dx'\, G_0^{(4)}\Big(\Pi(u_2', u_2), \Pi_4(u_2', u_3, x'), \Pi_5(u_2', u_3, x')\Big)\, \mathcal{H}\Big(\Pi_4(u_2', u_3, x'), \Pi_5(u_2', u_3, x')\Big) \Big]
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\mathcal{H}(v_2, v_3) &= 4\pi\, \tau(\epsilon^*) \int dv_3' v_3'^2 \\
&\times \Big[\int dx\, G_0^{(4)}\Big(v_3, \Pi_6(v_2, v_3', x), \Pi_7(v_2, v_3', x)\Big)\, \mathcal{K}\Big(\Pi_6(v_2, v_3', x), \Pi_7(v_2, v_3', x)\Big) \\
&+ G_0^{(4)}\Big(v_3, v_2, v_3'\Big)\, \mathcal{H}\Big(v_2, v_3'\Big) \Big]
\end{aligned}
$$

Subtracted Green's Functions:

$$
G_0^{(N)} = \frac{1}{E - H_0} - \frac{1}{-\mu_N^2 - H_0} \tag{12}
$$

μ_3: 3B scale $\quad$ μ_4: 4B scale

Numerical solution algorithm

$$|K\rangle = G_0 t_{12} P \Big[\Big(1 + P_{34}\Big)|K\rangle + |H\rangle \Big]$$
$$|H\rangle = G_0 t_{12} \tilde{P} \Big[\Big(1 + P_{34}\Big)|K\rangle + |H\rangle \Big] \quad (13)$$

Standard eigenvalue problem

$$\lambda(E).\psi = K(E).\psi; \quad \psi = \begin{pmatrix} K \\ H \end{pmatrix} \quad (14)$$

Searching E to get the solution of coupled Yakubovsky integral equations with $\lambda = 1$.

Integration: Gaussian quadrature

- magnitudes of Jacobi momenta: 80-160 mesh points
- polar angles: 40 mesh points

Eigenvalue problem: Lanczos type method

- Iterative orthogonal vectors (IOV) (Stadler PRC44 2319)
- ARPACK Fortran library (http://www.caam.rice.edu/software/ARPACK/)

Dimension of eigenvalue problem after using Lanczos technique:
# of iterations-1 $\sim$ 10!

Multidimensional interpolations: Cubic-Hermit Splines

- high computational speed and accuracy (Huber FBS22 107)

Example

Convergence of 1^{st} and 2^{nd} exited tetramer energies for $\frac{\mu_4}{\mu_3} = 300$:

$$u_i = \frac{1 + x_i}{c_1(1 - x_i) + c_2 x_i}; \quad c_1 \equiv \frac{\mu_4}{\mu_3},\ c_2 = 0.4$$

$$x_i \in [-1, +1] \implies u_i \in [0, 0.003] + [0.003, 5]$$