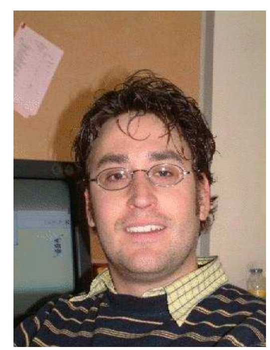
Leaky Repeated Interaction Quantum Systems *

Alain JOYE

* Joint work with

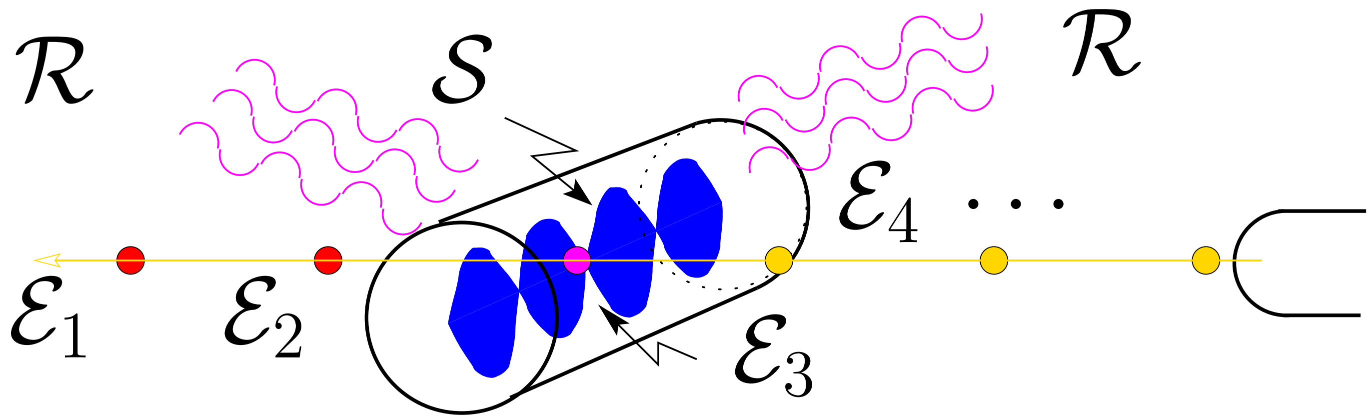
AHP 2010



Laurent BRUNEAU (Université de Cergy) & Marco MERKLI (Memorial University)

Motivation

One-atom maser — Walther et al '85, Haroche et al '92



- $\mathcal{S}$: one mode of E-M field in a cavity
- $\mathcal{E}_k$: atom $\# k$ interacting with the mode
- $\mathcal{C}$: sequence of atoms passing through the cavity
- $\mathcal{R}$: environment responsible for losses

Ideal RIQS used as simple models — Vogel et al 93, Wellens et al 00, BJM 06, Bruneau Pillet 09

Random RIQS to model fluctuations — BJM 08

Leaky RIQS to account for losses

The Formal Model

Quantum system $\mathcal{S}$:

- Finite dimensional system, driven by Hamiltonian $H_{\mathcal{S}}$ on $\mathfrak{H}_{\mathcal{S}}$, s.t. $\sigma(H_{\mathcal{S}}) = \{e_1, \cdots, e_d\}$.

Chain $\mathcal{C}$ of identical quantum sub-systems $\mathcal{E}_k \equiv \mathcal{E}$, $k = 1, 2, \cdots$:

$$\mathcal{C} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 + \cdots$$

- Each $\mathcal{E}_k$ is driven by the Hamiltonian $H_{\mathcal{E}\,k} \equiv H_{\mathcal{E}}$ on $\mathfrak{H}_{\mathcal{E}_k} \equiv \mathfrak{H}_{\mathcal{E}}$, $\dim \mathfrak{H}_{\mathcal{E}} \leq \infty$
- The chain $\mathcal{C}$ is driven by $H_{\mathcal{C}} \equiv H_{\mathcal{E}\,1} + H_{\mathcal{E}\,2} + \cdots$ on $\mathfrak{H}_{\mathcal{C}} \equiv \mathfrak{H}_{\mathcal{E}_1} \otimes \mathfrak{H}_{\mathcal{E}_2} \otimes \cdots$, with $[H_{\mathcal{E}\,j}, H_{\mathcal{E}\,k}] = 0, \ \forall j, k.$

Fermionic reservoir $\mathcal{R}$:

- ∞ -ly extended gas of indep. fermions at temperature β^{-1} , driven by "$H_{\mathcal{R}}$" on "$\mathfrak{H}_{\mathcal{R}}$" .

The Formal Model

Complete system $\mathcal{S}+\mathcal{R}+\mathcal{C}$

- Formal Hilbert space $\mathfrak{H}_{\mathcal{S}} \otimes "\mathfrak{H}_{\mathcal{R}}" \otimes \mathfrak{H}_{\mathcal{C}}$

Interaction $\mathcal{S}-\mathcal{C}$

- $W_{\mathcal{S}\mathcal{E}}$ operator on $\mathfrak{H}_{\mathcal{S}} \otimes \mathfrak{H}_{\mathcal{E}_k}$, $k=1,2,\cdots$.

Interaction $\mathcal{S}-\mathcal{R}$

- $W_{\mathcal{S}\mathcal{R}}$ operator on $\mathfrak{H}_{\mathcal{S}} \otimes "\mathfrak{H}_{\mathcal{R}}"$.

The Formal Model

Complete system $\mathcal{S} + \mathcal{R} + \mathcal{C}$

- Formal Hilbert space $\mathfrak{H}_{\mathcal{S}} \otimes \text{"}\mathfrak{H}_{\mathcal{R}}\text{"} \otimes \mathfrak{H}_{\mathcal{C}}$

Interaction $\mathcal{S} - \mathcal{C}$

- $W_{\mathcal{S}\mathcal{E}}$ operator on $\mathfrak{H}_{\mathcal{S}} \otimes \mathfrak{H}_{\mathcal{E}_k}$, $k = 1, 2, \cdots$.

Interaction $\mathcal{S} - \mathcal{R}$

- $W_{\mathcal{S}\mathcal{R}}$ operator on $\mathfrak{H}_{\mathcal{S}} \otimes \text{"}\mathfrak{H}_{\mathcal{R}}\text{"}$.

Evolution Let $\tau > 0$ be a duration, $\lambda = (\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}}) \in \mathbb{R}^2$ be couplings

For $t = (m-1)\tau + s, \;\; 0 \leq s < \tau$,

- $\mathcal{S}$, $\mathcal{R}$ and $\mathcal{E}_m$ are driven by $H_{\mathcal{S}} + \text{"}H_{\mathcal{R}}\text{"} + H_{\mathcal{E}} + \lambda_{\mathcal{R}} W_{\mathcal{S}\mathcal{R}} + \lambda_{\mathcal{E}} W_{\mathcal{S}\mathcal{E}}$
- $\mathcal{E}_k$ evolve freely with $H_{\mathcal{E}}$, $\forall\, k \neq m$

Leaky Repeated Interactions Quantum Systems

Pictorially

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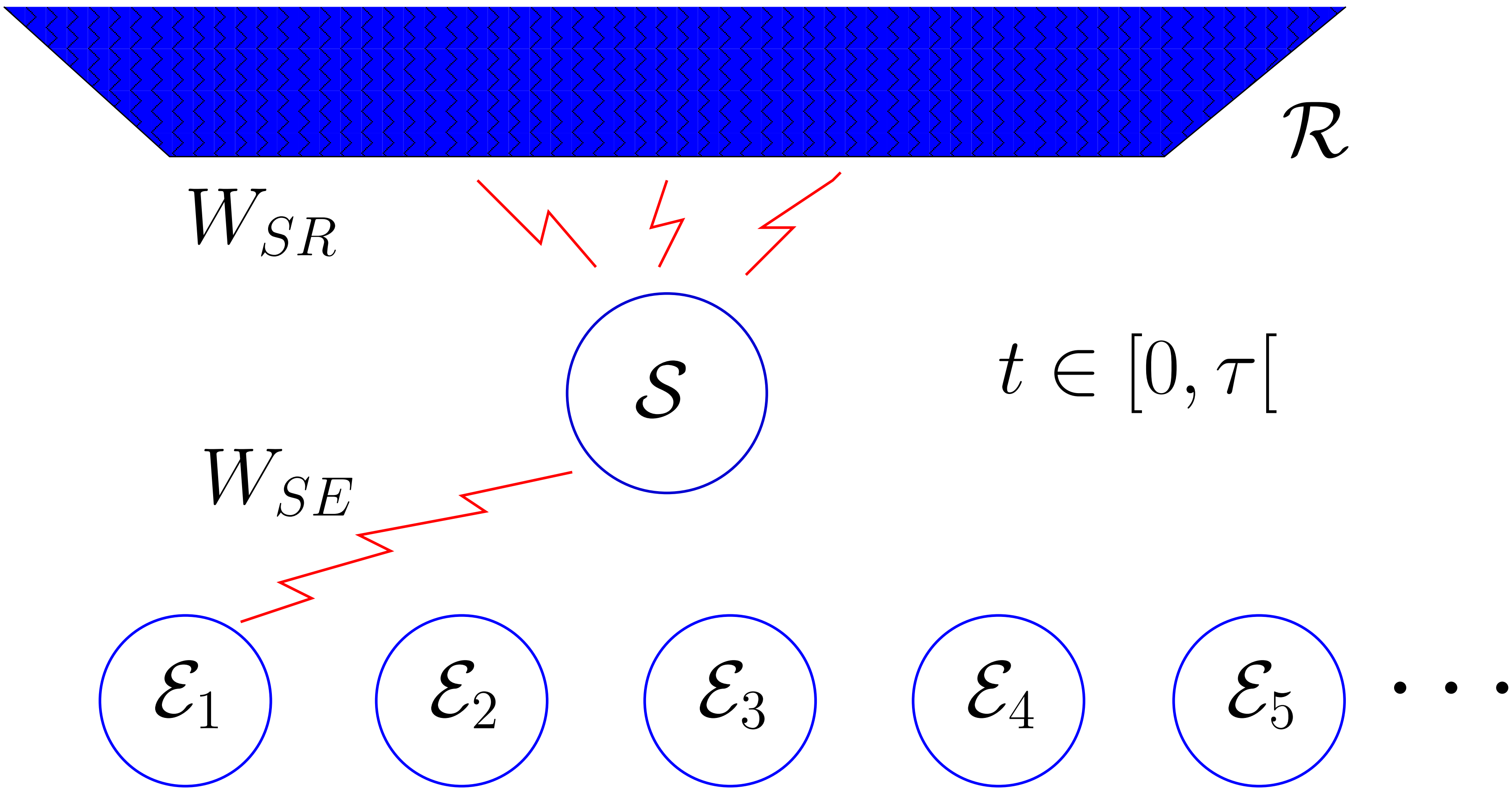

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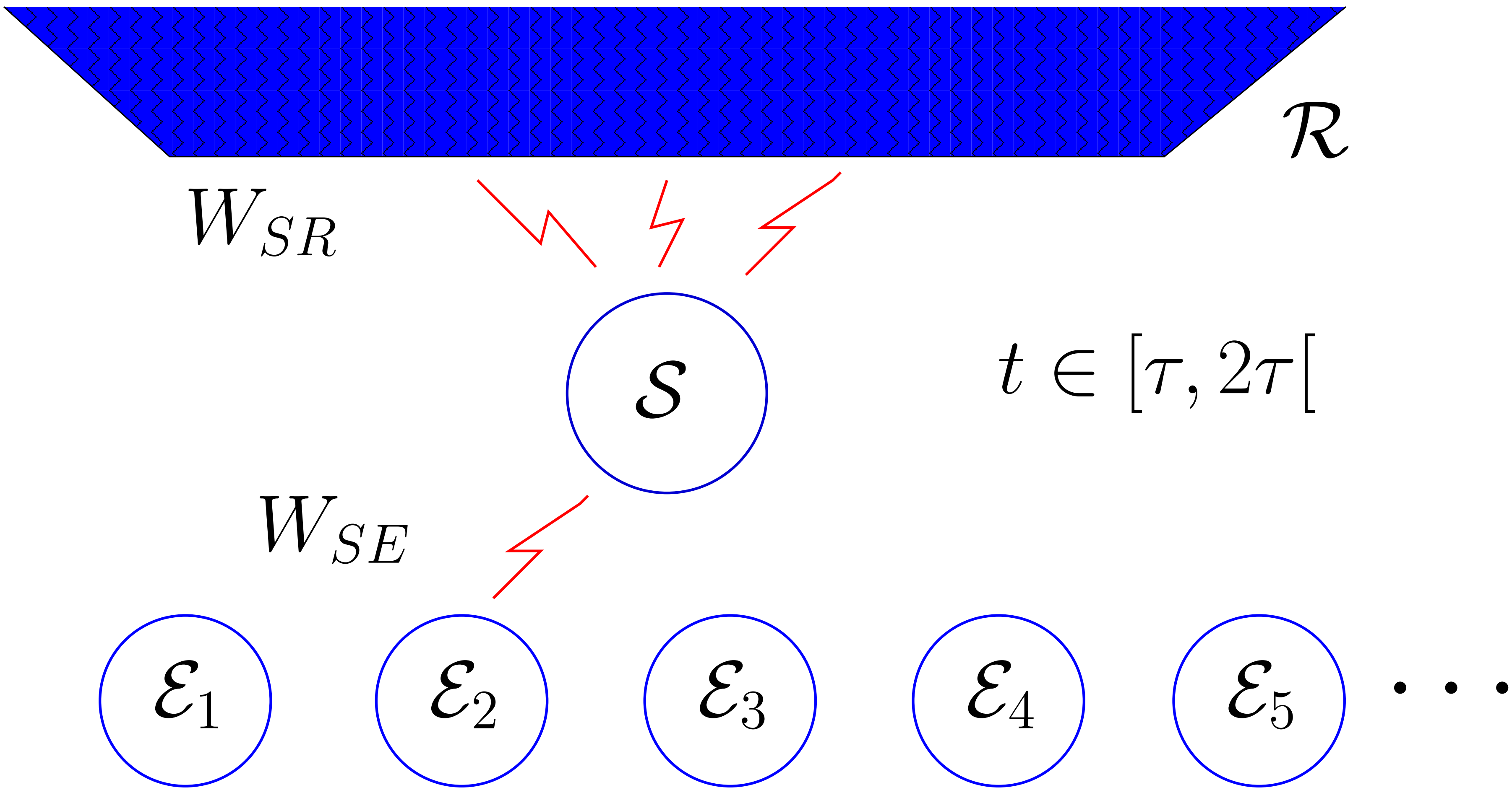

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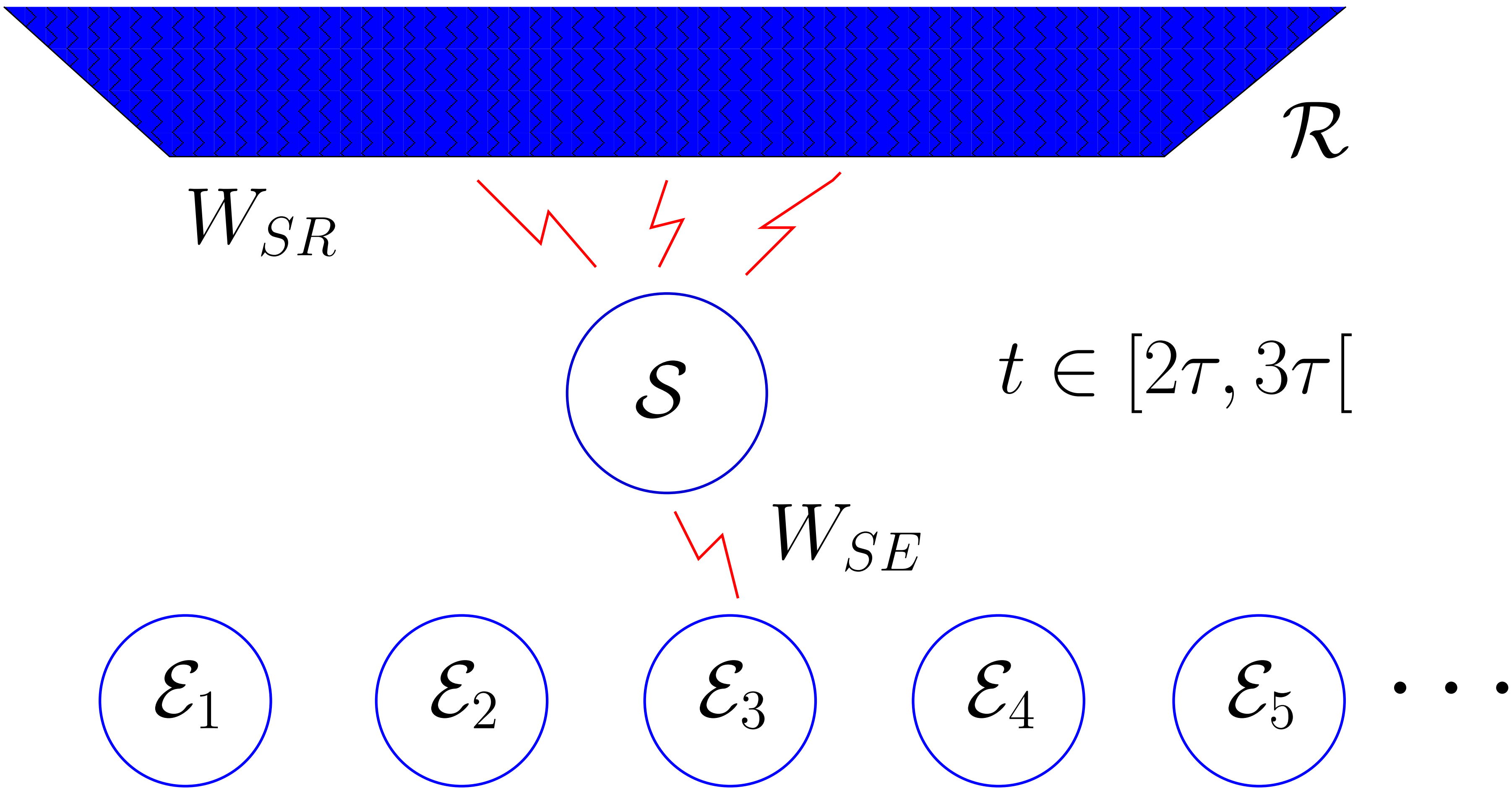

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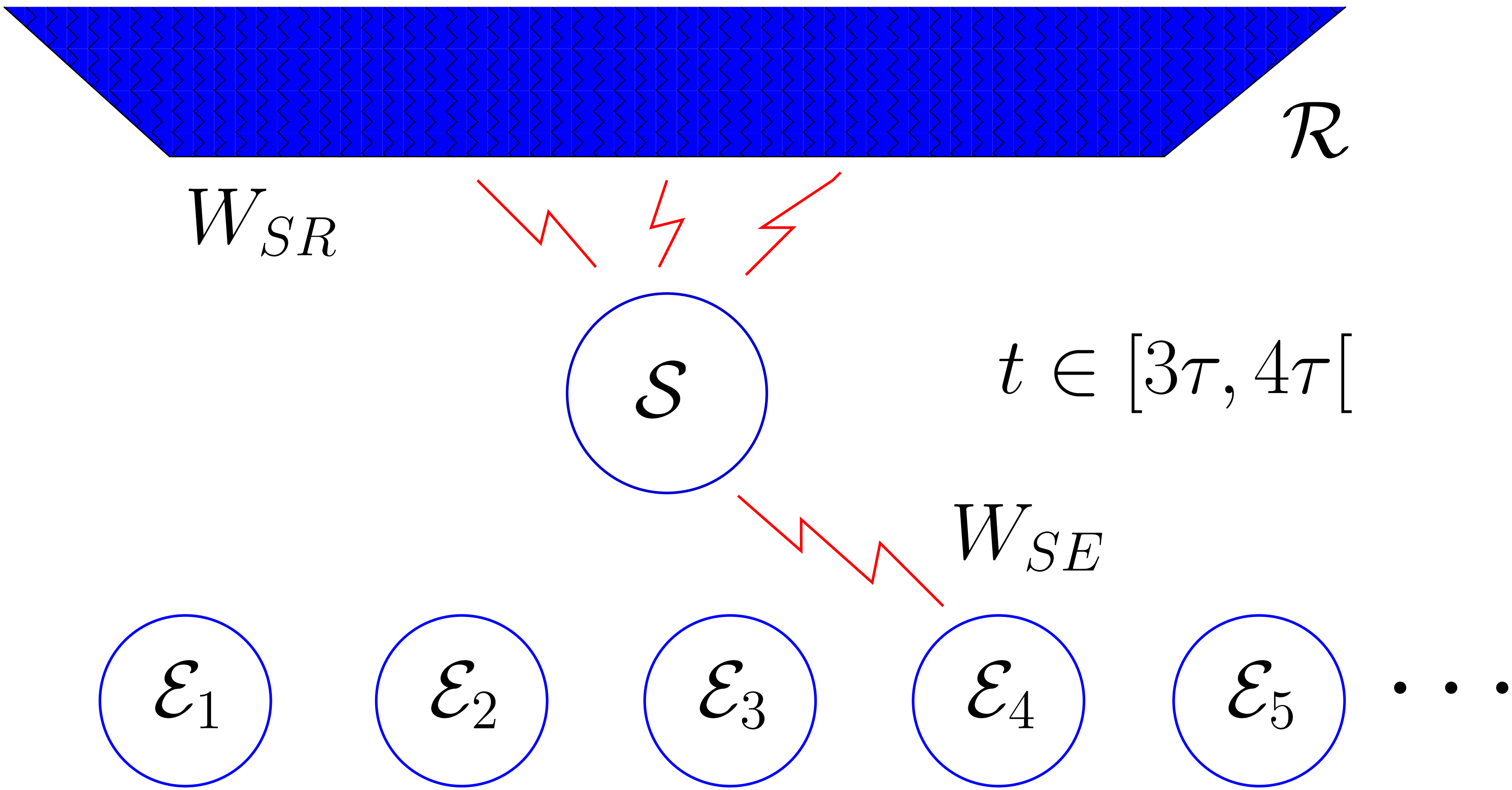

Questions

Large times asymptotics

Let $A = A_{SR} \otimes \mathbb{I}_{\mathcal{C}} \in \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes "\mathfrak{H}_{\mathcal{R}}" \otimes \mathfrak{H}_{\mathcal{C}})$ an observable acting on $\mathcal{S} + \mathcal{R}$

Let $\alpha^t(A)$ be its Heisenberg evolution, at time $t = m\tau$

Let $\rho : \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes "\mathfrak{H}_{\mathcal{R}}" \otimes \mathfrak{H}_{\mathcal{C}}) \to \mathbb{C}$ be a state (“density matrix”)

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Let $\rho : \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes \text{"}\mathfrak{H}_{\mathcal{R}}\text{"} \otimes \mathfrak{H}_{\mathcal{C}}) \to \mathbb{C}$ be a state (“density matrix”)

- Existence of $\lim_{m\to\infty} \rho \circ \alpha^{m\tau}(A) = \rho^+(A)$?

  Dependence of $\rho^+(A)$ on the coupling constants $\lambda = (\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})$?

- Exchanges between $\mathcal{R}$ and $\mathcal{C}$ through $\mathcal{S}$?

  Energy variations, Entropy production, 2 nd law of thermodynamics ?

- Non-trivial examples ?

Remark :

If $\lambda_{\mathcal{E}} = 0$, then $\mathcal{S} + \mathcal{R} \;\Rightarrow$ return to equilibrium — Jaksic-Pillet 96

If $\lambda_{\mathcal{R}} = 0$, then $\mathcal{S} + \mathcal{C} \;\Rightarrow$ convergence to a NESS — Bruneau-J.-Merkli 06

Liouvillian a.k.a. Positive Temperature Hamiltonians

Density matrix on $\mathfrak{H}$ $\quad\to\quad$ pure state on $\mathcal{H} = \mathfrak{H} \otimes \mathfrak{H}$:

state	$\rho = \sum \lambda_j \vert\varphi_j\rangle\langle\varphi_j\vert$	$\to$	$\Psi_\rho = \sum \sqrt{\lambda_j}\varphi_j \otimes \varphi_j$
observable	$A \in \mathcal{B}(\mathfrak{H})$	$\to$	$\Pi(A) = A \otimes \mathbb{I}_{\mathfrak{H}} \in \mathcal{B}(\mathcal{H})$
so that	$\mathrm{Tr}_{\mathfrak{H}}(\rho A)$	$=$	$\mathrm{Tr}_{\mathcal{H}}(\vert\Psi_\rho\rangle\langle\Psi_\rho\vert\Pi(A))$

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Dynamics

$$A \in \mathcal{B}(\mathfrak{H}) \quad \mapsto \quad \alpha^t(A) = e^{itH} A e^{-itH} \in \mathcal{B}(\mathfrak{H})$$

Liouville operator

$$L = H \otimes \mathbb{I}_{\mathfrak{H}} - \mathbb{I}_{\mathfrak{H}} \otimes H$$

s.t.

$$\Pi(\alpha^t(A)) = e^{itL}\Pi(A)e^{-itL}$$

Invariant state $\rho \leftrightarrow \Psi_\rho$ s.t.

$$L\Psi_\rho = 0$$

Notation $\quad \Pi(A) \simeq A \in \mathfrak{M}$

Quasi-free Fermi gas at β^{-1} Araki-Wyss 64 + Jaksic-Pillet 02

Ingredients: $\tilde{\mathfrak{h}} = L^2(\mathbb{R}^+, L^2(S^2, d\sigma))$ one particle Hilbert space

Hamiltonian $\tilde{h}$: $(\tilde{h}\tilde{f})(r,\sigma) = r^2 \tilde{f}(r,\sigma)$, $(r,\sigma) \in \mathbb{R}^+ \times S^2$,

State ω_β on $\Gamma_-(\tilde{\mathfrak{h}})$ fully characterized by

$$\omega_\beta(a^*(\tilde{g})a(\tilde{f})) = \langle \tilde{f} | (1 + e^{\beta \tilde{h}})^{-1} \tilde{g} \rangle$$

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Enlarged Hilbert space

$$\mathcal{H}_\mathcal{R} = \Gamma_-(\mathfrak{h}),\ \mathfrak{h} = L^2(\mathbb{R}, L^2(S^2, d\sigma))$$

β-dep. creat., annih. op's $a^*_\beta(g)$, $a_\beta(g)$, where $g \in \mathfrak{h} \leftrightarrow \tilde{g} \in \tilde{\mathfrak{h}}$

Liouvillean

$L_\mathcal{R} = d\Gamma(h)$, with h s.t.

$$(hf)(s,\sigma) = sf(s,\sigma),\ s \in \mathbb{R},\ \forall f \in \mathfrak{h} = L^2(\mathbb{R}, L^2(S^2, d\sigma))$$

Quasi-free Equilibrium State

$|\Psi_R\rangle\langle\Psi_\mathcal{R}|$, $\Psi_\mathcal{R}$ vacuum of $\Gamma_-(\mathfrak{h})$

Dynamics

Repeated interaction Schrödinger dynamics

For any $m \in \mathbb{N}$, if $t = m\tau$ and $\psi \in \mathcal{H}$,

$$U(m)\psi := e^{-i\tilde{L}_m} e^{-i\tilde{L}_{m-1}} \cdots e^{-i\tilde{L}_1} \psi$$

where the generator for the duration τ is

$$\tilde{L}_m = \tau L_m + \tau \sum_{k \neq m} L_{\mathcal{E},k}$$

with

$$\begin{cases} L_m = L_{\mathcal{S}} + L_{\mathcal{R}} + L_{\mathcal{E}} + V_m & \text{on } \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}} \otimes \mathcal{H}_{\mathcal{E}_m} \quad \text{coupled} \\ V_m = \lambda_{\mathcal{R}} V_{\mathcal{S}\mathcal{R}} + \lambda_{\mathcal{E}} V_{\mathcal{S}\mathcal{E}} & \\ L_{\mathcal{E},k} = L_{\mathcal{E}} & \text{on } \mathcal{H}_{\mathcal{E}_k} \quad \text{free} \end{cases}$$

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To be studied

Let $\varrho \in \mathcal{B}_1(\mathcal{H})$ be a state on $\mathcal{H}$ and $A_{\mathcal{S}\mathcal{R}}$ an observable on $\mathcal{S} + \mathcal{R}$

$$m \mapsto \varrho(U^*(m) A_{\mathcal{S}\mathcal{R}} U(m)) \equiv \varrho(\alpha^{m\tau}(A_{\mathcal{S}\mathcal{R}})), \quad \text{as} \quad m \to \infty$$

Tracing out the atoms

- Initial state $\varrho_0 \leftrightarrow \Psi_0 = \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{R}} \otimes \Psi_{\mathcal{C}}$ with
  $$\Psi_{\mathcal{C}} = \Psi_{\mathcal{E}_1} \otimes \Psi_{\mathcal{E}_2} \otimes \cdots \in \mathcal{H}_{\mathcal{C}}$$
- $U(m) = e^{-i\tilde{L}_m} \cdots e^{-i\tilde{L}_1}$
- $\tilde{L}_j \longrightarrow K_j := L_{\mathcal{S}} + L_{\mathcal{R}} + L_{\mathcal{E}} + \tilde{V}_j$ with $\tilde{V}_j$ explicit s.t.
  K_j implements the same dynamics and $K_j \Psi_0 = 0$
- Set $\Psi_{\mathcal{SR}} = \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{R}} \in \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}} \equiv \mathcal{H}_{\mathcal{SR}}$
  $P = \mathbb{I}_{\mathcal{S}R} \otimes |\Psi_{\mathcal{C}}\rangle\langle\Psi_{\mathcal{C}}|$ proj. on $\mathcal{H}_{\mathcal{SR}}$
- $A_{\mathcal{S}R}$ acts on $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}$ only
- Identical atoms

$$\boxed{\varrho_0(\alpha^{n\tau}(A_{\mathcal{S}R})) = \langle \Psi_{\mathcal{S}R} | M^n A_{\mathcal{S}R} \Psi_{\mathcal{S}R} \rangle}$$

Markov evolution on $\mathcal{H}_{\mathcal{S}R}$

$M \simeq P e^{iK} P$ on $\mathcal{H}_{\mathcal{S}R}$ with $K = L_{\mathcal{S}} + L_{\mathcal{R}} + L_{\mathcal{E}} + \lambda_{\mathcal{R}} \tilde{V}_{\mathcal{SR}} + \lambda_{\mathcal{E}} \tilde{V}_{\mathcal{SE}}$

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Reduced Dynamical Operators

$M \in \mathcal{B}(\mathcal{H}_{\mathcal{SR}})$ s.t. $\begin{cases} M\Psi_{\mathcal{SR}} = \Psi_{\mathcal{SR}} \\ \|M^n \varphi\| \le C(\varphi), \quad \forall n \in \mathbb{N}, \quad \forall \varphi \text{ in a dense set} \end{cases}$

Spectral Properties of RDO's

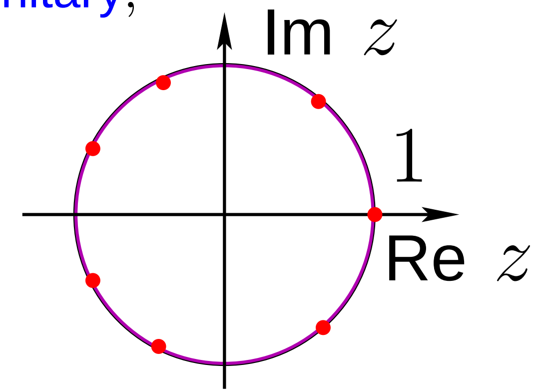
RDO

$$M = M_{(\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})}$$

Uncoupled case

$$M_{(0,0)} = e^{i\tau(L_{\mathcal{S}}+L_{\mathcal{R}})} \text{ unitary},$$

- eigenvalues of $M_{(0,0)}$: $\{e^{i\tau(e_k - e_l)}\}_{k,l}$
- 1 is $\dim \mathfrak{h}_{\mathcal{S}}$-fold degenerate
- ess spec $M_{(0,0)} = \mathbb{S}^1$



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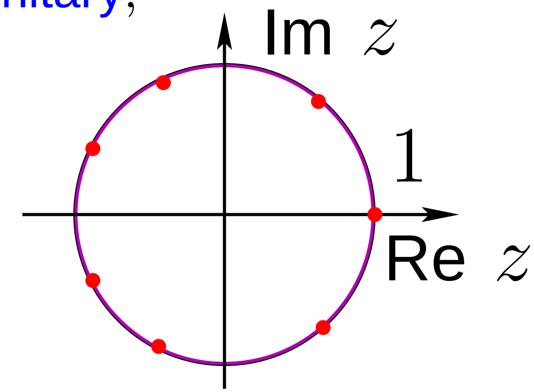

$(\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}}) \neq (0,0)$ $\Rightarrow$ Perturbation of embedded eigenvalues

$L_{\mathcal{R}} = d\Gamma(h)$ with h mult. by s on $L^2(\mathbb{R}, L^2(S^2, d\sigma))$ is suitable for translation analyticity Avron-Herbst 77

Translation Analyticity

Translation Group

$$\mathbb{R} \ni \theta \mapsto (e^{-\theta\partial_s} f)(s) = f(s-\theta), \quad \forall f \in L^2(\mathbb{R}, L^2(S^2, d\sigma))$$

s.t. $T(\theta) = \Gamma(e^{-\theta\partial_s})$ on $\Gamma_-(L^2(\mathbb{R}, L^2(S^2, d\sigma)))$

Assumption (A)

$\mathbb{R} \ni \theta \mapsto \tilde{V}_{\mathcal{SR}}(\theta) := T(\theta)^{-1}\tilde{V}_{\mathcal{SR}}T(\theta)$ admits an analytic extension to $\kappa_{\theta_0} = \{z \in \mathbb{C} \mid 0 < \text{Im } z < \theta_0\}$

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Recall

$$M = P \exp(iK) P, \quad \text{where}$$
$$K = \tau(L_0 + \lambda_{\mathcal{R}} \tilde{V}_{\mathcal{SR}} + \lambda_{\mathcal{E}} \tilde{V}_{\mathcal{SE}}), \quad L_0 = L_{\mathcal{S}} + L_{\mathcal{R}} + L_{\mathcal{E}}$$

Theorem

The following op's are analytic $\forall \theta \in \kappa_{\theta_0}$

$$K(\theta) = \tau(L_0 + \theta N + \lambda_{\mathcal{R}} \tilde{V}_{\mathcal{SR}}(\theta) + \lambda_{\mathcal{E}} \tilde{V}_{\mathcal{SE}}) \quad \text{on } D(L_0) \cap D(N),$$
$$M(\theta) = P \exp(iK(\theta)) P \in \mathcal{B}(\mathcal{H}_{\mathcal{SR}})$$

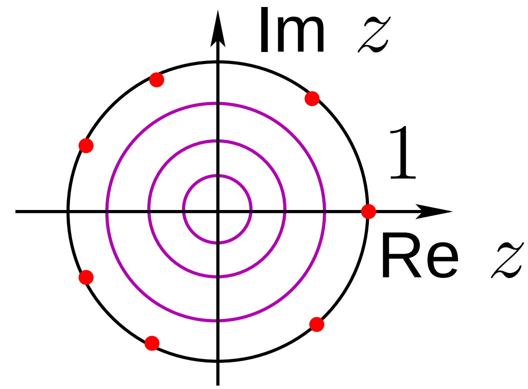
Translation Analyticity

Consequences

Discrete e.v. of $M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta)$ are θ-independent

Spectrum of $M_{(0,0)}(\theta) = \exp(i\tau(L_{\mathcal{S}} + L_{\mathcal{R}} + \theta N))$

$$\begin{cases} \text{eigenvalues of } M_{(0,0)}(\theta) : \quad \{e^{i\tau(e_k - e_l)}\}_{k,l} \\ 1 \text{ is } \dim \mathfrak{h}_{\mathcal{S}}\text{-fold degenerate} \\ \text{ess spec } M_{(0,0)}(\theta) = \cup_{n=1}^{\infty}\{|z| = e^{-n\tau \text{Im}\theta}\} \end{cases}$$



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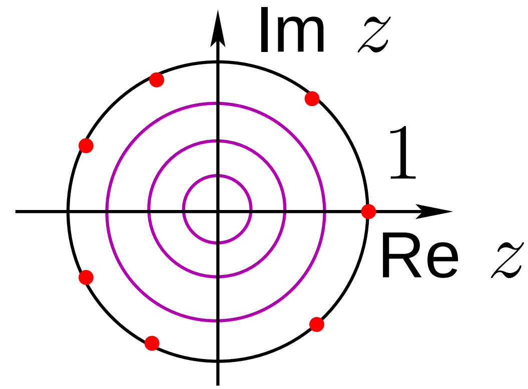
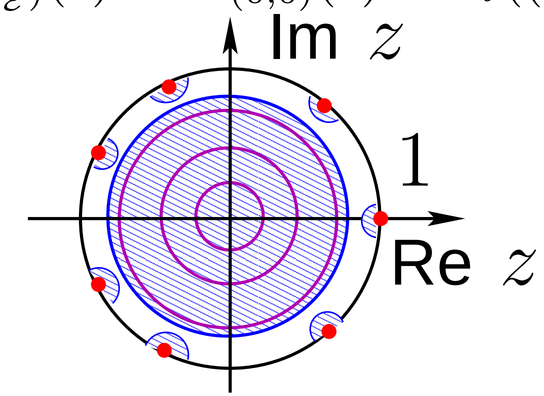


Perturbative approach

$$M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta) = M_{(0,0)}(\theta) + O_{\theta}((\lambda_{\mathcal{R}},\lambda_{\mathcal{E}}))$$



Lemma

$\|(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})\| < \lambda_0(\theta) \ \Rightarrow \sigma(M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta))$:

Asymptotic State

Analytic observables

$$A_{\mathcal{SR}} \text{ s.t. } A_{\mathcal{SR}}(\theta) = T(\theta)^{-1} A_{\mathcal{SR}} T(\theta) \text{ analytic in } \kappa_{\theta_0}$$

Note: For $A_{\mathcal{SR}}$ analytic,

$$\begin{aligned} \varrho_0(\alpha^{m\tau}(A_{SR})) &= \langle \Psi_{SR} | M^m A_{SR} \Psi_{SR} \rangle \\ &= \langle \Psi_{SR} | M(\theta)^m A_{SR}(\theta) \Psi_{SR} \rangle \end{aligned}$$

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Assumption (FGR)

$$\exists \theta_1 \in \kappa_{\theta_0}, \lambda_0(\theta_1) > 0 \text{ s.t. } \|(\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})\| < \lambda_0(\theta_1) \text{ implies}$$

$$\sigma(M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1)) \cap \mathbb{S} = \{1\} \text{ and } 1 \text{ is simple}$$

Consequences

$$\lim_{n\to\infty} M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1)^n = P_{1,M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1)} = |\Psi_{\mathcal{SR}}\rangle\langle\psi^*_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1)|$$

Main Result

Theorem

Assume (A) and (FRG). For any state ϱ on $\mathcal{H}_{\mathcal{SR}} \otimes \mathcal{H}_{\mathcal{C}}$ and any analytic observable A_{SR}

$$\begin{aligned}\lim_{n\to\infty} \varrho(\alpha^{\tau n}_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(A_{SR})) &= \langle \psi^*_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1)|A_{SR}(\theta_1)\Psi_{SR}\rangle \\ &\equiv \rho^+_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(A_{SR}).\end{aligned}$$

Instantaneous Observables

$\exists$ Upgrade to more general observables !

Application

- $\mathcal{S}$ and $\mathcal{E}$ two-level syst. with e.v. $\{0, E_\mathcal{S}\}$, resp. $\{0, E_\mathcal{E}\}$
- $\mathcal{R}$ Fermi gas at $\beta_\mathcal{R}$, equil. state $\omega_{\beta_\mathcal{R}}$
- $W_{\mathcal{S}\mathcal{E}} = a_S \otimes a_E^* + a_S^* \otimes a_E$
- $\omega_\mathcal{S} = \mathbb{I}$, i.e. "trace", $\omega_{\beta,\mathcal{E}} = e^{-\beta_\mathcal{E} H_\mathcal{E}}/Z_{\beta_\mathcal{E}}$
- $W_{\mathcal{S}\mathcal{R}} = \sigma_x \otimes (a_R^*(f) + a_R(f))$, $f \in L^2(\mathbb{R}^+, L^2(S^2, d\sigma))$ “regular”.

Perturbation theory

1) If $\|f(\sqrt{E_\mathcal{S}})\| > 0$ and $\tau(E_\mathcal{S} - E_\mathcal{E}) \neq 2\pi\mathbb{Z}^*$, then (FGR) holds

2) The asymptotic state ω_+ is given by

$$\omega_+ = (\gamma\, \omega_{\beta_\mathcal{R},\mathcal{S}} + (1-\gamma)\, \omega_{\tilde{\beta}_\mathcal{E},\mathcal{S}}) \otimes \omega_{\beta_\mathcal{R}} + \mathcal{O}_{\theta_1,\beta_\mathcal{R},\cdots}(\|(\lambda_\mathcal{R}, \lambda_\mathcal{E})\|)$$

with

$$\gamma = \frac{\lambda_\mathcal{R}^2 \pi \sqrt{E_\mathcal{S}}\|f(\sqrt{E_\mathcal{S}})\|^2}{\lambda_\mathcal{R}^2 \pi \sqrt{E_\mathcal{S}}\|f(\sqrt{E_\mathcal{S}})\|^2 + \lambda_\mathcal{E}^2 \tau \,\text{sinc}^2(\tau(E_\mathcal{S} - E_\mathcal{E})/2)}, \qquad \tilde{\beta}_\mathcal{E} = \beta_\mathcal{E} \frac{E_\mathcal{E}}{E_\mathcal{S}}.$$

Energy

$\alpha^m(\tilde{L}_m)$ represents the "total energy" for times $t \in [(m-1)\tau, m\tau)$.

Variation between $(m+1)\tau$ and $m\tau$,

$$\Delta E^{tot}(m) \quad = \quad \alpha^{m+1}(\tilde{L}_{m+1}) - \alpha^m(\tilde{L}_m) = \alpha^m(V_{m+1} - V_m)$$

Similarly

$$\begin{aligned} \Delta E^S(m) &= \alpha^{m+1}(L_{\mathcal{S}}) - \alpha^m(L_{\mathcal{S}}) \\ \Delta E^R(m) &= \alpha^{m+1}(L_{\mathcal{R}}) - \alpha^m(L_{\mathcal{R}}) \\ \Delta E^C(m) &= \alpha^{m+1}(L_{\mathcal{E}_{m+1}}) - \alpha^m(L_{\mathcal{E}_m}) \end{aligned}$$

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Asymptotic energy variation per unit time

$$dE_+^\# := \lim_{N\to\infty} \rho\left(\frac{\sum_{m=1}^N \Delta E^\#(m)}{N}\right)$$ exists under (A) and (FRG)

Property

$$dE_+^S = 0, \quad dE_+^{tot} = dE_+^R + dE_+^C$$

Entropy production

Let $\Psi_{\mathcal{S}}$ and $\Psi_{\mathcal{E}}$ correspond to Gibbs states at temperatures $\beta_{\mathcal{S}}$ and $\beta_{\mathcal{E}}$

Relative entropy ϱ and ϱ_0 are states on $\mathfrak{M}$, generalization of

$$Ent(\varrho|\varrho_0) = \text{Tr}\,(\rho(\ln\rho - \ln\rho_0)) \geq 0$$ Araki '75

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Variation of relative entropy w.r.t. KMS states Jaksic, Pillet '03

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$$\begin{aligned} \Delta S(m) &:= Ent(\varrho \circ \alpha^m|\varrho_0) - Ent(\varrho|\varrho_0) \\ &= \varrho(\alpha^m[\beta_{\mathcal{S}} L_{\mathcal{S}} + \beta_{\mathcal{R}} L_{\mathcal{R}} + \beta_{\mathcal{E}} \sum_{j=1}^{m} L_{\mathcal{E},j}] - \beta_{\mathcal{S}} L_{\mathcal{S}} - \beta_{\mathcal{R}} L_{\mathcal{R}} - \sum_{j=1}^{m} \beta_{\mathcal{E}_j} L_{\mathcal{E},j}) \end{aligned}$$

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Asymptotic entropy production rate

$$dS^+ := \lim_{N\to\infty}\frac{\Delta S(N)}{N} \quad \text{exists and}$$

$$dS^+ = \beta_{\mathcal{E}}dE_+^{\mathcal{C}} + \beta_{\mathcal{R}}dE_+^{\mathcal{R}} \quad 2^{nd} \text{ law}$$

For the Model

With $\tilde{\beta}_{\mathcal{E}} = \beta_{\mathcal{E}} \frac{E_{\mathcal{E}}}{E_{\mathcal{S}}}$ and $\lambda = (\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})$ small

$$\begin{aligned} dE_+^{\mathcal{C}} &= \kappa E_{\mathcal{E}} \left(e^{-\beta_{\mathcal{R}} E_{\mathcal{S}}} - e^{-\tilde{\beta}_{\mathcal{E}} E_{\mathcal{S}}} \right) + O(\lambda^3), \\ dE_+^{\mathcal{R}} &= \kappa E_{\mathcal{S}} \left(e^{-\tilde{\beta}_{\mathcal{E}} E_{\mathcal{S}}} - e^{-\beta_{\mathcal{R}} E_{\mathcal{S}}} \right) + O(\lambda^3), \\ dE_+^{\mathrm{tot}} &= \kappa (E_{\mathcal{E}} - E_{\mathcal{S}}) \left(e^{-\beta_{\mathcal{R}} E_{\mathcal{S}}} - e^{-\tilde{\beta}_{\mathcal{E}} E_{\mathcal{S}}} \right) + O(\lambda^3), \\ dS_+ &= \kappa (\tilde{\beta}_{\mathcal{E}} E_{\mathcal{S}} - \beta_{\mathcal{R}} E_{\mathcal{S}}) \left(e^{-\beta_{\mathcal{R}} E_{\mathcal{S}}} - e^{-\tilde{\beta}_{\mathcal{E}} E_{\mathcal{S}}} \right) + O(\lambda^3), \end{aligned}$$

where

$$\kappa = Z^{-1}_{\beta_{\mathcal{R}},\mathcal{S}} Z^{-1}_{\tilde{\beta}_{\mathcal{E}},\mathcal{S}} \frac{\lambda_{\mathcal{R}}^2 \frac{\pi}{2} \sqrt{E_{\mathcal{S}}} \|f(\sqrt{E_{\mathcal{S}}})\|^2 \lambda_{\mathcal{E}}^2 \tau \mathrm{sinc}^2(\tau(E_{\mathcal{S}} - E_{\mathcal{E}})/2)}{\lambda_{\mathcal{R}}^2 \pi \sqrt{E_{\mathcal{S}}} \|f(\sqrt{E_{\mathcal{S}}})\|^2 + \lambda_{\mathcal{E}}^2 \tau \mathrm{sinc}^2(\tau(E_{\mathcal{S}} - E_{\mathcal{E}})/2)}$$

Remarks:

- $\kappa > 0$ and $\kappa = 0 \Leftrightarrow \lambda_{\mathcal{R}} \lambda_{\mathcal{E}} = 0$
- $dE_+^{\mathcal{C}} > 0$ if and only $T_{\mathcal{R}} = \beta_{\mathcal{R}}^{-1} > \tilde{T}_{\mathcal{E}} = \tilde{\beta}_{\mathcal{E}}^{-1}$ (leading order).
- $dS^+ \geq 0$ and $dS^+ = 0$ if and only if $T_{\mathcal{R}} = \tilde{T}_{\mathcal{E}}$ (leading order).
- dE_+^{tot} has no sign.