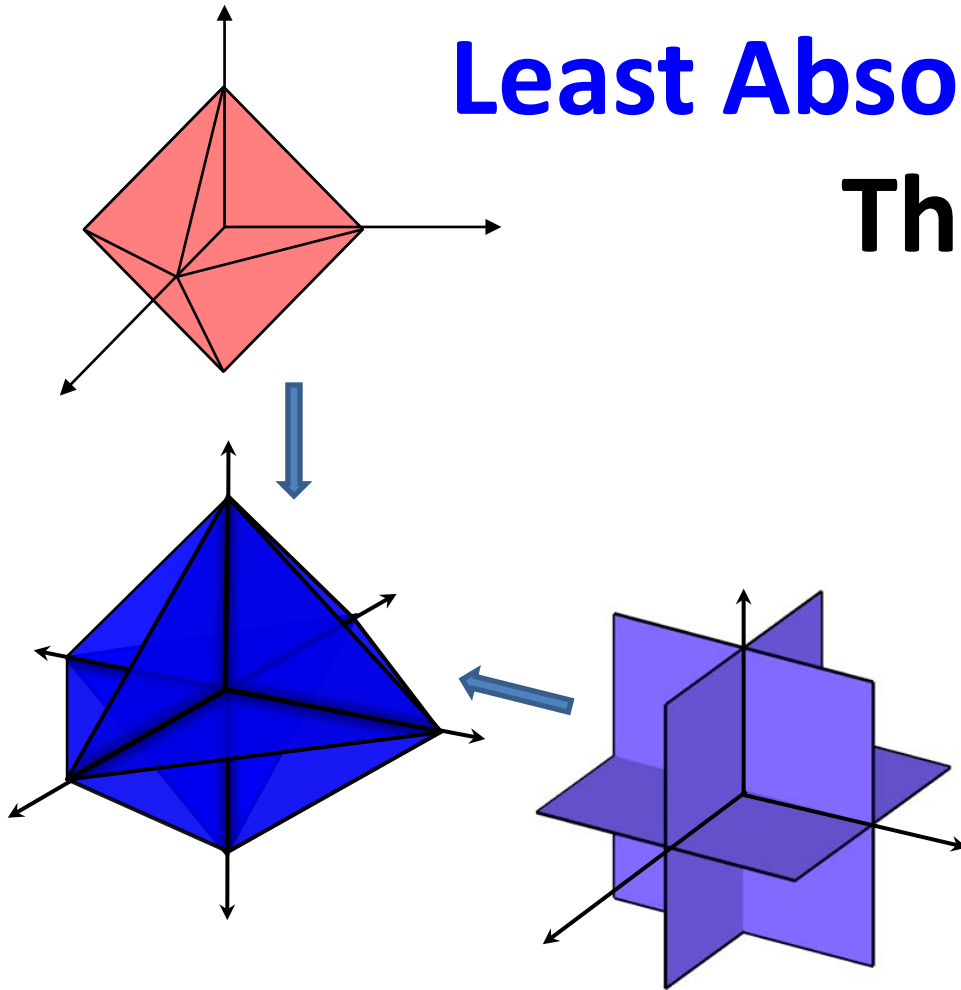


Combinatorial Selection and Least Absolute Shrinkage via The *CLASH* Operator



Volkan Cevher

*Laboratory for Information and Inference
Systems – LIONS / EPFL*

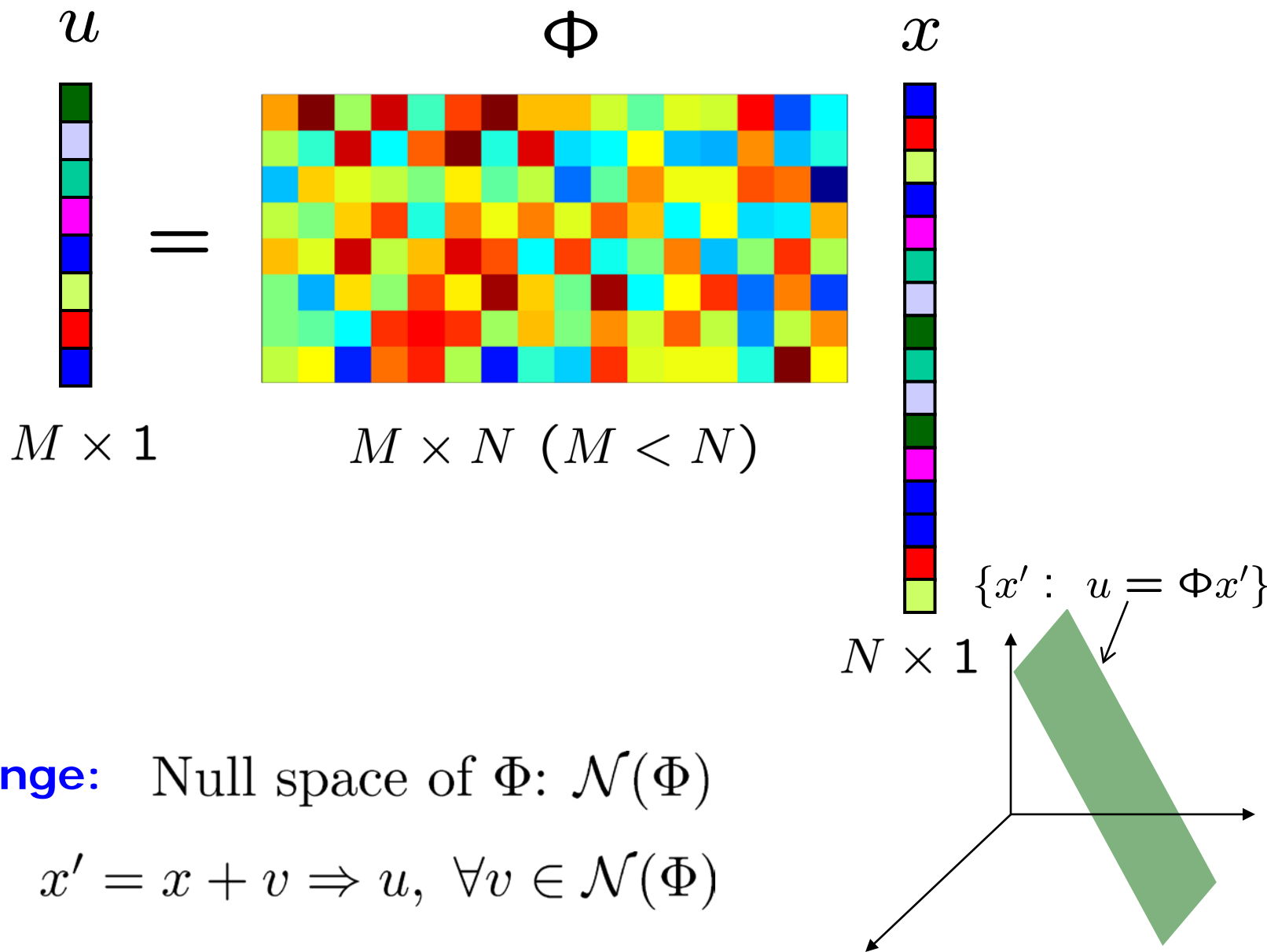
<http://lions.epfl.ch>

& Idiap Research Institute

joint work with my PhD student

Anastasios Kyrillidis @ EPFL

Linear Inverse Problems



Approaches



Deterministic

Probabilistic

Prior

 parsity
compressibility

$f(x)$

Metric

ℓ_p -norm*

likelihood
function

* : $\|x\|_p = (\sum_i |x_i|^p)^{1/p}$

A Deterministic View

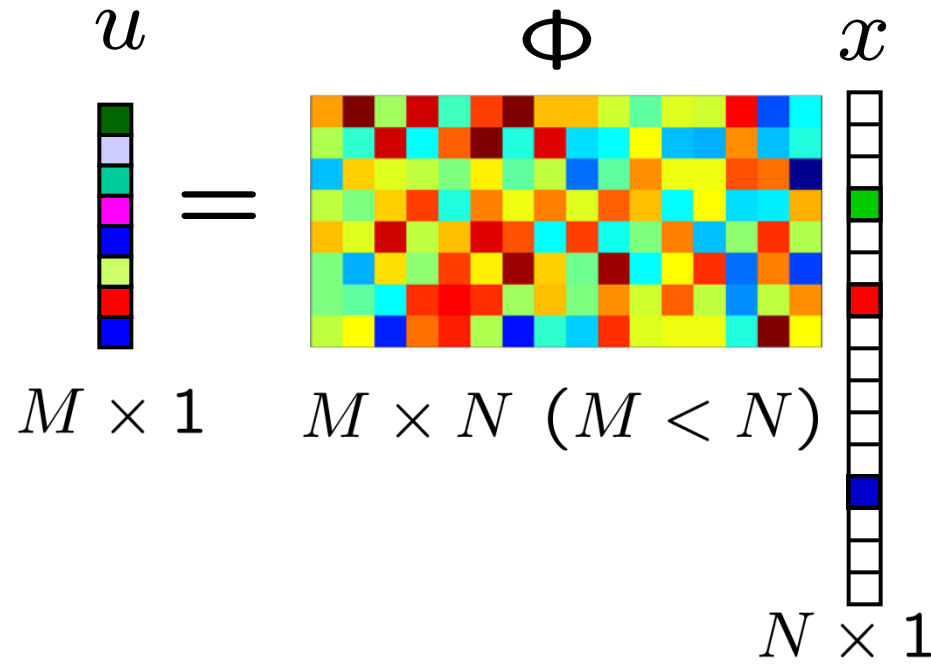
Model-based CS

(circa Aug 2008)



My Insights on Compressive Sensing

1. Sparse or compressible x
not sufficient alone

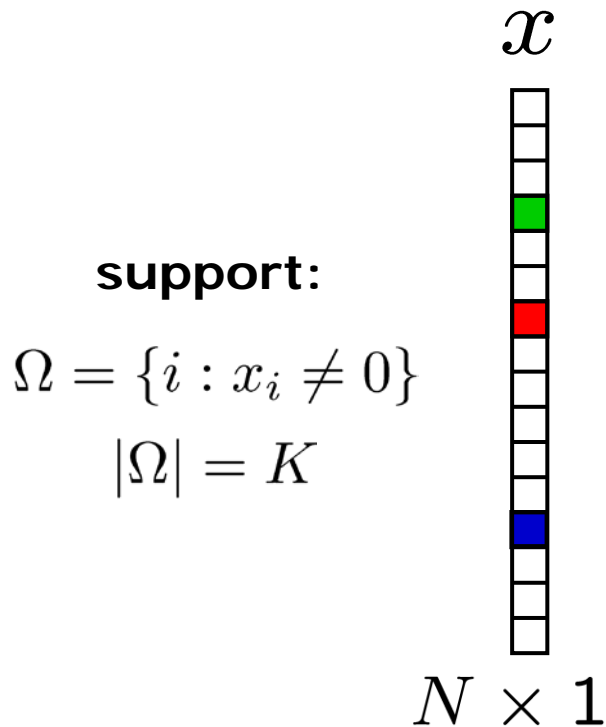


2. Projection Φ
information preserving
(stable embedding / special null space)

3. Decoding algorithms
tractable

Signal Priors

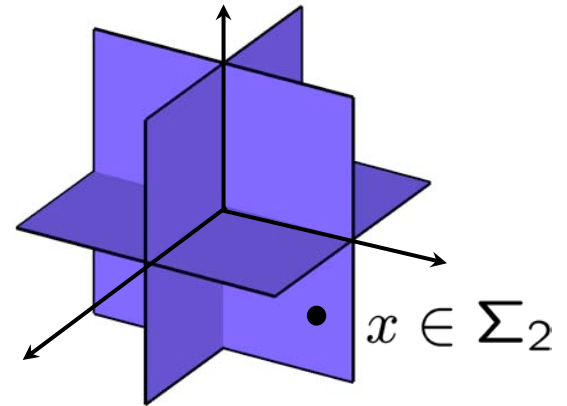
- **Sparse** signal: only K out of N coordinates nonzero
 - model: union of all K -dimensional subspaces aligned w/ coordinate axes



Example: 2-sparse in 3-dimensions

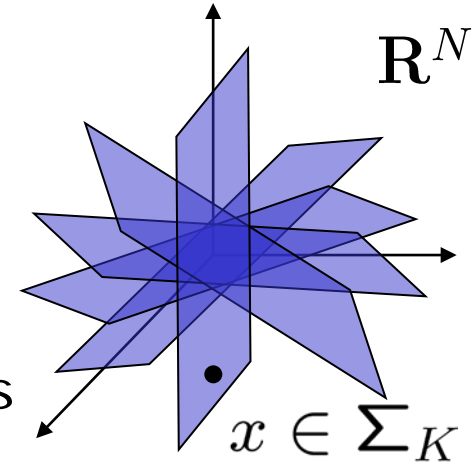
$$K = 2$$

$$\mathbb{R}^3$$

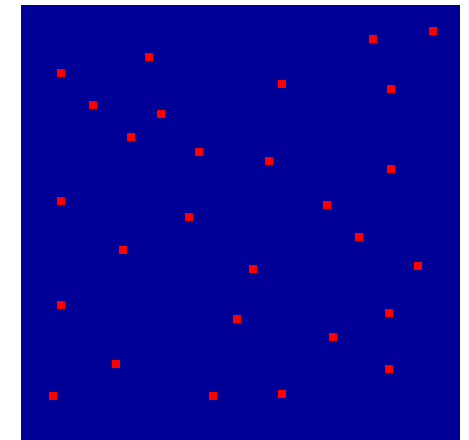
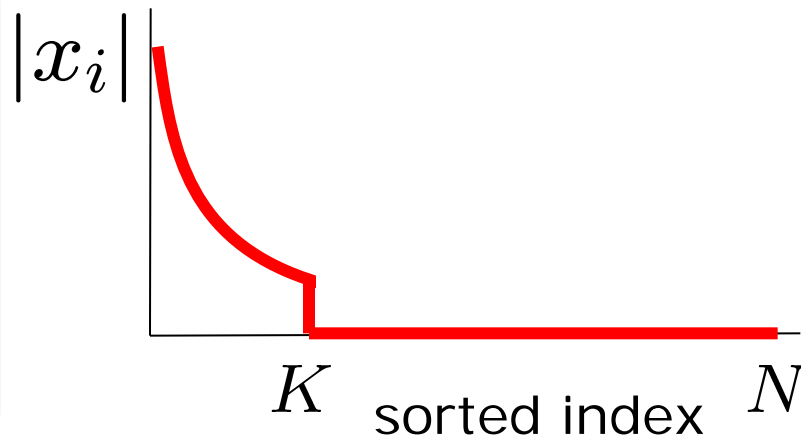
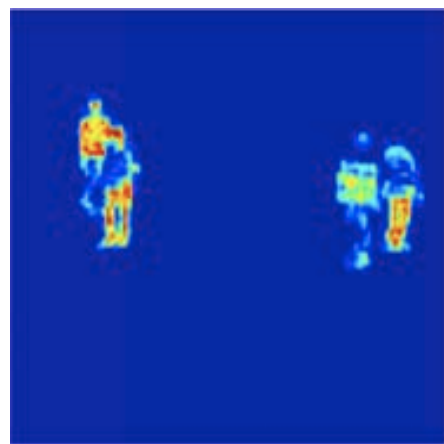


Signal Priors

- **Sparse** signal: only K out of N coordinates nonzero
 - model: union of all K -dimensional subspaces aligned w/ coordinate axes

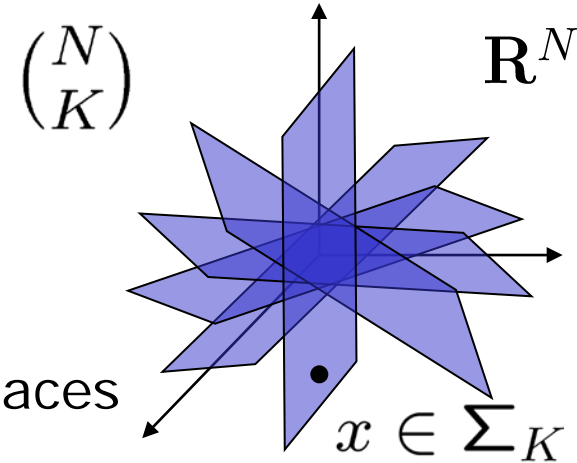


- **Structured sparse** signal: reduced set of subspaces (or model-sparse)
 - model: a particular union of subspaces
ex: clustered or dispersed sparse patterns

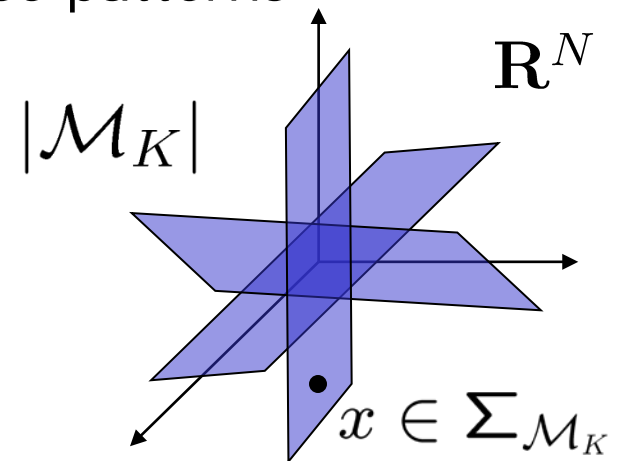
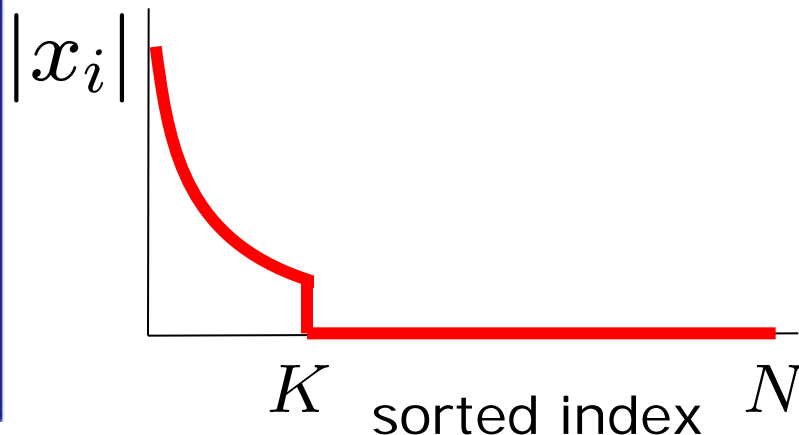
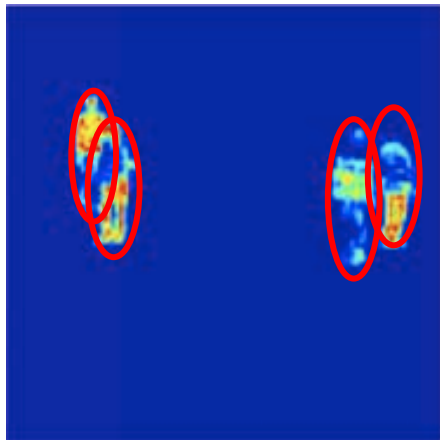


Signal Priors

- **Sparse** signal: only K out of N coordinates nonzero
 - model: union of all K -dimensional subspaces aligned w/ coordinate axes



- **Structured sparse** signal: reduced set of subspaces (or model-sparse)
 - model: a particular union of subspaces
ex: clustered or dispersed sparse patterns



Sparse Recovery Algorithms

- **Goal:** given $u = \Phi x + n$
recover x

- $\ell_{q:q \leq 1}$ and convex optimization formulations

– basis pursuit, Lasso, BP denoising...

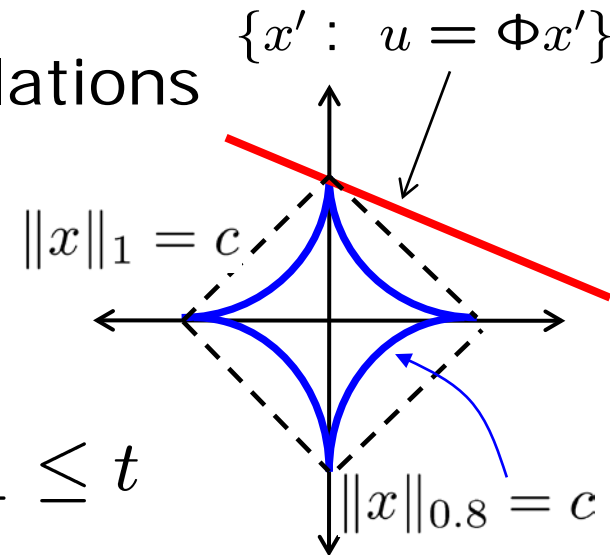
$$\hat{x} = \arg \min \|x\|_1 \text{ s.t. } u = \Phi x$$

$$\hat{x} = \arg \min \|u - \Phi x\|_2 \text{ s.t. } \|x\|_1 \leq t$$



$$\hat{x} = \arg \min \|u - \Phi x\|_2^2 + \mu \|x\|_1$$

– iterative re-weighted ℓ_1 & ℓ_2 algorithms

- Hard thresholding algorithms: ALPS, CoSaMP, SP, ...
- Greedy algorithms: OMP, MP, ...





Sparse Recovery Algorithms

	Geometric 	Combinatorial $\binom{N}{K}$	Probabilistic 
Encoding	atomic norm / convex relaxation	non-convex union-of-subspaces	compressible / sparse priors
Example	$\min_{x: \ x\ _1 \leq \lambda} \ u - \Phi x\ ^2$	$\min_{x: \ x\ _0 \leq K} \ u - \Phi x\ ^2$	$E\{x u\}$
Algorithm	Basis pursuit, Lasso, basis pursuit denoising...	IHT, CoSaMP, SP, ALPS, OMP...	Variational Bayes, EP, Approximate message passing (AMP)...

$$\|x\|_0 = \#\{x_i \neq 0\}$$

Sparse Recovery Algorithms

The Clash Operator

	Geometric 	Combinatorial $\binom{N}{K}$	Probabilistic 
Encoding	atomic norm / convex relaxation	non-convex union-of-subspaces	compressible / sparse priors
Example	$\min_{x:\ x\ _1 \leq \lambda} \ u - \Phi x\ ^2$	$\min_{x:\ x\ _0 \leq K} \ u - \Phi x\ ^2$	$E\{x u\}$
Algorithm	Basis pursuit, Lasso, basis pursuit denoising...	IHT, CoSaMP, SP, ALPS, OMP...	Variational Bayes, EP, Approximate message passing (AMP)...

$$\hat{x}_{\text{Clash}} = \arg \min_{x:\|x\|_1 \leq \lambda, \|x\|_0 \leq K} \|u - \Phi x\|^2$$

$$\|x\|_0 = \#\{x_i \neq 0\}$$

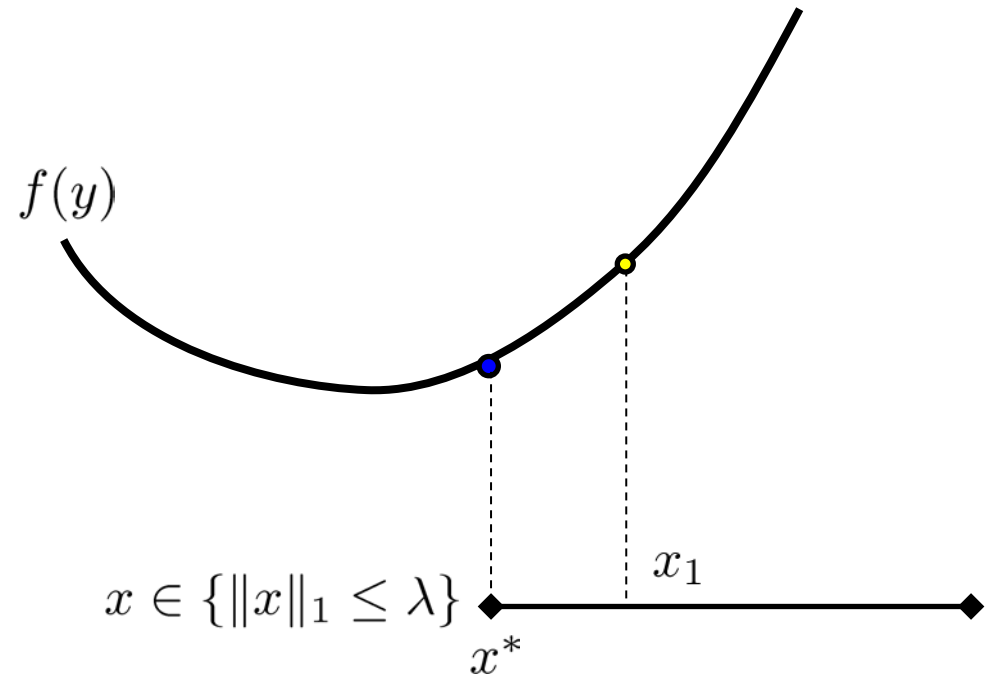


A Tale of Two Algorithms

- Soft thresholding

$$f(x) = \|u - \Phi x\|^2$$

$$\min_{x: \|x\|_1 \leq \lambda} f(x)$$



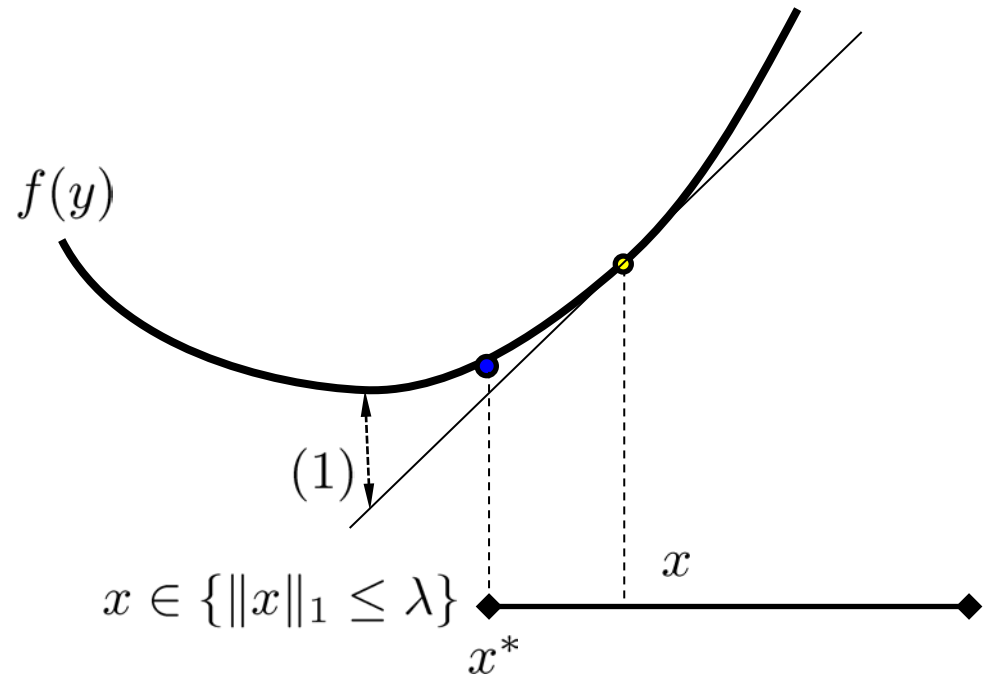


A Tale of Two Algorithms

- Soft thresholding

$$f(x) = \|u - \Phi x\|^2$$

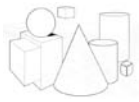
$$\min_{x: \|x\|_1 \leq \lambda} f(x)$$



Structure in optimization:

Bregman distance

$$(1) \quad \overbrace{f(y) - f(x) - \langle \nabla f(x), y - x \rangle}^{\text{Bregman distance}} = \|\Phi(y - x)\|^2 \quad \forall x, y \in \mathcal{R}^N,$$



A Tale of Two Algorithms

- Soft thresholding

$$f(x) = \|u - \Phi x\|^2$$

$$\min_{x: \|x\|_1 \leq \lambda} f(x)$$

$$U(x_2, x_1) = f(x_1) + \langle \nabla f(x_1), x_2 - x_1 \rangle + \frac{L}{2} \|x_2 - x_1\|^2$$

majorization-minimization

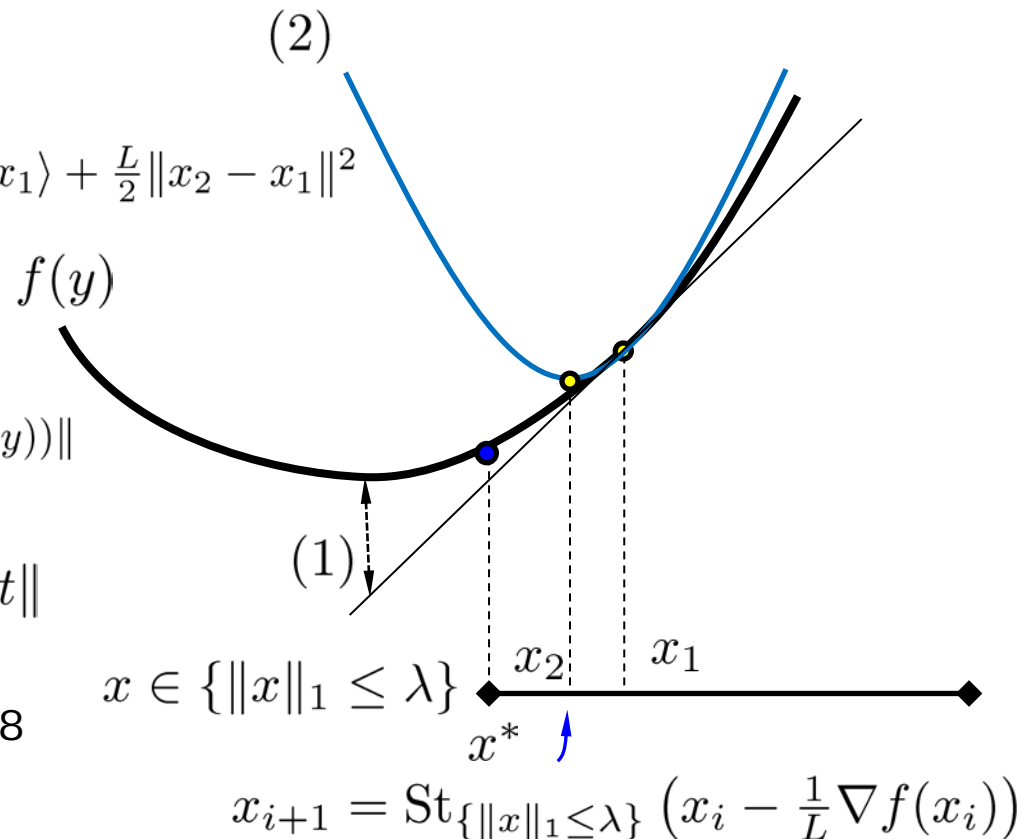
$$\arg \min_{\|x\|_1 \leq \lambda} U(x, y) = \arg \min_{\|x\|_1 \leq \lambda} \|x - (y - \frac{1}{L} \nabla f(y))\|$$

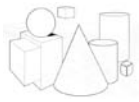
$$\text{St}_{\{\|x\|_1 \leq \lambda\}}(t) = \arg \min_{\|x\|_1 \leq \lambda} \|x - t\|$$

ALGO: cf. J. Duchi et al. ICML 2008

Bregman distance

$$\begin{aligned} (1) \quad & f(y) - f(x) - \langle \nabla f(x), y - x \rangle = \|\Phi(y - x)\|^2 \quad \forall x, y \in \mathcal{R}^N, \\ (2) \quad & f(y) - f(x) - \langle \nabla f(x), y - x \rangle \leq \frac{L}{2} \|y - x\|^2 \quad L = 2\|\Phi\|, \forall x, y \in \mathcal{R}^N, \end{aligned}$$





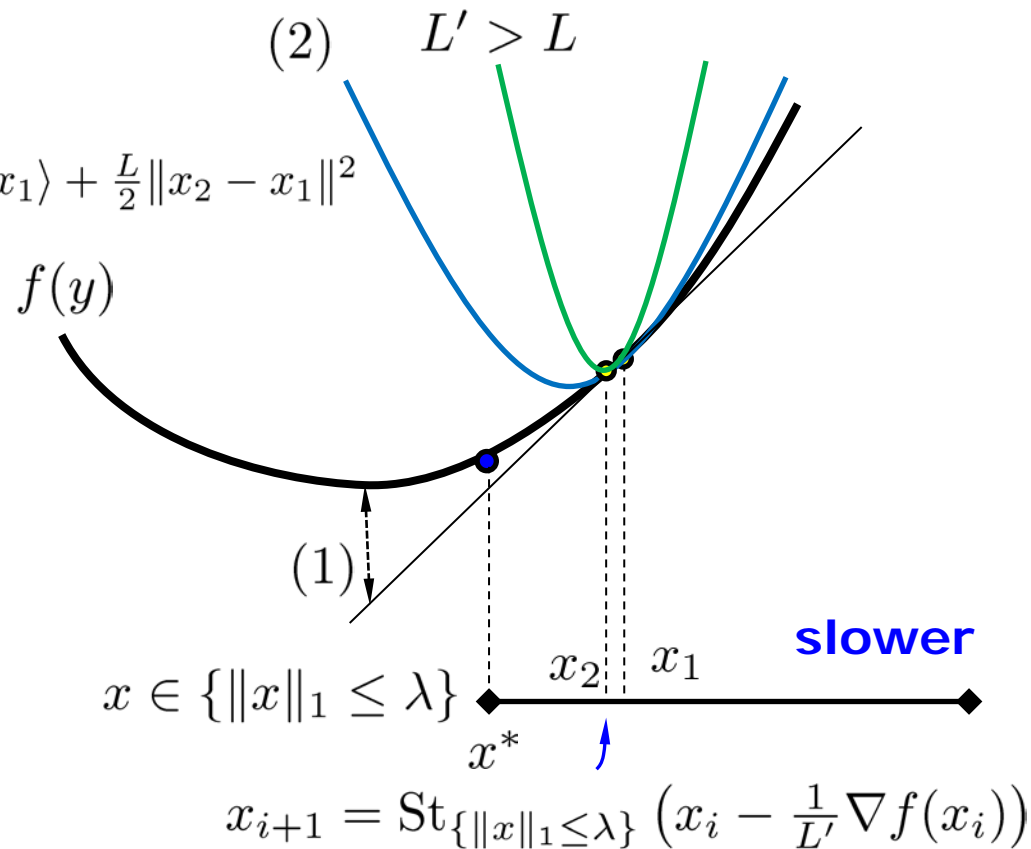
A Tale of Two Algorithms

- Soft thresholding

$$\min_{x: \|x\|_1 \leq \lambda} f(x)$$

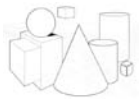
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Bregman distance

$$\begin{aligned} (1) \quad & f(y) - f(x) - \langle \nabla f(x), y - x \rangle = \|\Phi(y - x)\|^2 \quad \forall x, y \in \mathcal{R}^N, \\ (2) \quad & f(y) - f(x) - \langle \nabla f(x), y - x \rangle \leq \frac{L}{2} \|y - x\|^2 \quad L = 2\|\Phi\|, \forall x, y \in \mathcal{R}^N, \end{aligned}$$



A Tale of Two Algorithms

- Soft thresholding

$$\min_{x: \|x\|_1 \leq \lambda} f(x)$$

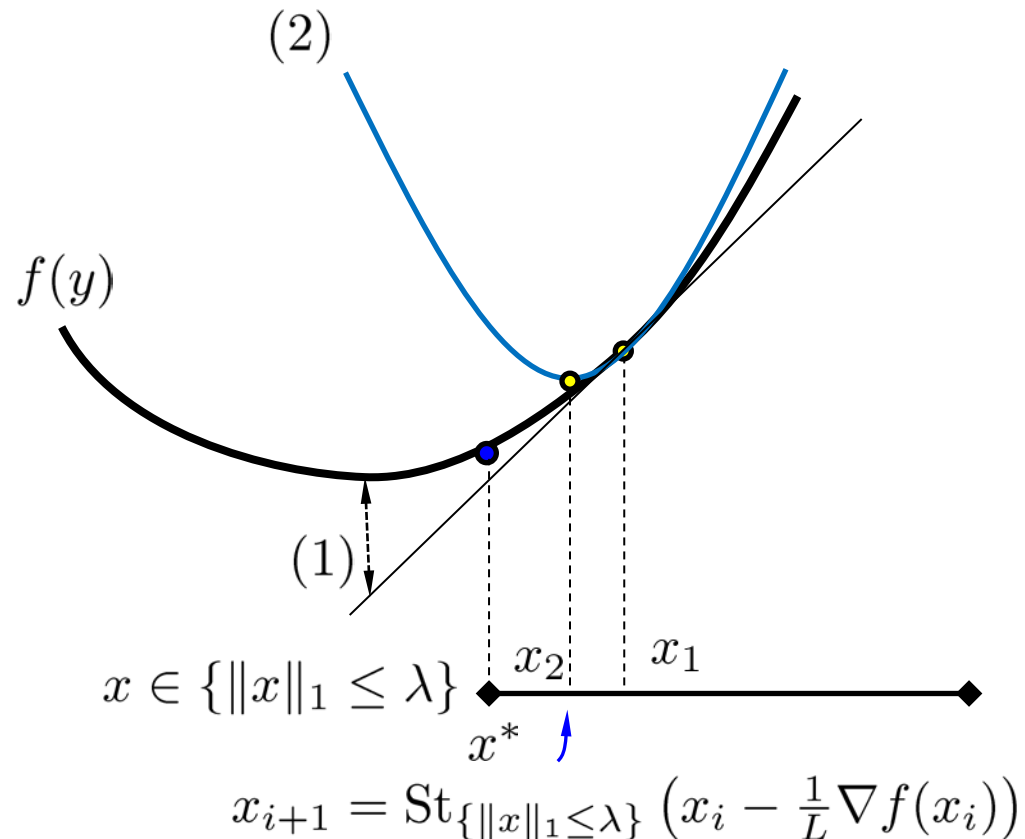
- Is x^* what we are looking for?

local “unverifiable” assumptions:

- ERC/URC condition
- compatibility condition ...

(local \rightarrow global / dual certification / random signal models)

$$f(x) = \|u - \Phi x\|^2$$



$\binom{N}{K}$ A Tale of Two Algorithms

- Hard thresholding

$$f(x) = \|u - \Phi x\|^2$$

$$\min_{x: \|x\|_0 \leq K} f(x)$$

(2)

$$U(x_2, x_1)$$

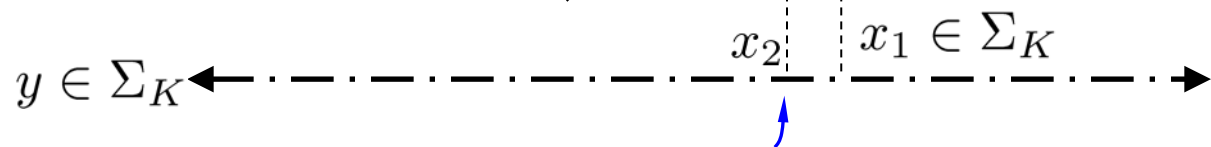
$$\arg \min_{\|x\|_0 \leq K} U(x, y) = \arg \min_{\|x\|_0 \leq K} \|x - (y - \frac{1}{L} \nabla f(y))\|$$

$$H_{\{\|x\|_0 \leq K\}}(t) = \arg \min_{\|x\|_0 \leq K} \|x - t\|$$

$f(y)$

ALGO: sort and pick the largest K

(1)



x_2

$x_1 \in \Sigma_K$

$y \in \Sigma_K$

$$x_{i+1} = H_{\{\|x\|_0 \leq K\}} \left(x_i - \frac{1}{L} \nabla f(x_i) \right)$$

$\binom{N}{K}$ A Tale of Two Algorithms

- Hard thresholding

$$\min_{x: \|x\|_0 \leq K} f(x)$$

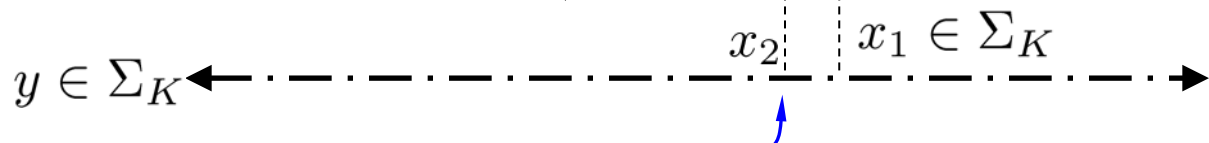
$$f(x) = \|u - \Phi x\|^2$$

Global “unverifiable” assumption:

$$(1 - \delta_K) \leq \frac{\|\Phi x\|_2^2}{\|x\|_2^2} \leq (1 + \delta_K), \quad \forall x \in \Sigma_K$$

⇒ we can tiptoe among percolations!

$$\delta_{2K} < 1/3$$

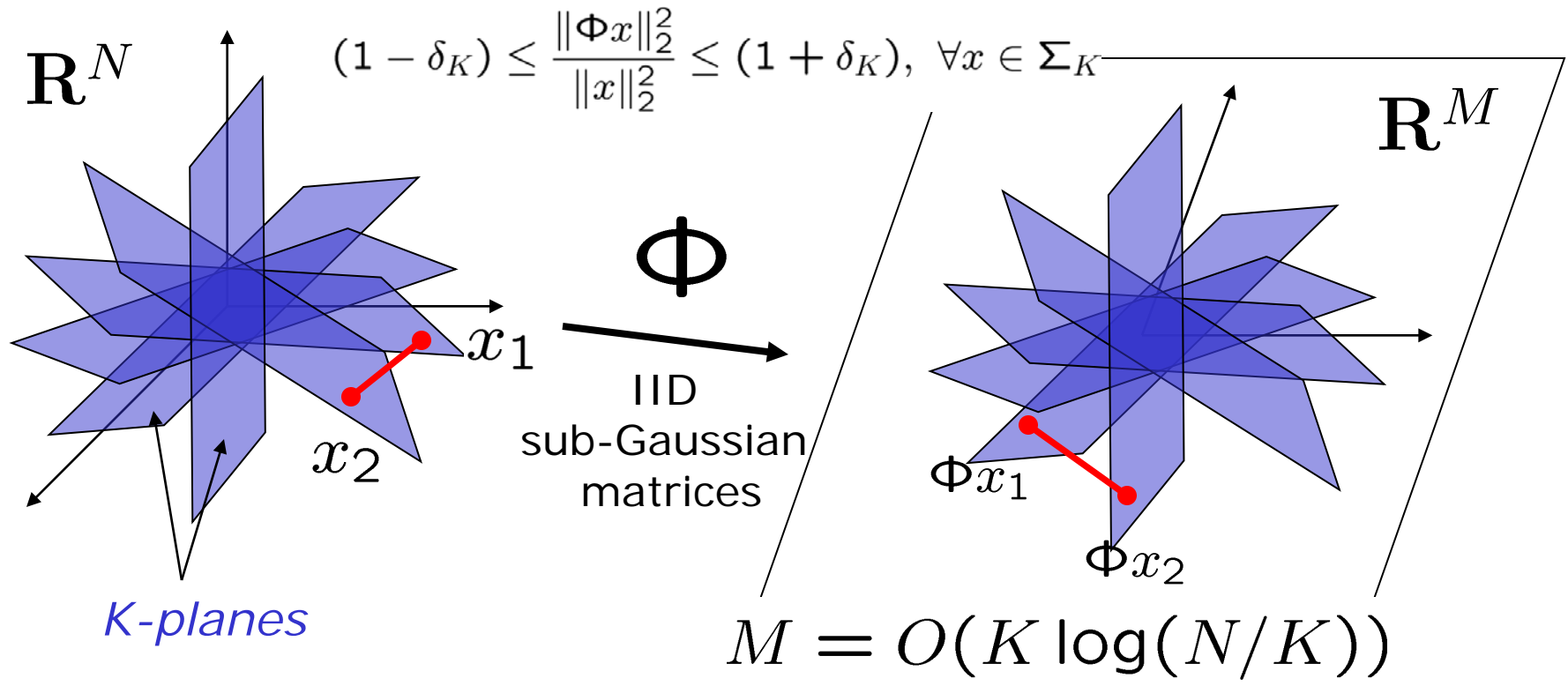
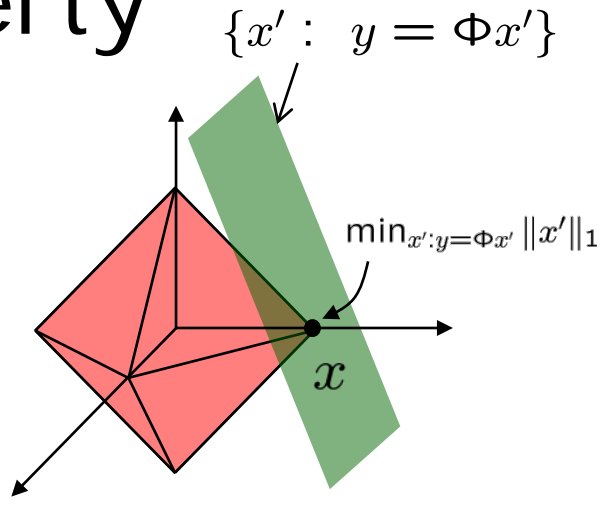


another variant has $\delta_{3K} < 1/2$ GraDes: $x_{i+1} = \mathbb{H}_{\{\|x\|_0 \leq K\}} \left(x_i - \frac{1}{L_{2K}} \nabla f(x_i) \right)$

$$\begin{aligned} (1) \quad f(y) - f(x) - \langle \nabla f(x), y - x \rangle &= \|\Phi(y - x)\|^2 & \forall x, y \in \mathcal{R}^N, \\ (2) \quad f(y) - f(x) - \langle \nabla f(x), y - x \rangle &\leq \frac{L_{2K}}{2} \|y - x\|^2 & L_{2K} = 2(1 + \delta_{2K}), \forall x, y \in \Sigma_K, \\ (3) \quad f(y) - f(x) - \langle \nabla f(x), y - x \rangle &\geq \frac{\mu_{2K}}{2} \|y - x\|^2 & \mu_{2K} = 2(1 - \delta_{2K}), \forall x, y \in \Sigma_K, \end{aligned}$$

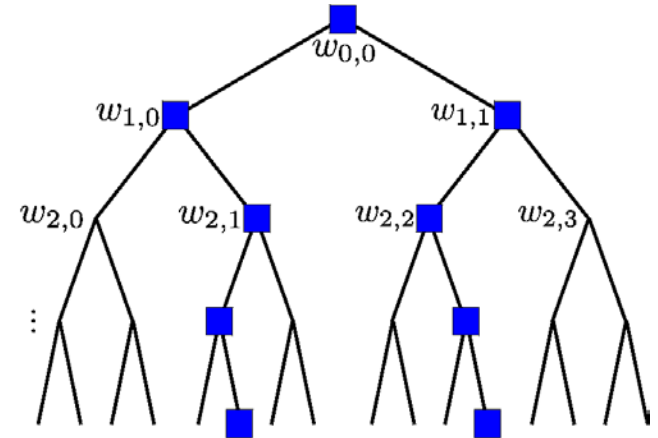
Restricted Isometry Property

- **Model:** K -sparse coefficients
- **Remark:** implies convergence of convex relaxations also e.g., $\delta_{2K} < .465$ is sufficient for BP
- **RIP:** stable embedding



Tree-Sparse

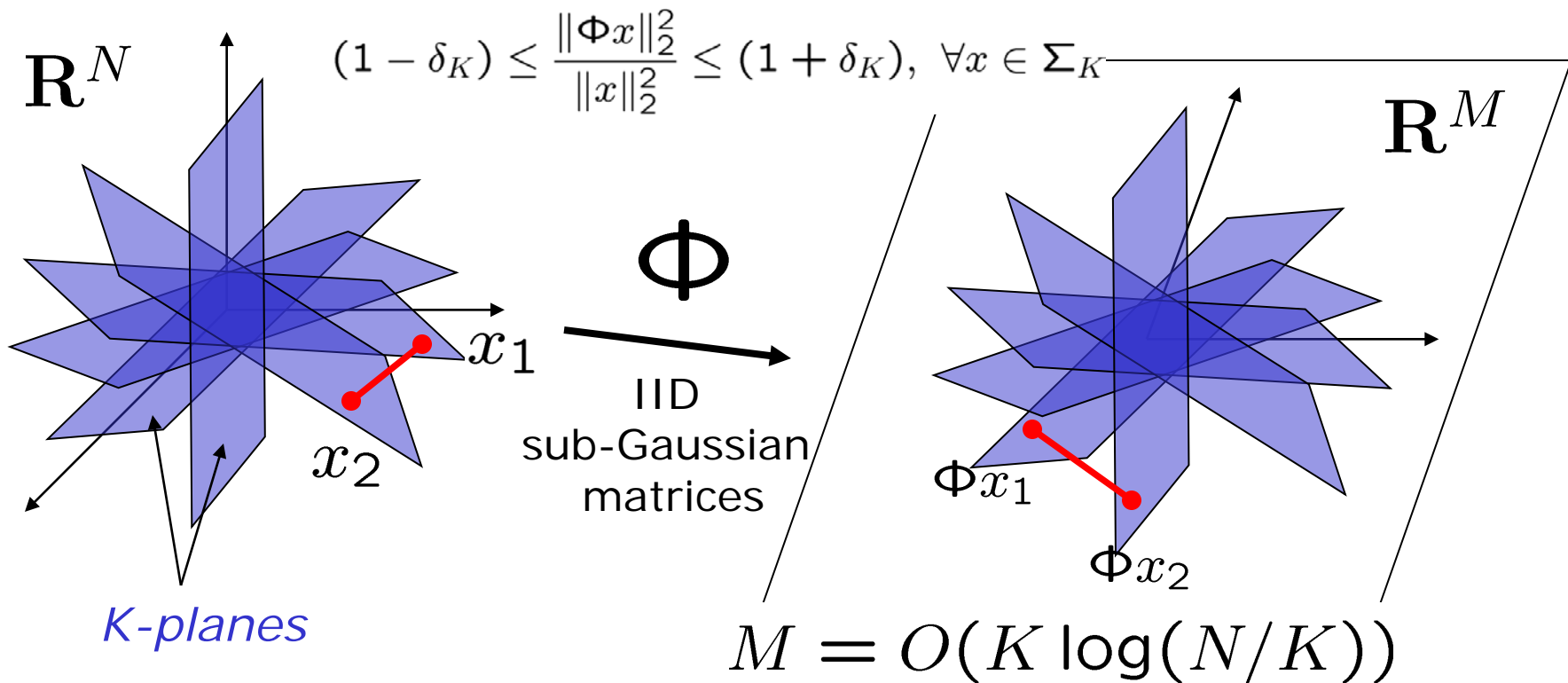
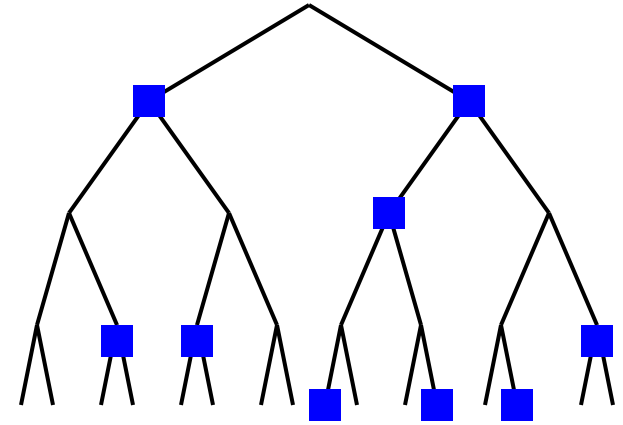
- **Model:** K -sparse coefficients
+ significant coefficients
lie on a rooted subtree



- **Sparse approx:** find **best set** of coefficients
 - sorting
 - hard thresholding
- **Tree-sparse approx:** find **best rooted subtree** of coefficients
 - condensing sort and select [Baraniuk]
 - dynamic programming [Donoho]

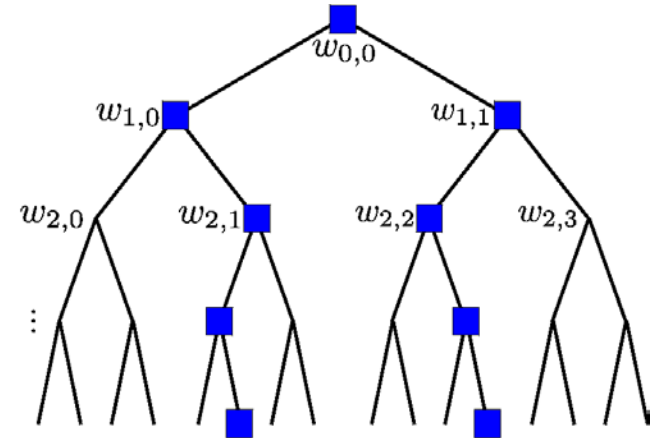
Sparse

- **Model:** K -sparse coefficients
- **RIP:** stable embedding

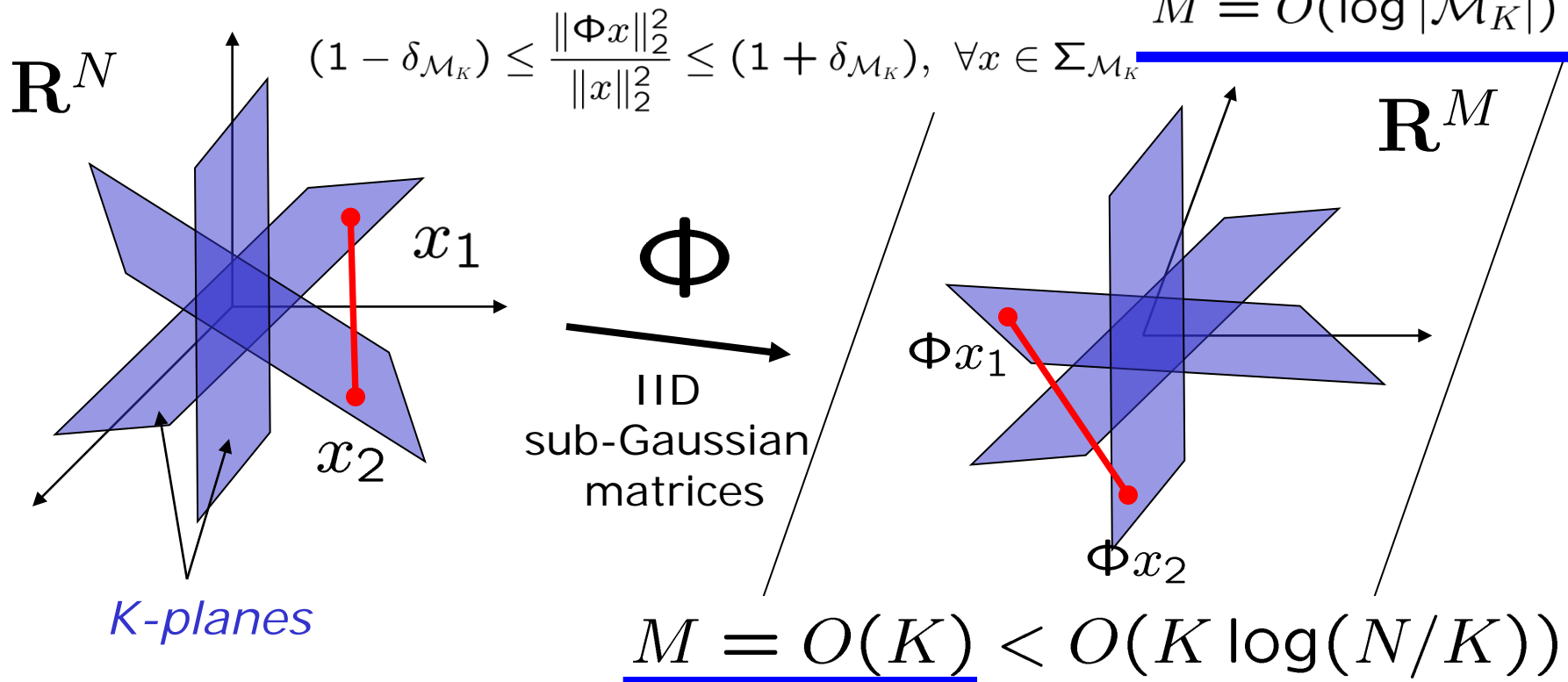


Tree-Sparse

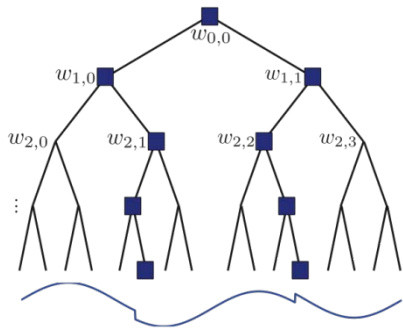
- **Model:** K -sparse coefficients + significant coefficients lie on a rooted subtree
- **Tree-RIP:** stable embedding



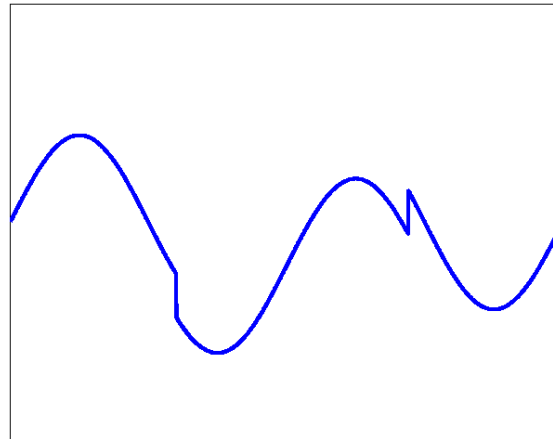
$$M = O(\log |\mathcal{M}_K|)$$



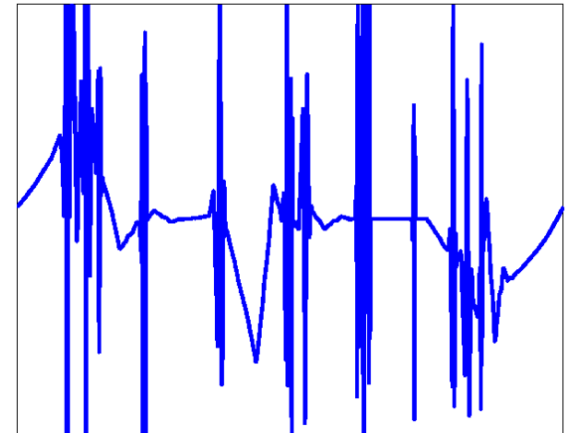
Tree-Sparse Signal Recovery



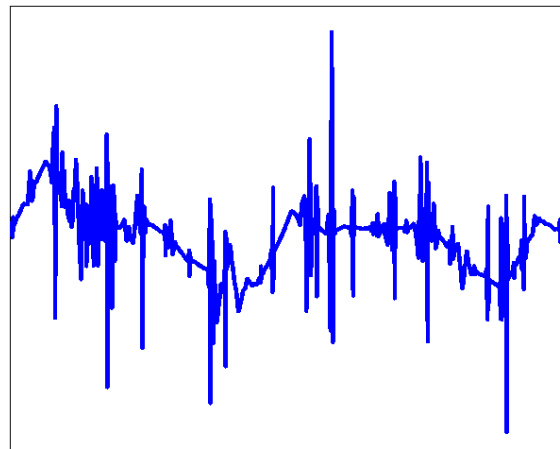
$N=1024$
 $M=80$



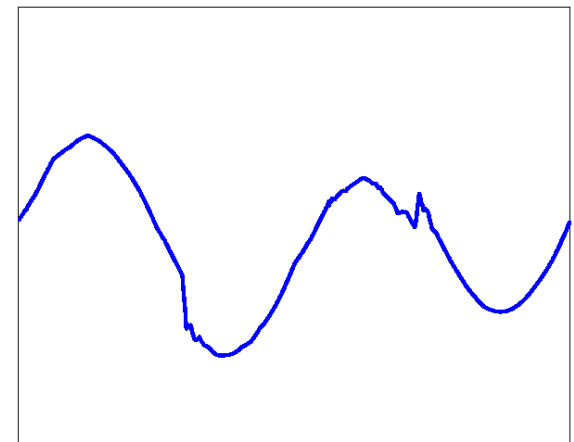
target signal



CoSaMP,
(MSE=1.12)



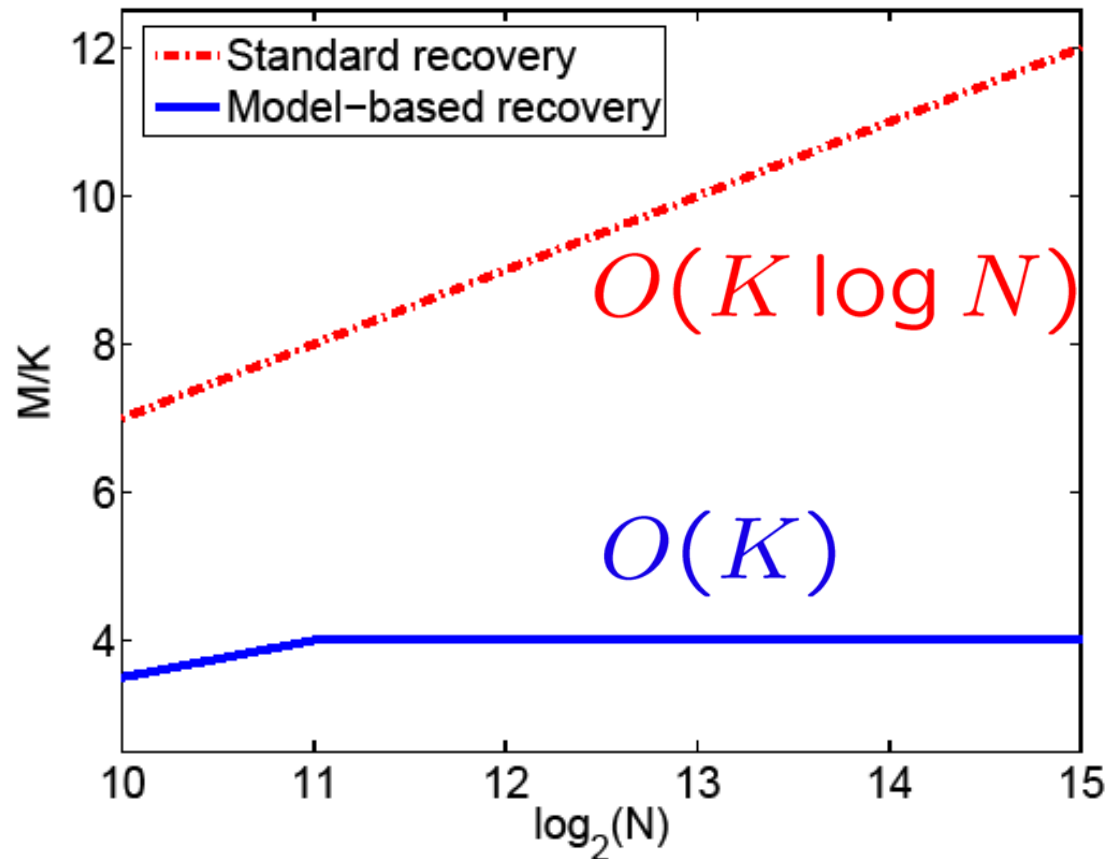
L1-minimization
(MSE=0.751)



Tree-sparse CoSaMP
(MSE=0.037)

Tree-Sparse Signal Recovery

- Number samples for correct recovery
- Piecewise cubic signals + wavelets
- Models/algorithms:
 - compressible (CoSaMP)
 - tree-compressible (tree-CoSaMP)



Model CS in Context

- Basis pursuit and Lasso

exploit geometry < > interplay of ℓ_1 ball and ℓ_2 error
arbitrary selection < > *difficulty of interpretation*
cannot leverage further structure

- Structured-sparsity **inducing** norms

“customize” geometry < > “mixing” of norms over groups /
for selection Lovasz extension of submodular
set functions
inexact selections

- Structured-sparsity via OMP / Model-CS

greedy selection < > *cannot leverage geometry*
exact selection < > *cannot leverage geometry*

Model CS in Context

- Basis pursuit and Lasso

exploit geometry < > interplay of ℓ_1 ball and ℓ_2 error
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- Structured-sparsity **inducing** norms

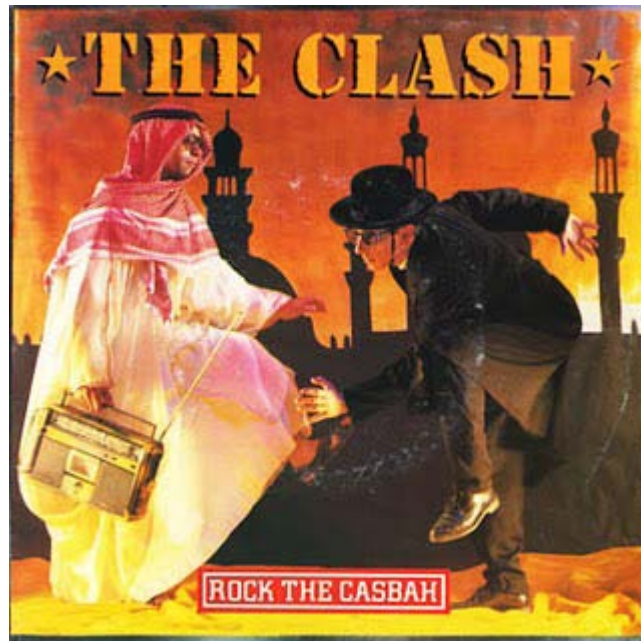
“customize” geometry < > “mixing” of norms over groups /
for selection Lovasz extension of submodular
set functions
inexact selections

- Structured-sparsity via OMP / Model-CS

greedy selection < > *cannot leverage geometry*
exact selection < > **Or, can it?**

Enter CLASH

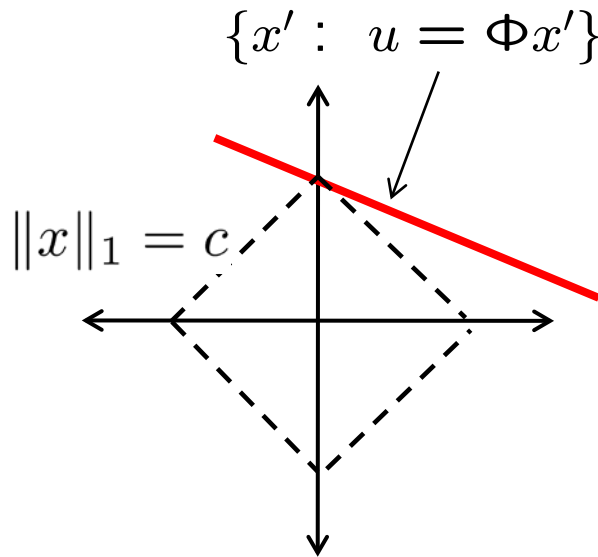
<http://lions.epfl.ch/CLASH>



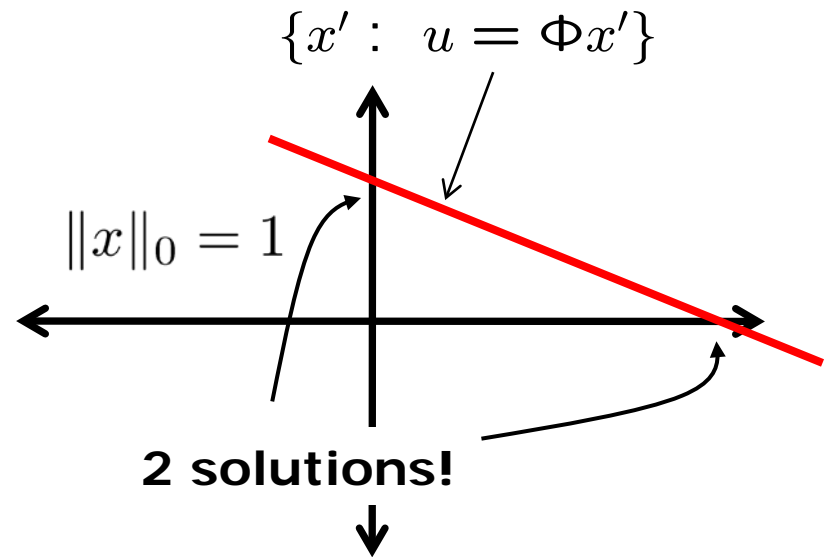
Importance of Geometry

- A subtle issue

$$\hat{x} = \arg \min \|x\|_1 \text{ s.t. } u = \Phi x$$



$$\hat{x} = \arg \min \|x\|_0 \text{ s.t. } u = \Phi x$$

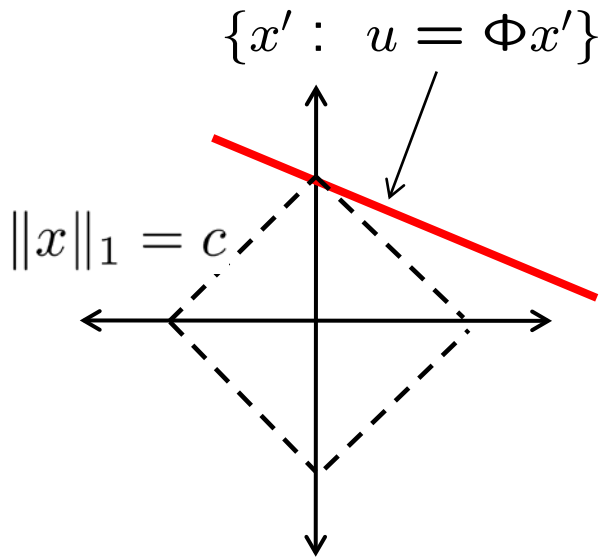


Which one is correct?

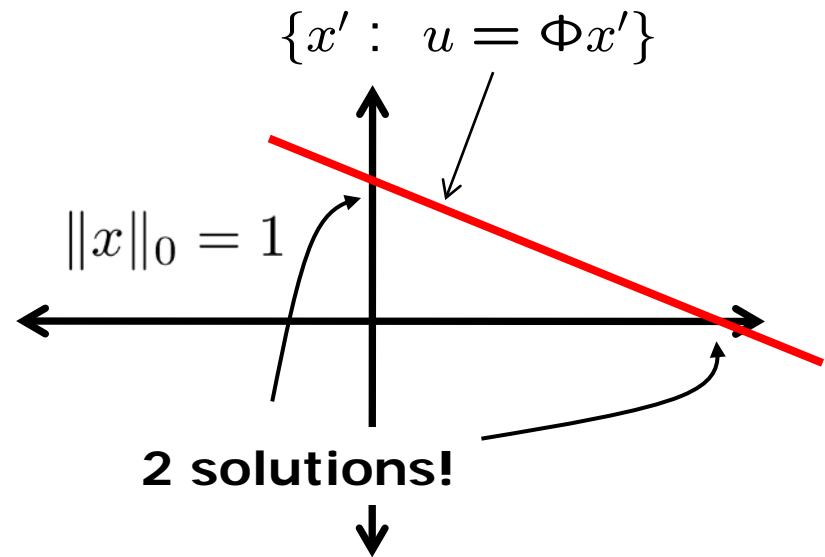
Importance of Geometry

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$$\hat{x} = \arg \min \|x\|_0 \text{ s.t. } u = \Phi x$$



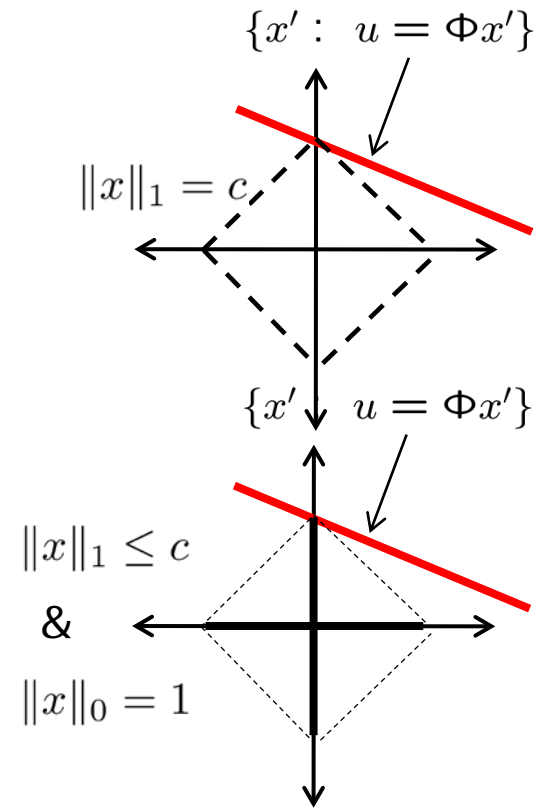
Which one is correct?

EPIC FAIL

CLASH Pseudocode

- Algorithm code @ <http://lions.epfl.ch/CLASH>

- Active set expansion
- Greedy descend
- Combinatorial selection
- Least absolute shrinkage
- De-bias with convex constraint

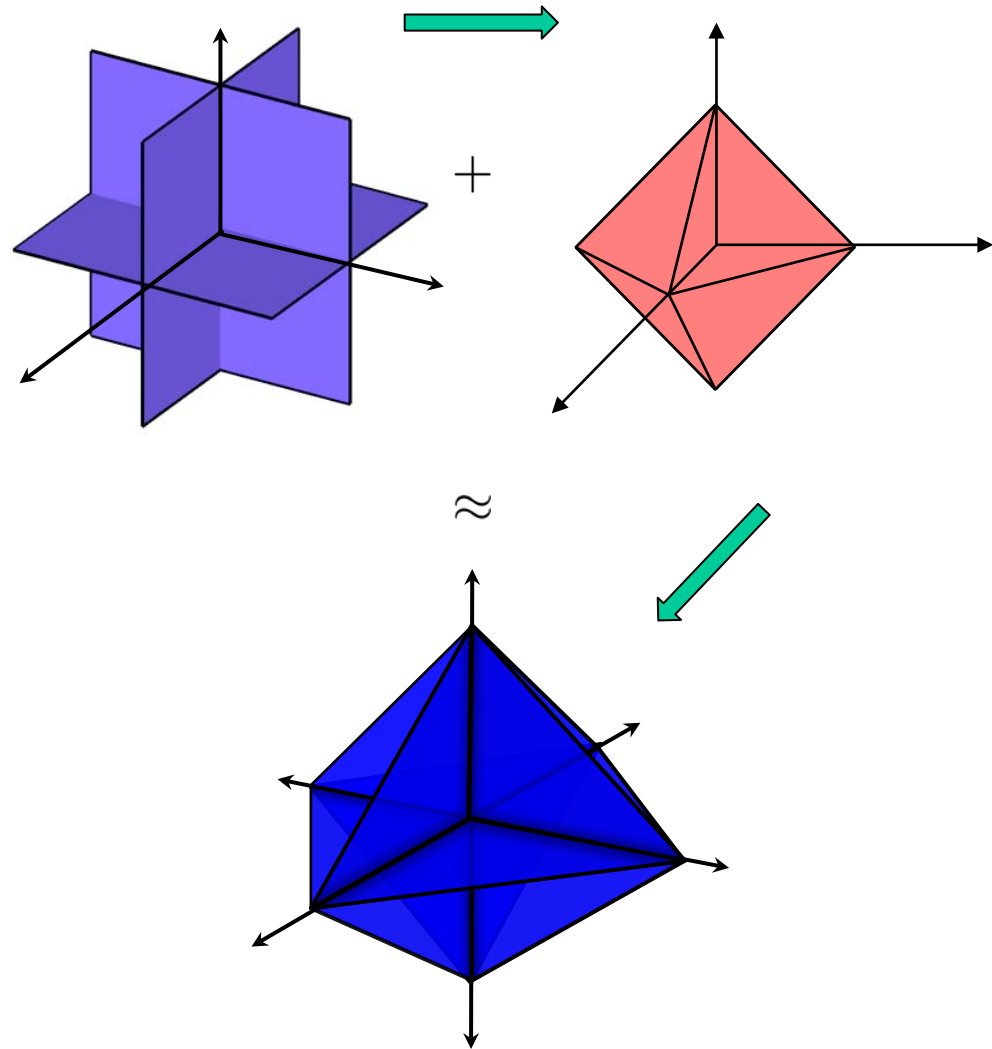


Minimum 1-norm solution still makes sense!

CLASH Pseudocode

- Algorithm code @ <http://lions.epfl.ch/CLASH>

- Active set expansion
- Greedy descend
- Combinatorial selection
- Least absolute shrinkage
- De-bias with convex constraint

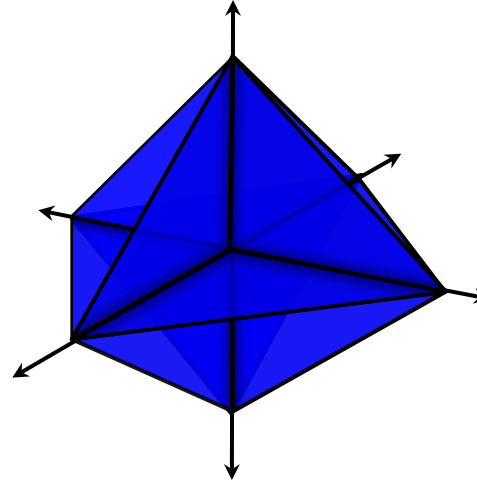
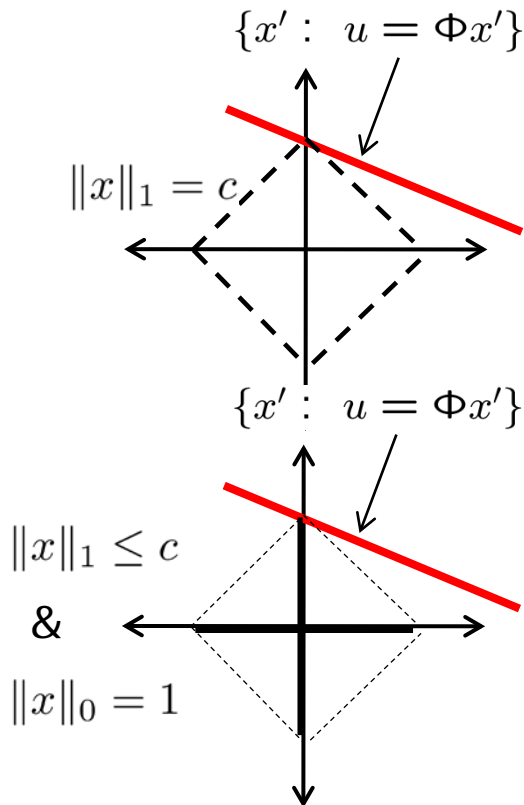


Geometry of CLASH

- Combinatorial selection
+
least absolute shrinkage

$$H_{\{\|x\|_0 \leq K\}}(t) = \arg \min_{\|x\|_0 \leq K} \|x - t\|$$

$$St_{\{\|x\|_1 \leq \lambda\}}(t) = \arg \min_{\|x\|_1 \leq \lambda} \|x - t\|$$

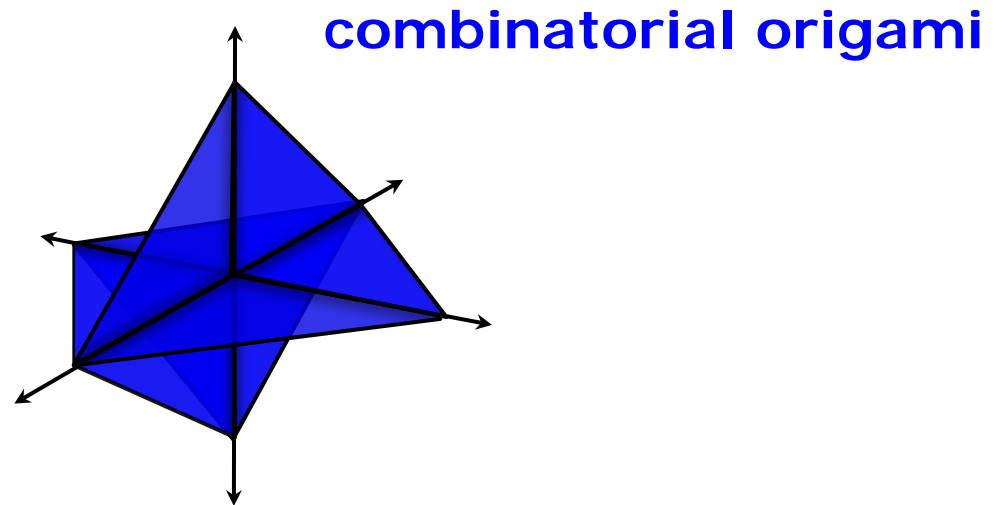
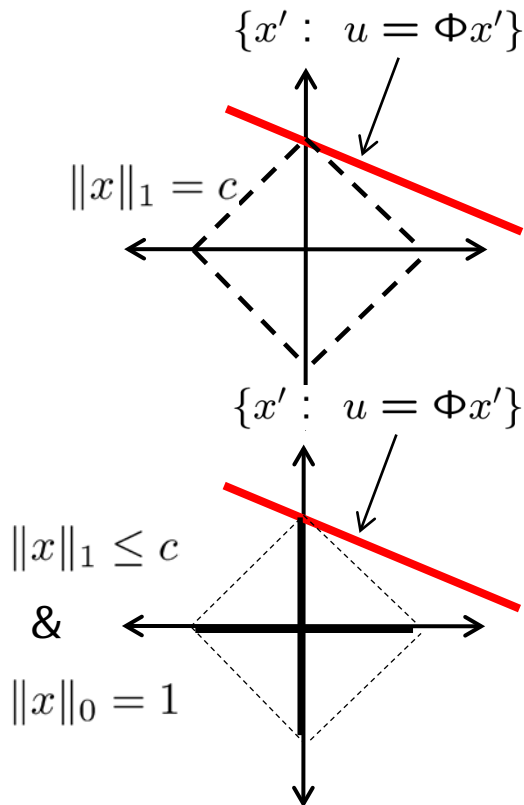


Geometry of CLASH

- Combinatorial selection
+
least absolute shrinkage

$$\underline{H_{\Sigma_{\mathcal{M}_K}}(y) = \arg \min_{x: x \in \Sigma_{\mathcal{M}_K}} \|x - y\|}$$

$$\text{St}_{\{\|x\|_1 \leq \lambda\}}(t) = \arg \min_{\|x\|_1 \leq \lambda} \|x - t\|$$



Combinatorial Selection

- A different view of the model-CS workhorse

$$H_{\Sigma \mathcal{M}_K}(y) = \arg \min_{x: x \in \Sigma \mathcal{M}_K} \|x - y\|$$

(Lemma) support of the solution \leftrightarrow modular approximation problem

$$\text{supp} \left(\arg \min_{x: \text{supp}(x) \in \mathcal{M}_K} \|x - y\|_2^2 \right) = \arg \max_{S: S \in \bar{\mathcal{M}}_K} F(S; y)$$

where $F(S; y) = \sum_{i \in S} |y_i|^2$.

indexing set



PMAP

- An algorithmic generalization of union-of-subspaces

Polynomial time modular epsilon-approximation property: PMAP_ϵ

- Sets with PMAP-0 $F(\hat{\mathcal{S}}_\epsilon; y) \geq (1 - \epsilon) \max_{\mathcal{S} \in \bar{\mathcal{M}}_K} F(\mathcal{S}; y)$

- **Matroids**

uniform matroids	< >	regular sparsity
partition matroids	< >	block sparsity (disjoint groups)
cographic matroids	< >	rooted connected tree group adapted hull model

- **Totally unimodular systems**

mutual exclusivity	< >	neuronal spike model
interval constraints	< >	sparsity within groups

Model-CS is applicable for all these cases!

PMAP

- An algorithmic generalization of union-of-subspaces

Polynomial time modular epsilon-approximation property: PMAP_ϵ

- Sets with PMAP-epsilon $F(\hat{\mathcal{S}}_\epsilon; y) \geq (1 - \epsilon) \max_{\mathcal{S} \in \bar{\mathcal{M}}_K} F(\mathcal{S}; y)$

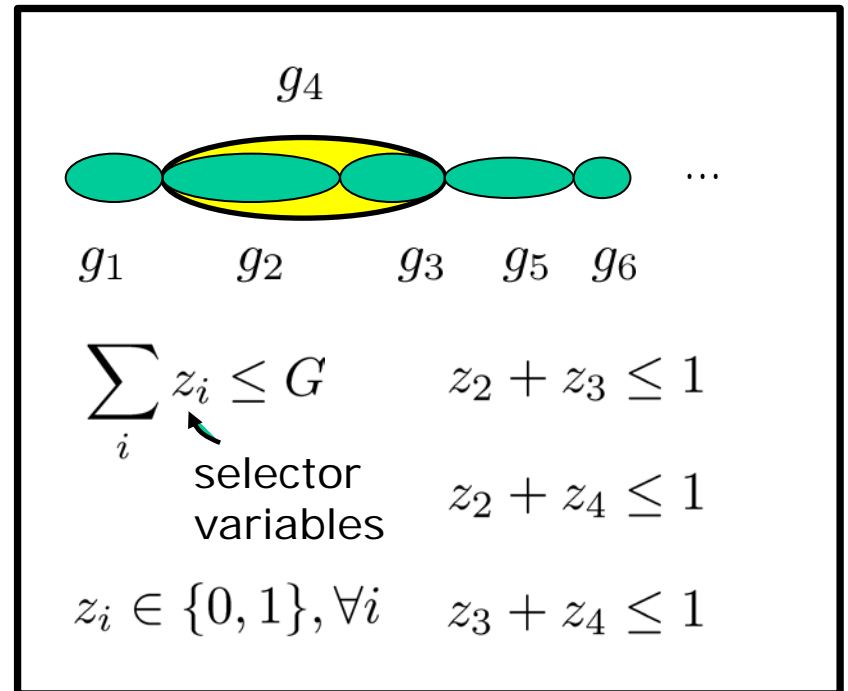
- **Knapsack**

multi-knapsack constraints

weighted multi-knapsack

quadratic knapsack (?)

- Define algorithmically!



PMAP

- An algorithmic generalization of union-of-subspaces

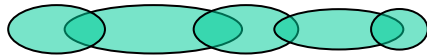
Polynomial time modular epsilon-approximation property: PMAP_ϵ

- Sets with PMAP-epsilon $F(\hat{\mathcal{S}}_\epsilon; y) \geq (1 - \epsilon) \max_{\mathcal{S} \in \bar{\mathcal{M}}_K} F(\mathcal{S}; y)$

- **Knapsack**

- Define algorithmically!

- Sets with PMAP-???



- pairwise overlapping groups $\langle \rangle$ mincut with cardinality constraint

$$\max_{\mathcal{S}: \mathcal{S} \in \bar{\mathcal{M}}_K} F(\mathcal{S}; \beta) = - \min \left\{ \sum_{i>j} \|(\beta)_{g_i \cap g_j}\|_2^2 z_i z_j - \sum_i \|(\beta)_{g_i}\|_2^2 z_i : \sum_i z_i \leq G \right\}.$$

CLASH Approximation Guarantees

- PMAP / downward compatibility

$$\text{SNR} = \frac{\|x^*\|}{\sqrt{f(x^*)}}$$

$$\frac{\|x_{i+1} - x^*\|_2}{\|x^*\|_2} \leq \rho \frac{\|x_i - x^*\|_2}{\|x^*\|_2} + \frac{c_1(\delta_{2K}, \delta_{3K}, \epsilon)}{\text{SNR}} + c_2(\delta_{2K}, \delta_{3K}, \epsilon) + c_3(\delta_{2K}, \delta_{3K}, \epsilon) \sqrt{\frac{1}{\text{SNR}}}$$

$$\rho = \frac{\delta_{3K} + \delta_{2K} + \sqrt{\epsilon}(1 + \delta_{2K})}{\sqrt{1 - \delta_{2K}^2}} \sqrt{\frac{1 + \left((1 - \epsilon) + 2\sqrt{1 - \epsilon} \right) \delta_{3K}^2 + 2\delta_{3K} \sqrt{\epsilon} + \epsilon}{1 - \delta_{3K}^2}}$$

$$c_2(\delta_{2K}, \delta_{3K}, \epsilon) = O(\delta_{3K} \sqrt{\epsilon} + \epsilon)$$

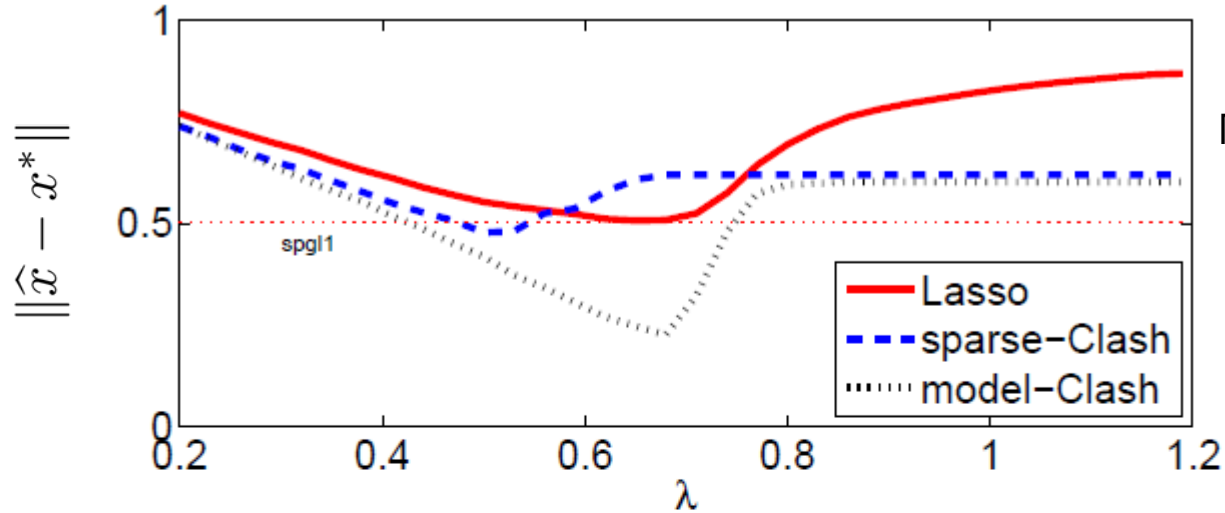
– precise formulae are in the paper

<http://lions.epfl.ch/CLASH>

- Isometry requirement (PMAP-0) $\Leftrightarrow \delta_{3K} < 0.3658$

Examples

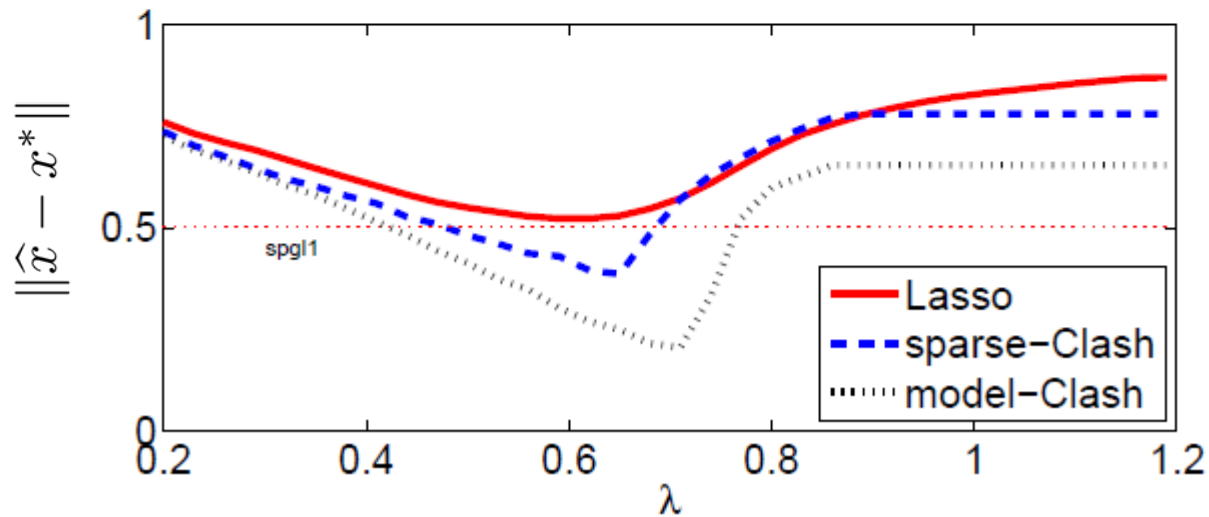
sparse matrix



Model: (K,C)-clustered model

$O(KCN)$ – per iteration

~10-15 iterations

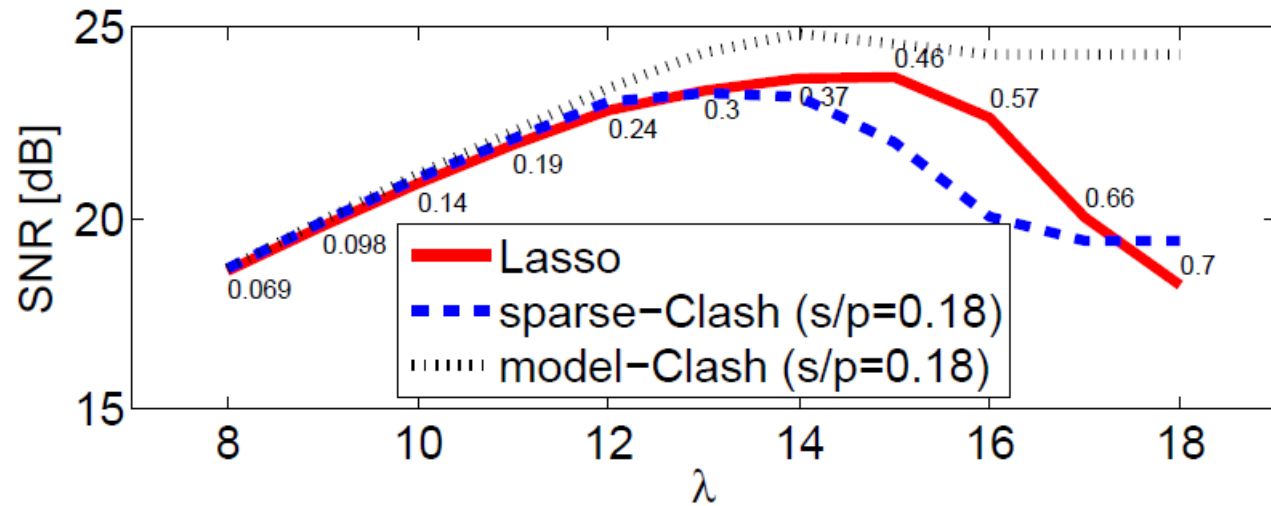
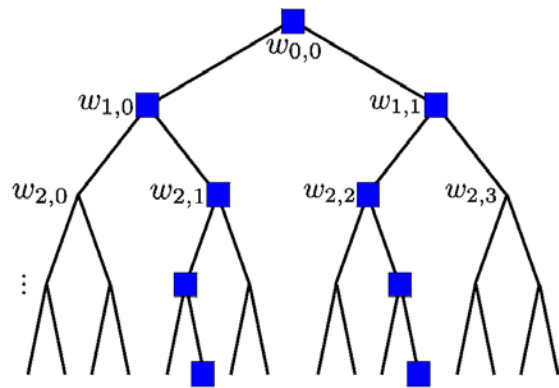


Model: partition model / TU

LP – per iteration



~20-25 iterations

Examples



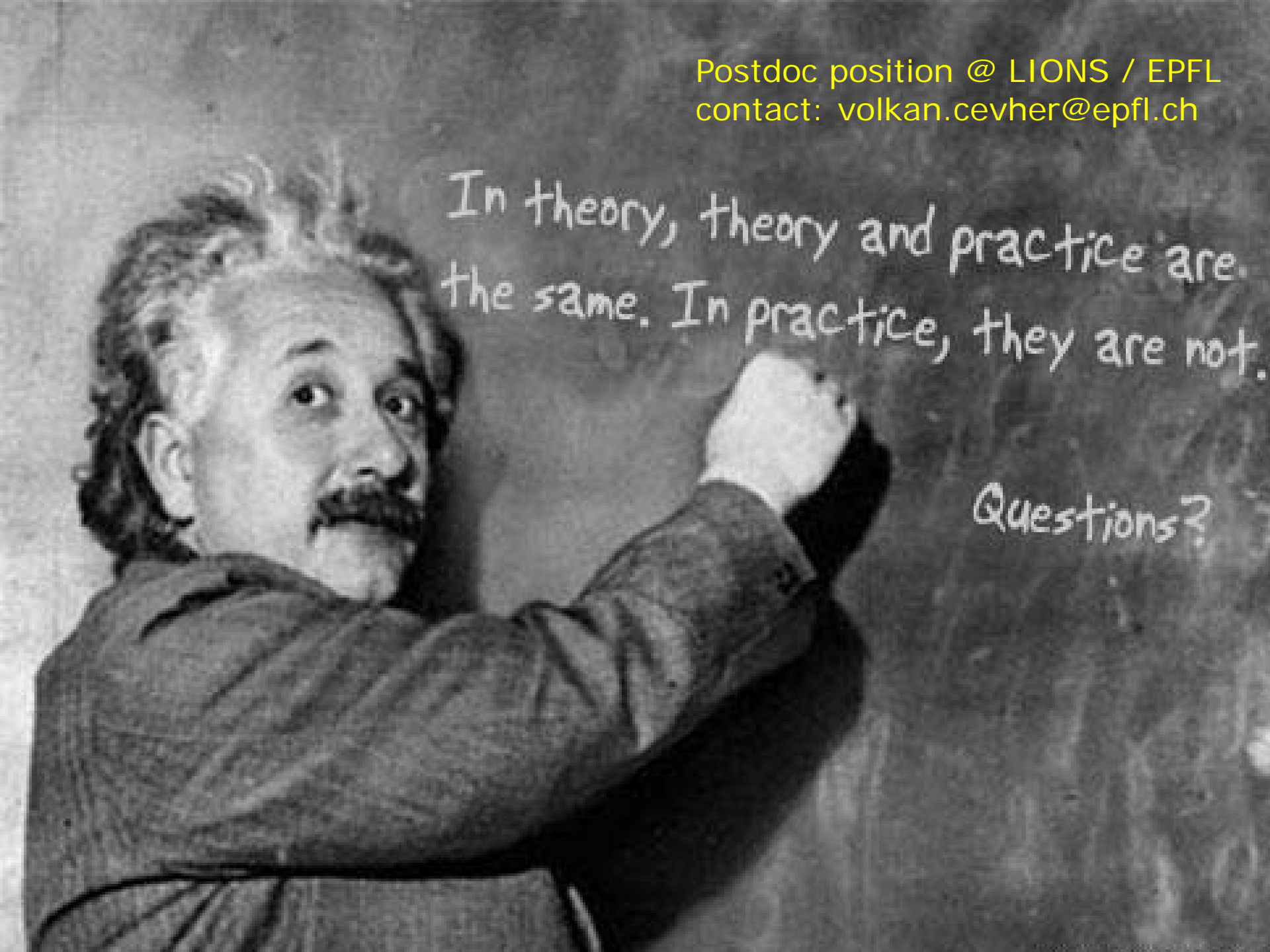
CCD array readout via noiselets

Conclusions

- CLASH  $\langle \rangle$ combinatorial selection
+
convex geometry
 $\lambda \rightarrow \infty \Rightarrow$ model-CS
- PMAP-epsilon  $\langle \rangle$ inherent difficulty in
combinatorial selection
 - beyond simple selection towards provable solution quality
+
runtime/space bounds
 - algorithmic definition of sparsity + many models
matroids, TU, knapsack,...
- Postdoc position @ LIONS / EPFL
contact: volkan.cevher@epfl.ch



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In theory, theory and practice are
the same. In practice, they are not.

Questions?